CHINESE MATHEMATICS COMPETITIONS AND OLYMPIADS 1993-2001

BOOK 2

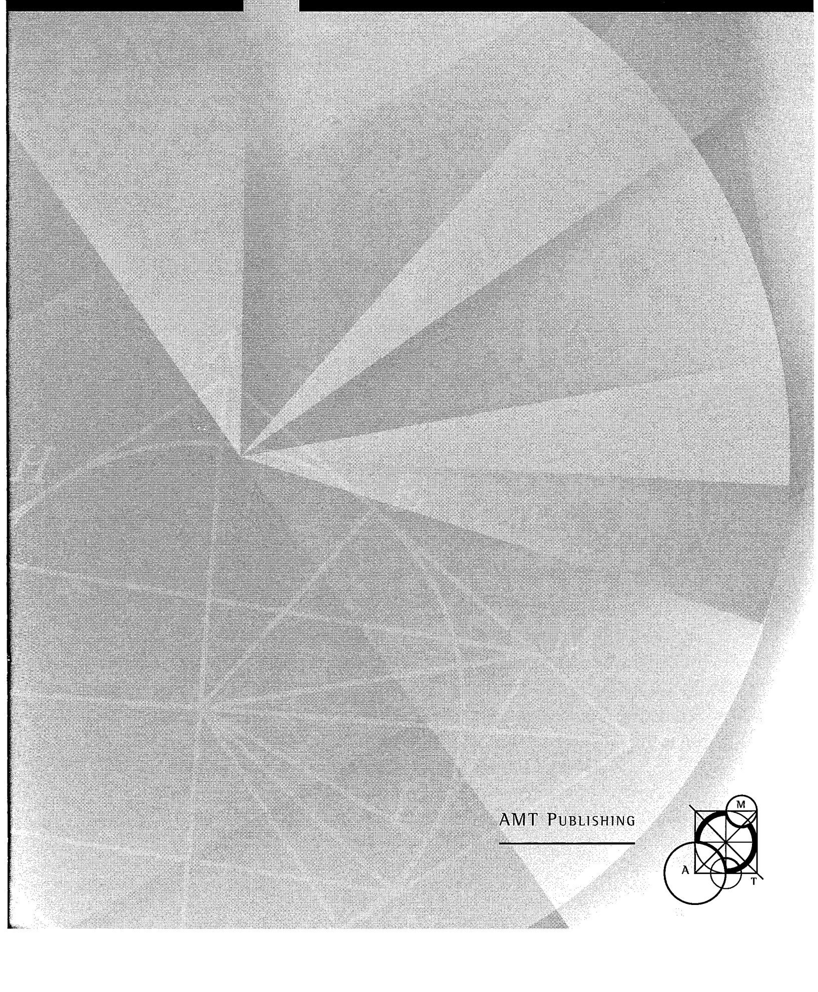
A LIU

AMT PUBLISHING

CHINESE MATHEMATICS COMPETITIONS AND OLYMPIADS 1993-2001

BOOK 2

A LIU



AMT PUBLISHING

Published by

AMT Publishing

Australian Mathematics Trust
University of Canberra ACT 2601
AUSTRALIA

Telephone: +61 2 6201 5137
www.amt.edu.au

AMTT Limited ACN 083 950 341

National Library of Australia Card Number and ISSN
Australian Mathematics Trust Enrichment Series ISSN 1326-0170
Chinese Mathematics Competitions and Olympiads 1993-2001 Book 2
ISBN 1 876420 16 2

The Australian Mathematics Trust

Enrichment Series

The books in this series are selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solutions. Typically, the problems have occurred in either national or international contests at the secondary school level.

They are intended to be sufficiently detailed at an elementary level for the mathematically inclined or interested to understand but, at the same time, be interesting and sometimes challenging to the undergraduate and the more advanced mathematician. It is believed that these mathematics competition problems are a positive influence on the learning and enrichment of mathematics.

THE AUSTRALIAN MATHEMATICS TRUST

ENRICHMENT SERIES

BOOKS IN THE SERIES

1 AUSTRALIAN MATHEMATICS COMPETITION BOOK 1 1978–1984
JD Edwards, DJ King & PJ O'Halloran

2 MATHEMATICAL TOOLCHEST
AW Plank & NH Williams

3 TOURNAMENT OF TOWNS QUESTIONS AND SOLUTIONS 1984–1989
PJ Taylor

4 AUSTRALIAN MATHEMATICS COMPETITION BOOK 2 1985–1991
PJ O'Halloran, G Pollard & PJ Taylor

5 PROBLEM SOLVING VIA THE AMC
W Atkins

6 TOURNAMENT OF TOWNS QUESTIONS AND SOLUTIONS 1980–1984
PJ Taylor

7 TOURNAMENT OF TOWNS QUESTIONS AND SOLUTIONS 1989–1993
PJ Taylor

8 ASIAN PACIFIC MATHEMATICS OLYMPIADS 1989-2000
H Lausch & C Bosch Giral

9 METHODS OF PROBLEM SOLVING BOOK 1
JB Tabov & PJ Taylor

10 CHALLENGE! 1991-1995
JB Henry, J Dowsey, AR Edwards, LJ Mottershead, A Nakos & G Vardaro

11 USSR MATHEMATICAL OLYMPIADS 1989-1992
AM Slinko

12 AUSTRALIAN MATHEMATICAL OLYMPIADS 1979-1995
H Lausch & PJ Taylor

13 CHINESE MATHEMATICS COMPETITIONS AND OLYMPIADS 1981-1993
A Liu

14 POLISH & AUSTRIAN MATHEMATICAL OLYMPIADS 1981– 1995
ME Kuczma & E Windischbacher

15 TOURNAMENT OF TOWNS QUESTIONS AND SOLUTIONS 1993–1997
PJ Taylor & AM Storozhev

16 AUSTRALIAN MATHEMATICS COMPETITION BOOK 3 1992–1998
WJ Atkins, JE Munro & PJ Taylor

17 SEEKING SOLUTIONS
JC Burns

18 101 PROBLEMS IN ALGEBRA
T Andreescu & Z Feng

19 METHODS OF PROBLEM SOLVING BOOK 2
JB Tabov & PJ Taylor

20 HUNGARY-ISRAEL MATHEMATICS COMPETITION: THE FIRST TWELVE YEARS
S Gueron

21 BULGARIAN MATHEMATICS COMPETITION 1992-2001
BJ Lazarov, JB Tabov, PJ Taylor & A Storozhev

22 CHINESE MATHEMATICS COMPETITIONS AND OLYMPIADS 1993-2001 BOOK 2
A Liu

CONTENTS

FOREWORD

It has been a real pleasure for the Australian Mathematics Trust to be associated with the publication of this second book on the national Chinese Olympiads, adding further to the amount of problems material available through our enrichment series.

It is the policy of the Trust, through this series, to publish high quality problems from national and international competitions, and clearly this material from China, one of the consistently strong countries through its few years in the International Mathematical Olympiad, will add a richness to the English speaking literature which did not previously exist.

The book was made particularly possible after an agreement between the Chiu Chang Publishers and us, which will also give Chiu Chang the right to publish our International Mathematics Tournament of Towns books in Chinese. This book is based on the personal translations from Chinese by Andy Liu.

It has also been a particular pleasure for me to work again with Andy Liu. On a personal basis, he had collaborated with me closely in the publication of our Tournament of Towns books and continues to do so with my colleague Andrei Storozhev.

In conclusion, publication of this second book has been a very happy experience.

Peter Taylor
Canberra
12 May 2004

PREFACE

This is a continuation of the earlier volume titled *Chinese Mathematics Competitions and Olympiads 1981 – 1993*. The source material comes from the *Journal of High School Mathematics*, published by the Tianjin Normal University.

As in the earlier volume, I wish to acknowledge the direct and indirect contributions from CHEUNG Pak-Hong, SUN Wen-Hsien, QIU Zong-Hu, Peter TAYLOR and Murray KLAMKIN.

Andy Liu
Edmonton
2004

Notes to the Reader

All problems in Papers II and the Olympiad Papers require full solutions. From 1994/95 to 1996/97, Paper I did not have Section 3 (Questions requiring Full Solutions).

Here are a few notations and terminology that may be somewhat unfamiliar. The traditional greatest integer function $[x]$, denoting the greatest integer less than or equal to the real number x, is modernized into the *floor* function $\lfloor x \rfloor$ which has the same meaning. We use the notation $[P]$ to denote the area of a polygon P. The symbols Z, R and C denote respectively the sets of integers, real numbers and complex numbers. By a lattice point is meant a point all coordinates of which are integers.

THE PROBLEMS

THE PROBLEMS

1993/94

Paper I.

Section 1. Questions with Multiple Choices.

1. What is the number of pairs (x, y) of real numbers satisfying

$$|\tan \pi y| + \sin^2 \pi x = 0 \quad \text{and} \quad x^2 + y^2 \leq 2?$$

(a) 4 (b) 5 (c) 8 (d) 9

2. Let a and b be real numbers and let

$$f(x) = a \sin x + b\sqrt[3]{x} + 4.$$

If

$$f(\log \log_3 10) = 5,$$

what is the value of $f(\log \log 3)$?

(a) -5 (b) -3 (c) 3 (d) dependent on a and b

3. If $A \neq B$, then $(A, B) \neq (B, A)$. What is the number of pairs (A, B) of sets such that

$$A \cup B = \{a_1, a_2, a_3\}?$$

(a) 8 (b) 9 (c) 26 (d) 27

4. Let a be a variable parameter. What is the minimum length of the chord of the curve

$$(x - \arcsin a)(x - \arccos a) + (y - \arcsin a)(y + \arccos a) = 0$$

along the line $x = \frac{\pi}{4}$?

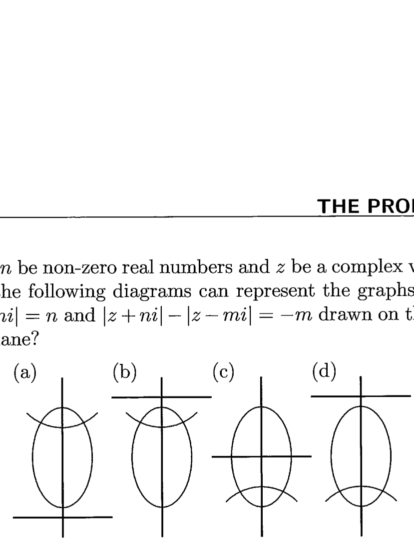
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

5. The length of the altitude on the side AC of triangle ABC is equal to $AB - BC$. What is the value of

$$\sin \frac{C - A}{2} + \cos \frac{C + A}{2} \quad ?$$

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) -1

6. Let m and n be non-zero real numbers and z be a complex variable. Which of the following diagrams can represent the graphs of $|z+ni|+|z-mi|=n$ and $|z+ni|-|z-mi|=-m$ drawn on the same complex plane?



Section 2. Questions requiring Answers Only.

1. If
$$(1-i)x^2+(\lambda+i)x+(1+i\lambda)=0$$
has two imaginary roots, what is the range of the real number λ?

2. Let x and y be real numbers such that $4x^2-5xy+4y^2=5$. What is the sum of the reciprocals of the maximum and the minimum values of x^2+y^2?

3. For which complex number z do we have $\arg(z^2-4)=\frac{5\pi}{6}$ and $\arg(z^2+4)=\frac{\pi}{3}$?

4. What are the last two digits of the greatest integer less than
$$\frac{10^{93}}{10^{31}+3}?$$

5. Let $x_0>x_1>x_2>x_3$ be any positive real numbers. What is the largest value of the real number k such that
$$\log_{\frac{x_0}{x_1}}1993+\log_{\frac{x_1}{x_2}}1993+\log_{\frac{x_2}{x_3}}1993\geq k\log_{\frac{x_0}{x_3}}1993?$$

6. The three-digit numbers from 100 to 999 are printed on 900 cards, each on a different card. When inverted, the digits 0, 1 and 8 remain unchanged, the digits 6 and 9 turn into each other, while the remaining digits are unintelligible. How many pairs of cards are identical up to inversion?

Section 3. Questions requiring Full Solutions.

1. In the tetrahedron $SABC$, SA, SB and SC are perpendicular to one another. M is the centroid of triangle ABC and D is the midpoint of AB. Let DD' be a line parallel to SC.

(a) Prove that DD' intersects SM at some point O.

(b) Prove that O is the circumcentre of $SABC$.

2. Let a and b be real numbers such that $0<a<b$. A variable line ℓ passes through the fixed point $(a,0)$ and a variable line m passes through the fixed point $(b,0)$, such that they intersect the parabola $y^2=x$ again at four distinct concyclic points. Determine the locus of the point of intersection of ℓ and m.

3. The sequence $\{a_n\}$ of real numbers is defined by $a_0=a_1=1$ and for $n\geq 2$,
$$\sqrt{a_na_{n-2}}-\sqrt{a_{n-1}a_{n-2}}=2a_{n-1}.$$
Find a formula for a_n independent of $a_0,\ a_1,\ \ldots,\ a_{n-1}$.

Paper II

1. Only one interior angle of the convex quadrilateral $ABCD$ is obtuse. $ABCD$ is to be partitioned into n obtuse triangles whose vertices other than A, B, C and D are inside $ABCD$. Prove that this is possible if and only if $n\geq 4$.

2. None of the subsets $A_1,\ A_2,\ \ldots,\ A_m$ of an n-element set contains another. Prove that

(a) $\displaystyle\sum_{i=1}^{m}\frac{1}{\binom{n}{|A_i|}}\leq 1$;

(b) $\displaystyle\sum_{i=1}^{m}\binom{n}{|A_i|}\geq m^2$.

3. A line m passes through the centre of a circle. A, B and C are three points outside the circle. They lie on a line ℓ perpendicular to m, on the same side of m with A farthest from m and C nearest to it. AP, BQ and CR are tangents to the circle.

(a) Prove that $AB\cdot CR+BC\cdot AP=CA\cdot BQ$ if ℓ is tangent to the circle.

(b) Prove that $AB\cdot CR+BC\cdot AP<CA\cdot BQ$ if ℓ intersects the circle in two points.

(c) Prove that $AB\cdot CR+BC\cdot AP>CA\cdot BQ$ if ℓ is disjoint from the circle.

Olympiad Paper I

1. Let $ABCD$ be a quadrilateral with AB parallel to DC. Let E be a point on AB and F a point on CD. The segments AF and DE intersect at G, while the segments BF and CE intersect at H.

(a) Prove that the area of $EGFH$ is at most one-quarter that of $ABCD$.

(b) Is this conclusion still valid if $ABCD$ is an arbitrary convex quadrilateral?

2. There are at least 4 smarties randomly distributed among at least 4 boxes. In each move, remove 1 smarty from each of 2 boxes and put both of them into a third box. Is it always possible to have all the smarties in 1 box?

3. Determine all functions $f : [1, \infty) \to [1, \infty)$ such that for all $x \geq 1$,

$$f(x) \leq 2(x + 1)$$

and

$$f(x+1) = \frac{1}{x}((f(x))^2 - 1).$$

Olympiad Paper II

1. The coefficients of the n-th degree polynomial

$$f(z) = c_0 z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_{n-1} z + c_n$$

are complex numbers. Prove that there exists a complex number z_0 such that $|z_0| \leq 1$ while $|f(z_0)| \geq |c_0| + |c_n|$.

2. Prove that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} = \binom{2n+1}{n}$$

for any positive integer n.

3. Let p be a prime. Determine the number of right triangles such that the incentre is (0,0), the vertex of the right angle is $(1994p, 7 \cdot 1994p)$, and the other two vertices have integer coordinates.

1994/95

Paper I.

Section 1. Questions with Multiple Choices.

1. Let a, b and c be real numbers. What is a necessary and sufficient condition for $a \sin x + b \cos x + c > 0$ where x is any real number?

(a) $a = b = 0, c > 0$ (b) $\sqrt{a^2 + b^2} = c$ (c) $\sqrt{a^2 + b^2} < c$
(d) $\sqrt{a^2 + b^2} > 0$

2. Let a, b and c be complex numbers. Consider the following two statements.

(P) If $a^2 + b^2 > c^2$, then $a^2 + b^2 - c^2 > 0$.
(Q) If $a^2 + b^2 - c^2 > 0$, then $a^2 + b^2 > c^2$.

Which of them is or are correct?

(a) Both are correct. (b) Only P is correct. (c) Neither is correct.
(d) Only Q is correct.

3. The sequence $\{a_n\}$ is defined by $a_1 = 9$ and $3a_{n+1} + a_n = 4$ for $n \geq 1$. Let S_n denote the sum of the first n terms. What is the smallest positive integer n such that

$$|S_n - n - 6| < \frac{1}{125} \quad ?$$

(a) 5 (b) 6 (c) 7 (d) 8

4. Let

$$\begin{aligned} x &= (\sin\alpha)^{\log_b \sin\alpha} \\ y &= (\cos\alpha)^{\log_b \cos\alpha} \\ \text{and} \quad z &= (\sin\alpha)^{\log_b \cos\alpha} \end{aligned}$$

where $0 < b < 1$ and $0 < \alpha < \frac{\pi}{4}$. What are the relative sizes of x, y and z?

(a) $x < z < y$ (b) $y < z < x$ (c) $z < x < y$ (d) $x < y < z$

5. What is the range of the dihedral angle between adjacent lateral faces of a right pyramid whose base is a regular n-gon?

(a) $\left(\frac{(n-2)\pi}{n}, \pi\right)$ (b) $\left(\frac{(n-1)\pi}{n}, \pi\right)$ (c) $\left(0, \frac{\pi}{2}\right)$
(d) $\left(\frac{(n-2)\pi}{n}, \frac{(n-1)\pi}{n}\right)$

6. Let a and b be distinct positive numbers. What is the graph of

$$\frac{|x+y|}{2a}+\frac{|x-y|}{2b}=1 \quad ?$$

(a) triangle (b) square (c) non-square rectangle

(d) non-square rhombus

Section 2. Questions requiring Answers Only.

1. Let P be the point $(-1,1)$ and Q be the point (2,2). What is the range of m if the line $x+my+m=0$ intersects the extension of PQ?

2. Let $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ and $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$. Let a be a real number such that

$$x^3+\sin x-2a=0$$

and

$$4y^3+\sin y\cos y+a=0.$$

What is the value of $\cos(x+2y)$ in terms of a?

3. Let A be the set

$$\{(x,y)|(x-3)^2+(y-4)^2 \le \left(\frac{5}{2}\right)^2\}$$

and B be the set

$$\{(x,y)|(x-4)^2+(y-5)^2 > \left(\frac{5}{2}\right)^2\}.$$

How many lattice points does $A \cap B$ contain?

4. Let $0<\theta<\pi$. What is the maximum value of $\sin\frac{\theta}{2}(1+\cos\theta)$?

5. The angle between a plane and each of the 12 edges of a cube is α. What is the value of $\sin\alpha$?

6. Each of the numbers $a_1, a_2, \ldots, a_{95}$ is ± 1. What is the smallest positive value of the sum of $a_i a_j$, $1 \le i < j \le 95$?

Paper II

1. Let z_1 and z_2 be complex numbers such that

$$z_1^2-4z_2=16+20i.$$

Suppose the roots α and β of

$$x^2+z_1x+z_2+m=0$$

for some complex number m satisfy

$$|\alpha-\beta|=2\sqrt{7}.$$

(a) Determine the maximum value of $|m|$.

(b) Determine the minimum value of $|m|$.

2. If all positive integers relatively prime to 105 are arranged in ascending order, determine the 1000-th term.

3. Let O and I be the circumcentre and the incentre of triangle ABC respectively. If $\angle B=60°$, $\angle A<\angle C$ and the exterior bisector of $\angle A$ meets the circumcircle again at E, prove that

(a) $IO=AE$;

(b) $2R<IO+IA+IC<(1+\sqrt{3})R$, where R is the circumradius of ABC.

4. On the plane are 1994 points, no three collinear. They are to be partitioned into 83 sets, each with at least three points. Three points in the same set form a triangle.

(a) What is the maximum number of triangles?

(b) In a partition which yields the maximum number of triangles, prove that each segment joining two points in the same set can be painted in one of four colours such that no triangles have three sides with the same colour.

Olympiad Paper I

1. Let $n \geq 3$ be an integer and let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n,$$

$0 < a_1 = a_2$, $0 < b_1 \leq b_2$ and for $1 \leq i \leq n-2$,

$$a_i + a_{i+1} = a_{i+2}$$

and

$$b_i + b_{i+1} < b_{i+2}.$$

Prove that

$$a_{n-1} + a_n \leq b_{n-1} + b_n.$$

2. Let f be a function from the set of positive integers to itself such that $f(1) = 1$ and, for each positive integer n, $f(2n) < 6f(n)$ and

$$3f(n)f(2n+1) = f(2n)(1 + 3f(n)).$$

Determine all pairs (k, ℓ) such that

$$f(k) + f(\ell) = 293$$

and $k < \ell$.

3. Determine the minimum value of

$$\sum_{i=1}^{10}\sum_{j=1}^{10}\sum_{k=1}^{10} |k(x + y - 10i)(3x - 6y - 36j)(19x + 95y - 95k)|$$

where x and y range over all real numbers.

Olympiad Paper II

1. The radii of four spheres are 2, 2, 3 and 3 respectively. Each is externally tangent to the three others. If a smaller sphere is tangent to each of these 4 spheres, determine the radius of the smaller sphere.

2. Let a_1, a_2, $\ldots$, a_{10} be ten distinct positive integers whose sum is 1995. Determine the minimum value of

$$a_1a_2 + a_2a_3 + \cdots + a_9a_{10} + a_{10}a_1.$$

3. Let $n > 1$ be an odd integer. Suppose

$$X_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}) = (1, 0, 0, \ldots, 0, 1).$$

For $1 \leq k \leq n$, let

$$x_i^{(k)} = \begin{cases} 0 & \text{if } x_i^{(k-1)} = x_{i+1}^{(k-1)}; \\ 1 & \text{if } x_i^{(k-1)} \neq x_{i+1}^{(k-1)}. \end{cases}$$

We take $x_{n+1}^{(k-1)} = x_1^{(k-1)}$. Let

$$X_k = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}).$$

If the positive integer m satisfies $X_m = X_0$, prove that m is a multiple of n.

1995/96

Paper I.

Section 1. Questions with Multiple Choices.

1. In the arithmetic progression $\{a_n\}$, $a_1 > 0$ and $3a_8 = 5a_{13}$. Let S_n be the sum of the first n terms. For which value of n is S_n maximum?

(a) 10 (b) 11 (c) 20 (d) 21

2. The complex numbers $z_1, z_2, \ldots, z_{20}$ represent the vertices of a regular 20-gon inscribed in the unit circle of the complex plane. What is the number of distinct points represented by the complex numbers $z_1^{1995}, z_2^{1995}, \ldots, z_{20}^{1995}$?

(a) 4 (b) 5 (c) 10 (d) 20

3. A is said to be no weaker than B if A is either taller or heavier than B. Among 100 people, if someone is no weaker than any of the others, then this person is said to be strong. Of the 100 people, at most how many can be strong?

(a) 1 (b) 2 (c) 50 (d) 100

4. Let n be a positive integer. If $|x - 2n| = k\sqrt{x}$ has two unequal real roots in the interval $(2n-1, 2n+1]$, what is the range of k?

(a) $k > 0$ (b) $0 < k \le \dfrac{1}{\sqrt{2n+1}}$ (c) $\dfrac{1}{2n+1} < k \le \dfrac{1}{\sqrt{2n+1}}$

(d) none of these

5. If

$$\begin{aligned} w &= \log_{\sin 1} \cos 1, \\ x &= \log_{\sin 1} \tan 1, \\ y &= \log_{\cos 1} \sin 1 \\ \text{and} \quad z &= \log_{\cos 1} \tan 1, \end{aligned}$$

what are the relative sizes of w, x, y and z?

(a) $w < y < x < z$ (b) $y < z < w < x$ (c) $x < z < y < w$

(d) $z < x < w < y$

6. Let ABC be an equilateral triangle with centre O. P is a point such that OP is perpendicular to the plane ABC. A variable plane through O intersects the rays PA, PB and PC at Q, R and S respectively. What can be said about

$$\frac{1}{PQ} + \frac{1}{PR} + \frac{1}{PS}?$$

(a) It has a maximum but no minimum.

(b) It has a minimum but no maximum.

(c) It has both maximum and minimum which are distinct.

(d) It is constant.

Section 2. Questions requiring Answers Only.

1. Let α and β be conjugate complex numbers such that $\frac{\alpha}{\beta^2}$ is a real number and $|\alpha - \beta| = 2\sqrt{3}$. What is the value of $|\alpha|$?

2. What is the ratio of the volume of a sphere to the maximum volume of a cone inscribed in the sphere?

3. What is the number of real roots of

$$(\log x)^2 - \lfloor \log x \rfloor - 2 = 0?$$

4. How many lattice points are inside the region which is defined by $y \le 3x$, $y \ge \frac{x}{3}$ and $x + y \le 100$?

5. In how many ways can the vertices of a square pyramid be painted in 5 colours if adjacent vertices must have different colours?

6. A is a subset of $\{1, 2, \ldots, 1995\}$ such that whenever x is in A, then $15x$ is not. What is the maximum number of elements in A?

Paper II

1. The equation

$$2(2\sin\theta - \cos\theta + 3)x^2 - (8\sin\theta + \cos\theta + 1)y = 0$$

represents a family of parabolas with parameter θ. Determine the maximum length of the chord along the line $y = 2x$ of a parabola in the family.

2. Determine all real numbers p such that the three roots of

$$5x^3 - 5(p+1)x^2 + (71p-1)x - (66p-1) = 0$$

are positive integers.

3. A circle is tangent to the sides AB, BC, CD and DA of a rhombus $ABCD$ at E, F, G and H respectively. M, N, P and Q are points on AB, BC, CD and DA respectively such that MN is tangent to the arc EF and PQ is tangent to the arc GH. Prove that MQ is parallel to NP.

4. Each point of the plane is painted in one of two colours. Prove that there exist two similar triangles such that the lengths of the sides of one are 1995 times the lengths of the corresponding sides of the other, and all three vertices of each triangle have the same colour.

Olympiad Paper I

1. Let H be the orthocentre of an acute triangle ABC. From A draw two tangent lines AP and AQ to the circle whose diameter is BC, the points of tangency being P and Q respectively. Prove that P, H and Q are collinear.

2. Let $S = \{1, 2, \ldots, 50\}$. Determine the smallest positive integer k such that for any k-element subset of S, there are two different elements a and b for which $a+b$ divides ab.

3. A function f from the set of real numbers to itself satisfies

$$f(x^3+y^3) = (x+y)((f(x))^2 - f(x)f(y) + (f(y))^2),$$

where x and y are arbitrary real numbers. Prove that for any real number x,

$$f(1996x) = 1996f(x).$$

Olympiad Paper II

1. Eight singers take part in a festival. The organizer wants to plan a number of concerts with four singers performing in each. The number of concerts in which a pair of singers performs together is the same for every pair. Determine the minimum number of concerts.

2. Let n be a positive integer. For $1 \le i \le n$, let x_i be a positive real number, where $x_1 + x_2 + \cdots + x_n = 1$. Take $x_0 = 0$. Prove that

$$1 \le \sum_{i=1}^{n} \frac{x_i}{\sqrt{1+x_0+x_1+\cdots+x_{i-1}}\sqrt{x_i+x_{i+1}+\cdots+x_n}} < \frac{\pi}{2}.$$

3. In triangle ABC, $\angle C = 90^\circ$, $\angle A = 30^\circ$ and $BC = 1$. Determine the minimum length of the longest sides of all triangles whose vertices lie respectively on the three sides of triangle ABC.

1996/97
Paper I.

Section 1. Questions with Multiple Choices.

1. What is the convex hull of the points of intersection of the circle

$$x^2+(y-1)^2=1$$

and the ellipse

$$9x^2+(y+1)^2=9?$$

(a) segment (b) non-equilateral triangle (c) equilateral triangle

(d) quadrilateral

2. The geometric progression $\{a_n\}$ has first term $a_1=1536$ and common ratio $-\frac{1}{2}$. What is the value of n for which the product of the first n terms is maximum?

(a) 9 (b) 11 (c) 12 (d) 13

3. For how many prime numbers p does there exist a positive integer n such that $\sqrt{p+n}+\sqrt{n}$ is an integer?

(a) 0 (b) 1 (c) infinite (d) greater than 1 but finite

4. Let

$$\begin{aligned} a_1 &= \cos(\sin \pi x), \\ a_2 &= \sin(\cos \pi x), \\ a_3 &= \cos(\pi(x+1)), \end{aligned}$$

where $-\frac{1}{2}<x<0$. What are the relative sizes of a_1, a_2 and a_3?

(a) $a_3<a_2<a_1$ (b) $a_1<a_3<a_2$ (c) $a_3<a_1<a_2$

(d) $a_2<a_3<a_1$

5. On the interval [1,2], the functions $f(x)=x^2+px+q$ and $g(x)=x+\frac{1}{x^2}$ take the same minimum value at the same point. What is the maximum value of $f(x)$ on this interval?

(a) $4+\frac{1}{2}\sqrt[3]{2}+\sqrt[3]{4}$ (b) $4-\frac{5}{2}\sqrt[3]{3}+\sqrt[3]{4}$ (c) $1-\frac{1}{2}\sqrt[3]{2}+\sqrt[3]{4}$

(d) none of these

6. A hollow inverted right circular cone has height $6+2\sqrt{2}$. A sphere of radius 2 is resting at the bottom inside the cone. A sphere of radius 3 is tangent to the first sphere, and to the lateral and top faces of the cone. How many more spheres of radius 3 can fit inside the cone?

(a) 1 (b) 2 (c) 3 (d) 4

Section 2. Questions requiring Answers Only.

1. What is the number of non-empty subsets of

$$\{x \mid -1\le \log_{\frac{1}{x}} 10 < -\frac{1}{2}, x \text{ integer}\}?$$

2. The points represented by the complex numbers z_1 and z_2 lie on a circle in the complex plane, with centre represented by i and radius 1. The real part of $\overline{z_1}z_2$ is 0 and $\arg z_1=\frac{\pi}{6}$. What is the value of z_2?

3. The polar curve $r=1+\cos\theta$ is rotated once around the point with polar coordinates [2,0]. What is the area of the region it sweeps over?

4. A pyramid has an equilateral triangle as base and three lateral edges of equal length. Two congruent copies are glued together along their common base to produce a hexahedron in which every dihedral angle between two adjacent faces is the same. If the shortest edge of the hexahedron is 2, what is the greatest distance between two of its vertices?

5. Each face of a cube is to be painted with one of 6 colours such that every two adjacent faces have different colours. In how many ways can these faces be painted?

6. How many lattice points lie on the circle with centre (199,0) and radius 199?

Paper II

1. The sum of the first n terms of the sequence $\{a_k\}$ is $2a_n-1$ for all $n\ge 1$. The sequence $\{b_k\}$ is defined by $b_1=3$ and for $k\ge 1$, $b_{k+1}=a_k+b_k$. Determine the sum of the first n terms of $\{b_n\}$.

2. Determine the range of values of the real number a such that

$$(x+3+2\sin\theta\cos\theta)^2+(x+a\sin\theta+a\cos\theta)^2\ge\frac{1}{8}$$

for any real numbers x and θ where $0\le\theta\le\frac{\pi}{2}$.

3. The excircle of triangle ABC opposite C is tangent to the line BC at E and the line CA at G. The excircle of ABC opposite B is tangent to the line BC at F and the line AB at H. If the lines EG and FH intersect at P, prove that AP is perpendicular to BC.

4. There are $n \geq 6$ people at a party. Each is a mutual acquaintance of at least $\lfloor \frac{n}{2} \rfloor$ others. Among any $\lfloor \frac{n}{2} \rfloor$ of them, either two of them are mutual acquaintances, or two of the remaining $n - \lfloor \frac{n}{2} \rfloor$ are mutual acquaintances. Prove that there are three people at the party such that each pair among them are mutual acquaintances.

Olympiad Paper I

1. Let $x_1, x_2, \ldots, x_{1997}$ be real numbers such that $-\frac{1}{\sqrt{3}} \leq x_i \leq \sqrt{3}$ for $1 \leq i \leq 1997$ and

$$x_1 + x_2 + \cdots + x_{1997} = -318\sqrt{3}.$$

Determine the maximum value of

$$x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}.$$

2. $A_1B_1C_1D_1$ is any convex quadrilateral. P is a point inside such that any line joining P to a vertex forms an acute angle with each of the two sides meeting at that vertex. Suppose A_{k-1}, B_{k-1}, C_{k-1} and D_{k-1} have been defined. Let A_k, B_k, C_k and D_k be the respective reflections of P across $A_{k-1}B_{k-1}$, $B_{k-1}C_{k-1}$, $C_{k-1}D_{k-1}$ and $D_{k-1}A_{k-1}$.

(a) Which of $A_iB_iC_iD_i$, $1 \leq i \leq 12$, is necessarily similar to $A_{1997}B_{1997}C_{1997}D_{1997}$?

(b) Which of $A_iB_iC_iD_i$, $1 \leq i \leq 12$, is necessarily cyclic if $A_{1997}B_{1997}C_{1997}D_{1997}$ is?

3. Prove that there exist infintely many positive integers n for which the integers $1, 2, \ldots, 3n$ can be arranged in a $3 \times n$ array such that all rows have the same sum, all columns have the same sum, and both sums are divisible by 6.

Olympiad Paper II

1. $ABCD$ is a quadrilateral inscribed in a circle. The extensions of AB and DC meet at P, and the extensions of AD and BC meet at Q. The tangents from Q to the circle touch it at E and F. Prove that P, E and F are collinear.

2. Let $A = \{0, 1, \ldots, 16\}$. For any mapping $f : A \to A$, define

$$f^{(1)}(x) = f(x)$$

and for any $n \geq 1$,

$$f^{(n+1)}(x) = f(f^{(n)}(x)).$$

Interpret $f^{(n)}(17)$ as $f^{(n)}(0)$. Suppose that for a bijection $f : A \to A$, there exists a positive integer M such that

$$f^{(M)}(i+1) - f^{(M)}(i) \equiv \pm 1 \pmod{17}$$

for $0 \leq i \leq 16$ and for $m < M$, we have

$$f^{(m)}(i+1) - f^{(m)}(i) \not\equiv \pm 1 \pmod{17}$$

for $0 \leq i \leq 16$. Determine the maximum value of M taken over all bijections $f : A \to A$ with the above properties.

3. Let $\{a_1, a_2, \ldots\}$ be a sequence of non-negative numbers such that $a_{n+m} \leq a_n + a_m$ for all n and m.

Prove that for all $n \geq m$,

$$a_n \leq ma_1 + \left(\frac{n}{m} - 1\right) a_m.$$

1997/98

Paper I.

Section 1. Questions with Multiple Choices.

1. The sequence $\{x_n\}$ is such that $x_1 = a,\ x_2 = b$ and for $n \geq 2$,

$$x_{n+1} = x_n - x_{n-1}.$$

Let S_{100} denote the sum of the first 100 terms. Which of the following statements is correct?

(a) $x_{100} = -a$ and $S_{100} = 2b - a$

(b) $x_{100} = -b$ and $S_{100} = 2b - a$

(c) $x_{100} = -b$ and $S_{100} = b - a$

(d) $x_{100} = -a$ and $S_{100} = b - a$

2. Let λ be a positive number. E is a point on the side AB, and F is a point on the side CD, of a regular tetrahedron $ABCD$ such that

$$\frac{AE}{EB} = \frac{CF}{FD} = \lambda.$$

Let $f(\lambda)$ denote the sum of the non-obtuse angle beween EF and AC and the angle between EF and BD. How does $f(\lambda)$ behave on $(0, \infty)$?

(a) increasing (b) decreasing (c) constant

(d) increasing on (0,1) and decreasing on $(1, \infty)$

3. How many finite arithmetic progressions of length at least 3 are there, with the first term and the common difference being positive integers, such that the sum of all the terms is 97^2?

(a) 2 (b) 3 (c) 4 (d) 5

4. What is the range of m if

$$m(x^2 + y^2 + 2y + 1) = (x - 2y + 3)^2$$

is an ellipse?

(a) (0,1) (b) $(1, \infty)$ (c) (0,5) (d) $(5, \infty)$

5. Let

$$\begin{aligned} f(x) &= x^2 - \pi x, \\ \alpha &= \arcsin\frac{1}{3}, \\ \beta &= \arctan\frac{5}{4}, \\ \gamma &= \arccos\left(-\frac{1}{3}\right) \\ \text{and}\quad \delta &= \operatorname{arccot}\left(-\frac{5}{4}\right). \end{aligned}$$

What are the relative sizes of $f(\alpha)$, $f(\beta)$, $f(\gamma)$ and $f(\delta)$?

(a) $f(\alpha) > f(\beta) > f(\delta) > f(\gamma)$ (b) $f(\alpha) > f(\delta) > f(\beta) > f(\gamma)$

(c) $f(\delta) > f(\alpha) > f(\beta) > f(\gamma)$ (d) $f(\delta) > f(\alpha) > f(\gamma) > f(\beta)$

6. Three mutually skew lines are given in space. How many lines can intersect all of them?

(a) 0 (b) 1 (c) infinite (d) greater than 1 but finite

Section 2. Questions requiring Answers Only.

1. Let x and y be real numbers such that

$$(x-1)^3 + 1997(x-1) = -1$$

and

$$(y-1)^3 + 1997(y-1) = 1.$$

What is the value of $x + y$?

2. A line through the right focus of the hyperbola

$$x^2 - \frac{y^2}{2} = 1$$

intersects the hyperbola at A and B. If the number of such lines for which $AB = \lambda$ is exactly 3, what is the value of λ?

3. Let z be a complex number such that

$$\left|2z + \frac{1}{z}\right| = 1.$$

What is the range of $\arg z$?

4. ABC is a right isosceles triangle with hypotenuse $AB = 2$. $SABC$ is a tetrahedron with circumcentre O. What is the distance from O to the plane ABC if $SA = SB = SC = 2$?

5. A frog starts at vertex A of a regular hexagon $ABCDEF$ and hops from vertex to adjacent vertex. It stops either when it reaches vertex D or when it has made five hops, whichever is sooner. What is the number of different sequences of hops?

6. Let x, y and z be positive numbers. Let M be the largest of

$$\log z + \log\left(\frac{x}{yz} + 1\right),$$
$$\log\frac{1}{x} + \log(xyz + 1)$$
$$\text{and} \quad \log y + \log\left(\frac{1}{xyz} + 1\right).$$

What is the minimum value of M?

Section 3. Questions requiring Full Solutions.

1. Let x, y and z be real numbers such that $x \geq y \geq z \geq \frac{\pi}{12}$ and

$$x + y + z = \frac{\pi}{2}.$$

(a) Determine the maximum value of $\cos x \sin y \cos z$.
(b) Determine the minimum value of $\cos x \sin y \cos z$.

2. (a) Prove that not all of the vertices of an equilateral triangle can lie on the same branch of the hyperbola $xy = 1$.
(b) If one vertex of an equilateral triangle is at $(-1, -1)$ and the other two lie on the branch of $xy = 1$ for which $x > 0$, determine the coordinates of the other two vertices.

3. Let a_1, a_2, a_3, a_4 and a_5 be non-zero complex numbers such that

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \frac{a_5}{a_4}$$

and

$$a_1 + a_2 + a_3 + a_4 + a_5 = 4\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5}\right) = S,$$

where S is a real number such that $|S| \leq 2$.

Prove that the points on the complex plane representing a_1, a_2, a_3, a_4 and a_5 are concyclic.

Paper II

1. Two circles with unequal radii intersect at M and N. They are inside a circle with centre O and tangent to it at S and T. Prove that OM is perpendicular to MN if and only if S, N and T are collinear.

2. What conditions must the real numbers $x_0, x_1, \ldots, x_n$ satisfy to guarantee the existence of real numbers $y_0, y_1, \ldots, y_n$ such that

$$(x_0 + iy_0)^2 = \sum_{k=1}^{n}(x_k + iy_k)^2?$$

3. Each entry of a 100×25 array is a non-negative real number such that the sum of the 25 numbers in each row is at most 1. The 100 numbers in each column are rearranged from top to bottom in descending order. Determine the smallest value of k such that the sum of the 25 numbers in each row from the k-th row on down will always be at most 1.

Olympiad Paper I

1. Let ABC be a non-obtuse triangle with circumcentre O and incentre I. If $AB > AC$, $\angle B = 45°$ and

$$\sqrt{2}OI = AB - AC,$$

determine $\sin A$.

2. Let n be an integer greater than 1. Do there always exist $2n$ distinct positive integers $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ such that

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$$

and

$$n - 1 > \sum_{i=1}^{n} \frac{a_i - b_i}{a_i + b_i} > n - 1 - \frac{1}{1998}?$$

3. Let $S = \{1, 2, \ldots, 98\}$. Determine the smallest positive integer n for which any subset of S of size n contains 10 elements such that no matter how they are divided into two subsets of size 5, one subset contains an element relatively prime to each of the other four, while the other subset contains an element not relatively prime to any of the other four.

Olympiad Paper II

1. Determine all integers $n > 3$ such that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$$

divides 2^{2000}.

2. Let D be a point inside an acute triangle ABC. Characterize geometrically the set of possible locations of the point D if it satisfies

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA.$$

3. Let $n \geq 2$ be an integer. Let $x_1, x_2, \ldots, x_n$ be real numbers such that

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n-1} x_i x_{i+1} = 1.$$

For any fixed k, $1 \leq k \leq n$, determine the maximum value of $|x_k|$.

1998/99

Paper I.

Section 1. Questions with Multiple Choices.

1. Let $a > 1$ and $b > 1$ be such that

$$\log(a+b) = \log a + \log b.$$

What can be said about the value of

$$\log(a-1) + \log(b-1)?$$

(a) equal to $\log 2$ (b) equal to 1 (c) equal to 0

(d) dependent on a and b

2. Let $A = \{x | 2a + 1 \leq x \leq 3a - 5\}$ and $B = \{x | 3 \leq x \leq 22\}$. What is the set of values of a for which $A \neq \emptyset$ and $A \subseteq A \cap B$?

(a) $\{a | 1 \leq a \leq 9\}$ (b) $\{a | 6 \leq a \leq 9\}$ (c) $\{a | a \leq 9\}$ (d) $\emptyset$

3. Let S_n be the sum of the first n terms of a geometric progression. If $S_{10} = 10$ and $S_{30} = 70$, what is the set of values of S_{40}?

(a) $\{400, -50\}$ (b) $\{150, -200\}$ (c) $\{-200\}$ (d) $\{150\}$

4. Consider the following two statements.

(P) The inequality

$$a_1 x^2 + b_1 x + c_1 > 0$$

has the same solution as the inequality

$$a_2 x^2 + b_2 x + c_2 > 0.$$

(Q) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

What is the relationship between P and Q?

(a) Q is a necessary and sufficient condition for P.

(b) Q is a sufficient condition for P, but not necessary.

(c) Q is a necessary condition for P, but not sufficient.

(d) Q is neither necessary nor sufficient for P.

5. Let E, F and G be the midpoints of the sides AB, BC and CD, respectively, of a regular tetrahedron $ABCD$. What is the dihedral angle between the planes EFG and CFG?

(a) $\arcsin\frac{\sqrt{6}}{3}$ (b) $\frac{\pi}{2}+\arccos\frac{\sqrt{3}}{3}$ (c) $\frac{\pi}{2}-\arctan\sqrt{2}$

(d) $\pi-\operatorname{arccot}\frac{\sqrt{2}}{2}$

6. The 8 vertices, the midpoints of the 12 edges, the centres of the 6 faces plus the centre of the cube form a set of 27 points. How many subsets consist of 3 collinear points?

(a) 57 (b) 49 (c) 43 (d) 37

Section 2. Questions requiring Answers Only.

1. Let $f(x)$ be an even periodic function with period 2. On the interval [0,1], $f(x)=x^{\frac{1}{1998}}$. What are the relative sizes of $f(\frac{98}{19})$, $f(\frac{101}{17})$ and $f(\frac{104}{15})$?

2. The points P, Q and R are represented by the complex numbers z, $(1+i)z$ and $2\overline{z}$, where $z=\cos\theta+i\sin\theta$ with $0\le\theta\le\pi$. When they are not collinear, let S be the fourth vertex of the parallelogram $PQSR$. What is the maximum distance between S and the origin of the complex plane?

3. How many subsets of size 3 of $\{0,1,2,3,4,5,6,7,8,9\}$ have even sums which are at least 10?

4. What is the maximum number of terms in an arithmetic progression with common difference 4 such that the square of the first term plus the sum of all the other terms is at most 100?

5. What is the range of a if the ellipse $x^2+4(y-a)^2=4$ intersects the parabola $x^2=2y$?

6. In triangle ABC, $\angle C=90°$, $\angle B=30°$ and $AC=2$. M is the midpoint of AB. The triangle is folded along CM until the distance between A and B is $2\sqrt{2}$. What is the volume of the tetrahedron $ABCM$?

Section 3. Questions requiring Full Solutions.

1. Let θ be a real number such that $\frac{\pi}{2}<\theta<\pi$. If

$$1-\sin\theta-i\cos\theta=r(\cos\phi+i\sin\phi)$$

for real numbers r and ϕ such that $0<\phi<2\pi$, determine ϕ.

2. For any real number $a<0$, there exists a largest positive real number $\ell(a)$ such that $|ax^2+8x+3|\le 5$ for all real numbers x satisfying $0\le x\le\ell(a)$.

(a) Determine the maximum value of $\ell(a)$.
(b) Determine a for which $\ell(a)$ is maximum.

3. Let p be a positive number. Let a and b be real numbers such that $ab\ne 0$ and $b^2\ne 2pa$. M is a variable point on the parabola $y^2=2px$. The lines joining M to $A(a,b)$ and $B(-a,0)$ intersect the parabola again at M_1 and M_2 respectively.

(a) Prove that as long as M_1 and M_2 exist and do not coincide, then the line M_1M_2 passes through a fixed point.
(b) Determine the coordinates of this point.

Paper II

1. The circumcentre and incentre of triangle ABC are O and I respectively. The line OI cuts the side BC at D, and AD is perpendicular to BC. Prove that the circumradius of ABC is equal to the radius of the excircle of ABC opposite A.

2. Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be real numbers in [1,2] such that $\sum_{i=1}^n a_i^2=\sum_{i=1}^n b_i^2$.

(a) Prove that $\sum_{i=1}^n \frac{a_i^3}{b_i}\le\frac{17}{10}\sum_{i=1}^n a_i^2$.
(b) Determine a necessary and sufficient condition for equality to hold.

3. For positive integers a and n, let $a=qn+r$ where q and r are non-negative integers with $r<n$. Define $F_n(a)=q+r$. Determine the largest positive integer A such that for all positive integers $a\le A$, $F_{n_6}(F_{n_5}(F_{n_4}(F_{n_3}(F_{n_2}(F_{n_1}(a))))))=1$ for some positive integers n_1, n_2, n_3, n_4, n_5 and n_6.

Olympiad Paper I

1. In an acute triangle ABC, $\angle C>\angle B$. D is a point on BC such that $\angle ADB$ is obtuse. H is the orthocentre of triangle BAD. F is a point inside triangle ABC and on the circumcircle of triangle BAD. Prove that F is the orthocentre of triangle ABC if and only if CF is parallel to HD and H is on the circumcircle of triangle ABC.

2. Let a be a fixed real number. A sequence of polynomials $\{f_n(x)\}$ is defined by $f_0(x) = 1$ and for $n = 0, 1, 2, \ldots$,

$$f_{n+1}(x) = xf_n(x) + f_n(ax).$$

(a) Prove that $f_n(x) = x^n f_n(\frac{1}{x})$ for $n = 0, 1, 2, \ldots$.

(b) Find an explicit expression for $f_n(x)$.

3. A space city consists of 99 space stations. Every two stations are connected by a space highway. All highways are one-way except for 99 which are two-way. A *group* is defined as a set of four stations such that we can travel from any one to any other of the four along the highways. Determine the maximum number of groups in a space city.

Olympiad Paper II

1. For any integer m, prove that $2m$ can be expressed in the form $a^{19} + b^{99} + k \cdot 2^{1999}$, where a and b are odd integers and k is a non-negative integer.

2. Let $f(x) = x^3 + ax^2 + bx + c$ be any cubic polynomial.

(a) Determine the maximum value of λ if $f(x) \geq \lambda(x-a)^3$ for all $x \geq 0$ whenever $f(x)$ has three non-negative roots.

(b) Determine all x for which $f(x) = \lambda(x-a)^3$ for the maximum value of λ.

3. Determine the number of ways of constructing a $4 \times 4 \times 4$ block from 64 unit cubes, exactly 16 of which are red, so that there is exactly one red cube within each $1 \times 1 \times 4$ subblock in any orientation.

1999/00

Paper I.

Section 1. Questions with Multiple Choices.

1. Let $\{a_n\}$ be a geometric progression with common ratio $q \neq 1$, and let

$$b_n = a_{3n-2} + a_{3n-1} + a_{3n}$$

for $n \geq 1$. What kind of sequence is $\{b_n\}$?

(a) arithmetic progression

(b) geometric progression with common ratio q

(c) geometric progression with common ratio q^3

(d) neither arithemtic nor geometric progression

2. How many lattice points are contained in the region defined by

$$(|x|-1)^2 + (|y|-1)^2 < 2?$$

(a) 16 (b) 17 (c) 18 (d) 25

3. If

$$(\log_2 3)^x - (\log_5 3)^x \geq (\log_2 3)^{-y} - (\log_5 3)^{-y},$$

what can be said about x and y?

(a) $x + y \geq 0$ (b) $x + y \leq 0$ (c) $x \leq y$ (d) $y \leq x$

4. Consider the following two statements.

(P) The planes α and β intersect along the line c. The lines a on α and b on β are skew lines. Then c can intersect at most one of a and b.

(Q) There do not exist infintely many lines which are pairwise skew.

Which of them is or are true?

(a) only P (b) only Q (c) both (d) neither

5. In a table-tennis round-robin tournament, three participants withdrew after having played 2 games each. If 50 games in all were played, how many games were played among these three participants before their withdrawal?

(a) 0 (b) 1 (c) 2 (d) 3

6. Let A be the point (1,2). Let B and C be the points of intersection of the parabola $y^2 = 4x$ with a variable line through the point $(5, -2)$. What kind of triangle is ABC?

(a) acute (b) obtuse (c) right (d) not uniquely determined

Section 2. Questions requiring Answers Only.

1. How many positive integers not exceeding 2000 are the sums of at least 60 consecutive positive integers?

2. If $\theta = \arctan \frac{5}{12}$, what is the value of

$$\arg \frac{\cos 2\theta + i \sin 2\theta}{239 + i} \quad ?$$

3. In triangle ABC, if

$$9BC^2 + 9CA^2 - 19AB^2 = 0,$$

what is the value of

$$\frac{\cot C}{\cot A + \cot B} \quad ?$$

4. P is a point on the hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

such that its distance to the right directrix is the average of its distances to the two foci. What is the x-coordinate of P?

5. How many subsets $\{a, b, c\}$ of $\{-3, -2, -1, 0, 1, 2, 3\}$ are there such that the line $ax + by + c = 0$ makes an acute angle with the positive x-axis?

6. ABC is an equilateral triangle. In the tetrahedron $SABC$ with $SA = 2\sqrt{3}$, the projection H of A onto the plane SBC is the orthocentre of triangle SBC. If the dihedral angle between the planes HAB and ABC is 30°, what is the volume of $SABC$?

Section 3. Questions requiring Full Solutions.

1. Determine the range of the real number θ such that for $0 \le x \le 1$,

$$x^2 \cos\theta - x(1 - x) + (1 - x)^2 \sin\theta > 0.$$

2. B is a variable point on the ellipse

$$\frac{x^2}{16} + \frac{y^2}{25} = 1.$$

A is the point $(-2, 2)$ and F is the focus of the ellipse whose x-coordinate is negative. Determine the coordinates of B if $AB + \frac{5}{3}BF$ is minimum.

3. Let n be a positive integer and M be a positive real number. Among all arithmetic progressions $\{a_k\}$ satisfying

$$a_1^2 + a_{n+1}^2 \le M,$$

determine the maximum value of

$$a_{n+1} + a_{n+2} + \cdots + a_{2n+1}.$$

Paper II

1. The diagonal AC of the quadrilateral $ABCD$ bisects $\angle BAD$. E is a point on the side CD. BE cuts AC at F, and the line DF cuts BC at G. Prove that $\angle CAE = \angle CAG$.

2. Let a, b and c be real numbers. Let z_1, z_2 and z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = 1$ and

$$\frac{z_1}{z_2} + \frac{z_2}{z_3} + \frac{z_3}{z_1} = 1.$$

Determine all possible values of $|az_1 + bz_2 + cz_3|$.

3. Let n be a positive integer. We wish to construct a set of tokens, the weight of each being an integral number of grams, such that any object whose weight is an integral number of grams up to n can be balanced by a subset of the tokens. Some of the tokens may be placed in the same pan as the object.

(a) Determine in terms of n the minimum value of the number of tokens.

(b) For what values of n is the minimal set of tokens unique?

Olympiad Paper I

1. In triangle ABC, $a \le b \le c$ where $a = BC$, $b = CA$ and $c = AB$. The circumradius is R and the inradius is r. What can be said about $\angle C$ if $a + b - 2R - 2r$ is

(a) positive;

(b) zero;

(c) negative?

2. The sequence $\{a_n\}$ is defined by $a_1 = 0,\ a_2 = 1$ and for $n \geq 3$,

$$a_n = \frac{n}{2}a_{n-1} + \frac{n(n-1)}{2}a_{n-2} + (-1)^n(1 - \frac{n}{2}).$$

Simplify

$$a_n + 2\binom{n}{1}a_{n-1} + 3\binom{n}{2}a_{n-2} + \cdots + (n-1)\binom{n}{n-2}a_2 + n\binom{n}{n-1}a_1.$$

3. In a table-tennis tournament, all games are between pairs of participants. Each participant is a member of at most two pairs. No participant ever plays against another if the two form a pair. Two pairs play exactly once against each other as long as the preceding rule is not violated. A set $\{a_1, a_2, \ldots, a_k\}$ is given, where k is a positive integer and $0 < a_1 < a_2 < \cdots < a_k$ are multiples of 6.

What is the minimum number of participants so that at the end of the tournament, the number of games played by each participant is a_i for some i, and for each i, at least one participant has played exactly a_i games?

Olympiad Paper II

1. Let $\langle a_1, a_2, \ldots, a_n\rangle$ be any permutation of $1, 2, \ldots, n$. For $k = 1, 2, \ldots, n$, define $b_k = \max\{a_i : 1 \leq i \leq k\}$. Determine the average value of the first term a_1 of all permutations for which the sequence $\{b_1, b_2, \ldots, b_n\}$ takes on exactly two distinct values.

2. Find all positive integers n for which there exist k integers n_1, n_2, $\ldots$, n_k, each greater than 3, such that

$$n = n_1 n_2 \cdots n_k = \sqrt[2^k]{2^{(n_1-1)(n_2-1)\cdots(n_k-1)}} - 1.$$

3. A multiple-choice examination has 5 questions, each with 4 choices. Each of 2000 students picks exactly 1 choice for each question. Among any n students for some positive integer n, there exist 4 such that any 2 of them give the same answers to at most 3 questions. Determine the minimum value of n.

2000/01
Paper I.

Section 1. Questions with Multiple Choices.

1. Let $A = \{x|\sqrt{x-2} \leq 0\}$ and $B = \{x|10^{x^2-2} = 10^x\}$. What is $A \cap \overline{B}$?

(a) $\{2\}$ (b) $\{-1\}$ (c) $\{x|x \leq 2\}$ (d) $\emptyset$

2. Let $\sin\alpha > 0$, $\cos\alpha < 0$ and $\sin\frac{\alpha}{3} > \cos\frac{\alpha}{3}$. What is the range of $\frac{\alpha}{3}$?

(a) $(2k\pi + \frac{\pi}{6}, 2k\pi + \frac{\pi}{3}), k \in \mathbb{Z}$

(b) $(\frac{2k\pi}{3} + \frac{\pi}{6}, \frac{2k\pi}{3} + \frac{\pi}{3}), k \in \mathbb{Z}$

(c) $(2k\pi + \frac{5\pi}{6}, 2k\pi + \pi), k \in \mathbb{Z}$

(d) $(2k\pi + \frac{\pi}{4}, 2k\pi + \frac{\pi}{3}) \cup (2k\pi + \frac{5\pi}{6}, 2k\pi + \pi), k \in \mathbb{Z}$

3. A is the left focus of the hyperbola $x^2 - y^2 = 1$. B and C are points on the right branch of the hyperbola such that ABC is an equilateral triangle. What is the area of ABC?

(a) $\frac{\sqrt{3}}{3}$ (b) $\frac{3\sqrt{3}}{2}$ (c) $3\sqrt{3}$ (d) $6\sqrt{3}$

4. Let p, q, a, b and c be positive numbers with $p \neq q$ such that p, a and q form a geometric progression while p, b, c and q form an arithmetic progression. What can be said about the roots of $bx^2 - 2ax + c = 0$?

(a) two complex roots

(b) repeated real root

(c) distinct real roots of the same sign

(d) real roots of opposite signs

5. What is the minimum distance from any lattice point to the line $y = \frac{5}{3}x + \frac{4}{5}$?

(a) $\dfrac{\sqrt{34}}{170}$ (b) $\dfrac{\sqrt{34}}{85}$ (c) $\dfrac{1}{20}$ (d) $\dfrac{1}{30}$

6. Let $\omega = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$. Which quartic polynomial has ω, ω^3, ω^7 and ω^9 as its roots?

(a) $x^4 + x^3 + x^2 + x + 1$ (b) $x^4 - x^3 + x^2 - x + 1$

(c) $x^4 - x^3 - x^2 + x + 1$ (d) $x^4 + x^3 + x^2 - x - 1$

Section 2. Questions requiring Answers Only.

1. What is the value of $\arcsin(\sin 2000°)$?

2. For $n \geq 2$, let a_n be the coefficient of the linear term x in the expansion of $(3-\sqrt{x})^n$. What is the limiting value of

$$\frac{3^2}{a_2}+\frac{3^3}{a_3}+\cdots+\frac{3^n}{a_n}$$

as n tends to infinity?

3. Let a be any real number. If $a+\log_2 3,\ a+\log_4 3$ and $a+\log_8 3$ form a geometric progression, what is its common ratio?

4. Let $a>b$ be positive numbers. The eccentricity of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is $\frac{\sqrt{5}-1}{2}$. Let F be the left focus, A be the point $(a,0)$ and B be the point $(0,b)$. What is the measure of $\angle ABF$?

5. What is the volume of a sphere which is tangent to all six sides of a regular tetrahedron of side-length a?

6. Each digit of a four-digit number is one of 1, 2, 3 and 4. Every two adjacent digits are different. The first and the last digits are also different. Moreover, the first digit is no greater than any other digit. How many such four-digit numbers are there?

Section 3. Questions requiring Full Solutions.

1. For any positive integer n, let $S_n=1+2+\cdots+n$. Determine the maximum value of

$$\frac{S_n}{(n+32)S_{n+1}}.$$

2. Determine the real numbers $a<b$ such that for $a\leq x\leq b$, the minimum value of $\frac{1}{2}(13-x^2)$ is $2a$ and the maximum value is $2b$.

3. What conditions must the real numbers $a>b>0$ satisfy such that for any point P on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

there exists a parallelogram having P as a vertex, which is inscribed in the ellipse and circumscribed about the unit circle?

Paper II

1. E and F are points on the side BC of an acute triangle ABC, with B closer to E than to F, such that $\angle BAE=\angle CAF$. Perpendiculars FM and FN are dropped from F onto the sides AB and AC, respectively. The line AE intersects the circumcircle of ABC again at D. Prove that the area of ABC is equal to the area of the quadrilateral $AMDN$.

2. The sequences $\{a_n\}$ and $\{b_n\}$ are defined by $a_0=1,\ b_0=0,$

$$a_{n+1}=7a_n+6b_n-3$$

and

$$b_{n+1}=8a_n+7b_n-4$$

for $n\geq 0$. Prove that a_n is the square of an integer for all $n\geq 0$.

3. Any two of n friends have a phone conversation at most once. Among any $n-2$ of them, the total number of phone conversations is a positive constant power of 3. Determine all values of n for which this is possible.

Olympiad Paper I

1. The quadrilateral $ABCD$ is inscribed in the unit circle such that it contains the centre of the circle and the length of its shortest side is $\sqrt{4-a^2}$ and that of its longest side is a, where $\sqrt{2}<a<2$. Let $A'B'C'D'$ be the quadrilateral determined by the tangents to the circle at A, B, C and D.

(a) Determine the minimum value of the ratio of the area of $A'B'C'D'$ to that of $ABCD$.

(b) Determine the maximum value of the ratio of the area of $A'B'C'D'$ to that of $ABCD$.

2. Determine the smallest positive integer m such that every subset of $\{1,2,\ldots,2001\}$ of size m contains two elements, not necessarily distinct, such that their sum is a power of 2.

3. On each vertex of a regular n-gon is a blue jay. They fly away and then return, again one blue jay on each vertex, but not necessarily to their original positions. Prove that there exist three blue jays such that the triangle determined by their earlier positions and the triangle determined by their later positions are of the same type, that is, both acute, both right or both obtuse.

Olympiad Paper II

1. Let a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ be seven distinct prime numbers such that $a+b=800$. Determine the maximum value of the difference between the largest and the smallest of these seven numbers.

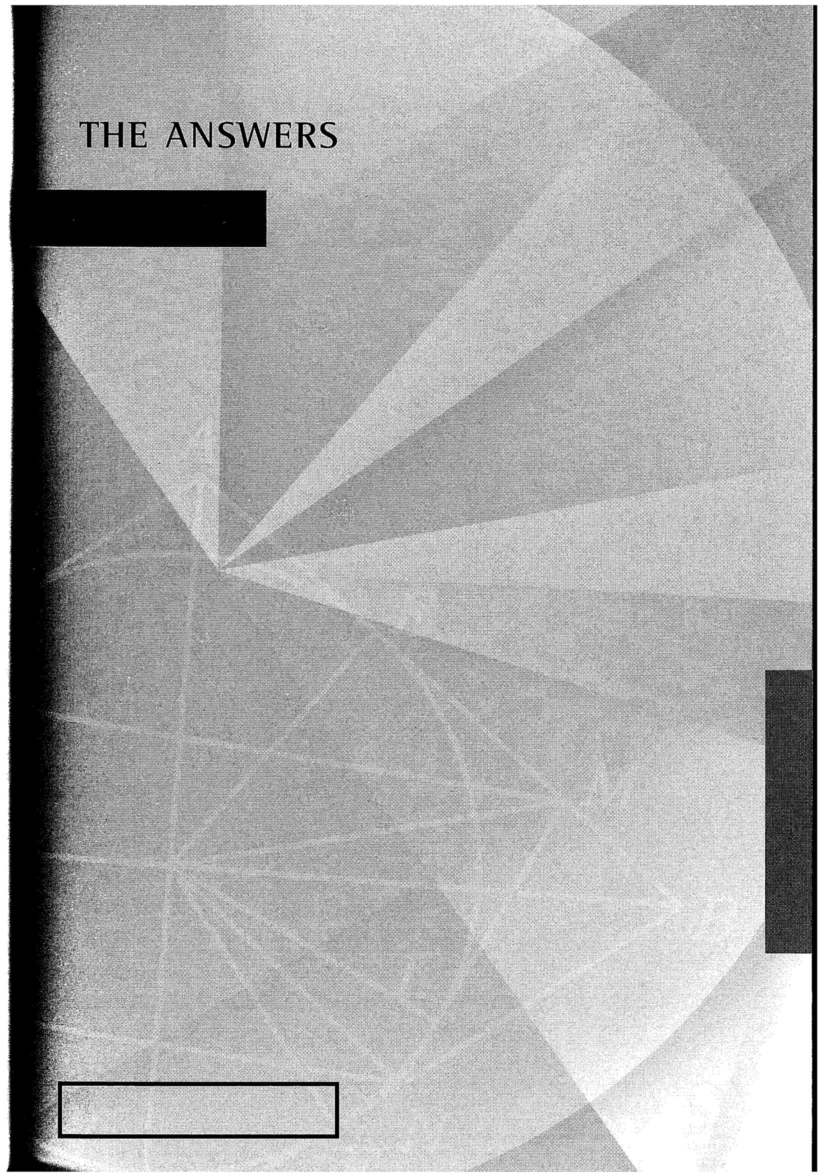
2. On the circumference of a circle are 24 points which divide it into 24 arcs of length 1. In how many ways can we choose 8 of these points such that neither arc determined by any two chosen points has length 3 or 8?

3. Let m and n be positive integers such that $4002m - m^2 - n^2$ is divisible by $2n$, $m < 4002$ and

$$n^2 - m^2 + 2mn \leq 4002(n-m).$$

(a) Determine the minimum value of $\dfrac{4002m - m^2 - mn}{n}$.

(b) Determine the maximum value of $\dfrac{4002m - m^2 - mn}{n}$.



THE ANSWERS

Paper I.

Section 1. Questions with Multiple Choices.

Question 1.

1993/94 (d)	**1994/95** (c)	**1995/96** (c)	**1996/97** (c)
1997/98 (a)	**1998/99** (c)	**1999/00** (c)	**2000/01** (d)

Question 2.

1993/94 (c)	**1994/95** (b)	**1995/96** (a)	**1996/97** (c)
1997/98 (c)	**1998/99** (b)	**1999/00** (a)	**2000/01** (d)

Question 3.

1993/94 (d)	**1994/95** (c)	**1995/96** (d)	**1996/97** (c)
1997/98 (c)	**1998/99** (d)	**1999/00** (a)	**2000/01** (c)

Question 4.

1993/94 (c)	**1994/95** (a)	**1995/96** (b)	**1996/97** (a)
1997/98 (d)	**1998/99** (d)	**1999/00** (d)	**2000/01** (a)

Question 5.

1993/94 (a)	**1994/95** (a)	**1995/96** (c)	**1996/97** (b)
1997/98 (b)	**1998/99** (d)	**1999/00** (b)	**2000/01** (b)

Question 6.

1993/94 (b)	**1994/95** (d)	**1995/96** (d)	**1996/97** (b)
1997/98 (c)	**1998/99** (b)	**1999/00** (c)	**2000/01** (b)

Section 2. Questions requiring Answers Only.

Question 1.

1993/94 $\lambda \neq 2$. **1994/95** $-3 < m < -\frac{2}{3}$. **1995/96** 2.
1996/97 $2^{90} - 1$. **1997/98** 2.
1998/99 $f(\frac{101}{17}) < f(\frac{98}{19}) < f(\frac{104}{15})$.
1999/00 6. **2000/01** $-\frac{\pi}{9}$.

Question 2.

1993/94 $\frac{8}{5}$. **1994/95** 1. **1995/96** 8:27. **1996/97** $\frac{1}{2}(-\sqrt{3} + 3i)$.
1997/98 4. **1998/99** 3. **1999/00** $\frac{\pi}{4}$. **2000/01** 18.

Question 3.
1993/94 $\pm(1+\sqrt{3}i)$. **1994/95** 7. **1995/96** 3. **1996/97** $\frac{16\pi}{3}$.
1997/98 $[\frac{\pi}{2}-\frac{1}{2}\arccos\frac{3}{4}, \frac{\pi}{2}+\frac{1}{2}\arccos\frac{3}{4}]\cup[\frac{3\pi}{2}-\frac{1}{2}\arccos\frac{3}{4}, \frac{3\pi}{2}+\frac{1}{2}\arccos\frac{3}{4}]$.
1998/99 51. **1999/00** $\frac{5}{9}$. **2000/01** $\frac{1}{3}$.

Question 4.
1993/94 08. **1994/95** $\frac{4\sqrt{3}}{9}$. **1995/96** 2551. **1996/97** 3.
1997/98 $\frac{\sqrt{3}}{3}$. **1998/99** 8. **1999/00** $-\frac{64}{5}$. **2000/01** 90°.

Question 5.
1993/94 9. **1994/95** $\frac{\sqrt{3}}{3}$. **1995/96** 420.
1996/97 230. **1997/98** 26. **1998/99** $-1 \le a \le \frac{17}{8}$.
1999/00 43. **2000/01** $\frac{\sqrt{2}\pi}{24}a^3$.

Question 6.
1993/94 34. **1994/95** 13. **1995/96** 1879. **1996/97** 4.
1997/98 $\log 2$. **1998/99** $\frac{2\sqrt{2}}{3}$. **1999/00** $\frac{9\sqrt{3}}{4}$. **2000/01** 28.

Section 3. Questions requiring Full Solutions.

Question 1.
1997/98 (a) $\frac{2+\sqrt{3}}{8}$; (b) $\frac{1}{8}$. **1998/99** $\frac{3\pi}{4}-\frac{\theta}{2}$.
1999/00 $2k\pi+\frac{\pi}{12} < \theta < 2k\pi+\frac{5\pi}{12}$ for each integer k.
2000/01 $\frac{1}{50}$.

Question 2.
1993/94 The line $x = \frac{a+b}{2}$.
1997/98 (b) $(2-\sqrt{3}, 2+\sqrt{3}), (2+\sqrt{3}, 2-\sqrt{3})$.
1998/99 (a) $\frac{\sqrt{5}+1}{2}$; (b) -8. **1999/00** $(-\frac{5\sqrt{3}}{3}, 2)$.
2000/01 (1,3), $(-2-\sqrt{7}, \frac{13}{4})$.

Question 3.
1993/94 $a_n = (2^n-1)^2(2^{n-1}-1)^2\cdots(2^2-1)^2(2-1)^2$ for $n \ge 1$ and $a_0 = 1$.
1998/99 (b) $(a, \frac{2pa}{b})$. **1999/00** $(n+1)\sqrt{\frac{5M}{2}}$.
2000/01 $\frac{1}{a^2}+\frac{1}{b^2}=1$.

Paper II

Problem 1.
1994/95 (a) $7+\sqrt{41}$; (b) $7-\sqrt{41}$. **1995/96** $8\sqrt{5}$.
1996/97 $2^n - 1 + 2n$.

Problem 2.
1994/95 2186. **1995/96** 76.
1996/97 $a \ge \frac{7}{2}$ or $a \le \sqrt{6}$.
1997/98 $x_0^2 \le \sum_{i=1}^{n} x_i^2$.
1998/99 For any i, either $a_i = 1$ and $b_i = 2$ or $a_i = 2$ and $b_i = 1$. Moreover, n is even and $a_i = 1$ half of the time.
1999/00 $\sqrt{(b+c)^2+a^2}$, $\sqrt{(c+a)^2+b^2}$, $\sqrt{(a+b)^2+c^2}$.

Problem 3.
1997/98 97. **1998/99** 53590.
1999/00 (a) m where $\frac{3^{m-1}-1}{2} < n \le \frac{3^m-1}{2}$; (b) $\frac{3^m-1}{2}$.
2000/01 5.

Problem 4.
1994/95 (a) 168 544.

Olympiad Paper I

Problem 1.
1993/94 (b) no. **1996/97** 189 548. **1997/98** $\sqrt{\sqrt{2}-\frac{1}{2}}$.
1999/00 (a) acute; (b) right; (c) obtuse.
2000/01 (a) $\frac{4}{a\sqrt{4-a^2}}$; (b) $\frac{8}{a^2(4-a^2)}$.

Problem 2.
1993/94 yes. **1994/95** (5,47), (7,45), (13,39), (15,37).
1995/96 39. **1996/97** (a) 1, 5, 9. (b) 1, 3, 5, 7, 9, 11. **1997/98** yes.
1998/99 (b) $f_n(x) = \sum_{i=0}^{n} \frac{(a^n-1)(a^{n-1}-1)\cdots(a^{n-i+1}-1)}{(a^i-1)(a^{i-1}-1)\cdots(a-1)}$.
1999/00 $2n! - 1 - n$. **2000/01** 999.

Problem 3.
1993/84 $f(x) = x+1$. **1994/95** 2394000000. **1997/98** 50.
1998/99 2052072. **1999/00** $\frac{a_k}{2}+3$.

Olympiad Paper II

Problem 1.

1994/95 $\frac{6}{11}$. **1995/96** 14. **1997/98** $2^6, 2^{11}$.

1999/00 $n-(n-1)\left(\sum_{m=1}^{n-1}\frac{1}{n-m}\right)^{-1}$. **2000/01** 1594.

Problem 2.

1994/95 6044. **1996/97** 8. **1997/98** orthocentre of ABC.

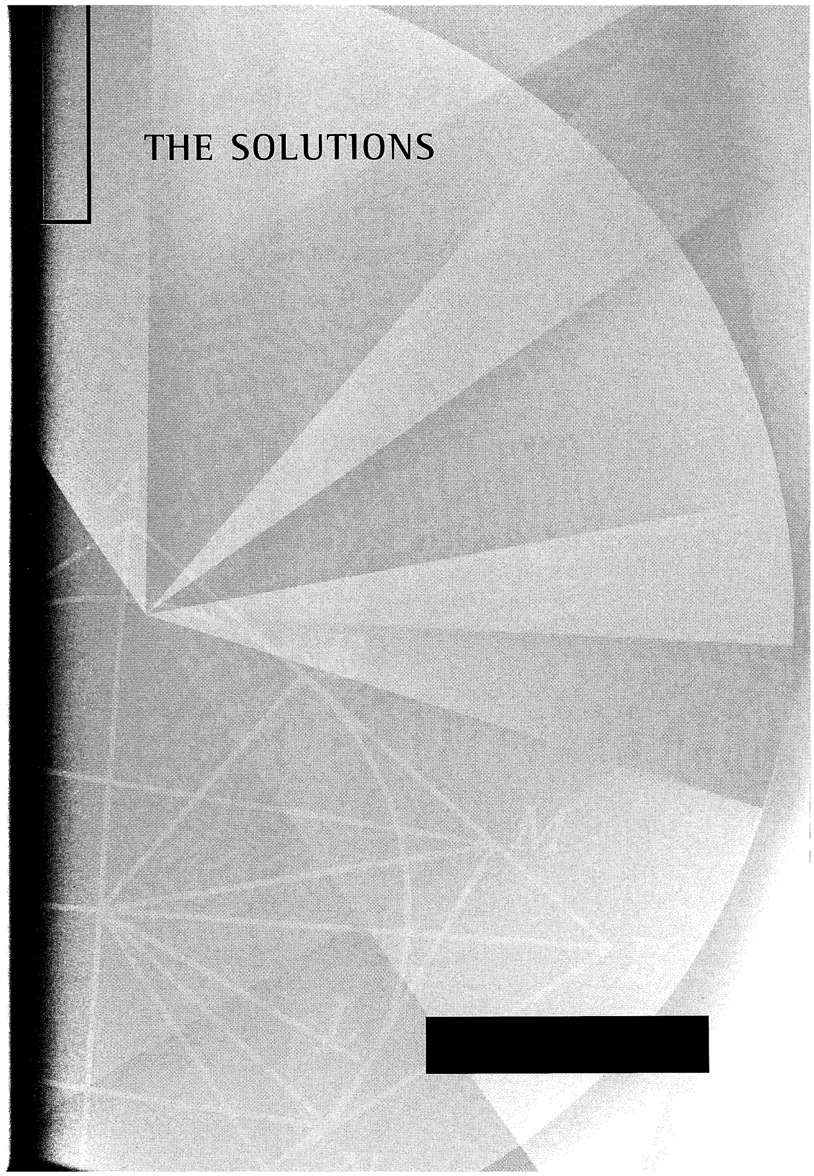
1998/99 (a) $-\frac{1}{27}$; (b) 0, $\frac{\gamma}{2}$, where γ is the largest root of $f(x)$.

1999/00 7. **2000/01** 258.

Problem 3.

1993/94 36. **1995/96** $\sqrt{\frac{3}{7}}$. **1997/98** $\sqrt{\frac{2k(n-k+1)}{n+1}}$.

1998/99 576. **1999/00** 25. **2000/01** (a) 2; (b) 3750.



THE SOLUTIONS

THE SOLUTIONS

1993/94

Paper I.

Section 1. Questions with Multiple Choices.

1. Since $|\tan \pi y| \geq 0$ and $\sin^2 \pi y \geq 0$, each is 0, so that both x and y are integers. From $x^2 + y^2 \leq 2$, we have $x, y \in \{-1, 0, 1\}$. Hence there are 9 such points (x, y).

2. The function $g(x) = f(x) - 4$ is odd. Now

$$\log \log_3 10 = \log \left(\frac{1}{\log 3} \right) = -\log \log 3.$$

It follows that

$$\begin{aligned} f(\log \log 3) &= g(\log \log 3) + 4 \\ &= g(-\log \log_3 10) + 4 \\ &= -g(\log \log_3 10) + 4 \\ &= -f(\log \log_3 10) + 8 \\ &= 3. \end{aligned}$$

3. For each a_i, there are three possibilities. It may be in A only, in B only, or in both. Hence there are $3^3 = 27$ pairs (A, B) for which $A \cup B = \{a_1, a_2, a_3\}$.

4. The equation simplifies to

$$\begin{aligned} 0 &= x^2 - (\arccos \alpha + \arcsin \alpha)x \\ &\quad + y^2 + (\arccos \alpha - \arcsin \alpha)y \\ &= x^2 + y^2 - \frac{\pi}{2}x + (\arccos \alpha - \arcsin \alpha)y. \end{aligned}$$

This represents a circle which passes through (0,0). Moreover, the chord which runs along the line $x = \frac{\pi}{4}$ is a diameter. To minimize its length, the centre of the circle should be as close to (0,0) as possible. Hence it must be at $(\frac{\pi}{4}, 0)$, occuring when $\arccos \alpha = \arcsin \alpha = \frac{\pi}{4}$. The length of this minimal diameter is $\frac{\pi}{2}$.

5. Let h be the altitude on AC. Then

$$h = AB - BC = \frac{h}{\sin A} - \frac{h}{\sin C}$$

so that

$$\sin C - \sin A = \sin C \sin A.$$

This is equivalent to

$$\begin{aligned} 2\cos\frac{C+A}{2}\sin\frac{C-A}{2} &= \frac{1}{2}(\cos(C-A) - \cos(C+A)) \\ &= \frac{1}{2}\left(1 - 2\sin^2\frac{C-A}{2}\right) \\ &\quad - \frac{1}{2}\left(2\cos^2\frac{C+A}{2} - 1\right) \\ &= 1 - \sin^2\frac{C-A}{2} - \cos^2\frac{C+A}{2}. \end{aligned}$$

It follows that

$$\left(\cos\frac{C+A}{2} + \sin\frac{C-A}{2}\right)^2 = 1.$$

This in turn implies that

$$\cos\frac{C+A}{2} + \sin\frac{C-A}{2} = \pm 1.$$

Since

$$0 < \frac{C-A}{2} < \frac{C+A}{2} < \frac{\pi}{2},$$

we have

$$\cos\frac{C+A}{2} + \sin\frac{C-A}{2} = 1.$$

6. Note that $n = |z+ni| + |z-mi| > 0$. Also,

$$\begin{aligned} n &= |z+ni| + |mi - z| \\ &\geq |z + ni + mi - z| \\ &= |(m+n)i| \\ &= |m+n|. \end{aligned}$$

This implies that $m < 0$. Finally,

$$n = |z+ni| + |z-mi| > |z+ni| - |z-mi| = -m.$$

Now $|z+ni| + |z-mi| = n$ is the equation of an ellipse with foci $(0,-n)$ and $(0,m)$. Hence both foci are below the x-axis. On the other hand, $|z+ni| - |z-mi| = -m$ is the equation of the branch of the hyperbola with foci $(0,-n)$ and $(0,m)$ which is closer to $(0,-m)$. Thus it opens upward.

Section 2. Questions requiring Answers Only:

1. Suppose there is a real root r. Then

$$\begin{aligned} 0 &= (1-i)r^2 + (\lambda + i)r + (1 + \lambda i) \\ &= (r^2 + \lambda r + 1) + (-r^2 + r + \lambda). \end{aligned}$$

Hence

$$r^2 + \lambda r + 1 = 0 = r^2 - r - \lambda,$$

so that

$$(\lambda + 1)(r + 1) = 0.$$

If $\lambda = -1$, then

$$r^2 - r + 1 = 0$$

but r is not real. Hence $r = -1$ and $\lambda = 2$.

In order for the equation not to have a real root, all we need is $\lambda \neq 2$.

2. Let $r = x^2 + y^2 > 0$. Let $x = \sqrt{r}\cos\theta$ and $y = \sqrt{r}\sin\theta$.

Then

$$4r\cos^2\theta - 5r\cos\theta\sin\theta + 4r\sin^2\theta = 5,$$

so that

$$\sin 2\theta = \frac{8r - 10}{5r}.$$

Since $|\sin 2\theta| \leq 1$, we have

$$-5r \leq 8r - 10 \leq 5r.$$

It follows that

$$\frac{10}{13} \leq r \leq \frac{10}{3}.$$

The minimum value is attained when $\theta = \frac{3\pi}{4}$ and the maximum value is attained when $\theta = \frac{\pi}{4}$. The sum of their reciprocals is

$$\frac{13}{10} + \frac{3}{10} = \frac{8}{5}.$$

3. In the complex plane, let O be the origin, B be the point z^2, A be the point z^2-4 and C be the point z^2+4. Then $AB=BC=4$. Moreover, OA makes an angle of $\frac{5\pi}{6}$ and OC makes an angle of $\frac{\pi}{3}$ with the positive x-axis. Hence OBC is an equilateral triangle, so that

$$z^2 = 4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}).$$

It follows that

$$z = \pm 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = \pm(1+\sqrt{3}i).$$

4. Long division yields

$$\left\lfloor \frac{10^{93}}{10^{31}+3} \right\rfloor = \left\lfloor 10^{62} - 3\cdot 10^{31} + 9 - \frac{27}{10^{31}+3} \right\rfloor = 10^{62} - 3\cdot 10^{31} + 8.$$

The last two digits of this number are 0 and 8.

5. Let

$$y_i = \log_{1993}\frac{x_{i-1}}{x_i}$$

for $i=1,2,3$. Then the given inequality may be rewritten as

$$\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3} \geq \frac{k}{y_1+y_1+y_3}.$$

By the Arithmetic-Geometric Means Inequality,

$$\left(\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}\right)(y_1+y_2+y_3) \geq 3\left(\sqrt[3]{\frac{1}{y_1y_2y_3}}\right)3(\sqrt[3]{y_1y_2y_3}) = 9.$$

Equality holds if and only if $y_1=y_2=y_3$, or x_0, x_1, x_2, x_3 forms a geometric progression. Hence the maximum value of k is 9.

6. A three-digit number which is still a number when inverted must have 0, 1, 6, 8 or 9 as its tens-digit, and 1, 6, 8 or 9 as its hundreds-digit and units-digit. Thus there are $4^2.5 = 80$ invertible three-digit numbers.

However, some of them are the same either way up. Such a number must have 0, 1 or 8 as its tens-digit, and for each possible hundreds-digit, there is a unique units-digit which makes the number self-inverted. There are $3\cdot 4 = 12$ self-inverted three-digit numbers.

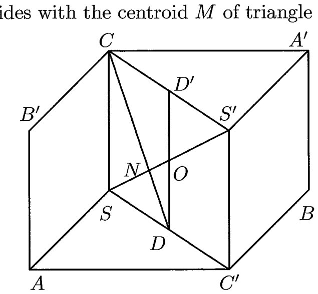
Hence the number of invertible pairs of three-digit numbers is $\frac{1}{2}(80-12)=34$.

Section 3. Questions requiring Full Solutions:

1. Complete the rectangles $ASBC'$, $SAB'C$, $SBA'C$ and the rectangular block $SAC'BCB'S'A'$. D is therefore also the midpoint of SC', and we may take D' to be the midpoint of CS'. Now SS' intersects CD at some point N and DD' at its midpoint O. Triangles CSN and DON are similar, so that

$$\frac{CN}{ND} = \frac{CS}{DO} = 2.$$

Hence N coincides with the centroid M of triangle ABC.



(a) Since SN intersects DD' at O, so does SM.

(b) O is the circumcentre of the rectangular block. Hence it is also the circumcentre of the tetrahedron $SABC$.

2. Let the equations of ℓ and m be $y-kx+ka=0$ and $y-hx+hb=0$ respectively. Then the equation of the conic section passing through the desired four points of intersection is

$$(y^2-x)+\lambda(y-kx+ka)(y-hx+hb)=0.$$

This may be rewritten as

$$(1+\lambda)y^2 - \lambda(k+h)xy + \lambda khx^2$$
$$+\lambda(ka+hb)y - (\lambda kh(a+b)+1)x + \lambda khab = 0.$$

This is a circle if and only if $1+\lambda = \lambda kh$ and $\lambda(k+h)=0$. The first equation shows that $\lambda \neq 0$. From the second equation, we then have $h=-k$. Solving for the point of intersection of ℓ and m, we obtain

$$\left(\frac{a+b}{2}, \frac{k(b-a)}{2}\right),$$

and its locus is the line $x = \dfrac{a+b}{2}$.

3. Let

$$b_n = \sqrt{\frac{a_n}{a_{n-1}}}.$$

Then we have $b_n - 2b_{n-1} = 1$. In particular,

$$b_1 = \sqrt{\frac{a_1}{a_0}} = 1 \quad \text{and} \quad b_2 = 2b_1 + 1 = 3.$$

Now

$$b_{n-1} - 2b_{n-2} = 1,$$

and subtraction yields

$$b_n - 3b_{n-1} + 2b_{n-2} = 0.$$

The characteristic equation is

$$x^2 - 3x + 2 = (x-1)(x-2) = 0$$

with characteristic roots 1 and 2. Hence $b_n = c_1 + c_2 2^n$. From $1 = b_1 = c_1 + 2c_2$ and $3 = b_2 = c_1 + 4c_2$, we have $c_1 = -1$ and $c_2 = 1$ so that

$$b_n = 2^n - 1.$$

Finally, for $n \geq 1$, we have

$$\begin{aligned} a_n &= a_{n-1}(2^n-1)^2 \\ &= a_{n-2}(2^n-1)^2(2^{n-1}-1)^2 \\ &= \cdots \\ &= a_0(2^n-1)^2(2^{n-1}-1)^2 \cdots (2^2-1)^2(2-1)^2. \end{aligned}$$

Hence $a_0 = 1$ and

$$a_n = (2^n-1)^2(2^{n-1}-1)^2 \cdots (2^2-1)^2(2-1)^2$$

for all $n \geq 1$.

Paper II

1. An obtuse triangle may be divided into n obtuse triangles for any $n \geq 1$. Let AB_0B_n be such that $\angle AB_0B_n > 90°$. Take points $B_1, B_2, \ldots, B_{n-1}$ on B_0B_n and join each of them to A. Then we have divided AB_0B_n into n obtuse triangles. A non-obtuse triangle cannot be divided into n obtuse triangles for $n \leq 2$.

The case $n = 1$ is trivial.

In the case $n = 2$, the cut must be from one vertex to the opposite side. There are no obtuse angles to begin with, and at most one can be created where the cut meets the opposite side. On the other hand, a non-obtuse triangle can be divided into 3 obtuse triangles. Let the angle at A be the largest in a non-obtuse triangle. Drop a perpendicular from A to BC until it reaches some point D inside the semicircle with diameter BC. Joining D to the vertices of ABC will result in 3 obtuse triangles. In the given quadrilateral, we can first cut off the obtuse triangle CAD. Then we cut the non-obtuse triangle CAB into 3 obtuse triangles. Thus $ABCD$ can be cut into 4 obtuse triangles.

By cutting up one of the obtuse triangles in the manner described above, we can obtain n obtuse triangles for any $n \geq 4$, establishing sufficiency.

We now establish necessity. If each side of $ABCD$ belongs to a different triangle, we already have $n \geq 4$. Hence some triangle must contain two adjacent sides of $ABCD$, implying that we cut along one of the diagonals.

If we cut along AC, we have one obtuse triangle CAD, but it is not possible to cut the non-obtuse triangle CAB into 2 obtuse triangles.

If the cut is along BD, at least one of triangles BAD and BCD is non-obtuse, and it cannot be cut into 2 obtuse triangles.

2. (a) Note that

$$\frac{1}{\binom{n}{|A_i|}} = \frac{|A_i|!(n-|A_i|)!}{n!}$$

so that the desired result is equivalent to

$$\sum_{i-1}^{m} |A_i|!(n-|A_i|)! \leq n!.$$

Since there are $n!$ permutations of the n elements, we have to prove that A_i accounts for

$$|A_i|!(n-|A_i|)!$$

of them, without overlap. The $|A_i|$ elements in A_i are listed first, and they can be permuted among themselves in $|A_i|!$ ways. They are followed by the elements not in A_i, which can be permuted among themselves in $(n-|A_i|)!$ ways.

Suppose the same permutation is generated by A_i and A_j. We may assume that $|A_i| \leq |A_j|$. Then A_i must be a subset of A_j, which contradicts the hypothesis.

(b) The desired result may be derived from Cauchy's Inequality as follows:

$$
\begin{aligned}
m^2 &= \left(\sum_{i=1}^{m} \frac{1}{\sqrt{\binom{n}{|A_i|}}}\sqrt{\binom{n}{|A_i|}}\right)^2 \\
&\le \left(\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}}\right)\left(\sum_{i=1}^{m}\binom{n}{|A_i|}\right) \\
&\le \sum_{i=1}^{m}\binom{n}{|A_i|}.
\end{aligned}
$$

3. Denote by M the point of intersection of ℓ and m.

(a) If ℓ is tangent to the circle, then $AP = AM$, $BQ = BM$ and $CR = CM$. We have

$$
\begin{aligned}
AB\cdot CR + AP\cdot BC &= AB\cdot CM + (AC+CM)BC \\
&= (AB+BC)CM + AC\cdot BC \\
&= AC(CM+BC) \\
&= AC\cdot BQ.
\end{aligned}
$$

This is known as Ptolemy's Theorem for collinear points.

(b) Suppose ℓ intersects the circle at D and E. Then we have $AP^2 = AD\cdot AE$, $BQ^2 = BD\cdot BE$ and $CR^2 = CD\cdot CE$. Using Ptolemy's Theorem for collinear points, we have

$$
\begin{aligned}
& AC^2BQ^2 - (AB\cdot CR + BC\cdot AP)^2 \\
= \;& AC\cdot BD(AC\cdot BE) - AB^2CD\cdot CE - BC^2AD\cdot AE \\
& -2AB\cdot BC\sqrt{CD\cdot CE\cdot AD\cdot AE} \\
= \;& AC\cdot BD(AB\cdot CE + AE\cdot BC) \\
& -AB\cdot CE(AB\cdot CD) - BC\cdot AE(BC\cdot AD) \\
& -2AB\cdot BC\sqrt{CD\cdot CE\cdot AD\cdot AE} \\
= \;& AB\cdot CE(AC\cdot BD - AB\cdot CD) \\
& +AE\cdot BC(AC\cdot BD - BC\cdot AD) \\
& -2AB\cdot BC\sqrt{CD\cdot CE\cdot AD\cdot AE} \\
= \;& AB\cdot CE(AD\cdot BC) + AE\cdot BC(AB\cdot AD) \\
& -2AB\cdot BC\sqrt{CD\cdot CE\cdot AD\cdot AE} \\
= \;& AB\cdot BC(\sqrt{AD\cdot CE} - \sqrt{CD\cdot AE})^2 \\
\ge \;& 0.
\end{aligned}
$$

Since

$$AD\cdot CE = AC\cdot DE + CD\cdot AE > CD\cdot AE,$$

the inequality is strict. Hence

$$AC\cdot BQ > AB\cdot CR + BC\cdot AP.$$

(c) Suppose ℓ is disjoint from the circle. Let MS be a tangent and let T be the point on m inside the circle such that $MT = MS$. Let O be the centre of the circle and r the radius. Then

$$
\begin{aligned}
AP^2 &= OA^2 - r^2 \\
&= AM^2 + OM^2 - r^2 \\
&= AM^2 + MS^2 \\
&= AM^2 + MT^2 \\
&= AT^2.
\end{aligned}
$$

Hence $AP = AT$. Similarly, $BQ = BT$ and $CR = CT$. Since T, A, B and C are neither collinear nor concyclic, Ptolemy's Inequality yields

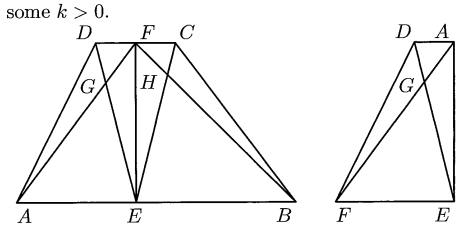
$$AB\cdot CT + BC\cdot AT > AC\cdot BT.$$

Olympiad Paper I

1. (a) In the diagram on the left, let

$$\frac{FD}{EA} = \frac{1}{k}$$

for some $k > 0$.



If we take $[DFG] = 1$, then $[EFG] = [DAG] = k$ and $[AGE] = k^2$.

By the Arithmetic-Geometric Mean Inequality, $\frac{1}{2}(k^2+1) \ge k$.

It follows that $[EFG] \le \frac{1}{4}[ADFE]$.

Similarly, we can prove that $[EFH] \le \frac{1}{4}[BCFE]$, so that $[EGFH] \le \frac{1}{4}[ABCD]$.

(b) The answer is negative and we construct a counter-example as follows. In the diagram on the right, we have switched the labels A and F in $ADFE$. Then $[EFG] > \frac{1}{4}[ADFE]$.

Extend AE to B and DF to C so that

$$\frac{1}{4}[BCFE] < [EFG] - \frac{1}{4}[ADFE].$$

Then we have $[EGFH] > [EFG] > \frac{1}{4}[ABCD]$.

2. Solution 1

Choose any box and call it B. If there are no smarties outside B, the task is accomplished. Suppose there are at least two non-empty boxes other than B. Take 1 smartie from each and put them in B.

Eventually, there is at most one other non-empty box X. If it has at least 2 smarties, take 1 from X and 1 from B and put them in a third box. Then take another 1 from X and one from this third box and put them in B. So the number of smarties in B increases by 1.

Eventually, there is only 1 smartie outside B. The task can then be accomplished with the following transformation:

$$\begin{aligned}(0,0,1,n) &\to (2,0,0,n-1)\\ &\to (1,2,0,n-2)\\ &\to (0,2,2,n-3)\\ &\to (0,1,1,n-1)\\ &\to (0,0,0,n+1).\end{aligned}$$

Solution 2

We use induction on the number n of smarties to prove that the answer is affirmative.

For $n = 4$, we have

$$\begin{aligned}(1,1,1,1) &\to (0,0,3,1)\\ &\to (0,2,2,0)\\ &\to (0,1,1,2)\\ &\to (0,0,0,4).\end{aligned}$$

Since this sequence includes all possible distributions of 4 smarties, the basis is established. Suppose the result holds for some $n \geq 4$.

Consider the next case with $n+1$ smarties. For now, treat one of them as non-existent. By the induction hypothesis, the other n can be put into 1 box. If the $(n+1)$-st smarty is there as well, we have nothing further to do.

Otherwise, we perform

$$\begin{aligned}(0,0,1,n) &\to (2,0,0,n-1)\\ &\to (1,2,0,n-2)\\ &\to (0,2,2,n-3)\\ &\to (0,1,1,n-1)\\ &\to (0,0,0,n+1).\end{aligned}$$

This completes the induction argument.

3. Solution 1

We have

$$(f(x))^2 = xf(x+1) + 1.$$

Subtracting $(x+1)^2$ from both sides,

$$(f(x) - (x+1))(f(x) + x + 1) = x(f(x+1) - (x+2)).$$

Since $1 \leq f(x) \leq 2(x+1)$ and $1 \leq f(x+n) \leq 2(x+n+1)$, we have

$$\begin{aligned}|f(x) - (x+1)| &\leq \frac{x}{x+2}|f(x+1) - (x+2)|\\ &\leq \frac{x}{x+2}\cdot\frac{x+1}{x+3}|f(x+2) - (x+3)|\\ &\leq \cdots\\ &\leq \frac{x(x+1)}{(x+n)(x+n+1)}|f(x+n) - (x+n+1)|\\ &\leq \frac{x(x+1)}{x+n}.\end{aligned}$$

Since n may be arbitrarily large, we must have $f(x) = x+1$, and it is easy to verify that this function has all the desired properties.

Solution 2

Let

$$g(x) = \frac{f(x)}{x+1}.$$

Then

$$\frac{1}{x+1} \le g(x) \le 2$$

and

$$\begin{aligned} g(x+1) - (g(x))^2 &= \frac{f(x+1)}{x+2} - \left(\frac{f(x)}{x+1}\right)^2 \\ &= \frac{(f(x))^2-1}{x(x+2)} - \left(\frac{f(x)}{x+1}\right)^2 \\ &= \frac{(f(x))^2-(x+1)^2}{x(x+2)(x+1)^2} \\ &= \frac{(g(x))^2-1}{x(x+2)}. \end{aligned}$$

If $g(x) > 1$ for some $x \ge 1$, then we have

$$g(x+1) > (g(x))^2 > 1.$$

Iteration yields

$$g(x+n) > (g(x))^{2^n}.$$

Since n may be arbitrarily large, this contradicts $g(x) \le 2$. If $g(x) < 1$ for some $x \ge 1$, then we have

$$g(x+1) < (g(x))^2 < 1.$$

Iterating this now yields

$$g(x+n) < (g(x))^{2^n}$$

so that

$$\left(\frac{1}{x+n+1}\right)^{2^{-k}} < (g(x+n))^{2^{-k}} < g(x) < 1.$$

However, the leftmost term may be made arbitrarily close to 1 by increasing n, and we have a contradiction. It follows that $g(x) = 1$ for all $x \ge 1$. Hence $f(x) = x+1$, and it is easy to verify that this function has all the desired properties.

Olympiad Paper II

1. **Solution 1**

Let

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

We have

$$1 + \omega^j + \cdots + \omega^{(n-1)j} = n$$

for $j = 0$ or n, but this sum is zero for $j = 1, 2, \ldots, n-1$. For any complex number λ,

$$\sum_{k=0}^{n-1} f(\lambda\omega^k) = n(c_0\lambda + c_n).$$

Choose λ so that $|\lambda| = 1$ and $c_0\lambda^n$ has the same argument as c_n. Then

$$\frac{1}{n}\sum_{k=0}^{n-1} |f(\lambda\omega^k)| \ge \frac{1}{n}\left|\sum_{k=0}^{n-1} f(\lambda\omega^k)\right| = |c_0\lambda^n + c_n| = |c_0| + |c_n|.$$

It follows that for some k, $0 \le k \le n-1$, $z_0 = \lambda\omega^k$ satisfies

$$|f(z_0)| \ge |c_0| + |c_n|.$$

Solution 2

First we suppose that $c_n \ne 0$. Let

$$g(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_{n-1}z - \frac{|c_0|c_n}{|c_n|}.$$

Let its roots be $z_1,\ z_2,\ \ldots,\ z_n$. Then

$$|z_1 z_2 \cdots z_n| = \left|\frac{|c_0|c_n}{|c_n|c_0}\right| = 1.$$

It follows that at least one root z_0 satisfies $|z_0| \le 1$ and

$$f(z_0) = g(z_0) + \left(\frac{|c_0|}{|c_n|} + 1\right)c_n.$$

Hence $|f(z_0)| = |c_0| + |c_n|$. We now suppose that $c_n = 0$. Let

$$g(z) = c_0 z^n + c_1 z^{n-1} + \cdots + c_{n-1}z - c_0.$$

Let its roots be $z_1,\ z_2,\ \ldots,\ z_n$. Then

$$|z_1z_2\cdots z_n| = \left|\frac{c_0}{c_0}\right| = 1.$$

It follows that at least one root z_0 satisfies $|z_0| \le 1$ and

$$|f(z_0)| = |g(z_0) + c_0| = |c_0| = |c_0| + |c_n|.$$

2. Solution 1

From the Binomial Theorem, we have

$$\begin{aligned}\sum_{i=0}^{2n}\binom{2n}{i}x^i &= (1+x)^{2n}\\ &= (x^2+(1+2x))^n\\ &= \sum_{i=0}^{n}\binom{n}{i}x^{2i}(1+2x)^{n-i}\\ &= \sum_{i=0}^{n}\binom{n}{i}x^{2i}\sum_{j=0}^{n-i}2^j\binom{n-i}{j}x^j.\end{aligned}$$

Comparing the coefficients of x^n on both sides, we have

$$\binom{2n}{n} = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}2^{n-2i}\binom{n}{i}\binom{n-i}{n-2i} = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}2^{n-2i}\binom{n}{2i}\binom{2i}{i}.$$

Comparing the coefficients of x^{n-1} on both sides, we have

$$\begin{aligned}\binom{2n}{n-1} &= \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}2^{n-2i-1}\binom{n}{i}\binom{n-i}{n-2i-1}\\ &= \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}2^{n-2i-1}\binom{n}{2i+1}\binom{2i+1}{i}.\end{aligned}$$

Adding the last two equations, we have

$$\begin{aligned}\binom{2n+1}{n} &= \sum_{k=0}^{n}2^{n-k}\binom{n}{k}\binom{k}{\lfloor\frac{k}{2}\rfloor}\\ &= \sum_{k=0}^{n}2^k\binom{n}{n-k}\binom{n-k}{\lfloor\frac{n-k}{2}\rfloor}\\ &= \sum_{k=0}^{n}2^k\binom{n}{k}\binom{n-k}{\lfloor\frac{n-k}{2}\rfloor}.\end{aligned}$$

Solution 2

Consider the number of ways of choosing n of the $2n+1$ objects $a_1,\ b_1,\ a_2,\ b_2,\ \ldots,\ a_n,\ b_n$ and c. First we choose k of the pairs $\{a_i, b_i\}$ and take exactly one from each pair. The number of ways of doing this is $2^k\binom{n}{k}$. The remaining $n-k$ objects are chosen as pairs $\{a_i, b_i\}$ plus c if $n-k$ is odd.

This can be done in

$$\binom{n-k}{\lfloor\frac{n-k}{2}\rfloor}$$

ways. Since the total number of ways is obviously $\binom{2n+1}{n}$, we have

$$\binom{2n+1}{n} = \sum_{k=0}^{n}2^k\binom{n}{k}\binom{n-k}{\lfloor\frac{n-k}{2}\rfloor}.$$

3. Performing a half-turn about the point $(997p, 7\cdot 997p)$, we may instead take $I(1994p, 7\cdot 1994p)$ as the incentre and $O(0,0)$ as the vertex of the right angle of triangle OAB. Since the slope of OI is 7, the slope of one side, say OA, is

$$\tan(\arctan 7 - 45°) = \frac{7-1}{1+7} = \frac{3}{4}.$$

Then the slope of the other side OB is $-\frac{4}{3}$.

We can deduce from

$$(3k)^2 + (4k)^2 = (5k)^2$$

that the distance of any lattice point on OA or OB from O is a multiple of 5. Rotate about O so that OA falls on the positive x-axis and OB on the positive y-axis. Take as a new unit of length 5 times the old one. Then the new coordinates of I are (r, r) where $r = 1994p$.

Let the new coordinates of A and B be $(r+s, 0)$ and $(0, r+t)$, respectively. Then $AB = s+t$ and we have

$$(r+s)^2 + (r+t)^2 = (s+t)^2.$$

This is equivalent to $2r^2 = (s-r)(t-r)$. Since the hypotenuse is the longest side in a right triangle, both $u = s-r$ and $v = t-r$ are positive and we have $2r^2 = uv$.

We claim that for any pair (u, v) of positive integers such that $2r^2 = uv$, the triangle OAB with O at (0,0), A at $(2r+u, 0)$ and B at $(0, 2r+v)$ has incentre $I(r, r)$. Since

$$(2r+u)^2 + (2r+v)^2 = (2r+u+v)^2,$$

we have $AB = 2r + u + v$. The inradius is given by

$$\frac{1}{2}(OA + OB - AB) = r.$$

This justifies the claim. The problem now becomes finding the number of pairs (u, v) of positive integers such that

$$uv = 2r^2 = 2^3 997^2 p^2.$$

For $p = 2$, the number of pairs of positive divisors of $2^5 997^2$ is

$$(5+1)(2+1) = 18.$$

For $p = 997$, the number of pairs of positive divisors of $2^3 997^4$ is

$$(3+1)(4+1) = 20.$$

For any prime p other than 2 and 997, the number of pairs of positive divisors of $2^3 997^2 p^2$ is

$$(3+1)(2+1)(2+1) = 36.$$

1994/95

Paper I.

Section 1. Questions with Multiple Choices.

1. Suppose $a^2 + b^2 > 0$. Let θ be the angle such that

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}} \quad \text{and} \quad \cos\theta = \frac{a}{\sqrt{a^2+b^2}}.$$

Then

$$a\sin x + b\cos x + c = \sqrt{a^2+b^2}\sin(x+\theta) + c.$$

This is positive if and only if

$$\sin(x+\theta) > -\frac{c}{\sqrt{a^2+b^2}}.$$

Now $\sin(x+\theta) \geq -1$, with equality if $x = \pi - \theta$. Hence the necessary and sufficient condition is

$$-1 > -\frac{c}{\sqrt{a^2+b^2}}$$

or $c > \sqrt{a^2+b^2}$.

If $a = b = 0$, then the condition is $c > 0 = \sqrt{a^2+b^2}$.

2. If $a^2 + b^2 > c^2$, then both $a^2 + b^2$ and c^2 are real, and we certainly have $a^2 + b^2 - c^2 > 0$. On the other hand, if $a^2 + b^2 - c^2 > 0$, all we can say is that $a^2 + b^2 - c^2$ is real. We may have $a = 2+i$, $b = i$ and $c = \sqrt{2}(1+i)$. Then $a^2 + b^2 - c^2 = 2 > 0$, but neither $a^2 + b^2 = 2+4i$ nor $c^2 = 4i$ is real. Thus we cannot conclude that $a^2 + b^2 > c^2$.

3. Let $b_n = a_n - 1$. Then $b_1 = 8$ and $b_{n+1} = -\frac{1}{3}b_n$. Hence $\{b_n\}$ is a geometric progression with common ratio $-\frac{1}{3}$. Summing the first n terms yields

$$S_n - n = 8\left(1 - \left(-\frac{1}{3}\right)^n\right) \div \left(1 - \left(-\frac{1}{3}\right)\right) = 6 - 6\left(-\frac{1}{3}\right)^n.$$

Hence

$$|S_n - n - 6| = 6\left(\frac{1}{3}\right)^n.$$

Now

$$6\left(\frac{1}{3}\right)^6 = \frac{2}{243} > \frac{1}{125} > \frac{2}{729} = 6\left(\frac{1}{3}\right)^7.$$

It follows that the smallest value of n with the desired property is 7.

4. Since $0<b<1$, $\log_b x$ is a decreasing function. Since $0<\alpha<\frac{\pi}{4}$, we have $0<\sin\alpha<\cos\alpha<1$. Hence

$$\log_b \sin\alpha > \log_b \cos\alpha > 0$$

and

$$(\sin\alpha)^{\log_b \sin\alpha} < (\sin\alpha)^{\log_b \cos\alpha} < (\cos\alpha)^{\log_b \cos\alpha}.$$

5. Let the regular n-gon at the base be fixed and let the vertex move up and down along the axis. At one end, the pyramid degenerates into its base, and the dihedral angles are π. At the other end, the pyramid degenerates into an infinite prism, and the dihedral angles are

$$\frac{(n-2)\pi}{n}.$$

6. Use the lines $y = x$ and $y = -x$ to divide the plane into four quadrants. The equation becomes

$$\frac{x+y}{2a} + \frac{y-x}{2b} = 1$$

in the north quadrant. This is a line which intersects $y = x$ at (a, a) and $y = -x$ at $(-b, b)$. In the east quadrant, we have a line joining (a, a) to $(b, -b)$. In the south quadrant, the line joins $(b, -b)$ to $(-a, -a)$, and the rhombus is closed in the west quadrant. Since $a \neq b$, the rhombus is not a square.

Section 2. Questions requiring Answers Only:

1. The line $x + my + m = 0$ passes through the point $R(0, -1)$ and has slope $-\frac{1}{m}$. The line PQ has slope $\frac{1}{3}$ and the line QR has slope $\frac{3}{2}$. In order for $x + my + m = 0$ to intersect the extension of PQ, we must have $\frac{1}{3} < -\frac{1}{m} < \frac{3}{2}$ or $-3 < m < -\frac{2}{3}$.

2. Let $f(t) = t^3 + \sin t$. Then

$$f(x) = 2a = (-2y)^3 + \sin(-2y) = f(-2y).$$

Since $f(t)$ is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we must have $x = -2y$ or $x + 2y = 0$. Hence $\cos(x + 2y) = 1$.

3. If (x, y) is a lattice point inside circle A, then $1 \leq x \leq 5$. For $x = 1$, $3 \leq y \leq 5$ and all three are outside circle B. For $x = 2$, $2 \leq y \leq 6$ but only (2,2) and (2,3) are outside. For $x = 3$ or 4, $2 \leq y \leq 6$ but only $(x, 2)$ is outside. For $x = 5$, $3 \leq y \leq 5$ but all three are inside circle B. Hence $A \cap B$ contains 3+2+1+1=7 lattice points.

4. We have

$$y=\sin\frac{\theta}{2}(1+\cos\theta)=\sin\frac{\theta}{2}(2\cos^2\frac{\theta}{2})=2\sin\frac{\theta}{2}(1-\sin^2\frac{\theta}{2}).$$

Let $x = \sin\frac{\theta}{2}$. Then

$$\frac{y^2}{2} = 2x^2(1-x^2)(1-x^2).$$

The three factors on the right sum to 2. Hence the maximum value of $\frac{y^2}{2}$, and of y, occurs at $2x^2 = \frac{2}{3}$ or $x = \frac{1}{\sqrt{3}}$. It follows that the maximum value of y is

$$\frac{2}{\sqrt{3}}\left(1-\frac{1}{3}\right) = \frac{4\sqrt{3}}{9}.$$

5. Let D be a vertex of a unit cube. Let A, B and C be the vertices adjacent to D.

Let the plane be horizontal, and let the cube lie entirely above it. Clearly none of the edges can be parallel to this plane. Hence there is a vertex D which rises the highest above this plane. Let A, B and C be the vertices adjacent to D. Since $DA = DB = DC$, D must be at the same vertical height above each of A, B and C. Hence the plane in question is parallel to ABC.

Let F be the vertex opposite to D, and let DF intersect the plane ABC at K. Then

$$DK = \frac{1}{3}DF = \frac{\sqrt{3}}{3}$$

and it follows that

$$\sin\alpha = \frac{DK}{AD} = \frac{\sqrt{3}}{3}.$$

There are altogether four families of parallel planes with the desired property, each perpendicular to one of the space diagonals of the cube.

6. Let the number of +1s be m and the number of −1s be n. Then $m + n = 95$ and

$$a_1^2 + a_2^2 + \cdots + a_{95}^2 = 95.$$

Let S denote the sum in question. Then

$$2S + 95 = (a_1 + a_2 + \cdots + a_{95})^2 = (m-n)^2.$$

The smallest positive S which makes a square when added to 95 is 13, with $|m - n| = 11$.

This minimum value is attained when

$$(m,n) = (53,42) \quad \text{or} \quad (42,53).$$

Paper II

1. We have $\alpha+\beta = -z_1$ while $\alpha\beta = z_2 + m$. Hence

$$\begin{aligned}(\alpha-\beta)^2 &= (\alpha+\beta)^2 - 4\alpha\beta \\ &= z_1^2 - 4z_2 - 4m \\ &= 16 + 20i - 4m.\end{aligned}$$

Since $|\alpha-\beta| = 2\sqrt{7}$, we have $|4+5i-m| = 7$. Hence the point M representing m in the complex plane lies within a circle of radius 7 and centred at $C(4,5)$. Let O be the origin and AB be the diameter of the circle through O, with A closer to O. Then

$$OC = \sqrt{4^2+5^2} = \sqrt{41}.$$

(a) The maximum value of m occurs when $M = B$, and we have $m = OM = 7 + \sqrt{41}$.

(b) The minimum value of m occurs when $M = A$, and we have $m = OM = 7 - \sqrt{41}$.

2. The number of positive integers less than 105 that are relatively prime to $105 = 3 \times 5 \times 7$ is

$$105\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right) = 48.$$

When 1000 is divided by 48, the quotient is 20 and the remainder is 40. The positive integers relatively prime to 105 start with 1, 2, 4, 8, 11, 13, 16 and 17, with 19 being the 9-th. Thus the 40-th is

$$105 - 19 = 86,$$

and the 1000-th is

$$105 \times 20 + 86 = 2186.$$

3. Let $\alpha = \angle CAB$ and $\gamma = \angle BCA$. Then $\alpha + \gamma = 120°$. Extend AI and BI to cut the circumcircle again at F and M, respectively.

(a) Since $\angle MOA = 2\angle MBA = 60°$ and $OA = OM$, triangle MOA is equilateral. Hence $AM = R$. Since

$$\begin{aligned}\angle MIA &= \angle MBA + \angle BAI \\ &= 30° + \frac{\alpha}{2} \\ &= \angle MAC + \angle CAI \\ &= \angle MAI,\end{aligned}$$

we have $MI = MA = R$. Now

$$\angle IMO = \angle AMB - \angle AMO = \gamma - 60°.$$

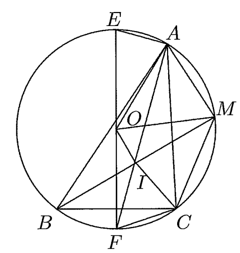
On the other hand,

$$\angle AEO = \angle ABF = 60° + \frac{\alpha}{2}$$

so that

$$\angle AOE = 180° - 2(60° + \frac{\alpha}{2}) = \gamma - 60°.$$

It follows that triangles AOE and IMO are congruent, so that $AE = IO$.



(b) Since $\angle IFC = \angle ABC = 60°$ and

$$\angle ICF = \angle ICB + \angle BCF = \frac{\gamma}{2} + \frac{\alpha}{2} = 60°,$$

triangle ICF is equilateral. It follows that

$$IO + IA + IC = AE + AF > EF = 2R.$$

On the other hand,

$$\begin{aligned}AE + AF &= 2R(\cos AFE + \sin AFE) \\ &= 2\sqrt{2}R\sin(45° + \frac{1}{2}\angle AOE) \\ &= 2\sqrt{2}R\sin(15° + \frac{\gamma}{2}) \\ &< 2\sqrt{2}R\sin 75°.\end{aligned}$$

We have

$$\begin{aligned}\cos 15^\circ &= \sin(60^\circ + 15^\circ)\\ &= \frac{\sqrt{3}}{2}\cos 15^\circ + \frac{1}{2}\sqrt{1-\cos^2 15^\circ}.\end{aligned}$$

This simplifies to

$$(2-\sqrt{3})^2\cos^2 15^\circ = 1 - \cos^2 15^\circ,$$

so that

$$\cos^2 15^\circ = \frac{1}{8-4\sqrt{3}} = \frac{8+4\sqrt{3}}{16} = \frac{4+2\sqrt{3}}{8}.$$

It follows that

$$\sin 75^\circ = \cos 15^\circ = \frac{1+\sqrt{3}}{2\sqrt{2}}.$$

Hence

$$IO + IA + IC < (1+\sqrt{3})R.$$

4. (a) Suppose $m - n \geq 2$. We claim that

$$\binom{m}{3} + \binom{n}{3} \quad \text{is larger than} \quad \binom{m-1}{3} + \binom{n+1}{3}.$$

Indeed, their difference is

$$\binom{m-1}{2} - \binom{n}{2} > 0$$

since $m - 1 > n$. It follows that the 1994 points should be distributed among the 83 sets as evenly as possible. When 1994 is divided by 83, the quotient is 24 and the remainder is 2. Thus we should have 81 sets of size 24 and 2 sets of size 25, and the mimimum number of triangles is

$$81\binom{24}{3} + 2\binom{25}{3} = 168544.$$

(b) All that is needed is to show that the task can be accomplished in a set with 25 points. The same method can then be applied to each of the other sets, suppressing an arbitrary point if there are only 24 points. Label the points (x, y) where $0 \leq x, y \leq 4$.

For two points (x_1, y_1) and (x_2, y_2), if $x_1 - x_2 \equiv \pm 1 \pmod 5$, colour the segment joining them red. If $x_1 - x_2 \equiv \pm 2 \pmod 5$, colour the segment green. If $x_1 \equiv x_2 \pmod 5$, colour it yellow if $y_1 - y_2 \equiv \pm 1 \pmod 5$, and colour it blue if $y_1 - y_2 \equiv \pm 2 \pmod 5$.

Consider any three points (x_1, y_1), (x_2, y_2) and (x_3, y_3). If they have different x-coordinates, then all three sides are red or green. However,

$$(x_1 - x_2) + (x_2 - x_3) + (x_3 - x_1) = 0.$$

Since no combination of three ± 1 and no combination of three ± 2 can be equal to 0, there is at least one red side and one green side. If exactly two x-coordinates are the same, the side joining these two points will be blue or yellow while the other two sides will be green or red.

Finally, if they have the same x-coordinates, then all three sides are yellow or blue, but there must be at least one of each colour since

$$(y_1 - y_2) + (y_2 - y_3) + (y_3 - y_1) = 0.$$

Olympiad Paper I

1. Let $c_i = b_i - a_i$ for $1 \leq i \leq n$. Then

$$c_1 + c_2 + \cdots + c_n = 0,$$

$c_1 \leq c_2$ and $c_i + c_{i+1} \leq c_{i+2}$ for $1 \leq i \leq n-2$.

We prove by induction on n that $c_{n-1} + c_n \geq 0$, weakening the condition

$$c_1 + c_2 + \cdots + c_n = 0$$

to

$$c_1 + c_2 + \cdots + c_n \geq 0.$$

For $n = 2$, this yields $c_1 + c_2 \geq 0$ immediately. For $n = 3$, if $c_1 < 0$, then

$$c_2 + c_3 = (c_1 + c_2 + c_3) - c_1 > 0.$$

If $c_1 \geq 0$, then $c_2 \geq c_1 \geq 0$ and $c_3 \geq c_1 + c_2 \geq 0$, so that $c_2 + c_3 \geq 0$.

Suppose that the result holds for $n - 1$ and $n \geq 3$. Consider the next case with the numbers $c_1, c_2, \ldots, c_{n+1}$. If

$$c_1 + c_2 + \cdots + c_{n-1} < 0,$$

then

$$c_n + c_{n+1} = (c_1 + c_2 + \cdots + c_{n+1}) - (c_1 + c_2 + \cdots + c_{n-1}) > 0.$$

If

$$c_1 + c_2 + \cdots + c_{n-1} \geq 0,$$

then

$$c_n \geq c_{n-2} + c_{n-1} \geq 0$$

by the induction hypothesis.

Hence

$$c_1 + c_2 + \cdots + c_n \geq 0,$$

and

$$c_{n+1} \geq c_{n-1} + c_n \geq 0$$

by the induction hypothesis. It follows that $c_n + c_{n+1} \geq 0$.

2. We have

$$3f(n)(f(2n+1) - f(2n)) = f(2n) < 6f(n).$$

It follows that $f(2n+1) - f(2n) < 2$. Since $f(n)$ and $f(2n)$ are both positive, $f(2n+1) - f(2n) \geq 1$. It follows that $f(2n) = 3f(n)$ and $f(2n+1) = f(2n)+1$ for all positive integers n. We now prove by induction on n that if $n = 2^{a_r} + 2^{a_{r-1}} + \cdots + 2^{a_0}$, then

$$f(n) = 3^{a_r} + 3^{a_{r-1}} + \cdots + 3^{a_0}.$$

For $n = 1$, we have $f(2^0) = f(1) = 1 = 3^0$. Suppose the result holds for 1, 2, …, $n-1$ for some $n \geq 2$. Consider

$$n = 2^{a_r} + 2^{a_{r-1}} + \cdots + 2^{a_0},$$

where $a_r > a_{r-1} > ... > a_1 > a_0 \geq 0$.

If $a_0 \geq 1$, let $m = 2^{a_r-1} + 2^{a_{r-1}-1} + \cdots + 2^{a_0-1}$. Then

$$\begin{aligned} f(n) &= f(2m) \\ &= 3f(m) \\ &= 3(3^{a_r-1} + 3^{a_{r-1}-1} + \cdots + 3^{a_0-1}) \\ &= 3^{a_r} + 3^{a_{r-1}} + \cdots + 3^{a_0}. \end{aligned}$$

If $a_0 = 0$, let $m = 2^{a_r-1} + 2^{a_{r-1}-1} + \cdots + 2^{a_1-1}$. Then

$$\begin{aligned} f(n) &= f(2m+1) \\ &= 3f(m) + 1 \\ &= 3(3^{a_r-1} + 3^{a_{r-1}-1} + \cdots + 3^{a_1-1}) + 3^0 \\ &= 3^{a_r} + 3^{a_{r-1}} + \cdots + 3^{a_0}. \end{aligned}$$

This completes the induction argument. Now let

$$k = \sum_{b \in B} 2^b \quad \text{and} \quad \ell = \sum_{c \in C} 2^c.$$

Then

$$f(k) + f(\ell) = 2 \sum_{a \in B \cap C} 3^a + \sum_{b \in B-C} 3^b + \sum_{c \in C-B} 3^c.$$

If

$$f(k) + f(\ell) = 293 = 3^5 + 3^3 + 2 \cdot 3^2 + 3^1 + 2 \cdot 3^0,$$

we must have $B \cap C = \{0, 2\}$. Since $k < \ell$, we must have $5 \in C$. There are four ways to distribute 1 and 3 between B and C, yielding the following four solutions:

$$\begin{array}{ll} k = 2^2 + 2^0 = 5, & \ell = 2^5+2^3+2^2+2^1+2^0 = 47, \\ k = 2^2 + 2^1 + 2^0 = 7, & \ell = 2^5 + 2^3 + 2^2 + 2^0 = 45, \\ k = 2^3 + 2^2 + 2^0 = 13, & \ell = 2^5 + 2^2 + 2^1 + 2^0 = 39, \\ k = 2^3 + 2^2 + 2^1 + 2^0 = 15, & \ell = 2^5 + 2^2 + 2^0 = 37. \end{array}$$

3. Let P_1, P_2, …, P_n be points on a line in that order. We allow $P_i = P_{i+1}$. We wish to minimize $XP_1 + XP_2 + \cdots + XP_n$ where X ranges over all points on that line. Note that we have $XP_1 + XP_n = P_1P_n$ if X is on the segment P_1P_n, and greater otherwise. Similarly, the minimum value of $XP_2 + XP_{n-1}$ is P_2P_{n-1}, attained if and only if X is on the segment P_2P_{n-1}. It follows that if $n = 2m+1$, the overall minimum is attained if and only if $X = P_{m+1}$, and this minimum is

$$P_1P_n + P_2P_{n-1} + \cdots + P_mP_{m+2}.$$

If $n = 2m$, then X can be any point on the segment P_mP_{m+1}, and the minimum sum is

$$P_1P_n + P_2P_{n-1} + \cdots + P_mP_{m+1}.$$

Consider now

$$f(x, y) = \sum_{i=1}^{10} |x + y - 10i|.$$

We have ten points 10, 20, …, 100. The minimum value of $f(x, y)$ is

$$10((10 + 9 + 8 + 7 + 6) - (1 + 2 + 3 + 4 + 5)) = 250,$$

attained if and only if $50 \leq x + y \leq 60$. For

$$g(x, y) = \sum_{j=1}^{10} |3x - 6y - 36j|,$$

we have ten points 12, 24, ..., 120. The minimum value of $g(x,y)$ is

$$36((10+9+8+7+6)-(1+2+3+4+5)) = 900,$$

attained if and only if $60 \le x-2y \le 72$. Finally, for

$$h(x,y) = \sum_{k=1}^{10} k|19x-95y-95k|,$$

we have fifty-five points consisting of one at 5, two at 10, three at 15 and so on, to ten at 50. The twenty-eighth point is the last one at 7. Hence the minimum value of $h(x,y)$ is

$$95((10^2+9^2+8^2)-(6\cdot 7+6^2+5^2+4^2+3^2+2^2+1^2)) = 10640,$$

attained if and only if $x+5y=35$. It is easy to verify that $(55,-4)$ satisfies all three conditions. Hence these minimum values can be attained simultaneously, so that the minimum value of $f(x,y)g(x,y)h(x,y)$ is

$$250\cdot 900\cdot 10640 = 2394000000.$$

Olympiad Paper II

1. Let the centre of the small sphere be O and its radius be r. Let A and B be the centres of the spheres of radii 3, and C and D be the centres of the spheres of radii 2. Let E be the midpoint of AB and F be the midpoint of CD. Now A and B are symmetric to each other with respect to the plane CDE, while C and D are symmetric with respect to the plane ABF. These two planes intersect along EF. By symmetry, O lies on the segment EF. Now

$$CE = \sqrt{AC^2-AE^2} = \sqrt{5^2-3^2} = 4$$

and

$$EF = \sqrt{CE^2-CF^2} = \sqrt{4^2-2^2} = 2\sqrt{3}.$$

We also have

$$OE = \sqrt{OA^2-AE^2} = \sqrt{(r+3)^2-3^2} = \sqrt{r^2+6r}$$

and

$$OF = \sqrt{OC^2-CE^2} = \sqrt{(r+2)^2-2^2} = \sqrt{r^2+4r}.$$

Squaring both sides of the equation

$$\sqrt{r^2+6r} = 2\sqrt{3}-\sqrt{r^2+4r},$$

we have

$$12-2r = 4\sqrt{3(r^2+4r)}.$$

Squaring both sides again, we have

$$11r^2+60r-36 = (11r-6)(r+6) = 0.$$

Hence $r = \frac{6}{11}$.

2. Let $\langle a_1, a_2, \ldots, a_{10}\rangle$ be the permutation of the fixed positive integers $x_1 < x_2 < \cdots < x_{10}$ which minimizes

$$a_1a_2 + a_2a_3 + \cdots + a_9a_{10} + a_{10}a_1.$$

We may assume that $a_{10} = x_{10}$. We claim that either a_1 or a_9 is x_1. Suppose on the contrary that $x_1 = a_j$ for some j, $1 < j < 9$. Define $b_i = a_{j+1-i}$ for $1 \le i \le j$ and $b_i = a_i$ for $j+1 \le i \le 10$. Then

$$\begin{aligned} & (a_1a_2+a_2a_3+\cdots+a_9a_{10}+a_{10}a_1) \\ & \quad -(b_1b_2+b_2b_3+\cdots+b_9b_{10}+b_{10}b_1) \\ = \ & (a_ja_{j+1}+a_{10}a_1)-(a_1a_{j+1}+a_ja_{10}) \\ = \ & (a_{10}-a_j)(a_1-a_j) \\ > \ & 0. \end{aligned}$$

This contradicts the minimality assumption on $\langle a_1, a_2, \ldots, a_n\rangle$, and the claim is justified. By symmetry, we may take $a_1 = x_1$. The same argument yields $a_9 = x_2$, $a_2 = x_9$, $a_8 = x_8$, $a_3 = x_3$, $a_7 = x_4$, $a_4 = x_7$, $a_6 = x_6$ and $a_5 = x_5$. We now determine positive integers $x_1 < x_2 < \cdots < x_{10}$ with sum 1995 such that

$$x_1x_9 + x_9x_3 + x_3x_7 + x_7x_5 + x_5x_6$$

$$+x_6x_4 + x_4x_8 + x_8x_2 + x_2x_{10} + x_{10}x_1$$

is a minimum. We claim that $x_i = i$ for $1 \le i \le 9$ and $x_{10} = 1950$. Suppose on the contrary that for some j, $1 \le j \le 9$, $x_i = i$ for $1 \le i \le j-1$ but $x_j \ne j$. Let the two terms in the sum involving x_j be x_px_j and x_jx_q. Then $x_p + x_q > x_1 + x_2$. Define $y_i = x_i$ for $1 \le i \le 9$ except for $y_j = x_j - 1$ and define $y_{10} = x_{10}+1$. Then

$$\begin{aligned} & (x_1x_9+x_9x_3+\cdots+x_2x_{10}+x_{10}x_1) \\ & \quad -(y_1y_9+y_9y_3+\cdots+y_2y_{10}+y_{10}y_1) \\ = \ & (x_p+x_q)-(x_1+x_2) \\ > \ & 0. \end{aligned}$$

This is a contradiction, and the claim is justified. It follows that the desired minimum is

$$1\cdot 9+9\cdot 3+3\cdot 7+7\cdot 5+5\cdot 6+6\cdot 4$$

$$+4\cdot 8+8\cdot 2+2\cdot 1950+1950\cdot 1=6044.$$

3. Let $A_1^{(0)}A_2^{(0)}\ldots A_n^{(0)}$ be a regular n-gon with centre O. Label $A_i^{(0)}$ with $x_i^{(0)}$ for $1\le i\le n$. The numerical pattern has a unique axis of symmetry passing through O and perpendicular to $A_1^{(0)}A_n^{(0)}$.

For $k\ge 1$, let $A_i^{(k)}$ be the midpoint of $A_i^{(k-1)}A_{i+1}^{(k-1)}$ for $1\le i\le n$, interpreting $A_{n+1}^{(k-1)}$ as $A_1^{(k-1)}$. Label $A_i^{(k)}$ with $x_i^{(k)}$, $1\le i\le n$, according to the given rule. Then the new numerical pattern has the same unique axis of symmetry. Now rotate the n-gon

$$A_1^{(k)}A_2^{(k)}\ldots A_n^{(k)}$$

about O through an angle of $\frac{\pi}{n}$ so that $A_i^{(k)}$ is collinear with O and $A_i^{(k-1)}$, $1\le i\le n$. Then delete the n-gon

$$A_1^{(k-1)}A_2^{(k-1)}\ldots A_n^{(k-1)}$$

along with its associated labels. Now the new numerical pattern still has a unique axis of symmetry, obtained from the old one by a rotation about O through an angle of $\frac{\pi}{n}$. Suppose $X_m=X_0$ for some positive integer m. Then the unique axis of symmetry must coincide with its initial position.

It follows that $m\frac{\pi}{n}=\ell\pi$ for some positive integer ℓ, so that $m=\ell n$ as desired.

1995/96
Paper I.

Section 1. Questions with Multiple Choices.

1. Since $a_8>a_{13}$, the common difference is a negative number $-d$ where $d>0$. We have

$$3(a_1-7d)=5(a_1-12d)$$

or

$$a_1=\frac{39d}{2}.$$

Hence

$$a_{20}=a_1-19d=\frac{d}{2}$$

while

$$a_{21}=a_1-20d=-\frac{d}{2}.$$

It follows that S_n is maximum when $n=20$.

2. Since the greatest common divisor of 1995 and 20 is 5, the number of distinct values is $20\div 5=4$.

3. Let the 100 people have different heights and different weights, with their heights in reverse order of their weights. Then all 100 are strong.

4. Clearly, $k\ge 0$. If $k=0$, then $x=2n$ is the only root. Hence $k>0$. Squaring yields

$$(x-2n)^2=k^2x.$$

Now the parabola $y=(x-2n)^2$ and the straight line $y=k^2x$ must intersect twice on the interval $(2n-1,2n+1)$. We must have

$$(x-2n)^2>k^2x$$

at $x=2n-1$ and

$$(x-2n)^2\ge k^2x$$

at $x=2n+1$. The stronger of these two inequalities is

$$\frac{1}{\sqrt{2n+1}}\ge k.$$

5. Since $\frac{\pi}{4} < 1$,

$$\cos 1 < \sin 1 < 1 < \tan 1.$$

Hence

$$\log_{\sin 1} \tan 1 < \log_{\cos 1} \tan 1 < 0,$$
$$0 < \log_{\cos 1} \sin 1 < \log_{\cos 1} \cos 1 = 1$$

and

$$1 = \log_{\sin 1} \sin 1 < \log_{\sin 1} \cos 1.$$

6. Since

$$\angle APB = \angle BPC = \angle CPA,$$

denote their common value by ϕ. Now O is equidistant from PAB, PBC and PCA. Denote this common distance by d. Finally, let θ be the angle made by one of PA, PB and PC with the plane determined by the other two. The area of triangle PRS is

$$\frac{1}{2} PR \cdot PS \sin\phi.$$

The distance from Q to PRS is $PQ \sin\theta$. Hence the volume of $PQRS$ is

$$\frac{1}{6} PQ \cdot PR \cdot PS \sin\phi \sin\theta.$$

We note $PQRS$ is also the disjoint union of $OPQR$, $OPRS$ and $OPSQ$. Another expression for the volume of $PQRS$ is

$$\frac{1}{6}(PQ \cdot PR + PR \cdot PS + PS \cdot PQ) d \sin\phi.$$

It follows that

$$\frac{1}{PQ} + \frac{1}{PR} + \frac{1}{PS} = \frac{\sin\theta}{d}.$$

Section 2. Questions requiring Answers Only:

1. Let $\alpha = a + bi$ and $\beta = a - bi$. From

$$|\alpha - \beta| = 2\sqrt{3},$$

we have $b = \sqrt{3}$. Since

$$\frac{\alpha}{\beta^2} = \frac{\alpha^3}{(\alpha\beta)^2}$$

is real and so is $\alpha\beta$, we conclude that α^3 is real. Since

$$(a + bi)^3 = a(a^2 - 3b^2) + b(3a^2 - b^2)i,$$

we have $3a^2 - b^2 = 0$ so that $|a| = 1$. Hence

$$|\alpha| = \sqrt{a^2 + b^2} = 2.$$

2. Let R be the radius of the sphere. Let r be the base radius of an inscribed cone and h be its height. Considering a plane section through the axis of the cone, we have $r^2 = h(2R - h)$. Now the volume of the cone is

$$\frac{1}{3}\pi r^2 h = \frac{\pi}{6} h \cdot h(4R - 2h).$$

Since

$$h + h + (4R - 2h) = 4R$$

is constant, the maximum volume occurs when $h = \frac{4R}{3}$, and its value is $\frac{32\pi R^3}{81}$. The volume of the sphere, on the other hand, is $\frac{4\pi R^3}{3}$. Hence the desired ratio is 8:27.

3. Since $\lfloor \log x \rfloor \le \log x$, we have

$$(\log x)^2 - \log x - 2 \le 0.$$

This is equivalent to $-1 \le \log x \le 2$.

When $-1 \le \log x < 0$, $\lfloor \log x \rfloor = -1$ so that $\log x = \pm 1$. However, $\log x = 1$ is not in the specified range. Hence $\log x = -1$ and $x = \frac{1}{10}$.

When $0 \le \log x < 1$, $\lfloor \log x \rfloor = 0$ so that $\log x = \pm\sqrt{2}$. Neither value is in range.

When $1 \le \log x < 2$, $\lfloor \log x \rfloor = 1$ so that $\log x = \pm\sqrt{3}$. From the only acceptable case $\log x = \sqrt{3}$, we have $x = 10^{\sqrt{3}}$.

Finally, when $\log x = 2$, $\lfloor \log x \rfloor = 2$ and the equation is satisfied. Thus $x = 100$ is the third real root.

4. The number of lattice points in the region bounded by $x = 0$, $y = 0$ and $x + y = 100$ is

$$\sum_{k=0}^{100} (k + 1) = 5151.$$

This includes those on the boundary.

The number of lattice points in the region bounded by $y = 0$, $y = \frac{x}{3}$ and $x = 75$ is

$$\sum_{k=1}^{75} \left\lfloor \frac{k-1}{3} \right\rfloor = 3\sum_{k=1}^{25} k = 975.$$

This includes those on the boundary other than $y = \frac{x}{3}$.

The number of lattice points in the region bounded by $x+y=100$, $x=75$ and $y=0$ is

$$\sum_{k=76}^{100}(101-k)=325.$$

This includes those on the boundary except $x=75$.

By symmetry, the number of lattice points in the region bounded by $y=\frac{x}{3}$, $y=3x$ and $x+y=100$ is

$$5151-2(975+325)=2551.$$

5. The top vertex can be painted in 5 ways and one of the bottom vertices in 4 ways. If the opposite vertex is painted the same colour, each of the other two can be painted in 3 ways. If not, the opposite vertex can be painted in 3 ways and each of the other two in 2 ways.

The total is

$$5\times4\times3\times3+5\times4\times3\times2\times2=420.$$

6. Remove first the

$$\left\lfloor\frac{1995}{15}\right\rfloor=133$$

multiples of 15. Then add back the

$$\left\lfloor\frac{1995}{15^2}\right\rfloor=8$$

multiples of 15^2. This yields a subset with 1870 elements which satisfies the hypothesis.

On the other hand, for $9\le k\le133$, both k and $15k$ are in the original set and at least one from each of the

$$133-9+1=125$$

pairs must be removed, so that the subset can have at most

$$1995-125=1879$$

elements.

Paper II

1. Since both the line and the parabola pass through the origin, we only have to determine their other point of intersection where $x\ne0$. Eliminating y, we have

$$(2\sin\theta-\cos\theta+3)x^2-(8\sin\theta-\cos\theta+1)x=0.$$

Since

$$2\sin\theta-\cos\theta+3=\sqrt{5}\sin\left(\theta-\arctan\frac{1}{2}\right)+3>0,$$

we have

$$x=\frac{8\sin\theta-\cos\theta+1}{2\sin\theta-\cos\theta+3}.$$

Let

$$\sin\theta=\frac{2t}{1+t^2}.$$

Then

$$\cos\theta=\frac{1-t^2}{1+t^2},$$

and we have

$$x=\frac{8t+1}{2t^2+2t+1}.$$

Hence

$$2xt^2+2(x-4)t+(x-1)=0.$$

Since t is real, the discriminant is

$$4(x-4)^2-4(x-1)2x\ge0.$$

This is equivalent to $(x+8)(x-2)\le0$, so that $-8\le x\le2$. The maximum value of $|x|$ is 8. Hence the maximum distance is

$$\sqrt{8^2+16^2}=8\sqrt{5}.$$

2. The given equation may be factored as

$$(x-1)(5x^2-5px+66p-1)=0.$$

Hence one of its roots is 1.

Let the other two be $u\le v$. Then

$$u+v=p\quad\text{and}\quad uv=\frac{66p-1}{5}.$$

Hence

$$25uv = 330(u+v) - 5$$

so that

$$(5u-66)(5v-66) = 4351 = 19 \cdot 229.$$

Since u and v are positive integers, $5u-66=19$ and $5v-66=229$.

Hence $u=17,\ v=59$ and

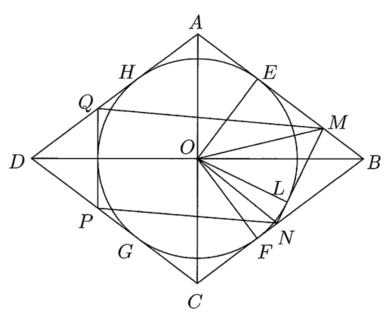
$$p = u+v = 76.$$

3. Let O be the centre of the circle and let L the point of tangency of the circle with MN. Since we have $\angle AOE = \angle COF$, $\angle EOM = \angle MOL$ and $\angle LON = \angle FON$,

$$2\angle AOE + 2\angle EOM + 2\angle FON = 180^\circ.$$

Hence

$$\angle FON = 90^\circ - \angle AOE - \angle EOM = \angle BOM.$$



It follows that

$$\angle AMO = \angle ABO + \angle BOM = \angle COF + \angle FON = \angle CON.$$

Along with $\angle MAO = \angle OCN$, triangles MAO and OCN are similar, so that

$$\frac{AM}{AO} = \frac{CO}{CN}$$

or $AM \cdot CN = AO \cdot CO$.

In the same way, we can prove that $AQ \cdot CP = AO \cdot CO$, so that

$$\frac{AM}{AQ} = \frac{CP}{CN}.$$

Along with $\angle MAQ = \angle PCN$, triangles MAQ and PCN are similar.

It follows that MQ is parallel to NP.

4. We first prove that there is a monochromatic right triangle. Let P and Q be any two points of the same colour and let $PQRS$ be any rectangle.

If either R or S has the same colour as P and Q, we have a monochromatic right triangle. Otherwise, any point on PS will form a monochromatic right triangle with either P and Q or R and S. So let (0,0), $(1995a, 0)$ and $(0, 1995b)$ be the vertices of a monochromatic right triangle T.

Expand it into a rectangle R by adding $(1995a, 1995b)$ as the fourth vertex, and divide it into 1995^2 small rectangles of equal sizes, all similar to R.

If any of them have three vertices of the same colour, then we have a monochromatic right triangle similar to T and having side lengths $\frac{1}{1995}$ the side lengths of T.

Suppose this is not the case. Then $(i, 0)$, $0 \le i \le 1995$, cannot be alternating in colour as otherwise (0,0) and $(1995a, 0)$ will have different colours.

Hence there exists a value $i, 0 \le i \le 1994$, such that $(ia, 0)$ and $((i+1)a, 0)$ have the same colour. Then (ia, b) and $((i+1)a, b)$ must both be of the other colour.

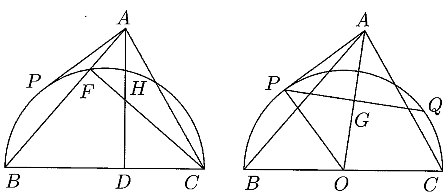
It follows that (ia, kb) and $((i+1)a, kb)$ have the same colour for $0 \le k \le 1995$. Similarly, there exists a value j, $0 \le j \le 1994$, such that $(0, jb)$ and $(0, (j+1)b)$ have the same colour, so that (ka, jb) and $(ka, (j+1)b)$ have the same colour for $0 \le k \le 1995$.

This is a contradiction since (ia, jb), $(ia, (j+1)b)$, $((i+1)a, jb)$ and $((i+1)a, (j+1)b)$ would all have the same colour.

Olympiad Paper I

1. Let AH cut BC at D and let AB cut the circle at F. Then $\angle BFC = 90° = \angle ADB$. Hence $BDHF$ is a cyclic quadrilateral, so that $AH \cdot AD = AF \cdot AB = AP^2$.

Let O be the midpoint of BC and let AO cut PQ at G.



Then $\angle APO = 90° = \angle AGP$. Hence triangles APO and AGP are similar, so that $AP^2 = AG \cdot AO$. From $AG \cdot AO = AH \cdot AD$, $DHGO$ is a cyclic quadrilateral. Since $\angle HDO = 90°$, we have $\angle HGO = 90° = \angle QGO$.

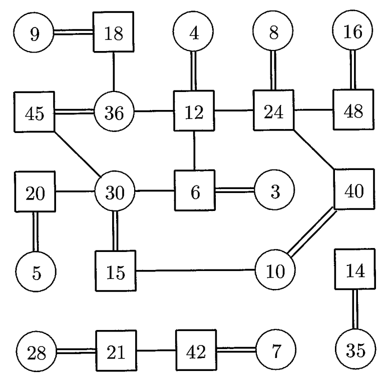
It follows that H lies on PQ.

2. Suppose a and b are positive integers such that $a+b$ divides ab. Let their greatest common divisor be d. Then $a = dk$ and $b = d\ell$ for some positive integers k and ℓ which are relatively prime. Hence $k+\ell$ is relatively prime to $k\ell$.

However, $d(k+\ell)$ divides $d^2k\ell$. Hence $k+\ell$ must divide d, so that $k+\ell \leq d$. Since $a, b \in S$, $d(k+\ell) = a+b \leq 99$. It follows that $3 \leq k+\ell \leq 9$. The following 23 pairs of distinct positive integers (a,b) in S have the desired property:

$k+\ell$	(a,b)
3	(3,6), (6,12), (9,18), (12,24), (15,30), (18,36), (21,42), (24,48);
4	(4,12), (8,24), (12,36), (16,48);
5	(5,20), (10,40), (10,15), (20,30), (30,45);
6	(6,36);
7	(7,42), (14,35), (21,28);
8	(24,40);
9	(36,45).

We construct a graph where the 23 edges represent the above pairs and the 24 vertices represent the individual numbers involved.



The twelve edges marked by double lines form an independent set, that is, no two share a vertex. If we take any subset of S of size $k \geq 39$, we will only be missing at most eleven numbers. Hence this subset must contain both vertices of one of the twelve independent edges, which means that it contains two distinct positive integers a and b such that $a+b$ divides ab.

On the other hand, the twelve vertices marked by squares form a covering set, that is, every edge has at least one of them as a vertex. If we remove these 12 elements from S, we will leave behind a subset of S of size 38 in which $a+b$ does not divide ab for any two distinct elements a and b.

It follows that the desired minimum value is $k = 39$.

3. We have $f(0) = 0$ by taking $x = y = 0$, and

$$f(x^3) = x(f(x))^2$$

by taking $y = 0$. This may be rewritten as

$$f(x) = \sqrt[3]{x}(f(\sqrt[3]{x}))^2,$$

showing that x and $f(x)$ have the same sign. Let S be the set of real numbers k such that $f(kx) = kf(x)$. Clearly, $1 \in S$. If $k \in S$, then

$$kx(f(x))^2 = kf(x^3) = f(kx^3) = f((\sqrt[3]{k}x)^3) = \sqrt[3]{k}x(f(\sqrt[3]{k}x))^2,$$

which is equivalent to

$$(\sqrt[3]{k}f(x))^2 = (f(\sqrt[3]{k}x))^2.$$

By the sign consideration discussed above, we have

$$\sqrt[3]{k}f(x) = f(\sqrt[3]{k}x),$$

so that $\sqrt[3]{k} \in S$. We claim that if $h, k \in S$, then $h + k \in S$. Indeed,

$$\begin{aligned} f((h+k)x) &= f((\sqrt[3]{hx})^3 + (\sqrt[3]{kx})^3) \\ &= (\sqrt[3]{hx} + \sqrt[3]{kx})((f(\sqrt[3]{hx})^2 \\ &\quad -f(\sqrt[3]{hx})f(\sqrt[3]{kx}) + (f(\sqrt[3]{kx}))^2) \\ &= (\sqrt[3]{h} + \sqrt[3]{k})\sqrt[3]{x}(\sqrt[3]{h^2} - \sqrt[3]{hk} + \sqrt[3]{k^2})(f(\sqrt[3]{x}))^2 \\ &= (h+k)f(x). \end{aligned}$$

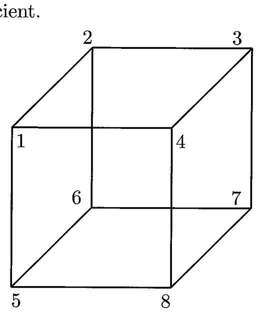
Since $1 \in S$, $1 + 1 = 2 \in S$. It follows that S contains all positive integers. In particular, $1996 \in S$ and $f(1996x) = 1996(f(x))$ for all x.

Olympiad Paper II

1. Let b be the number of concerts and λ be the number of concerts in which each pair of singers performs together. Counting the total number of appearances of such pairs in two different ways, we have

$$\lambda\binom{8}{2} = b\binom{4}{2}$$

or $14\lambda = 3b$. Since 3 and 14 are relatively prime, b must be divisible by 14, so that $b \geq 14$. We now construct a diagram which shows that $b = 14$ is sufficient.



Represent the 8 singers by the vertices of a cube. The 14 concerts are represented by the faces

$$(1,2,3,4), (5,6,7,8), (1,2,5,6), (3,4,7,8), (1,4,5,8), (2,3,6,7),$$

the cross-sections

$$(1,3,5,7), (2,4,6,8), (1,2,7,8), (3,4,5,6), (1,4,6,7), (2,3,5,8)$$

and the tetrahedra (1,3,6,8), (2,4,5,7).

2. By the Arithmetic-Geometric Mean Inequality, we have

$$\begin{aligned} &\sqrt{1 + x_0 + x_1 + \cdots + x_{i-1}}\sqrt{x_1 + x_{i+1} + \cdots + x_n} \\ &\leq \frac{1}{2}(1 + x_0 + x_1 + \cdots + x_n) = 1 \end{aligned}$$

for $1 \leq i \leq n$. Hence

$$\begin{aligned} &\sum_{i=1}^{n} \frac{x_i}{\sqrt{1 + x_0 + x_1 + \cdots + x_{i-1}}\sqrt{x_i + x_{i+1} + \cdots + x_n}} \\ &\geq \sum_{i=1}^{n} x_i = 1. \end{aligned}$$

For $0 \leq i \leq n$, since $0 \leq x_0 + x_1 + \cdots + x_i \leq 1$, we may let

$$\theta_i = \arcsin(x_0 + x_1 + \cdots + x_n).$$

Then $0 = \theta_0 < \theta_1 < \cdots < \theta_n = \frac{\pi}{2}$. Now

$$\begin{aligned} \cos\theta_{i-1} &= \sqrt{1 - \sin^2\theta_{i-1}} \\ &= \sqrt{1 - (x_0 + x_1 + \cdots + x_{i-1})^2} \\ &= \sqrt{1 + x_0 + x_1 + \cdots + x_{i-1}}\sqrt{x_i + x_{i+1} + \cdots + x_n}. \end{aligned}$$

On the other hand,

$$\begin{aligned} x_i &= \sin\theta_i - \sin\theta_{i-1} \\ &= 2\cos\frac{\theta_i + \theta_{i-1}}{2}\sin\frac{\theta_i - \theta_{i-1}}{2} \\ &< 2\cos\theta_{i-1}\sin\frac{\theta_i - \theta_{i-1}}{2} \\ &< 2\cos\theta_{i-1}\left(\frac{\theta_i - \theta_{i-1}}{2}\right). \end{aligned}$$

The last step follows from $\sin x < x$ whenever $0 \le x \le \frac{\pi}{2}$. Finally,

$$\sum_{i=1}^{n} \frac{x_i}{\sqrt{1+x_0+x_1+\cdots+x_{i-1}}\sqrt{x_i+x_{i+1}+\cdots+x_n}}$$

$$= \sum_{i=1}^{n} \frac{x_i}{\cos\theta_{i-1}} < \sum_{i=1}^{n}(\theta_i - \theta_{i-1}) = \frac{\pi}{2}.$$

3. Set up a coordinate system with C at (0,0), A at $(\sqrt{3}, 0)$ and B at (0,1). Let D be any point on BC and let $BD = d$. Take E on AC such that $CE = \frac{\sqrt{3}d}{2}$, and take F on AB such that $BF = 1 - \frac{d}{2}$. By Pythagoras' Theorem,

$$DE^2 = (1-d)^2 + \left(\frac{\sqrt{3}d}{2}\right)^2 = \frac{7}{4}d^2 - 2d + 1.$$

By the Law of Cosines,

$$\begin{aligned} DF^2 &= d^2 + \left(1 - \frac{d}{2}\right)^2 - 2d\left(1 - \frac{d}{2}\right)\cos 60^\circ \\ &= \frac{7}{4}d^2 + 2d - 1 \end{aligned}$$

and

$$\begin{aligned} EF^2 &= \left(\sqrt{3} - \frac{\sqrt{3}d}{2}\right)^2 + \left(1 + \frac{d}{2}\right)^2 \\ &\quad -2\left(\sqrt{3} - \frac{\sqrt{3}d}{2}\right)\left(1 + \frac{d}{2}\right)\cos 30^\circ \\ &= \frac{7}{4}d^2 + 2d - 1. \end{aligned}$$

Since d can take any value from 0 to 1, this means that we can always inscribe an equilateral triangle DEF in ABC from any point F on AB such that $\frac{1}{2} \le BF \le 1$. Now

$$\frac{7}{4}d^2 + 2d - 1 = \frac{7}{4}(d - \frac{4}{7})^2 + \frac{3}{7}.$$

It follows that the minimum value of $DE = EF = FD$ is $\sqrt{\frac{3}{7}}$. Let P be the point on AB with y-coordinate $\sqrt{\frac{3}{7}}$ and Q be the point on AB with x-coordinate $\sqrt{\frac{3}{7}}$. Then P lies on the segment BQ.

Let XYZ be any triangle inscribed in ABC, with X on BC, Y on CA and Z on AB. If Z lies on the segment BP, then

$$ZY \ge \sqrt{\frac{3}{7}}.$$

If Z lies on the segment AQ, then

$$ZX \ge \sqrt{\frac{3}{7}}.$$

Suppose Z lies on the segment PQ. Then $\frac{1}{2} < ZB < 1$. Hence we may inscribe an equilateral triangle DEZ in ABC as before. Now the y-coordinate of Z is greater than that of D, and the x-coordinate of Z is greater than that of E. If X lies on the segment CD, then

$$XZ \ge DZ \ge \sqrt{\frac{3}{7}}.$$

If Y lies on the segment CE, then

$$YZ \ge EZ \ge \sqrt{\frac{3}{7}}.$$

If X lies on the segment BD and Y lies on the segment AE, then

$$XY \ge DE \ge \sqrt{\frac{3}{7}}.$$

Hence $\sqrt{\frac{3}{7}}$ is the desired minimum.

1996/97
Paper I.

Section 1. Questions with Multiple Choices.

1. Eliminating x from

$$9-9(y-1)^2=9x^2=9-(y+1)^2,$$

we have

$$0=8y^2-20y+8=4(2y-1)(y-2).$$

When $y=2$, $x=0$. When $y=\frac{1}{2}$, $x=\pm\sqrt{3}$. It is easy to verify that these three points determine an equilateral triangle.

2. Let P_n denote the product of the first n terms. Since $1536=2^9 3$, we have $a_{10}=-3$, $a_{11}=\frac{3}{2}$ and $a_{12}=-\frac{3}{4}$. Hence

$$|P_1|<\cdots<|P_{11}|>|P_{12}|>|P_{13}|>\cdots.$$

Since P_{10} and P_{11} are negative while $P_{12}=\frac{27}{8}P_9>0$, P_n is maximum when $n=12$.

3. For any odd prime $p=2k+1$, take $n=k^2$. Then

$$\sqrt{p+n}+\sqrt{n}=\sqrt{(k+1)^2}+\sqrt{k^2}=k+1+k=p.$$

4. Let $y=-x$. Then we have $0<y<\frac{1}{2}$, $a_1=\cos(\sin\pi y)>0$, $a_2=\sin(\cos\pi y)>0$ and $a_3=\cos(1-y)\pi<0$. Since

$$\sin\pi y+\cos\pi y=\sqrt{2}\sin\left(y+\frac{1}{4}\right)\pi\le\sqrt{2}<\frac{\pi}{2},$$

we have $0<\cos\pi y<\frac{\pi}{2}-\sin\pi y<\frac{\pi}{2}$. It follows that

$$a_3<0<a_2<\sin\left(\frac{\pi}{2}-\sin\pi y\right)=a_1.$$

5. On [1,2],

$$g(x)=\frac{x}{2}+\frac{x}{2}+\frac{1}{x^2}\ge 3\left(\sqrt[3]{\frac{1}{4}}\right)=\frac{3}{2}(\sqrt[3]{2}),$$

with equality occurring at $\frac{x}{2}=\frac{1}{x^2}$ or $x=\sqrt[3]{2}$. Hence the minimum value of

$$f(x)=\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)$$

also occurs at $x=\sqrt[3]{2}$, which is not an endpoint. It follows that

$$\sqrt[3]{2}-\frac{p}{2}=0$$

so that

$$p=-2(\sqrt[3]{2}).$$

Moreover, the minimum value is

$$q-\frac{p^2}{4}=\frac{3}{2}(\sqrt[3]{2}),$$

so that

$$q=\frac{3}{2}(\sqrt[3]{2})+\sqrt[3]{4}.$$

Note that

$$f(1)=1-\frac{1}{2}(\sqrt[3]{2})+\sqrt[3]{4}$$

while

$$f(2)=4-\frac{5}{2}(\sqrt[3]{2})+\sqrt[3]{4}.$$

Since

$$f(2)-f(1)=3-2(\sqrt[3]{2})>0,$$

the maximum value of $f(x)$ on [1,2] is

$$4-\frac{5}{2}(\sqrt[3]{2})+\sqrt[3]{4}.$$

6. Let O be the centre of the small sphere, Q be the centre of the large sphere, and C be the point of intersection of the axis of the cone with the horizontal plane Π passing through Q. Since the large sphere is tangent to the top face of the inverted cone, the portion of the axis above C has length 3. The portion below O has length $2\sqrt{2}$. Hence $OC=3$.

Since $OQ=2+3=5$,

$$QC=\sqrt{5^2-3^2}=4.$$

Now the two tangents on Π from C to the large sphere form an angle θ such that

$$\sin\frac{\theta}{2}=\frac{3}{4}.$$

Since $\frac{1}{\sqrt{2}}<\frac{3}{4}<1$, we have

$$90^\circ<\theta<120^\circ.$$

The total number of large spheres that can fit is less than

$$\frac{360^\circ}{90^\circ} = 4$$

but can be as large as

$$\frac{360^\circ}{120^\circ} = 3.$$

Hence there is room for 2 more.

Section 2. Questions requiring Answers Only:

1. Since

$$\log_{\frac{1}{x}} 10 = -\frac{1}{\log x},$$

the inequality defining the set is equivalent to

$$1 \le \log x < 2$$

or

$$10 \le x < 100.$$

It follows that the set has 90 elements and $2^{90} - 1$ non-empty subsets.

2. In the complex plane, let O, C, Z_1 and Z_2 be represented by 0, i, z_1 and z_2 respectively. Then Z_1 and Z_2 lie on the circle with centre C and passing through O.

Since $\arg z_1 = \frac{\pi}{6}$,

$$\angle COZ_1 = \frac{\pi}{3} = \angle CZ_1O,$$

so that triangle COZ_1 is equilateral. Since the real part of $\overline{z_1}z_2$ is 0, $\arg z_2 - \frac{\pi}{6} = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, so that $\arg z_2 = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$. The latter is rejected since Z_2 is above the x-axis. Hence $\angle COZ_2 = \frac{\pi}{6}$ and $\angle Z_1OZ_2 = \frac{\pi}{2}$, so that

$$OZ_2 = \sqrt{Z_1Z_2^2 - OZ_1^2} = \sqrt{3}.$$

It follows that

$$z_2 = \sqrt{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2} + \frac{3}{2}i.$$

3. Let O be the origin and C be the point [2,0]. For any point P on the polar curve,

$$\begin{aligned} CP^2 &= OP^2 + OC^2 - 2OP \cdot OC\cos\theta \\ &= (1+\cos\theta)^2 + 4 - 4(1+\cos\theta)\cos\theta \\ &= \frac{16}{3} - 3\left(\cos\theta + \frac{1}{3}\right)^2. \end{aligned}$$

The maximum value of CP is $\frac{4}{\sqrt{3}}$, occurring at $\theta = \arccos(-\frac{1}{3})$. Now the region swept over is a circle of radius $\frac{4}{\sqrt{3}}$. Hence its area is $\frac{16\pi}{3}$.

4. Let the common base be an equilateral triangle ABC with side length $2a$. Let P and Q be the other two vertices. Then all other edges have a common length b. Let O be the centre of ABC.

Then $OA = \frac{2\sqrt{3}a}{3}$ so that

$$OP = \sqrt{PA^2 - OA^2} = \sqrt{b^2 - \frac{4a^2}{3}}.$$

Hence

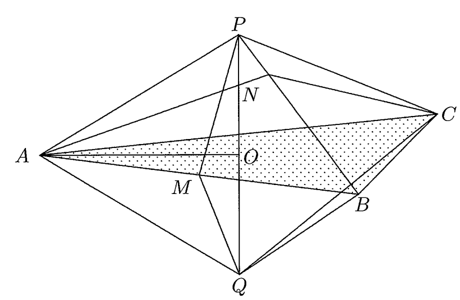
$$PQ = 2\sqrt{b^2 - \frac{4a}{3}}.$$

Let M be the midpoint of AB. Then $PM = QM$.

Let N be the foot of perpendicular from A to PB. Then $AN = CN$. Moreover,

$$\frac{PM}{AN} = \frac{PB}{AB} = \frac{b}{2a}.$$

The dihedral angle $\angle ANC$ between two faces on the same side of ABC is equal to the dihedral angle $\angle PMQ$ between two faces on opposite sides.



Hence the isosceles triangles ANC and PMQ are similar to each other. It follows that

$$\frac{2\sqrt{b^2-\frac{4a^2}{3}}}{2a}=\frac{PQ}{AC}=\frac{PM}{AN}=\frac{b}{2a}.$$

Hence

$$b^2-\frac{4a^2}{3}=\frac{b^2}{4}$$

so that

$$b=\frac{4a}{3}.$$

Since $b<2a$, we have $b=2$ and $2a=3$. Hence $PQ=2\sqrt{4-2}<3$, and the greatest distance between two vertices is $AB=3$.

5. If only three colours are used, each must be used to paint a pair of opposite faces. There are $\binom{6}{3}=20$ ways of choosing the colours, and it does not matter how they are applied. If only four colours are used, two must be used to paint two pairs of opposite faces. They can be chosen in $\binom{6}{2}=15$ ways. Two more colours are then chosen, in $\binom{4}{2}=6$ ways.

Since it does not matter how the colours are applied, the number of ways in this case is $15\times 6=90$. If five colours are used, one must be used to paint a pair of opposite faces. It can be chosen in 6 ways. The other four colours can be chosen in $\binom{5}{4}=5$ ways, and divided into two pairs in 3 ways, each pair being used to paint a pair of opposite faces. The number of ways in this case is $6\times 5\times 3=90$ also.

Finally, suppose all six colours are used. They can be divided into three pairs in $5\times 3=15$ ways, each pair being used to paint a pair of opposite faces. It is only when we come to the last pair that we have to make a distinction which colour is used on which face. Hence the number of ways in this case is $15\times 2=30$ ways. The total number of ways is therefore

$$20+90+90+30=230.$$

6. The four lattice points (0,0), (199,199), (398,0) and $(199,-199)$ obviously lie on the circle. Suppose (x,y) is another such lattice point. Then

$$(x-199)^2+y^2=199^2.$$

Note that 199 is prime. Hence $(x-199,y,199)$ is a primitive Pythagorean triple. It follows that

$$199=m^2+n^2$$

for some integers m and n.

However, we have a contradiction since $199\equiv 3 \pmod 4$. Hence there are only four lattice points on this circle.

Paper II

1. From $a_1=2a_1-1$, we have $a_1=1$. From

$$a_n=(2a_n-1)-(2a_{n-1}-1),$$

we have $a_n=2a_{n-1}$ so that $a_n=2^{n-1}$. Summing $b_{k+1}=a_k+b_k$ from $k=1$ to n, we have

$$b_{n+1}=2a_n-1+b_1=2^n+2.$$

Hence

$$b_1+b_2+\cdots+b_n=2^n-1+2n.$$

2. The inequality

$$(x+A)^2+(x+B)^2\ge\frac{1}{8}$$

may be rewritten as

$$x^2+(A+B)x+\frac{8A^2+8B^2-1}{16}\ge 0.$$

This holds for all real numbers x if and only if

$$(A+B)^2-\frac{8A^2+8B^2-1}{4}\le 0,$$

which is equivalent to

$$(A-B)^2\ge\frac{1}{4}.$$

Hence it is sufficient to deal with

$$(3+2\sin\theta\cos\theta-a\sin\theta-a\cos\theta)^2\ge\frac{1}{4}.$$

Consider first the case $3+2\sin\theta\cos\theta-a\sin\theta-a\cos\theta\ge\frac{1}{2}$. Then

$$\begin{aligned} a &\le \frac{3+2\sin\theta\cos\theta-\frac{1}{2}}{\sin\theta+\cos\theta}\\ &= \frac{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta+\frac{3}{2}}{\sin\theta+\cos\theta}\\ &= (\sin\theta+\cos\theta)+\frac{3}{2}\cdot\frac{1}{\sin\theta+\cos\theta}. \end{aligned}$$

By the AM-GM Inequality, the minimum value of the last expression is $2\sqrt{\frac{3}{2}}=\sqrt{6}$. Hence $a\le\sqrt{6}$.

Consider now the case

$$3+2\sin\theta\cos\theta-a\sin\theta-a\cos\theta\le-\frac{1}{2}.$$

Then

$$a\ge(\sin\theta+\cos\theta)+\frac{5}{2}\cdot\frac{1}{\sin\theta+\cos\theta}.$$

The minimum value of the function

$$f(x)=x+\frac{5}{2}\cdot\frac{1}{x}$$

occurs at $x=\sqrt{\frac{5}{2}}$.

Now

$$\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)=\sin\left(\theta+\frac{\pi}{4}\right).$$

For $0\le\theta\le\frac{\pi}{2}$,

$$\frac{1}{\sqrt{2}}\le\sin\left(\theta+\frac{\pi}{4}\right)\le 1.$$

Hence

$$1\le\sin\theta+\cos\theta\le\sqrt{2}.$$

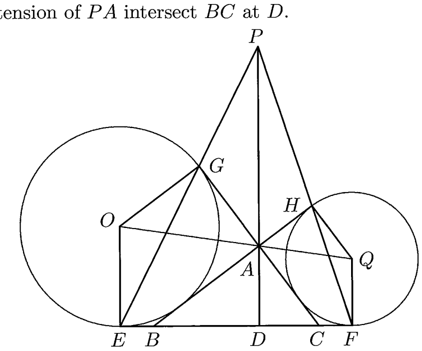
It follows that the maximum value of

$$(\sin\theta+\cos\theta)+\frac{5}{2}\cdot\frac{1}{\sin\theta+\cos\theta}$$

occurs at $\sin\theta+\cos\theta=1$, so that $a\ge\frac{7}{2}$.

In summary, either $a\le\sqrt{6}$ or $a\ge\frac{7}{2}$.

3. Let O be the excentre opposite C and Q be that opposite B, and let the extension of PA intersect BC at D.



Note that we have

$$\angle PGA=180^\circ-\angle AGE=180^\circ-\angle GEC.$$

Applying the Law of the Sines to triangle PGA,

$$\frac{AG}{\sin APG}=\frac{AP}{\sin PGA}=\frac{AP}{\sin GEC}.$$

Applying this law to triangle PDE, we have

$$\frac{DE}{\sin APG}=\frac{PD}{\sin GEC}.$$

Hence

$$\frac{DE}{AG}=\frac{PD}{AP}.$$

In the same way, we can prove that

$$\frac{DF}{AH}=\frac{PD}{AP},$$

so that

$$\frac{DE}{AG}=\frac{DF}{AH}.$$

Now triangles AGO and AHQ are similar, so that

$$\frac{AG}{AH}=\frac{AO}{AQ}.$$

Hence

$$\frac{DE}{DF}=\frac{AO}{AQ}.$$

Since OE and QF are parallel to each other, they are also parallel to AD. It follows that AP is perpendicular to BC.

4. Let a and b be two mutual acquaintances. Let A be the set of mutual acquaintances of a and B be that of b. If there exists $c \in A \cap B$, then a, b and c form a desired trio. Hence we may assume that $A \cap B = \emptyset$.

Since $|A| \geq \lfloor \frac{n}{2} \rfloor$ and $|B| \geq \lfloor \frac{n}{2} \rfloor$, we may assume that $|A| = \lfloor \frac{n}{2} \rfloor$ and either $|B| = n - |A|$ or

$$|B| = n - |A| - 1 = \lfloor \frac{n}{2} \rfloor.$$

In the former case, suppose there exist two mutual acquaintances in A. Then they form a desired trio with a. If no two mutual acquaintances exist in A, then we must have two in B. They form a desired trio with b. In the latter case, there exists $c \notin A \cup B$. If we still have two mutual acquaintances in A, or two in B, we can conclude as before.

Suppose they do not exist. Then we must have two mutual acquaintances in $A \cup \{c\}$ as well as in $B \cup \{c\}$. This means that c is a mutual acquaintance with some $a_1 \in A$ and some $b_1 \in B$. Since $n \geq 6$, c has at least 3 mutual acquaintances.

We may assume that a third one is $a_2 \in A$. Now b_1 is not a mutual acquaintance with anyone in B. Hence b_1 must be a mutual acquaintance with all but one of the others, which must include either a_1 or a_2. By symmetry, we may assume that it is a_1. Then a_1, b_1 and c form a desired trio.

Olympiad Paper I

1. For positive real numbers m and h,

$$(m+h)^{12} + (m-h)^{12} = \sum_{k=0}^{12} (1 + (-1)^k) \binom{12}{k} m^{12-k} h^k$$

is an increasing function of h. If we have

$$-\frac{1}{\sqrt{3}} < x_i \leq x_j < \sqrt{3},$$

we can increase the value of

$$x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$$

without changing the value of

$$x_1 + x_2 + \cdots + x_{1997}$$

by decreasing x_i and increasing x_j by the same amount, until either the smaller one becomes $-\frac{1}{\sqrt{3}}$ or the larger one becomes $\sqrt{3}$. It follows that if

$$x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$$

is maximum, then at most one of the numbers is strictly between $-\frac{1}{\sqrt{3}}$ and $\sqrt{3}$. Let there be u copies of $-\frac{1}{\sqrt{3}}$, v copies of $\sqrt{3}$ and w copies of numbers in between. We have already proved that $w = 0$ or 1, and if $w = 1$, let that number be t. Then $u + v + w = 1997$ and

$$-\frac{u}{\sqrt{3}} + \sqrt{3}v + wt = -318\sqrt{3}.$$

Eliminating u, we have

$$4v + (\sqrt{3}t + 1)w = 1043.$$

It follows from

$$0 \leq (\sqrt{3}t + 1)w < 4$$

and

$$1043 = 4(260) + 3$$

that $v = 260$, $w = 1$ and $t = \frac{2}{\sqrt{3}}$, so that $u = 1716$. Thus the maximum value of

$$x_1^{12} + x_2^{12} + \cdots + x_{1997}^{12}$$

is

$$\left(-\frac{1}{\sqrt{3}}\right)^{12} u + (\sqrt{3})^{12} v + t^{12} = 189548.$$

2. Instead of reflecting P across the sides, we simply project it onto the sides since the two quadrilaterals so obtained are homothetic to each other.

(a) Since

$$\angle PA_2A_1 = 90^\circ = \angle PD_2A_1,$$

$A_1A_2PD_2$ is a cyclic quadrilateral, as illustrated by the diagram on the left. Hence

$$\angle PA_1D_1 = \angle PA_2D_2 = \cdots = \angle PA_{1997}D_{1997}.$$

On the other hand,

$$\angle PA_1B_1 = \angle PD_2A_2 = \angle PC_3D_3 = \angle PD_4A_4$$

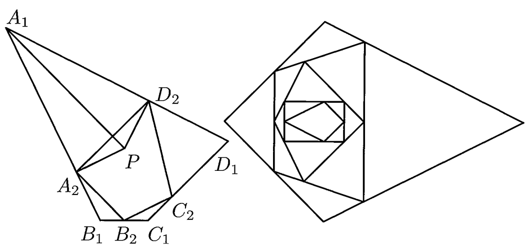
$$= \angle PA_5B_5 = \cdots = \angle PA_9B_9 = \cdots = \angle PA_{1997}B_{1997}.$$

It follows that

$$\angle B_1A_1D_1 = \angle B_5A_5D_5 = \cdots = \angle B_{1997}A_{1997}D_{1997}.$$

The same applies to the other three angles of the quadrilaterals. Thus $A_1B_1C_1D_1$, $A_5B_5C_5D_5$ and $A_9B_9C_9D_9$ are similar to $A_{1997}B_{1997}C_{1997}D_{1997}$ via spiral homothety from P.

The diagram on the right, generated by the point of intersection of the outermost kite, shows that $A_iB_iC_iD_i$ need not be similar to $A_{1997}B_{1997}C_{1997}D_{1997}$ for $i = 2, 3, 4, 6, 7, 8, 10, 11, 12$.



(b) As in (a), we have

$$\begin{aligned} & \angle B_1A_1D_1 + \angle D_1C_1B_1 \\ = \; & \angle B_3A_3D_3 + \angle D_3C_3B_3 \\ = \; & \cdots \\ = \; & \angle B_{11}A_{11}D_{11} + \angle D_{11}C_{11}B_{11} \\ = \; & \cdots \\ = \; & \angle B_{1997}A_{1997}D_{1997} + \angle D_{1997}C_{1997}B_{1997}. \end{aligned}$$

If $A_{1997}B_{1997}C_{1997}D_{1997}$ is cyclic, then so is $A_iB_iC_iD_i$ for $i = 1, 3, 5, 7, 9, 11$. The same counter-examples in (a) show that this is not necessarily so for $i = 2, 4, 6, 8, 10, 12$.

3. Solution 1

Let S be the set of positive integers n with the desired property. For such a $3 \times n$ array, let the sum of each row be $6s$ and that of each column be $6t$. Then

$$18s = \frac{3n(3n+1)}{2} = 6nt,$$

so that $n(3n+1) = 12s$ and $3n + 1 = 4t$.

Hence $n \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 4$, and it follows that $n \equiv 9 \pmod{12}$. We prove that $9 \in S$. Let $\alpha_1 = \langle 1 \rangle$, $\beta_1 = \langle 2 \rangle$ and $\gamma_1 = \langle 3 \rangle$. Consider the 3×3 array

$$\begin{bmatrix} \alpha_1 & \beta_1 + 6 & \gamma_1 + 3 \\ \beta_1 + 3 & \gamma_1 & \alpha_1 + 6 \\ \gamma_1 + 6 & \alpha_1 + 3 & \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & 8 & 6 \\ 5 & 3 & 7 \\ 9 & 4 & 2 \end{bmatrix}.$$

Note that $3 \notin S$ since the row sum 15 is not a multiple of 6. Let $\alpha_3 = \langle 1, 8, 6 \rangle$, $\beta_3 = \langle 5, 3, 7 \rangle$ and $\gamma_3 = \langle 9, 4, 2 \rangle$ be the rows of this magic square. Consider the 3×9 array

$$\begin{bmatrix} \alpha_3 & \beta_3 + 18 & \gamma_3 + 9 \\ \beta_3 + 9 & \gamma_3 & \alpha_3 + 18 \\ \gamma_3 + 18 & \alpha_3 + 9 & \beta_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 & 6 & 23 & 21 & 25 & 18 & 13 & 11 \\ 14 & 12 & 16 & 9 & 4 & 2 & 19 & 26 & 24 \\ 27 & 22 & 20 & 10 & 17 & 15 & 5 & 3 & 7 \end{bmatrix}.$$

It is easy to verify that it has all the desired properties. We claim that if $m \in S$, then $9m \in S$. It will then follow that since $9 \in S$, we have $9^k \in S$ for all positive integers k, so that S is indeed infinite. To justify the claim, let α_m, β_m and γ_m be the rows of a $3 \times m$ array with all the desired properties. As before, we first construct a $3 \times 3m$ array

$$\begin{bmatrix} \alpha_m & \beta_m + 6m & \gamma_m + 3m \\ \beta_m + 3m & \gamma_m & \alpha_m + 6m \\ \gamma_m + 6m & \alpha_m + 3m & \beta_m \end{bmatrix}.$$

It has all the desired properties except that the row and column sums are not divisible by 6. Let α_{3m}, β_{3m} and γ_{3m} be the rows of this array. It is easy to verify that the $3 \times 9m$ array

$$\begin{bmatrix} \alpha_{3m} & \beta_{3m} + 18m & \gamma_{3m} + 9m \\ \beta_{3m} + 9m & \gamma_{3m} & \alpha_{3m} + 18m \\ \gamma_{3m} + 18m & \alpha_{3m} + 9m & \beta_{3m} \end{bmatrix}$$

has all the desired properties. Thus the claim is justified.

Solution 2

Let S be the set of positive integers with the desired property. As in Solution 1, we focus on those of the form $12k+9$. We add the condition that $k \equiv 2 \pmod 9$, the reason for which will soon become clear. We first construct the following $3 \times (4k+3)$ array

$$\left[\begin{array}{ccccc} 1 & 4 & 7 & 10 & \dots \\ 6k+5 & 12k+8 & 6k+2 & 12k+5 & \dots \\ 12k+9 & 6k+3 & 12k+6 & 6k & \dots \end{array}\right.$$

$$\left.\begin{array}{cccc} 12k-2 & 12k+1 & 12k+4 & 12k+7 \\ 6k+11 & 5 & 6k+8 & 2 \\ 6 & 6k+9 & 3 & 6k+6 \end{array}\right].$$

Each column has sum $18k+15$. The sums of the three rows are $(4k+3)(6k+4)$, $(4k+3)(6k+5)$ and $(4k+3)(6k+6)$ respectively. Let $\ell = \frac{2k+5}{9}$. Since $k \equiv 2 \pmod 9$, ℓ is a positive integer. The 2ℓ-th term in the first row is

$$1+3(2\ell-1) = \frac{4k+4}{3}$$

while that in the third row is

$$6k+3-3(\ell-1) = \frac{16k+13}{3}.$$

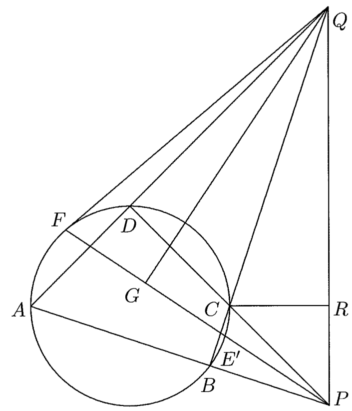
Switching these two terms will not change any column sum, but will change all three row sums to $(4k+3)(6k+5)$. Let α_{4k+3}, β_{4k+3} and γ_{4k+3} be the rows of this adjusted array. It is easy to verify that the $3 \times (12k+9)$ array

$$\left[\begin{array}{ccc} \alpha_{4k+3} & \beta_{4k+3}+6(4k+3) & \gamma_{4k+3}+3(4k+3) \\ \beta_{4k+3}+3(4k+3) & \gamma_{4k+3} & \alpha_{4k+3}+6(4k+3) \\ \gamma_{4k+3}+6(4k+3) & \alpha_{4k+3}+3(4k+3) & \beta_{4k+3} \end{array}\right]$$

has all the desired properties.

Olympiad Paper II

1. Let PF intersect the circle again at E' and let G be the foot of perpendicular from Q to PF. Now $\angle ADC = \angle PBC$ since $ABCD$ is cyclic. Let R be the point on PQ such that $\angle QRC$ has the same measure.



Then both $PRCB$ and $QDCR$ are cyclic quadrilaterals. It follows that

$$PQ \cdot PR = PC \cdot PD = PE' \cdot PF$$

and

$$PQ \cdot QR = QB \cdot QC = QF^2.$$

Addition yields

$$PQ(PR+QR) = PE' \cdot PF + QF^2.$$

Hence

$$PE' \cdot PF = PQ^2 - QF^2 = PG^2 - GF^2 = PF(PG-GF).$$

It follows that $PE' = PG - GF$ or

$$GF = PG - PE' = GE'.$$

This means that GQ passes through the centre of the circle and E' is symmetric to F with respect to GQ. Hence $E' = E$ and P, E and F are indeed collinear.

2. We first give an example where $M = 8$. Take $f(i) \equiv 3i-2 \pmod{17}$, with i and $3i-2$ both in A. If $f(i) = f(j)$, then $i \equiv j \pmod{17}$, so that f is indeed a bijection. Iteration yields $f^{(n)}(i) \equiv 3^n i - 3^n + 1 \pmod{17}$. Hence $f^{(n)}(i+1) - f^{(n)}(i) \equiv 3^n \pmod{17}$. In modulo 17, $3^1 \equiv 3$, $3^2 \equiv -8$, $3^3 \equiv -7$, $3^4 \equiv -4$, $3^5 \equiv 5$, $3^6 \equiv -2$, $3^7 \equiv -6$ and $3^8 \equiv -1$. Hence $M = 8$.

Now let $f : A \to A$ be a bijection with the largest possible M. Let $P_0P_1 \ldots P_{16}$ be a convex 17-gon. For any m, $1 \le m < M$, if $f^{(m)}(i+1) = a$ and $f^{(m)}(i) = b$, we connect P_a and P_b. Since $a-b \ne \pm 1$, P_aP_b is a diagonal. For each m, exactly 17 diagonals are drawn, and no two are identical. We claim that the diagonals are distinct for different values of m. Suppose on the contrary there exist p and q, $1 \le p < q < M$, such that $f^{(p)}(i) = f^{(q)}(j)$ and $f^{(p)}(i \pm 1) = f^{(q)}(j+1)$ for some i and j.

From $f^{(p)}(i) = f^{(q)}(j) = f^{(p)}(f^{(q-p)}(j))$, we have $f^{(q-p)}(j) \equiv i \pmod{17}$. Similarly, $f^{(q-p)}(j+1) \equiv i \pm 1 \pmod{17}$. Since $1 \le q-p < M$, this contradicts the definition of M. Thus the claim is justified. Now there are $M-1$ values of m satisfying $1 \le m < M$, so that $17(M-1)$ diagonals are drawn. The total number of diagonals, on the other hand, is $17 \cdot 7$. From $17(M-1) \le 17 \cdot 7$, we have $M \le 8$.

3. Solution 1

Note that $0 \le a_k \le a_{k-1} + a_1 \le a_{k-2} + 2a_1 \le \cdots \le ka_1$ for any k. If $n = m$, then

$$a_n \le na_1 = ma_1 + \left(\frac{n}{m} - 1\right) a_m.$$

Suppose $n > m$. Then

$$\begin{aligned}\frac{a_n}{n} - \frac{a_m}{m} &\le \frac{a_{n-m} + a_m}{n} - \frac{a_m}{m} \\ &= \frac{ma_{n-m} - (n-m)a_m}{mn} \\ &= \frac{n-m}{n}\left(\frac{a_{n-m}}{n-m} - \frac{a_m}{m}\right).\end{aligned}$$

If $n - m > m$, we can iterate this process. Eventually, there exists s, $1 \le s \le m$, such that

$$\frac{a_n}{n} - \frac{a_m}{m} \le \frac{s}{n}\left(\frac{a_s}{s} - \frac{a_m}{m}\right).$$

Since $a_s \le sa_1$ and $a_1 - \frac{a_m}{m} \ge 0$, we have

$$\frac{a_n}{n} - \frac{a_m}{m} \le \frac{s}{n}\left(1 - \frac{a_m}{m}\right) \le \frac{m}{n}\left(a_1 - \frac{a_m}{m}\right).$$

This is equivalent to

$$a_n \le ma_1 + \left(\frac{n}{m} - 1\right) a_m.$$

Solution 2

Note that $0 \le a_k \le a_{k-1} + a_1 \le a_{k-2} + 2a_1 \le \cdots \le ka_1$ for any k. Let m be fixed. We drop the condition $n \ge m$ and use induction to prove that the desired result holds for all n. For $n = 1$, we have

$$a_1 \le ma_1 + \left(\frac{1}{m} - 1\right) a_m$$

is equivalent to $a_m \le ma_1$, which we have already proved. Suppose

$$a_n \le ma_1 + \left(\frac{n}{m} - 1\right) a_m$$

for some $n \ge 1$. Consider a_{n+1}. Suppose $n < m$. Since we have $a_{n+1} \le (n+1)a_1$ and $a_1 - \frac{a_m}{m} \ge 0$,

$$\frac{a_{n+1}}{n+1} - \frac{a_m}{m} \le a_1 - \frac{a_m}{m} \le \frac{m}{n+1}\left(a_1 - \frac{a_m}{m}\right).$$

This is equivalent to

$$a_{n+1} \le ma_1 + \left(\frac{n+1}{m} - 1\right) a_m.$$

Suppose $n \ge m$. Then $n + 1 - m \ge 1$ and $a_{n+1} \le a_{n+1-m} + a_m$. By the induction hypothesis,

$$\begin{aligned}a_{n+1} &\le ma_1 + \left(\frac{n+1-m}{m} - 1\right) a_m + a_m \\ &= ma_1 + \left(\frac{n+1}{m} - 1\right) a_m.\end{aligned}$$

1997/98

Paper I.

Section 1. Questions with Multiple Choices.

1. We have $x_3 = b - a$, $x_4 = -a$, $x_5 = -b$, $x_6 = a - b$, $x_7 = a$ and $x_8 = b$. Hence for all $n \geq 1$, $x_{n+6} = x_n$ and $S_{6n} = 0$. It follows that $x_{100} = x_4 = -a$ and $S_{100} = S_4 = 2b - a$.

2. Let G be the point on BC such that $\frac{CG}{GB} = \lambda$. Then EG is parallel to AC and FG is parallel to BD. Since AC and BD are perpendicular, $\angle EGF = 90°$. Now the angle between EF and AC is $\angle GEF$ while the angle between EF and BD is $\angle GFE$. Their sum is $180° - \angle EGF = 90°$.

3. Let n be the number of terms, a be the first term and d the common difference. The sum of all terms is

$$na + \frac{n(n-1)}{2}d = 97^2,$$

so that

$$n(2a + (n-1)d) = 2 \times 97^2.$$

Since 97 is prime and $n \geq 3$, $n = 97$, 2×97, 97^2 or 2×97^2. If $d > 0$, then

$$2 \times 97^2 > n(n-1)d \geq n(n-1),$$

and only $n = 97$ is possible.

This leads to $(n, a, d) = (97, 49, 1)$ or (97,1,2). On the other hand, if $d = 0$, then $an = 97^2$, and this leads to $(n, a, d) = (97^2, 1, 0)$ or (97,97,0).

4. Note that $\sqrt{x^2 + y^2 + 2y + 1}$ is the distance from any point (x, y) on the curve to the point $(0, -1)$. The distance from (x, y) to the line $x - 2y + 3 = 0$ is

$$\frac{|x - 2y + 3|}{\sqrt{5}}.$$

It follows that the ratio of these two distances is the constant $\sqrt{\frac{5}{m}}$. In order for this to be an ellipse, we must have $\sqrt{\frac{5}{m}} < 1$ or $m > 5$.

5. The graph of $f(x)$ is a parabola which opens up and has $x = \frac{\pi}{2}$ as its axis. Hence $f(x)$ increases as $|x - \frac{\pi}{2}|$ increases. Note that

$$0 < \alpha < \frac{\pi}{6} < \frac{\pi}{4} < \beta < \frac{\pi}{3} < \frac{\pi}{2} < \gamma < \frac{2\pi}{3} < \frac{3\pi}{4} < \delta < \frac{5\pi}{6}.$$

Hence

$$\left|\gamma - \frac{\pi}{2}\right| < \frac{\pi}{6} < \left|\beta - \frac{\pi}{2}\right| < \frac{\pi}{4} < \left|\delta - \frac{\pi}{2}\right| < \frac{\pi}{3} < \left|\alpha - \frac{\pi}{2}\right|,$$

so that

$$f(\alpha) > f(\delta) > f(\beta) > f(\gamma).$$

6. Let the lines be ℓ_0, ℓ_1 and ℓ_2. We may choose a coordinate system so that ℓ_0 lies on the plane $z = 0$ and ℓ_1 lies on the plane $z = 1$. Now ℓ_2 has at most two points with z-coordinates equal to 0 or 1.

Let P be any other point on ℓ_2 and let Π be the plane determined by P and ℓ_0. The planes Π and $z = 1$ intersect at a line ℓ, which is not parallel to ℓ_1 as otherwise ℓ_0 and ℓ_1 would be parallel to each other. Hence ℓ_1 intersects Π at a point Q.

Since P and Q have different z-coordinates, PQ is not parallel to ℓ_0, and these two coplanar lines intersect at some point R. Now PQR is a line with P on ℓ_2, Q on ℓ_1 and R on ℓ_0.

Section 2. Questions requiring Answers Only:

1. Let $f(t) = t^3 + 1997t$. Then it is an increasing function. Since

$$f(x-1) = -1 = f(1-y),$$

we must have $x - 1 = 1 - y$ so that $x + y = 2$.

2. The x-coordinate of the right focus is $\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$.

From

$$(\sqrt{3})^2 - \frac{y^2}{2} = 1,$$

we have $y = \pm 2$ so that the length of the vertical focal chord is 4. This is the minimum length of any chord whose endpoints are on the same branch. By symmetry, one of the three lines is either the vertical focal chord or the horizontal axis. In the latter case, we must have $\lambda = 2$, the distance between the two vertices. However, there are no other chords of this length. In the former case, we have $\lambda = 4$, and there are two other chords, with endpoints on different branches, that have this length.

3. Let $z = r(\cos\theta + i\sin\theta)$. Then

$$\left|2z + \frac{1}{z}\right|^2 = 4r^2 + 4\cos 2\theta + \frac{1}{r^2} = 1.$$

This holds if and only if the two roots x_1 and x_2 of the quadratic equation

$$4x^2 - (1 - 4\cos 2\theta)x + 1 = 0$$

are both positive. Assume for now that they are real. Then they have the same sign as $x_1x_2 = \frac{1}{4} > 0$. In order for this sign to be positive, we need

$$0 < x_1 + x_2 = \frac{1}{4}(1 - 4\cos 2\theta) \quad \text{or} \quad 1 - 4\cos 2\theta > 0.$$

Now the discriminant of the quadratic equation is

$$(1 - 4\cos 2\theta)^2 - 4^2.$$

In order for it to be non-negative, we must have

$$1 - 4\cos 2\theta \geq 4 \quad \text{or} \quad 1 - 4\cos 2\theta \leq -4.$$

From the preliminary discussion, the latter is to be rejected. Hence $-\frac{3}{4} \geq \cos 2\theta$. It follows that the range of $\theta = \arg z$ is

$$\left[\frac{\pi}{2} - \frac{1}{2}\arccos\frac{3}{4}, \frac{\pi}{2} + \frac{1}{2}\arccos\frac{3}{4}\right]$$
$$\cup \left[\frac{3\pi}{2} - \frac{1}{2}\arccos\frac{3}{4}, \frac{3\pi}{2} + \frac{1}{2}\arccos\frac{3}{4}\right].$$

4. Let D be the midpoint of AB. Then D is the circumcentre of ABC. Since $SA = SB = SC$, D is also the projection of S onto ABC. Moreover, O lies on the segment DS. Since $OS = OA = OB$, O is the centre of the equilateral triangle SAB.

   Hence
$$DO = \frac{1}{3}DS = \frac{\sqrt{3}}{3}.$$

5. There are $2^3 = 8$ ways for the frog to make the first three moves. Two of the ways end in D. In the other six ways, the frog will make two more moves, in $2^2 = 4$ ways, since it cannot get to D on the fourth move. Hence the total number of ways is $2 + 6 \times 4 = 26$.

6. We are comparing the logarithms of $\frac{x}{y} + z$, $yz + \frac{1}{x}$ and $\frac{1}{xz} + y$. Let m denote the largest of them, so that $M = \log m$. We have
$$m^2 \geq \left(\frac{x}{y} + z\right)\left(\frac{1}{xz} + y\right) = \left(\frac{1}{yz} + yz\right) + \left(\frac{1}{x} + x\right) \geq 4.$$
   Hence $m \geq 2$, and when $x = y = z = 1$, $m = 2$. It follows that $M = \log 2$.

Section 3. Questions requiring Full Solutions:

1. (a) Note that
$$\cos^2\frac{\pi}{12} = \frac{1}{2}\left(1 + \cos\frac{\pi}{6}\right) = \frac{2+\sqrt{3}}{4}.$$
   Hence
$$\begin{aligned}\cos x \sin y \cos z &= \frac{1}{2}(\sin(x+y) - \sin(x-y))\cos z \\ &\leq \frac{1}{2}\sin(x+y)\cos z \\ &= \frac{1}{2}\cos^2 z \\ &\leq \frac{1}{2}\cos^2\frac{\pi}{12} \\ &= \frac{2+\sqrt{3}}{8}.\end{aligned}$$
   This is achieved by
$$\begin{aligned}&\cos\frac{5\pi}{24}\sin\frac{5\pi}{24}\cos\frac{\pi}{12} \\ =\ &\frac{1}{2}\sin\frac{5\pi}{12}\cos\frac{\pi}{12} \\ =\ &\frac{1}{2}\cos^2\frac{\pi}{12} \\ =\ &\frac{2+\sqrt{3}}{8}.\end{aligned}$$
   (b) Note that
$$x = \frac{\pi}{2} - (y+z) \leq \frac{\pi}{2} - \left(\frac{\pi}{12} + \frac{\pi}{12}\right) = \frac{\pi}{3}.$$
   Hence
$$\begin{aligned}\cos x \sin y \cos z &= \frac{1}{2}\cos x(\sin(y+z) + \sin(y-z)) \\ &\geq \frac{1}{2}\cos x \sin(y+z) \\ &= \frac{1}{2}\cos^2 x \\ &\geq \frac{1}{2}\cos^2\frac{\pi}{3} \\ &= \frac{1}{8}.\end{aligned}$$

This is achieved by

$$\cos\frac{\pi}{3}\sin\frac{\pi}{12}\cos\frac{\pi}{12}=\frac{1}{4}\sin\frac{\pi}{6}=\frac{1}{8}.$$

2. (a) Let $P(x_1,y_1)$, $Q(x_2,y_2)$ and $R(x_3,y_3)$ be three points on the same branch of the hyperbola $xy=1$. We may assume that $0<x_1<x_2<x_3$, so that $y_1>y_2>y_3>0$. Hence

$$\begin{aligned} & PQ^2+QR^2-RP^2 \\ = \; & (x_1-x_2)^2+(x_2-x_3)^2-(x_3-x_1)^2 \\ & +(y_1-y_2)^2+(y_2-y_3)^2-(y_3-y_1)^2 \\ = \; & 2(x_2-x_1)(x_2-x_3)+2(y_2-y_1)(y_2-y_3) \\ < \; & 0. \end{aligned}$$

Hence PQR is an obtuse triangle, and cannot be equilateral.

(b) Let $P(x_1,y_1)$, $Q(x_2,y_2)$ and $R(-1,-1)$ be the vertices of an equilateral triangle, with $0<x_1<x_2$. Then the slope of PQ is $-\frac{1}{x_1x_2}$. Hence the equation of the altitude from R to PQ is

$$\frac{y+1}{x+1}=x_1x_2.$$

Since it passes through the midpoint of PQ, we have

$$\frac{1}{2}\left(\frac{1}{x_1}+\frac{1}{x_2}\right)+1=x_1x_2\left(\frac{x_1+x_2}{2}+1\right),$$

or equivalently

$$(1-x_1x_2)((x_1+x_2)(1+x_1x_2)+2x_1x_2)=0.$$

Since the second factor is positive, we have $x_1x_2=1$ or $x_2=\frac{1}{x_1}$. Hence P and Q are symmetric about the line $y=x$. Now the slope of PR is equal to $\tan 75°=2+\sqrt{3}$. Solving $xy=1$ and

$$\frac{y+1}{x+1}=2+\sqrt{3},$$

we have $x=2-\sqrt{3}$ and $y=2+\sqrt{3}$. Hence the coordinates for P are $(2-\sqrt{3},2+\sqrt{3})$ while those of Q are $(2+\sqrt{3},2-\sqrt{3})$.

3. We have

$$S=a_1+a_2+a_3+a_4+a_5=a_1(1+q+q^2+q^3+q^4)$$

where $q=\frac{a_2}{a_1}$. On the other hand,

$$S=4\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}+\frac{1}{a_5}\right)=\frac{4}{a_1q^4}(1+q+q^2+q^3+q^4).$$

If

$$1+q+q^2+q^3+q^4=0,$$

then $q^5=1$. Hence $|q|=1$, so that

$$|a_1|=|a_2|=|a_3|=|a_4|=|a_5|,$$

which means that the points representing them all lie on some circle centred at the origin. Suppose

$$1+q+q^2+q^3+q^4\neq 0.$$

Then $a_1^2q^4=4$ or $a_3=\pm 2$. Moreover,

$$\frac{1}{q^2}+\frac{1}{q}+1+q+q^2=\pm\frac{S}{2}.$$

Let $x=q+\frac{1}{q}$. Then

$$x^2+x-1\mp\frac{S}{2}=0.$$

Let

$$f(x)=x^2+x-1\mp\frac{S}{2}.$$

Since $|S|\le 2$, we have

$$f(-2)=1\mp\frac{S}{2}\ge 0,$$

$$f\left(-\frac{1}{2}\right)=-\frac{1}{4}(5\pm 2S)<0$$

and

$$f(2)=5\mp\frac{S}{2}>0.$$

Hence both roots x satisfy $-2\le x<2$. For each of these roots, the equation $q^2-xq+1=0$ has discriminant $x^2-4\le 0$. If $x=-2$, we have repeated root $q=-1$ and $|q|=1$. If $|x|<2$, the two roots are conjugate imaginary numbers with product 1, so that again $|q|=1$. In either case,

$$|a_1|=|a_2|=|a_3|=|a_4|=|a_5|=2,$$

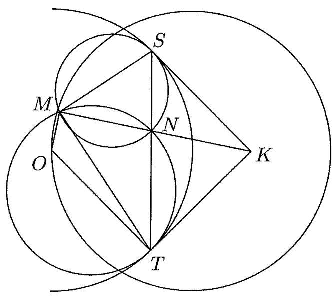
and the points representing them all lie on the circle centred at the origin with radius 2.

Paper II

1. If the common tangents at S and T are parallel, then S, O and T are collinear. It follows that N, S and T will not be collinear, and OM will not be perpendicular to MN if the two smaller circles are not of the same size. Hence we may assume that these two tangents meet at some point K. Then OK is a diameter of a circle passing through S and T. Moreover, we have

$$\angle SMT = \angle SMN + \angle NMT = \angle NSK + \angle NTK.$$

We claim that M, N and K are collinear. Otherwise, suppose the line KM intersects the first circle again at N_1 and the second circle again at N_2.



Then

$$KM \cdot KN_1 = KS^2 = KT^2 = KM \cdot KN_2,$$

which implies that N_1 and N_2 coincide at N. Suppose S, N and T are collinear. Then

$$\angle SMT + \angle SKT = \angle NSK + \angle NTK + \angle SKT = 180°.$$

Hence M also lies on the circle with diameter OK, so that OM is perpendicular to MN. Conversely, suppose OM is perpendicular to MN. Then M also lies on the circle with diameter OK. Hence

$$\angle NSK + \angle NTK + \angle SKT = \angle SMT + \angle SKT = 180°,$$

so that S, N and T are collinear.

2. We have $\sum_{i=1}^{n} x_i^2 - x_0^2 = \sum_{i=1}^{n} y_i^2 - y_0^2$ and $\sum_{i=1}^{n} x_i y_i = x_0 y_0$. Then by Cauchy's Inequality,

$$x_0^2 y_0^2 = \left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

If $x_0^2 > \sum_{i=1}^{n} x_i^2$, then $y_0^2 > \sum_{i=1}^{n} y_i^2$, but these contradict the above inequality.

Hence $x_0^2 \le \sum_{i=1}^{n} x_i^2$ is a necessary condition. Conversely, suppose it holds.

If $x_0^2 = \sum_{i=1}^{n} x_i^2$, we can simply take $y_i = x_i$ for $1 \le i \le n$.

Suppose $x_0^2 < \sum_{i=1}^{n} x_i^2$. Let $a^2 > 0$ be the difference. Since not all $x_i = 0$, we may assume that $x_n \ne 0$. Then we can take $y_i = 0$ for $1 \le i \le n-2$,

$$y_{n-1} = \frac{a x_n}{\sqrt{x_{n-1}^2 + x_n^2}} \quad \text{and} \quad y_n = -\frac{a x_{n-1}}{\sqrt{x_{n-1}^2 + x_n^2}}.$$

Hence $x_0^2 \le \sum_{i=1}^{n} x_i^2$ is also a sufficient condition.

3. We first construct an example to show that $k \ge 97$. Partition the array into four 25×25 subarrays. In each, set 25 entries equal to 0, with no two in the same row or the same column. Set the remaining entries equal to $\frac{1}{24}$. Then the sum of each row of the whole array is exactly 1.

After rearrangement, the one hundred 0s sink to the bottom 4 rows. The sum of the entries in each of the first 96 rows is $\frac{25}{24} > 1$. It follows that $k \ge 97$. We now prove that $k = 97$. Let the entries of the 97-th row after rearrangement be x_j, $1 \le j \le 25$. We wish to show that their sum is at most 1. Now the bottom 3 rows after rearrangement contain among them entries from at most 75 rows before.

Hence there is an original row such that each entry y_j, $1 \le j \le 25$, is in the top 97 rows after rearrangement. Then $x_j \le y_j$ for $1 \le j \le 25$, so that $x_1 + x_2 + \cdots + x_{25} \le y_1 + y_2 + \cdots + y_{25} \le 1$.

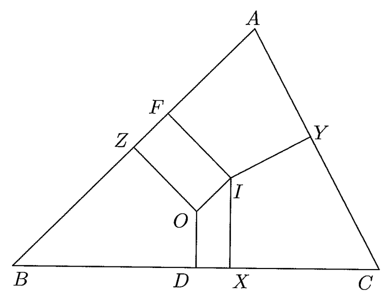
Olympiad Paper I

1. Solution 1

Let the incircle of triangle ABC touch BC at X, CA at Y and AB at Z. Let D be the midpoint of BC and F be the midpoint of AB. Then

$$\begin{aligned}
\sqrt{2}OI &= AB - AC \\
&= (AZ + ZB) - (AY + YC) \\
&= ZB - YC \\
&= XB - XC \\
&= \left(\frac{1}{2}BC + XD\right) - \left(\frac{1}{2}BC - XD\right) \\
&= 2XD.
\end{aligned}$$

Hence $OI = \sqrt{2}XD$. Since XD is the projection of OI onto BC, the angle between OI and BC is 45°. Since $\angle ABC = 45°$, OI is either perpendicular or parallel to AB. In the first case, Z and F coincide. By symmetry, $AC = BC$.



It follows that we have $\angle CAB = \angle ABC = 45°$, so that $\sin A = \frac{1}{\sqrt{2}}$. In the second case, as illustrated in the diagram above, let r be the inradius and R the circumradius. Then $\angle AOF = \angle ACB$, $OF = IZ = r$ and $AO = R$. Hence

$$\begin{aligned}
\cos C &= \frac{r}{R} \\
&= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\
&= 2\sin\frac{\pi}{8}\left(\cos\frac{C-A}{2} - \cos\frac{C+A}{2}\right)
\end{aligned}$$

$$\begin{aligned}
&= 2\sin\frac{\pi}{8}\left(\cos\left(C - \frac{3\pi}{8}\right) - \sin\frac{\pi}{8}\right) \\
&= \sin\left(\frac{\pi}{2} - C\right) + \sin\left(C - \frac{\pi}{4}\right) + \cos\frac{\pi}{4} - 1 \\
&= \cos C + \sin\left(C - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}} - 1.
\end{aligned}$$

Hence $\sin(C - \frac{\pi}{4}) = 1 - \frac{1}{\sqrt{2}}$ and

$$\begin{aligned}
\sin A &= \sin\left(\frac{\pi}{2} + \left(\frac{\pi}{4} - C\right)\right) \\
&= \cos\left(\frac{\pi}{4} - C\right) \\
&= \sqrt{1 - \sin^2\left(\frac{\pi}{4} - C\right)} \\
&= \sqrt{1 - \left(1 - \frac{1}{\sqrt{2}}\right)^2} \\
&= \sqrt{\sqrt{2} - \frac{1}{2}}.
\end{aligned}$$

Solution 2

Let r be the inradius and R be the circumradius of triangle ABC. By Euler's Formula and the Law of Sines,

$$\begin{aligned}
2(R^2 - 2Rr) &= (\sqrt{2}OI)^2 \\
&= (AB - AC)^2 \\
&= 4R^2(\sin C - \sin B)^2.
\end{aligned}$$

Also,

$$\begin{aligned}
r &= \frac{AB + BC - CA}{2}\tan\frac{B}{2} \\
&= R(\sqrt{2} - 1)(\sin C + \sin A - \sin B).
\end{aligned}$$

It follows that

$$1 - 2(\sqrt{2} - 1)\left(\sin C + \sin A - \frac{1}{\sqrt{2}}\right) = 2\left(\sin C - \frac{1}{\sqrt{2}}\right)^2.$$

Since

$$\sin C = \sin\left(\frac{3\pi}{4} - A\right) = \frac{1}{\sqrt{2}}(\sin A + \cos A),$$

we have

$$1 - (2-\sqrt{2})((\sqrt{2}+1)\sin A + \cos A - 1) = (\sin A + \cos A - 1)^2.$$

This is equivalent to

$$(\sqrt{2}\sin A - 1)(\sqrt{2}\cos A - \sqrt{2} + 1) = 0.$$

Hence either $\sin A = \frac{1}{\sqrt{2}}$ or $\cos A = 1 - \frac{1}{\sqrt{2}}$. The latter yields

$$\sin A = \sqrt{1 - \left(1 - \frac{1}{\sqrt{2}}\right)^2} = \sqrt{\sqrt{2} - \frac{1}{2}}.$$

2. We first prove that the upper bound holds for any $2n$ distinct positive integers satisfying the hypothesis. Note that we must have $a_j < b_j$ for some j, $1 \le j \le n$, so that $\frac{2b_j}{a_j+b_j} > 1$. It follows that

$$\begin{aligned}\sum_{i=1}^{n} \frac{a_i - b_i}{a_i + b_i} &= \sum_{i=1}^{n} \left(1 - \frac{2b_i}{a_i+b_i}\right) \\ &= n - \sum_{i=1}^{n} \frac{2b_i}{a_i+b_i} \\ &\ge n - \frac{2b_j}{a_j+b_j} \\ &> n-1.\end{aligned}$$

For $1 \le i \le n-1$, choose $a_i = 2Mi$ and $b_i = 2i$ where M is a sufficiently large integer. Then

$$\frac{a_i - b_i}{a_i + b_i} = \frac{M-1}{M+1} = 1 - \frac{2}{M+1}.$$

Choose

$$a_n = (M-1)^2 n(n-1) \quad \text{and} \quad b_n = M(M-1)n(n-1)$$

so that

$$a_1 + a_2 + \cdots + a_n = (M^2 - M + 1)n(n-1) = b_1 + b_2 + \cdots + b_n.$$

Then

$$\sum_{i=1}^{n} \frac{a_i - b_i}{a_i + b_i} = (n-1)\left(1 - \frac{2}{M+1}\right) - \frac{1}{2M-1}.$$

Since

$$\frac{1}{1998} > \frac{2(n-1)}{M+1} + \frac{1}{2M-1}$$

when M is sufficiently large, we have the desired lower bound.

3. Solution 1

Since S contains 49 even numbers, a subset with $n \le 49$ elements may consist only of even numbers, and will not satisfy the desired conditions. It follows that $n \ge 50$. We now prove that any subset T with 50 elements contains a subset A of size 10 which satisfies the desired conditions. Let e denote the number of even elements in T. For any odd number x in S, let $f(x)$ denote the number of even elements in S which are not relatively prime to x.

In particular, if x is prime, then

$$f(x) = \left\lfloor \frac{49}{x} \right\rfloor.$$

We claim that if $f(x) \le e - 9$ for any odd number x in T, then the subset A exists. This is because we can take x and 9 of the $e - f(x)$ even numbers in T relatively prime to x to form A. Thus the claim is justified.

Now in S there are $f(3) = 16$ odd numbers having 3 as the smallest prime divisor. There are $f(5) - 2 = 7$ odd numbers having 5 as the smallest prime divisor, the adjustment -2 reflecting the exclusion of $5 \cdot 3$ and $5 \cdot 3^2$.

Similarly, there are $f(7) - 3 = 4$ odd numbers having 7 as the smallest prime divisor, and only 1 odd number having p as the smallest prime divisor for any $p > 7$, namely, p itself. We consider five cases.

Case 1. $e \ge 25$.

For any odd number x in T, we have

$$f(x) \le f(3) = 16 = 25 - 9 \le e - 9.$$

The desired conclusion follows from our earlier claim.

Case 2. $16 \le e \le 24$.

The number of odd elements in T is at least

$$50 - 24 = 26 > 1 + 16 + 7,$$

the first term in the sum accounting for the number 1. Hence T contains an odd number x whose smallest prime divisor is at least 7. It follows that

$$f(x) \le f(7) = 7 = 16 - 9 \le e - 9,$$

and we may appeal to our earlier claim again.

Case 3. $10 \le e \le 15$.

The number of odd elements in T is at least

$$50 - 15 = 35 > 1 + 16 + 7 + 4 + 6 \cdot 1.$$

Hence T contains an odd number x whose smallest prime divisor is at least 31. It follows that $f(x) \le f(31) = 1 = 10 - 9 \le e - 9$.

Case 4. $e = 9$.

The number of odd elements in T is

$$50 - 9 = 41 > 1 + 16 + 7 + 4 + 12 \cdot 1.$$

Hence T contains an odd number x whose smallest prime divisor is at least 59. It follows that $f(x) \le f(59) = 0 = e - 9$.

Case 5. $e \le 8$

The number of odd elements in T is at least

$$50 - 8 = 42 > 1 + 16 + 7 + 4 + 13 \cdot 1.$$

Hence T contains an odd number x whose smallest prime divisor is at least 61. This means that $x \ge 61$ is a prime. At most $49 - (50 - 8) = 7$ odd numbers in S do not belong to T. Hence T contains at least 9 odd multiples of 3. They along with x form A.

Solution 2

Since S contains 49 even numbers, a subset with $n \le 49$ elements may consist only of even numbers, and will not satisfy the desired conditions. It follows that $n \ge 50$. We now prove that any subset T with 50 elements contains a subset A of size 10 which satisfies the desired conditions. Partition the odd numbers in S into the following five sets:

$$\begin{aligned} O_1 &= \{1, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}, \\ O_2 &= \{13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}, \\ O_3 &= \{49, 55, 65, 77, 85, 91, 95\}, \\ O_4 &= \{25, 33, 35, 39, 51, 57, 69, 87, 93\}, \\ O_5 &= \{3, 5, 7, 9, 11, 15, 21, 45, 63, 75, 81\}. \end{aligned}$$

We consider five cases.

Case 1. $T \cap O_1 \ne \emptyset$.

In S, there are 33 odd numbers not divisible by 3. Hence T contains at least 17 numbers which are either even numbers or odd multiples of 3. By the Pigeonhole Principle, T contains either 9 even numbers or 9 odd multiples of 3. We can take A to consist of these 9 numbers together with an element of $T \cap O_1$.

Case 2. $T \cap O_1 = \emptyset$ but $T \cap O_2 \ne \emptyset$.

T contains at most 38 odd numbers and hence at least 12 even numbers. Let $x \in T \cap O_2$. Then at most 3 of these 12 even numbers are multiples of 13, the worst case. Hence we can take A to consist of x and 9 even numbers which are not multiples of x.

Case 3. $T \cap (O_1 \cup O_2) = \emptyset$ but $T \cap O_3 \ne \emptyset$.

T contains at most 28 odd numbers and hence at least 22 even numbers. Let $x \in T \cap O_3$. Since

$$\left\lfloor \frac{49}{5} \right\rfloor + \left\lfloor \frac{49}{11} \right\rfloor - \left\lfloor \frac{49}{55} \right\rfloor = 13,$$

at least $22 - 13 = 9$ even numbers in T are relatively prime to 55, the worst case. Hence we may take A to consist of x and 9 of these even numbers.

Case 4. $T \cap (O_1 \cup O_2 \cup O_3) = \emptyset$ and $T \cap O_4 \ne \emptyset$.

T contains at most 21 odd numbers and hence at least 29 even numbers. Let $x \in T \cap O_4$. Since

$$\left\lfloor \frac{49}{3} \right\rfloor + \left\lfloor \frac{49}{11} \right\rfloor - \left\lfloor \frac{49}{33} \right\rfloor = 19,$$

at least $29 - 19 = 10$ even numbers in T are relatively prime to 33, the worst case. Hence we may take A to consist of x and 9 of these even numbers.

Case 5. $T \cap (O_1 \cup O_2 \cup O_3 \cup O_4) = \emptyset$.

T contains at most 12 odd numbers and hence at least 38 even numbers. Let $x \in T \cap O_5$. Since

$$\left\lfloor \frac{49}{3} \right\rfloor + \left\lfloor \frac{49}{5} \right\rfloor - \left\lfloor \frac{49}{15} \right\rfloor = 22,$$

at least $38 - 22 = 16$ even numbers in T are relatively prime to 15, the worst case. Hence we may take A to consist of x and 9 of these even numbers.

Olympiad Paper II

1. Note that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3} = \frac{(n+1)(n^2-n+6)}{6} = \frac{m(m^2-3m+8)}{6}$$

by setting $m=n+1>4$. In order to divide 2^{2000}, we must have

$$\frac{m(m^2-3m+8)}{6}=2^k$$

for some integer $k>3$. Hence

$$m(m^2-3m+8)=3\cdot 2^{k+1}.$$

Suppose that $m=2^u$ for some integer $u\geq 3$, so that

$$m^2-3m+8=3\cdot 2^v,$$

where $u+v=k+1$. If $u\geq 4$, then $8\equiv 3\cdot 2^v\equiv 2^v \pmod{16}$. This implies that $v=3$ so that $m^2-3m+8=24$ or $m(m-3)=16$. This is impossible. Hence $u=3$, $m=8$ and $n=7$.

Indeed,

$$\binom{7}{0}+\binom{7}{1}+\binom{7}{2}+\binom{7}{3}=2^6$$

divides 2^{2000}. Suppose that $m=3\cdot 2^u$ for some positive integer v. Then

$$m^2-3m+8=2^v$$

where $u+v=k+1$. If $u\geq 4$, we have $8\equiv 2^v \pmod{16}$. This implies that $v=3$ so that $m^2-3m+8=8$ or $m(m-3)=0$. This contradicts $m>4$. If $u=1$, then $m=6$ but $m^2-3m+8=26$ is not a power of 2. If $u=2$, then $m=12$ but $m^2-3m+8=116$ is not a power of 2 either.

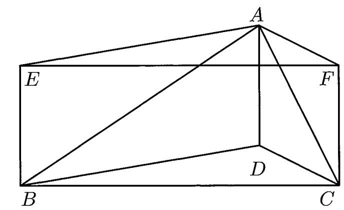
Hence $u=3$, $m=24$ and $n=23$. Indeed,

$$\binom{23}{0}+\binom{23}{1}+\binom{23}{2}+\binom{23}{3}=2^{11}$$

divides 2^{2000}.

2. Solution 1

We prove a stronger result that the left side of the given equation is always greater than or equal to the right side, with equality if and only if D is the orthocentre of triangle ABC. Complete the parallelograms $ADBE$ and $ADCF$. Then $BCFE$ is also a parallelogram.



Applying Ptolemy's Inequality to the quadrilaterals $ABCF$ and $AEBF$, we have

$$DC\cdot BC+AB\cdot DA=AF\cdot BC+AB\cdot CF\geq AC\cdot BF$$

and

$$DB\cdot BF+DC\cdot DA=AE\cdot BF+AF\cdot BE\geq AB\cdot EF=AB\cdot BC.$$

It follows that

$$\begin{aligned}&DA\cdot DB\cdot AB+DB\cdot DC\cdot BC+DC\cdot DA\cdot CA\\ =\;&DB(AB\cdot DA+BC\cdot DC)+DC\cdot DA\cdot CA\\ \geq\;&DB\cdot AC\cdot BF+DC\cdot DA\cdot CA\\ =\;&AC(DB\cdot BF+DC\cdot DA)\\ \geq\;&AC\cdot AB\cdot BC.\end{aligned}$$

Equality holds if and only if both quadrilaterals are cyclic, that is, $AEBCF$ is a cyclic pentagon. Since $BCFE$ is a parallelogram, it must be a rectangle, so that BF and CE are diameters of the circumcircle. Hence $\angle BAF=\angle CAE=90°$, so that AB is perpendicular to CD and AC is perpendicular to BD. This means that D is the orthocentre of triangle ABC.

Solution 2

In the complex plane, let the points A, B, C and D be represented by the complex numbers α, β, γ and δ respectively. Moreover, we may assume that $|\alpha|=|\beta|=|\gamma|=1$. In other words, triangle ABC

is inscribed in the unit circle. Thus its orthocentre H is represented by $\alpha+\beta+\gamma$. Define

$$f(z)=\frac{(z-\alpha)(z-\beta)}{(\gamma-\alpha)(\gamma-\beta)}+\frac{(z-\beta)(z-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(z-\gamma)(z-\alpha)}{(\beta-\gamma)(\beta-\alpha)}.$$

Note that this is a polynomial in z of degree at most 2. It follows from $f(\alpha)=f(\beta)=f(\gamma)=1$ that $f(z)=1$ for all z. In particular, $f(\delta)=1$. Now

$$\left|\frac{(\delta-\alpha)(\delta-\beta)}{(\gamma-\alpha)(\gamma-\beta)}\right|+\left|\frac{(\delta-\beta)(\delta-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}\right|+\left|\frac{(\delta-\gamma)(\delta-\alpha)}{(\beta-\gamma)(\beta-\alpha)}\right| \geq |f(\delta)|=1.$$

Hence

$$\left|\frac{DA\cdot DB}{CA\cdot CB}\right|+\left|\frac{DB\cdot DC}{AB\cdot AC}\right|+\left|\frac{DC\cdot DA}{BC\cdot BA}\right|\geq 1,$$

so that

$$DA\cdot DB\cdot AB+DB\cdot DC\cdot BC+DC\cdot DA\cdot CA\geq AB\cdot BC\cdot CA.$$

Equality holds if and only if

$$\left|\frac{(\delta-\alpha)(\delta-\beta)}{(\gamma-\alpha)(\gamma-\beta)}\right|+\left|\frac{(\delta-\beta)(\delta-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}\right|+\left|\frac{(\delta-\gamma)(\delta-\alpha)}{(\beta-\gamma)(\beta-\alpha)}\right|=1.$$

Hence $\delta=\alpha,\beta,\gamma$ or $\alpha+\beta+\gamma$. In other words, D coincides with A, B, C or H. Since D is inside triangle ABC, it cannot be one of the vertices, and must be the orthocentre.

3. Solution 1

First,

$$2=x_1^2+(x_1+x_2)^2+(x_2+x_3)^2+\cdots+(x_{n-1}+x_n)^2+x_n^2.$$

For a fixed k, $1\leq k\leq n$, it follows from the AM-GM Inequality that

$$\begin{aligned}&\sqrt{\frac{x_1^2+(x_1+x_2)^2+\cdots+(x_{k-1}+x_k)^2}{k}}\\ \geq\ &\frac{|x_1|+|x_1+x_2|+\cdots+|x_{k-1}+x_k|}{k}\\ \geq\ &\frac{|x_1-(x_1+x_2)+\cdots+(-1)^{k-1}(x_{k-1}+x_k)|}{k}\\ =\ &\frac{|x_k|}{k}.\end{aligned}$$

This is equivalent to

$$x_1^2+(x_1+x_2)^2+\cdots+(x_{k-1}+x_k)^2\geq\frac{x_k^2}{k}.$$

In the same way, we can prove that

$$(x_k+x_{k+1})^2+\cdots+(x_{n-1}+x_n)^2+x_n^2\geq\frac{x_k^2}{n-k+1}.$$

Now addition yields

$$2\geq\left(\frac{1}{k}+\frac{1}{n-k+1}\right)x_k^2\quad\text{or}\quad |x_k|\leq\sqrt{\frac{2k(n-k+1)}{n+1}}$$

for $1\leq k\leq n$. In the above inequalities, equality holds if and only if

$$x_1=-(x_1+x_2)=\cdots=(-1)^{k-1}(x_{k-1}+x_k)$$

and

$$x_k+x_{k+1}=-(x_{k+1}+x_{k+2})=\cdots=(-1)^{n-k}x_n.$$

In other words, the maximum of x_k is

$$\sqrt{\frac{2k(n-k+1)}{n+1}}$$

if we take

$$x_i=(-1)^{k-i}\frac{i}{k}x_k$$

for $1\leq i\leq k-1$ and

$$x_j=(-1)^{j-k}\frac{n-j+1}{n-k+1}x_k$$

for $k+1\leq j\leq n$.

Solution 2

We first determine the maximum value of x_n. Rewrite the given equation as

$$1=\sum_{i=1}^{n-1}(\sqrt{a_i}x_i+\sqrt{1-a_{i+1}}x_{i+1})^2+(1-(1-a_n))x_n^2$$

for suitably chosen constants a_1, a_2, $\ldots$, a_n. Taking into consideration that the coefficient of x_ix_{i+1} for all i is 1 in the original equation, we have $a_1=1$ and

$$2\sqrt{a_i}\sqrt{1-a_{i+1}}=1$$

for $1 \le i \le n-1$. In other words,

$$a_{i+1} = 1 - \frac{1}{4a_i}$$

so that

$$a_i = \frac{i+1}{2i}$$

for $1 \le i \le n$. From the displayed equation, we have

$$1 \ge (1-(1-a_n))x_n^2 = \frac{n+1}{2n}x_n^2$$

or

$$x_n \le \sqrt{\frac{2n}{n+1}}.$$

By symmetry,

$$x_1 \le \sqrt{\frac{2n}{n+1}}$$

also. For $2 \le k \le n-1$, we have

$$\begin{aligned} 1 &= \sum_{i=1}^{k-1}(\sqrt{a_i}x_i + \sqrt{1-a_{i+1}}x_{i+1})^2 \\ &\quad + \sum_{i=1}^{n-k}(\sqrt{a_i}x_{n-i+1} + \sqrt{1-a_{i+1}}x_{n-i})^2 \\ &\quad + (1-(1-a_k)-(1-a_{n-k+1}))x_k^2. \end{aligned}$$

It follows from

$$1 \ge (1-(1-a_k)-(1-a_{n-k+1}))x_k^2$$

that we have

$$x_k \le \sqrt{\frac{2k(n-k+1)}{n+1}}.$$

Note that this coincides with previous results when $k = n$ or 1. In all cases, the maximum can be attained when

$$\begin{aligned} 0 &= \sqrt{a_1}x_1 + \sqrt{1-a_2}x_2 \\ &= \sqrt{a_2}x_2 + \sqrt{1-a_3}x_3 \\ &= \cdots \\ &= \sqrt{a_{k-1}}x_{k-1} + \sqrt{1-a_k}x_k \\ &= \sqrt{a_{n-k}}x_{k+1} + \sqrt{1-a_{n-k+1}}x_k \\ &= \cdots \\ &= \sqrt{a_1}x_n + \sqrt{1-a_2}x_{n-1}. \end{aligned}$$

Starting with x_k in the middle, we can work our way outward and choose x_{k-1}, x_{k-2}, $\ldots$, x_1 as well as x_{k+1}, x_{k+2}, $\ldots$, x_n which satisfy the above equations.

1998/99
Paper I.

Section 1. Questions with Multiple Choices.

1. From
$$\log(a+b) = \log a + \log b = \log ab,$$
we have $a+b = ab$ so that
$$1 = ab - a - b + 1 = (a-1)(b-1).$$
Hence
$$\log(a-1) + \log(b-1) = \log(a-1)(b-1) = \log 1 = 0.$$

2. In order for $A \neq \emptyset$, we must have $2a+1 \leq 3a-5$ or $a \geq 6$. In order for $A \subseteq A \cap B$ or $A \subseteq B$, we must have $2a+1 \geq 3$ and $3a-5 \leq 22$. The former leads to $a \geq 1$ while the latter leads to $a \leq 9$. Hence $6 \leq a \leq 9$.

3. Let r be the common ratio of the progression. Let $S_0 = 0$ and
$$b_n = S_{10n} - S_{10(n-1)}$$
for $n \geq 1$. Then $\{b_n\}$ is a geometric progression with common ratio r^{10}. Since
$$70 = b_1 + b_2 + b_3 = b_1(1 + r^{10} + r^{20}),$$
we have
$$0 = r^{20} + r^{10} - 6 = (r^{10}+3)(r^{10}-2).$$
Since $r^{10} > 0$, we must have $r^{10} = 2$. Hence
$$S_{40} = b_1(1 + r^{10} + r^{20} + r^{30}) = 10(1+2+4+8) = 150.$$

4. Note that
$$\frac{1}{-1} = \frac{-3}{3} = \frac{2}{-2} = -1.$$
However, the solution set for $x^2 - 3x + 2 > 0$ is $(-\infty, 1) \cup (2, \infty)$ while that for $-x^2 + 3x - 2 > 0$ is (1,2). Hence Q is not sufficient for P. On the other hand, $x^2 + x + 1 > 0$ and $x^2 + x + 3 > 0$ have the same solution set $(-\infty, \infty)$, but the coefficients are not in proportion. Hence Q is not necessary for P either.

5. Let K be the point on BG such that EK is perpendicular to BG. Then EK is perpendicular to the plane BCD. Since EF is parallel to AC while FG is parallel to BD, $\angle EFG = 90°$. It follows that $\angle KFG = 90°$ also. Let H be the centre of triangle BCD. Taking $AC = 2$, we have $AG = BG = \sqrt{3}$ so that
$$BK = HK = HG = \frac{\sqrt{3}}{3}.$$
Hence
$$AH = \sqrt{AG^2 - HG^2} = \frac{2\sqrt{6}}{3},$$
so that
$$EK = \frac{1}{2}AH = \frac{\sqrt{6}}{3}.$$
Since $EF = \frac{1}{2}AC = 1$, we have
$$FK = \sqrt{EF^2 - EK^2} = \frac{\sqrt{3}}{3},$$
so that
$$\cot EFK = \frac{FK}{EK} = \frac{\sqrt{2}}{2}.$$
Since $\angle EFK$ is the dihedral angle between EFG and BFG, the dihedral angle between EFG and CFG is $\pi - \text{arccot}\frac{\sqrt{2}}{2}$.

6. The number of such lines joining two vertices of the cube is $\frac{8\times 7}{2} = 28$. The number of such lines joining the midpoints of two parallel edges is $\frac{12\times 3}{2} = 18$. The number of lines joining the centres of two opposite faces is $\frac{6\times 1}{2} = 3$. Since there are no other such lines, the total number is 28+18+3=49.

Section 2. Questions requiring Answers Only:

1. By the properties of f,
$$f\left(\frac{98}{19}\right) = f\left(6 - \frac{16}{19}\right) = f\left(-\frac{16}{19}\right) = f\left(\frac{16}{19}\right)$$
and
$$f\left(\frac{101}{17}\right) = f\left(6 - \frac{1}{17}\right) = f\left(-\frac{1}{17}\right) = f\left(\frac{1}{17}\right).$$
Similarly, we also have
$$f\left(\frac{104}{15}\right) = f\left(6 + \frac{14}{15}\right) = f\left(\frac{14}{15}\right).$$

Since $f(x)$ is increasing on [0,1] and $\frac{1}{17} < \frac{16}{19} < \frac{14}{15}$, we have

$$f\left(\frac{101}{17}\right) < f\left(\frac{98}{19}\right) < f\left(\frac{104}{15}\right).$$

2. Let S be represented by the complex number w. Since $PQSR$ is a parallelogram, $w + z = 2\overline{z} + (i+1)z$ or $w = 2\overline{z} + i^2$. Hence

$$|w|^2 = (2\overline{z} + iz)(2z - i\overline{z}) = 5 + 2i(z^2 - \overline{z}^2) = 5 - 4\sin 2\theta \le 9,$$

with equality when $\theta = \frac{3\pi}{4}$. It follows that the maximum distance of S from the origin is 3.

3. The number of ways of choosing three even digits is $\binom{5}{3} = 10$, and only (0,2,4) and (0,2,6) have sums less than 10. The number of ways of choosing one even and two odd digits is $\binom{5}{1}\binom{5}{2} = 50$, but (0,1,3), (0,1,5), (0,1,7), (0,3,5), (1,2,3), (1,2,5) and (1,3,4) have sums less than 10. Hence the total number of choices is $10 - 2 + 50 - 7 = 51$.

4. Let n be the number of terms and a be the first term. Then

$$a^2 + (n-1)a + 2n(n-1) \le 100.$$

In order that there exists at least one real number a which satisfies

$$a^2 + (n-1)a + (2n^2 - 2n - 100) \le 0,$$

we must have

$$(n-1)^2 - 4(2n^2 - 2n - 100) \ge 0$$

or, equivalently,

$$7n^2 - 6n - 401 \le 0.$$

This leads to

$$\frac{3 - \sqrt{2816}}{7} \le n \le \frac{3 + \sqrt{2816}}{7}.$$

Since

$$8 < \frac{3 + \sqrt{2816}}{7} < 9,$$

the maximium value of n is 8.

5. Solving

$$4 - 4(y-a)^2 = x^2 = 2y,$$

we have

$$y = \frac{8a - 2 \pm \sqrt{68 - 32a}}{8}.$$

In order for y to be real, we must have $68 - 32a \ge 0$ or $a \le \frac{17}{8}$. We also need y to be non-negative. This is certainly the case if $0 \le a \le \frac{17}{8}$. Suppose $a < 0$. Then we need

$$\sqrt{68 - 32a} \ge 2 - 8a,$$

which leads to $1 \ge a^2$, so that $-1 \le a < 0$. Thus the overall range is

$$-1 \le a \le \frac{17}{8}.$$

6. Let D be the midpoint of CM and let E be the point on BC such that DE and CM are perpendicular to each other. Note that we have $AM = BM = CM = 2$ so that $CD = 1$. Moreover, $AD = \sqrt{3}$, $DE = \frac{\sqrt{3}}{3}$, $CE = \frac{2\sqrt{3}}{3}$ and $BC = 2\sqrt{3}$. Hence $AB^2 + AC^2 = BC^2$, so that $\cos ACE = \frac{AC}{BC} = \frac{\sqrt{3}}{3}$. Now

$$AE^2 = AC^2 + CE^2 - 2AC \cdot CE \cos ACD = \frac{8}{3}.$$

Hence $AE^2 + CE^2 = AC^2$. Since we also have $AE^2 + DE^2 = AD^2$, The area of triangle MBC is $\frac{1}{4}AC \cdot BC = \sqrt{3}$. Since AE is perpendicular to the plane containing MBC, the volume of $ABCM$ is $\frac{1}{3}AE\sqrt{3} = \frac{2\sqrt{2}}{3}$.

Section 3. Questions requiring Full Solutions:

1. We have

$$\begin{aligned} z &= 1 - \sin\theta - i\cos\theta \\ &= 1 - \cos\left(\frac{\pi}{2} - \theta\right) - i\sin\left(\frac{\pi}{2} - \theta\right) \\ &= 2\sin^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) - 2i\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \\ &= -2\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\left(-\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + i\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right) \\ &= -2\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\left(\cos\left(\frac{3\pi}{4} - \frac{\theta}{2}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\theta}{2}\right)\right). \end{aligned}$$

Hence

$$\arg z = \frac{3\pi}{4} - \frac{\theta}{2}.$$

2. Let

$$f(x) = ax^2 + 8x + 3 = a\left(x + \frac{4}{a}\right)^2 + 3 - \frac{16}{a}.$$

It is an upsidedown parabola with maximum $3-\frac{16}{a}$ occurring at $x=-\frac{4}{a}$. Note that $f(0)=3$ lies between -5 and 5. Suppose $3-\frac{16}{a}>5$, so that $-8<a<0$. Then $|f(x)|<5$ until $f(x)=5$ for the first time after $x=0$. Hence $\ell(a)$ is the smaller root of the equation $ax^2+8x+3=5$, and we have

$$\ell(a)=\frac{-8+\sqrt{64+8a}}{2a}=\frac{2}{\sqrt{16+2a}+4}<\frac{1}{2}.$$

Suppose $3-\frac{16}{a}\le 5$, so that $a\le -8$. Then $|f(x)|<5$ until $f(x)=-5$ for the first time after $x=0$. Hence $\ell(a)$ is the larger root of the equation $ax^2+8x+3=-5$, and we have

$$\begin{aligned}\ell(a) &= \frac{-8-\sqrt{64-32a}}{2a}\\ &= \frac{4}{\sqrt{4-2a}-2}\\ &\le \frac{4}{\sqrt{4-2(-8)}-2}\\ &= \frac{\sqrt{5}+1}{2}.\end{aligned}$$

(a) Since $\frac{\sqrt{5}+1}{2}>\frac{1}{2}$, the maximum value of $\ell(a)$ is $\frac{\sqrt{5}+1}{2}$.

(b) The maximum value of $\ell(a)$ occurs at $a=-8$.

3. Let $M(x_0,y_0)$, $M_1(x_1,y_1)$ and $M_2(x_2,y_2)$ be three points on the parabola such that MM_1 passes through A and also MM_2 passes through B. Then

$$\frac{y_1-y_0}{x_1-x_0}=\frac{y_0-b}{x_0-a},$$

which leads to

$$y_1=\frac{by_0-2pa}{y_0-b}.$$

Similarly, we have

$$y_2=\frac{2pa}{y_0}.$$

The equation of the line M_1M_2 is

$$y_1y_2=(y_1+y_2)y-2px.$$

Eliminating y_1 and y_2, we have

$$(2px-by)y_0^2+2pb(a-x)y_0+2px(by-2pa)=0.$$

(a) The system

$$2px-by=2pb(a-x)=2px(by-2pa)=0$$

has a unique solution. Hence M_1M_2 passes through a fixed point.

(b) The unique solution is $x=a$ and $y=\frac{2pa}{b}$. Hence M_1M_2 passes through $(a,\frac{2pa}{b})$.

Paper II

1. Let $BC=a$, $CA=b$, $AB=c$, R be the circumradius and r_a be the exradius opposite A. The area of triangle ABC is given by $\frac{1}{2}bc\sin A$ as well as $\frac{1}{2}r_a(b+c-a)$. It follows that

$$r_a=\frac{bc\sin A}{b+c-a}=\frac{2R\sin A\sin B\sin C}{\sin B+\sin C-\sin A}.$$

Now

$$\begin{aligned}&\sin B+\sin C-\sin A\\ =\;& 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}-2\sin\frac{B+C}{2}\cos\frac{B+C}{2}\\ =\;& 2\sin\frac{B+C}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)\\ =\;& 4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\end{aligned}$$

Hence

$$r_a=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

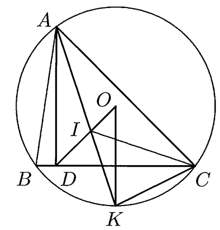
Let the extension of AI cut the circumcircle at K. Then K is the midpoint of the arc BC, and OK is perpendicular to BC. Hence triangles ADI and KOI are similar, so that

$$\frac{AI}{IK}=\frac{AD}{KO}=\frac{c\sin B}{R}=2\sin B\sin C.$$

On the other hand,

$$\frac{AI}{IK}=\frac{[CAI]}{[CKI]}$$

where $[T]$ denotes the area of triangle T.



Then we have

$$[CAI] = \frac{1}{2}AC \cdot CI \sin\frac{C}{2}.$$

Note that

$$\angle KCI = \angle KAB + \angle ICB = \frac{A+C}{2} = \frac{\pi}{2} - \frac{B}{2}.$$

It follows that

$$[CKI] = \frac{1}{2}CK \cdot CI \cos\frac{B}{2}.$$

Hence

$$2\sin B \sin C = \frac{AI}{IK} = \frac{AC}{CK}\frac{\sin\frac{C}{2}}{\cos\frac{B}{2}} = \frac{\sin B \sin\frac{C}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}}.$$

This reduces to

$$4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 1,$$

so that $r_a = R$.

2. (a) Note that we have

$$\frac{1}{2} \le \sqrt{\frac{a_i^3}{b_i}} \cdot \frac{1}{\sqrt{a_ib_i}} = \frac{a_i}{b_i} \le 2.$$

It follows that

$$\frac{\sqrt{a_ib_i}}{2} \le \sqrt{\frac{a_i^3}{b_i}} \le 2\sqrt{a_ib_i}.$$

Hence

$$\begin{aligned} 0 &\ge \left(\frac{\sqrt{a_ib_i}}{2} - \sqrt{\frac{a_i^3}{b_i}}\right)\left(2\sqrt{a_ib_i} - \sqrt{\frac{a_i^3}{b_i}}\right) \\ &= a_ib_i + \frac{a_i^3}{b_i} - \frac{5}{2}a_i^2. \end{aligned}$$

Similarly, $\frac{b_i}{2} \le a_i \le 2b_i$ leads to $0 \ge b_i^2 + a_i^2 - \frac{5}{2}a_ib_i$. It follows that

$$\begin{aligned} \sum_{i=1}^n \frac{a_i^3}{b_i} &\le \frac{5}{2}\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_ib_i \\ &\le \frac{5}{2}\sum_{i=1}^n a_i^2 - \frac{2}{5}\sum_{i=1}^n (a_i^2 + b_i^2) \\ &= \frac{17}{10}\sum_{i=1}^n a_i^2. \end{aligned}$$

(b) For equality to hold, we must have $\frac{a_i}{b_i} = \frac{1}{2}$ or 2 for every i. This means that one of a_i and b_1 is 1 and the other is 2. Since

$$\sum_{i=1}^n a_i^2 = \sum_{i=1}^n b_i^2,$$

n must be even and $a_i = 1$ occurs the same number of times as $a_i = 2$.

3. Let x_k be the largest integer such that there exist positive integers $n_1, n_2, \ldots, n_k$ for which

$$F_{n_k}(F_{n_{k-1}}(\cdots(F_{n_1}(a))\cdots)) = 1$$

for all positive integers $a \le x_k$. Then $A = x_6$. We shall determine x_k recursively and show that it is always even. For $k = 1$, choose $n_1 = 2$. Then $F_2(1) = F_2(2) = 1$ while $F_2(3) = 2$, so that $x_1 \ge 2$. On the other hand, $F_n(2) = 2$ for $n \ne 2$. Hence $x_1 = 2$ and it is indeed even.

Suppose x_k has been determined and is even. Then x_{k+1} is the largest integer such that there exists a positive integer n for which $F_n(a) \le x_k$ for all positive integers $a \le x_{k+1}$. Let $x_{k+1} = qn + r$ where $0 \le r \le n-1$. Then

$$F_n(x_{k+1}) = q + r \le x_k$$

so that

$$x_{k+1} \le x_k + q(n-1).$$

We also have

$$F_n(qn-1) = (q-1) + (n-1) \le x_k$$

so that

$$\begin{aligned} q(n-1) &\le \left\lfloor \left(\frac{q+n-1}{2}\right)^2 \right\rfloor \\ &\le \left\lfloor \left(\frac{x_k+1}{2}\right)^2 \right\rfloor \\ &= \left\lfloor \frac{x_k(x_k+2)}{4} + \frac{1}{4} \right\rfloor . \end{aligned}$$

Since x_k is even, we have

$$q(n-1) \le \frac{x_k(x_k+2)}{4}$$

so that

$$x_{k+1} \le x_k + q(n-1) \le \frac{x_k(x_k+6)}{4}.$$

Now take $n = \frac{x_k+4}{2}$. Then we have

$$\frac{x_k(x_k+6)}{4} = \frac{x_k}{2}n + \frac{x_k}{2}.$$

If $a \le \frac{x_k(x_k+6)}{4}$, write $a = qn + r$ with $0 \le r \le n-1$.

If $q = \frac{x_k}{2}$, then $r \le \frac{x_k}{2}$ so that $f_n(a) = q + r \le x_k$.

If $q \le \frac{x_k}{2} - 1$, then $r \le n - 1 = \frac{x_k}{2} + 1$ so that $F_n(a) = q + r \le x_k$.

It follows that

$$x_{k+1} = \frac{x_k(x_k+6)}{4}.$$

Since one of x_k and $x_k + 6$ is a multiple of 4 and the other is even, x_{k+1} is also even. From $x_1 = 2$, we have $x_2 = 4$, $x_3 = 10$, $x_4 = 40$, $x_5 = 460$ and $A = x_6 = 53590$.

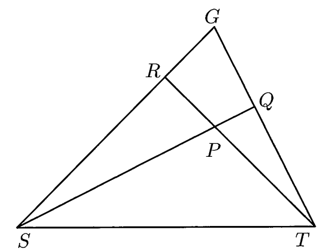
Olympiad Paper I

1. We first prove a preliminary result. Let GST be a triangle with acute angles at S and T. Let P be a point on the same side of ST as G such that GP is perpendicular to ST. Then P is the orthocentre of GST if and only if $\angle SPT + \angle SGT = 180°$.

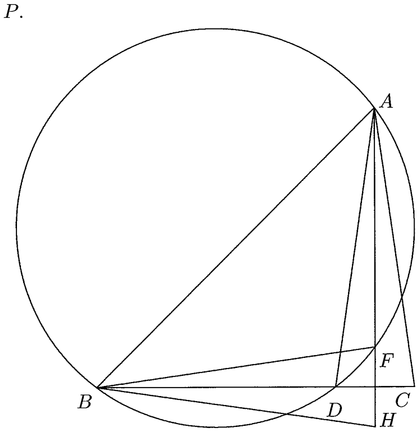
Suppose P is the orthocentre. First assume that it is inside GST. Let SP cut GT at Q, and let TP cut GS at R. Then $\angle SQT = 90° = \angle GRT$. Hence

$$\angle SPT = 90° + \angle GTR = 180° - \angle SGT.$$

If P is outside GST, then the same argument applies with the roles of G and P interchanged.



Conversely, suppose that $\angle SPT + \angle SGT = 180°$. Let the orthocentre of GST be some point P' which must lie on the line GP. Since $\angle GST$ and $\angle GTS$ are both acute, P' is on the same side of ST as G and P. Since $\angle SP'T = 180° - \angle SGT = \angle SPT$, we must have $P' = P$.



Suppose F is the orthocentre of triangle ABC. Then CF is perpendicular to AB, and hence parallel to DH. Note that we have $\angle ACB = 180° - \angle AFB$ and $\angle AHB = 180° - \angle ADB$ by our preliminary result.

Since $ABDF$ is cyclic, $\angle AFB = \angle ADB$ by Thales' Theorem. Hence $\angle ACB = \angle AHB$. By the converse of Thales' Theroem, $ABHC$ is cyclic. Conversely, suppose CF is parallel to DH and

$ABHC$ is cyclic. Then CF is perpendicular to AB. By Thales' Theorem and our preliminary result,

$$\angle ACB = \angle AHB = 180^\circ - \angle ADB = 180^\circ - \angle AFB.$$

By the converse of our preliminary result, F is the orthocentre of ABC.

2. (a) Since

$$f_1(x) = xf_0(x) + f_0(ax) = x + 1$$

we have

$$f_1(x) - f_0(x) = x.$$

We claim in general that

$$f_n(x) - f_{n-1}(x) = a^{n-1}xf_{n-1}\left(\frac{x}{a}\right).$$

We use induction on n, and the basis $n = 1$ holds since

$$a^{1-1}xf_0\left(\frac{x}{a}\right) = x.$$

Suppose the claim holds for some $n \geq 1$. Then

$$\begin{aligned} & f_{n+1}(x) - f_n(x) \\ = \; & (xf_n(x) + f_n(ax)) - (xf_{n-1}(x) + f_{n-1}(ax)) \\ = \; & x(f_n(x) - f_{n-1}(x)) + (f_n(ax) - f_{n-1}(ax)) \\ = \; & xa^{n-1}xf_{n-1}\left(\frac{x}{a}\right) + a^{n-1}(ax)f_{n-1}(x) \\ = \; & a^n x\left(\frac{x}{a}f_{n-1}\left(\frac{x}{a}\right) + f_{n-1}\left(a \cdot \frac{x}{a}\right)\right) \\ = \; & a^n x f_n\left(\frac{x}{a}\right). \end{aligned}$$

We now prove by induction on n that

$$f_n(x) = x^n f_n\left(\frac{1}{x}\right).$$

For $n = 0$, we have $f_0(x) = 1$ and $x^0 f_0\left(\frac{1}{x}\right) = 1$. Suppose the result holds for some $n \geq 0$. Then

$$\begin{aligned} f_{n+1}(x) &= f_n(x) + a^n x f_n\left(\frac{x}{a}\right) \\ &= x^n f_n\left(\frac{1}{x}\right) + a^n x\left(\frac{x}{a}\right)^n f_n\left(\frac{a}{x}\right) \\ &= x^{n+1}\left(\frac{1}{x}f_n\left(\frac{1}{x}\right) + f_n\left(a \cdot \frac{1}{x}\right)\right) \\ &= x^{n+1} f_{n+1}\left(\frac{1}{x}\right). \end{aligned}$$

(b) Let

$$f(x) = \sum_{i=0}^{n} b_i^{(n)} x^i.$$

From the given conditions, we have

$$b_n^{(n)} = b_{n-1}^{(n-1)} = \cdots = b_0^{(0)} = 1.$$

Comparing the coefficient of x^i in $f_n(x) = x^n f_n(\frac{1}{x})$, we have $b_i^{(n)} = b_{n-i}^{(n)}$. It follows that

$$b_0^{(n)} = b_0^{(n-1)} = \cdots = b_0^{(0)} = 1.$$

Comparing the coefficients of x^i in

$$f_n(x) = xf_{n-1}(x) + f_{n-1}(ax),$$

we have

$$b_i^{(n)} = b_{i-1}^{(n-1)} + a^i b_i^{(n-1)}.$$

Similarly,

$$b_{n-i}^{(n)} = b_{n-i-1}^{(n-1)} + a^{n-i} b_{n-i}^{(n-1)},$$

which is equivalent to

$$a^i b_i^{(n)} = a^i b_i^{(n-1)} + a^n b_{i-1}^{(n-1)}.$$

From the two displayed equations, we have

$$(a^i - 1)b_i^{(n)} = (a^n - 1)b_{i-1}^{(n-1)}.$$

It follows that

$$\begin{aligned} b_i^{(n)} &= \frac{a^n - 1}{a^i - 1} b_{i-1}^{(n-1)} \\ &= \frac{(a^n - 1)(a^{n-1} - 1)}{(a^i - 1)(a^{i-1} - 1)} b_{i-2}^{(n-2)} \\ &= \cdots \\ &= \frac{(a^n - 1)(a^{n-1} - 1)\cdots(a^{n-i+1} - 1)}{(a^i - 1)(a^{i-1} - 1)\cdots(a - 1)} b_0^{(n-i)} \\ &= \frac{(a^n - 1)(a^{n-1} - 1)\cdots(a^{n-i+1} - 1)}{(a^i - 1)(a^{i-1} - 1)\cdots(a - 1)}. \end{aligned}$$

Hence

$$f_n(x) = \sum_{i=0}^{n} \frac{(a^n - 1)(a^{n-1} - 1)\cdots(a^{n-i+1} - 1)}{(a^i - 1)(a^{i-1} - 1)\cdots(a - 1)} x^i.$$

3. If a set of four space stations is not a group, it must consist of two non-empty subsets X and Y such that all space-highways between X and Y are one-way going from X to Y. Such a set may be classified as type I, II or III according to whether $|X| = 1$, 2 or 3. Note that this classification is not mutually exclusive, so that a set may be of two or even all three types.

We solve a more general problem with n space stations, where $n > 3$ is an odd integer. For $1 \leq i \leq n$, let s_i be the number of one-way space highways from the i-th space station. Then the number of type I sets with this space station as the sole member of X is $\binom{s_i}{3}$, so that the number of sets of type I is

$$T = \sum_{i=1}^{n} \binom{s_i}{3}.$$

Suppose a and b are positive integers such that $a < b + 1$. By Pascal's Formula,

$$\binom{a}{3} - \binom{a-1}{3} = \binom{a-1}{2} < \binom{b}{2} = \binom{b+1}{3} - \binom{b}{3}.$$

In other words,

$$\binom{a}{3} + \binom{b}{3} < \binom{a-1}{3} + \binom{b+1}{3}.$$

It follows that to minimize T, we should make s_i, $1 \leq i \leq n$, equal to one another if possible. Now

$$\frac{1}{n}\sum_{i=1}^{n} s_i = \frac{1}{n}\left(\binom{n}{2} - n\right) = \frac{n-3}{2}.$$

Denote this value by m. If $s_1 = s_2 = \cdots = s_n = m$, then $T = n\binom{m}{3}$ and the number of groups is at most $\binom{n}{4} - n\binom{m}{3}$. For $n = 99$, we have $m = 48$ and

$$\binom{n}{4} - n\binom{m}{3} = 2052072.$$

We now show that this maximum can be attained. Let the space stations be $1, 2, \ldots, n$ clockwise round a circle. The space highways joining two adjacent space stations are two-way. For non-adjacent space stations i and j, the one-way space highway goes from i to j if and only if in going from i to j clockwise round the circle, the number of other space stations passed over is odd.

Since n is odd, this scheme can be applied consistently. By symmetry, there are exactly m one-way space highways from each space station. It follows that the number of sets of type I is exactly $n\binom{m}{3}$.

To complete the argument, we prove that all sets not of type I are groups. Suppose in $\{A, B, C, D\}$, A and B are joined by a two-way space highway. Suppose the space highways joining C to A and B are both one-way. Then the space highway between A to C goes from A to C if and only if the one between B and C goes from C to B. If either is two-way, it makes things even simpler.

In the same way, we can also go from D to A and B and vice versa. Suppose $\{A < B < C < D\}$ is of type II, with $X = \{A < B\}$ and $Y = \{C < D\}$. If X and Y separate each other round the circle, we may assume that A, C, B and D are in clockwise order. Then the numbers of space stations from A to C and from B to D must be odd.

Since n is odd, either the number of space stations from C to B or that from D to A must be even. This will contradict the direction of the one-way space highway between B and C or that between A and D. It follows that X and Y cannot separate each other round the circle, so that we may assume that A, B, C and D are in clockwise order. Now the number of space stations from A to B, from B to C and from C to D must all be odd.

This means that we may redefine $X = \{A\}$ and $Y = \{B < C < D\}$, so that the set is also of type I. Finally, suppose $\{A, B, C, D\}$ is of type III. In order for this not to be of type I, we must have exactly one one-way space highway from each of A, B and C to the other two. We may assume that A, B and C are in clockwise order. If the space highway between A and B goes from A to B, then the numbers of space stations from A to B, from B to C and from C to A must all be odd.

However, this is impossible since n is odd. Hence the space highway goes from B to A, and the numbers of space stations from A to B, from B to C and from C to A are all even. By symmetry, we may assume that D is one of those between A and B. Since a one-way space highway goes from A to D, the number of space stations from A to D is odd.

However, this implies that the number of space stations from D to B is even, contradicting the fact that a one-way space highway goes from B to D.

This completes the proof that all sets which are not of type I are groups.

Olympiad Paper II

1. Let $n = 2^{1999}$. We claim that $1^{19}, 3^{19}, \ldots, (n-1)^{19}$ are not congruent modulo n to one another. Let x and y be any odd numbers with $x \not\equiv y \pmod{n}$. We have

$$x^{19} - y^{19} = (x-y)(x^{18} + x^{17}y + \cdots + xy^{17} + y^{18}).$$

The second factor being odd, $x^{19} \equiv y^{19} \pmod{n}$ would imply $x \equiv y \pmod{n}$. This justifies the claim. It follows that for any integer m, $2m - 1 \equiv a^{19} \pmod{n}$ for some odd integer a, so that $2m = a^{19} + 1^{99} + kn$ for some integer k. If $k \geq 0$, we can simply take $b = 1$. If not, take $b_1 = 1$, $a_1 = a - hn$ and

$$k_1 = k + \frac{a^{19} - a_1^{19}}{n}.$$

Then $2m = a_1^{19} + b_1^{99} + k_1 n$, and if h is sufficiently large, we will have $k_1 \geq 0$.

2. Let the roots be $0 \leq \alpha \leq \beta \leq \gamma$. Then $\alpha + \beta + \gamma = -a$ and $f(x) = (x-\alpha)(x-\beta)(x-\gamma)$.

(a) When $0 \leq x \leq \alpha$, the Arithmetic-Geometric Means Inequality yields

$$\begin{aligned}
-f(x) &= (\alpha - x)(\beta - x)(\gamma - x) \\
&\leq \left(\frac{(\alpha - x) + (\beta - x) + (\gamma - x)}{3}\right)^3 \\
&= \frac{1}{27}(-3x - a)^3 \\
&\leq \frac{1}{27}(x - a)^3.
\end{aligned}$$

Hence $f(x) \geq -\frac{1}{27}(x-a)^3$. When $\beta \leq x \leq \gamma$. we have

$$\begin{aligned}
-f(x) &= (x - \alpha)(x - \beta)(\gamma - x) \\
&\leq \left(\frac{(x - \alpha) + (x - \beta) + (\gamma - x)}{3}\right)^3 \\
&\leq -\frac{1}{27}(x + \alpha + \beta + \gamma)^3 \\
&= -\frac{1}{27}(x - a)^3.
\end{aligned}$$

Hence $f(x) \geq -\frac{1}{27}(x-a)^3$.
When $\alpha < x < \beta$ or $\gamma < x$,

$$f(x) > 0 \geq -\frac{1}{27}(x-a)^3.$$

We can take $\lambda = -\frac{1}{27}$ since the cases of equality in (b) will show that no larger value of λ is possible.

(b) When $0 \leq x \leq \alpha$, the necessary and sufficient conditions for equality are $\alpha - x = \beta - x = \gamma - x$ and $-3x = x$, or equivalently $\alpha = \beta = \gamma$ and $x = 0$. When $\beta \leq x \leq \gamma$, the necessary and sufficient conditions for equality to hold are $x - \alpha = x - \beta = \gamma - x$ and $\alpha + \beta = -\alpha - \beta$, or equivalently $\alpha = \beta = 0$ and $2\gamma = x$. When $\alpha < x < \beta$ or $\gamma < x$, equality cannot occur.

In summary, we have two cases of equality,

$$f(0) = \frac{a^3}{27} = -\frac{1}{27}(0-a)^3$$

when $\alpha = \beta = \gamma$, and

$$f\left(\frac{\gamma}{2}\right) = -\frac{\gamma^3}{8} = -\frac{1}{27}\left(\frac{\gamma}{2} + \gamma\right)^3.$$

3. Clearly, each vertical $1 \times 1 \times 4$ stack contains exactly one red cube. All we have to decide is whether it is on level 1, 2, 3 or 4. If we represent the base of the cube as a 4×4 table, we fill in each square with a number indicating the level of the red cube on that stack.

Also, each horizontal $1 \times 1 \times 4$ slab contains exactly one red cube, and for this to happen, no two numbers in each row and each column in the table can be the same. The first row can be filled in 4! ways, and by symmetry we only need consider the case when the entries are 1, 2, 3 and 4 in that order.

The remaining entries of the first column are 2, 3 and 4 in some order. They can be permuted in 3! ways, and we need only consider the permutation 2, 3 and 4 in that order. Now the entry in the second row and second column is 1, 3 or 4. If it is 3 or 4, the remaining entries are determined uniquely. If it is 1, the remaining entries in the second row and second column are determined uniquely, but there are two ways to complete the table.

Thus there are four different tables with the elements in the first row and the first column being 1, 2, 3 and 4 in that order, and these are shown in the diagram below. The number of different tables is therefore $4! \times 3! \times 4 = 576$, which is also the number of different ways of assembling the cube.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

1999/2000

Paper I.

Section 1. Questions with Multiple Choices.

1. Let $a_n = a_1 q^{n-1}$. Then

$$\frac{b_{n+1}}{b_n} = \frac{a_1(1+q+q^2)q^{3n}}{a_1(1+q+q^2)q^{3n-3}} = q^3.$$

Hence $\{b_n\}$ is a geometric progression with common ratio q^3.

2. From

$$(|x|-1)^2 + (|y|-1)^2 \le 2,$$

we have $(|x|-1, |y|-1) = (0,0)$, $(1,0)$, $(-1,0)$, $(0,1)$ or $(0,-1)$. The first yields $(x,y) = (\pm1,\pm1)$, the second $(x,y) = (\pm2,\pm1)$, the third $(x,y) = (0,\pm1)$, the fourth $(x,y) = (\pm1,\pm2)$ and the fifth $(x,y) = (\pm1,0)$. The total number of such lattice points is

$$4+4+2+4+2 = 16.$$

3. Let

$$f(t) = (\log_2 3)^t - (\log_5 3)^t.$$

Then $f(t)$ is an increasing function. The given inequality states that $f(x) \ge f(-y)$. Hence $x \ge -y$ or $x+y \ge 0$.

4. Let a be a line on α which intersects c at a point A, and let b be a line on β which intersects c at a point B. If $A \ne B$, then a and b are skew lines both of which intersect c. Hence P is false. For any positive integer n, let the line ℓ_n lie in the plane $x = n$ and have slope n. These lines are pairwise skew. Hence Q is also false.

5. Let the number of games played among these three players be r. Then the total number of games played is

$$\binom{n-3}{2} + r + (6-2r) = 50,$$

so that

$$(n-3)(n-4) = 88 + 2r.$$

Since $0 \le r \le 3$, it is only when $r = 1$ that $88 + 2r = 90$ is the product of two consecutive integers.

6. Let the coordinates for B be $(t^2, 2t)$ and those for C be $(s^2, 2s)$, with $s \neq t \neq 1 \neq s$. Then the equation of BC is

$$\frac{y-2t}{2s-2t} = \frac{x-t^2}{s^2-t^2}$$

or $2x-(s+t)y+2st=0$. Since it passes through $(5,-2)$, we have $(s+1)(t+1)=-4$. Now the slope of AB is

$$\frac{2t-2}{t^2-1} = \frac{2}{t+1}$$

while that of AC is

$$\frac{2s-2}{s^2-1} = \frac{2}{s+1}.$$

Hence their product is equal to

$$\frac{4}{(t+1)(s+1)} = -1,$$

so that $\angle CAB = 90°$. In other words, ABC is a right triangle.

Section 2. Questions requiring Answers Only:

1. Let the number of consecutive integers be n, and let a be the first one. Then their sum

$$S = na + \frac{n(n-1)}{2} \geq \frac{n(n+1)}{2}.$$

From

$$\frac{n(n+1)}{2} \leq 2000,$$

we have $60 \leq n \leq 62$.

When $n=60$, we have $60a+30\times 59 \leq 2000$. Hence $a \leq 3$, yielding $S=1830$, 1890 and 1950. When $n=61$, we have $61a+30\times 61 \leq 2000$. Hence $a \leq 2$, yielding $S=1891$ and 1952.

Finally, when $n=62$, we have $62a+31\times 61 \leq 2000$. Hence $a \leq 1$, yielding $S=1953$.

2. We have

$$\arg z = \arg((12+5i)^2(239-i)) = \arg(28561+28561i) = \frac{\pi}{4}.$$

3. Note that

$$\cot A + \cot B = \frac{\cos A \sin B + \cos B \sin A}{\sin A \sin B} = \frac{\sin C}{\sin A \sin B}.$$

Hence

$$\frac{\cot C}{\cot A + \cot B} = \frac{\sin A}{\sin C}\left(\frac{\sin B}{\sin C}\right)\cos C.$$

By the Laws of the Sines and Cosines,

$$\frac{\cot C}{\cot A + \cot B} = \left(\frac{a}{c}\right)\left(\frac{b}{c}\right)\frac{a^2+b^2-c^2}{2ab} = \frac{9(a^2+b^2)-9c^2}{18c^2} = \frac{5}{9}.$$

4. Since $\sqrt{4^2+3^2}=5$, the equation of the right directrix is $x=\frac{16}{5}$. The right focus is at $F_1(5,0)$ and the left focus is at $F_2(-5,0)$. Let d be the distance from P to the right directrix. If P is on the right branch, then $PF_2 - PF_1 = 8$ while $PF_2 + PF_2 = 2d$. Hence $2PF_2 = 2d-8$.

However,

$$\frac{PF_1}{d} = \frac{d-8}{8} < 1,$$

contradicting the fact that we have a hyperbola. It follows that P is on the left branch. Now $PF_1 - PF_2 = 8$ and $2PF_1 = 2d+8$. Hence

$$\frac{PF_1}{d} = \frac{d+4}{d} = \frac{5}{4}.$$

It follows that $d=16$ and the x-coordinate of P is

$$\frac{16}{5} - 16 = -\frac{64}{5}.$$

5. The slope of the line is $-\frac{a}{b}$. We may assume that $a>0$ and $b<0$ so that $a \neq b$. We must also have $a \neq c \neq b$. If $c=0$, there are 3 ways to choose a and 3 ways to choose b.

However, the lines $x-y=0$, $2x-2y=0$ and $3x-3y=0$ are the same. Thus the number of lines in this case is $3^2-2=7$. If $c \neq 0$, there are 3 ways to choose a, 3 ways to choose b and 4 ways to choose c. Thus the number of lines in this case is $3^2 4 = 36$. The total is 7+36=43.

6. Let BH intersect SC at P. Since H is the orthocentre of triangle SBC, BP is perpendicular to SC. A must lie on the plane through BP perpendicular to SC, as otherwise AH will not be perpendicular to SBC. Note in particular that AB and SC are perpendicular

skew lines. Let the plane through SC perpendicular to AB cut it at F. Then CF is an altitude of triangle ABC, and the projection O of S lies on it.

Repeat the above argument starting with Q as the point of intersection of CH and SB. We can show that SB and AC are also perpendicular skew lines. If E is the point of intersection of AC with the plane through SB perpendicular to AC, then O also lies on the altitude BE of triangle ABC. Hence it is the orthocentre of ABC.

Since ABC is equilateral, we have $SA=SB=SC=2\sqrt{3}$.

Let F be the point of intersection of CO and AB. Since CF is perpendicular to AB, we have EF perpendicular to AB, so that the dihedral angle between the planes ABH and ABC is $\angle EFC=30°$.

Since EF is perpendicular to SC, $\angle ECF=60°$. It follows that

$$\begin{aligned} OC &= \frac{1}{2}SC = \sqrt{3}, \\ SO &= \sqrt{SC^2 - OC^2} = 3 \\ \text{and}\quad AB &= 2OC\left(\frac{\sqrt{3}}{2}\right) = 3, \end{aligned}$$

so that the volume of $SABC$ is given by

$$\frac{1}{3} \times 3 \times 3^2 \times \frac{\sqrt{3}}{4} = \frac{9\sqrt{3}}{4}.$$

Section 3. Questions requiring Full Solutions:

1. Let

$$\begin{aligned} f(x) &= x^2\cos\theta - x(1-x) + (1-x)^2\sin\theta \\ &= (x\sqrt{\cos\theta} - (1-x)\sqrt{\sin\theta})^2 \\ &\quad +x(1-x)(2\sqrt{\cos\theta\sin\theta} - 1). \end{aligned}$$

In the open interval (0,1), the equation

$$x\sqrt{\cos\theta} = (1-x)\sqrt{\sin\theta}$$

has a root

$$c = \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}+\sqrt{\sin\theta}}.$$

Hence

$$2\sqrt{\cos\theta\sin\theta} - 1 = \frac{f(c)}{c(1-c)} > 0.$$

Conversely, if

$$2\sqrt{\cos\theta\sin\theta} - 1 > 0,$$

then $f(x) > 0$ for $0 \le x \le 1$. This may be rewritten as $\sin 2\theta > \frac{1}{2}$. Now $\cos\theta = f(1) > 0$ and $\sin\theta = f(0) > 0$, so that $0 < \theta < \frac{\pi}{2}$ on the interval $[0, 2\pi]$. Hence $0 < 2\theta < \pi$.

It follows that $\frac{\pi}{6} < 2\theta < \frac{5\pi}{6}$ so that $\frac{\pi}{12} < \theta < \frac{5\pi}{12}$. Thus the entire range is

$$2k\pi + \frac{\pi}{12} < \theta < 2k\pi + \frac{5\pi}{12}$$

for each integer k.

2. The x-coordinate of the foci are $\pm\sqrt{5^2-4^2} = \pm 3$, the eccentricity is $\frac{3}{5}$ and the equation of the directrices are $x = \pm\frac{25}{3}$. Hence F is $(-3, 0)$.

Let M and N be the respective feet of perpendicular from A and B to the directrix $x = -\frac{25}{3}$. Then

$$AB + \frac{5}{3}BF = AB + BN \ge AN \ge AM,$$

which is fixed. Hence the minimum value occurs when B is the point of intersection of AM with the ellipse. Hence B is $(-\frac{5\sqrt{3}}{3}, 2)$.

3. Let $a_{n+1} = x$ and let the common difference of the arithmetic progression be d. Then we have

$$S = a_{n+1} + a_{n+2} + \cdots + a_{2n+1} = (n+1)x + \frac{n(n+1)}{2}d$$

so that

$$\frac{S}{n+1} = x + \frac{nd}{2}.$$

Let p be a real number to be chosen later. We have

$$\begin{aligned} a_1^2 + a_{n+1}^2 &= (x - nd)^2 + x^2 \\ &= p\left(\frac{S}{n+1}\right)^2 + (2-p)x^2 - (2+p)ndx \\ &\quad + \left(1 - \frac{p}{4}\right)n^2d^2. \end{aligned}$$

The last three terms can be combined into a square if the discriminant

$$(2+p)^2 - 4(2-p)\left(1-\frac{p}{4}\right) = 0.$$

This yields $p = \frac{2}{5}$ so that these terms become

$$\frac{8}{5}x^2 - \frac{12}{5}ndx + \frac{9}{10}n^2d^2 = \frac{1}{10}(4x-3nd)^2 \geq 0.$$

It follows that

$$M \geq a_1^2 + a_{n+1}^2 \geq \frac{2}{5}\left(\frac{S}{n+1}\right)^2$$

so that

$$S \leq \sqrt{\frac{5M}{2}}(n+1).$$

To show that this maximum value can be achieved, we choose $x = 3k$ and $d = \frac{4k}{n}$ so that $4x - 3nd = 0$, with k to be chosen later.

Then

$$S = 3k(n+1) + \frac{4k}{n} \cdot \frac{n(n+1)}{2} = 5k(n+1).$$

Hence

$$a_1^2 + a_{n+1}^2 = \frac{2}{5}\left(\frac{5k(n+1)}{n+1}\right)^2 = 10k^2.$$

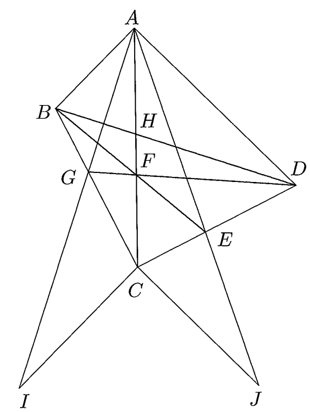
Setting this equal to M, we have $k = \sqrt{\frac{M}{10}}$. It follows that if $x = 3\sqrt{\frac{M}{10}}$ and $d = \frac{4}{n}\sqrt{\frac{M}{10}}$, then

$$S = \sqrt{\frac{5M}{2}}(n+1).$$

Paper II

1. Let BD intersect AC at H. Applying Ceva's Theorem to triangle BCD,

$$\frac{BH}{HD} \cdot \frac{DE}{EC} \cdot \frac{CG}{GB} = 1.$$



Since AH bisects $\angle BAD$, we have

$$\frac{BH}{HD} = \frac{BA}{AD}$$

so that

$$\frac{BA \cdot CG}{GB} = \frac{AD \cdot EC}{DE}.$$

Draw a line through C parallel to AB, cutting the extension of AG at I. Since triangles BAG and CIG are similar,

$$CI = \frac{BA \cdot CG}{BG}.$$

Draw a line through C parallel to AD, cutting the extension of AE at J.

As before, we have

$$CJ = \frac{AD \cdot EC}{DE}.$$

It follows that $CI = CJ$.

Along with $AC = AC$ and

$$\angle ACI = 180^\circ - \angle BAC = 180^\circ - \angle DAC = \angle ACJ,$$

triangles ACI and ACJ are congruent.

Hence $\angle CAG = \angle CAE$.

2. Let
$$\frac{z_1}{z_2} = \cos\theta + i\sin\theta$$
and
$$\frac{z_2}{z_3} = \cos\phi + i\sin\phi.$$
Then we have
$$\frac{z_3}{z_1} = \cos(-\theta-\phi) + i\sin(-\theta-\phi).$$
Since the sum of these numbers is 1, we have
$$\begin{aligned} 0 &= \sin\theta + \sin\phi - \sin(\theta+\phi) \\ &= 2\sin\frac{\theta+\phi}{3}\cos\frac{\theta-\phi}{2} - 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2} \\ &= 2\sin\frac{\theta+\phi}{2}\left(\cos\frac{\theta-\phi}{2} - \cos\frac{\theta+\phi}{2}\right) \\ &= 4\sin\frac{\theta+\phi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}. \end{aligned}$$
It follows that $\theta = 2k\pi$, $\phi = 2k\pi$ or $\theta+\phi = 2k\pi$ for some integer k. Hence either $z_1 = z_2$, $z_2 = z_3$ or $z_3 = z_1$. If $z_1 = z_2$, then
$$\frac{z_1}{z_3} + \frac{z_3}{z_1} = 0 \quad \text{or} \quad \left(\frac{z_3}{z_1}\right)^2 = -1.$$
Hence $\frac{z_3}{z_1} = \pm i$ so that
$$|az_1 + bz_2 + cz_3| = |z_1||a + b \pm ci| = \sqrt{(a+b)^2 + c^2}.$$
The other possible values are
$$\sqrt{(b+c)^2 + a^2} \quad \text{and} \quad \sqrt{(c+a)^2 + b^2},$$
arising from the cases $z_2 = z_3$ and $z_3 = z_1$ respectively.

3. (a) Let the weights of the tokens be the positive integers $a_1, a_2, \dots, a_k$ in non-decreasing order. Then the weight of an object that may be balanced is
$$w = \sum_{i=1}^{k} x_i a_i,$$
where $x_i = -1$, 0 or 1 for $1 \le i \le k$. The possible values of w must include the $2n+1$ integers from $-n$ to n. It follows that $2n+1 \le 3^k$ so that $n \le \frac{3^k-1}{2}$.

Let m be such that
$$\frac{3^{m-1}-1}{2} < n \le \frac{3^m-1}{2}.$$
Then $k \ge m$. We claim that we may have $k = m$, that by taking $a_i = 3^{i-1}$ for $1 \le i \le m$, we can balance any object of weight $w \le n$.

Note that
$$w + \sum_{i=1}^{m} 3^{i-1} \le 3^m - 1.$$
Let its base 3 representation be
$$\sum_{i=1}^{m} y_i 3^{i-1},$$
where $y_i = 0$, 1 or 2. Then
$$w = \sum_{i=1}^{m} x_i 3^{i-1},$$
where $x_i = -1$, 0 or 1.

(b) Let $\frac{3^{m-1}-1}{2} < n < \frac{3^m-1}{2}$. We know that the set of tokens of weight 1, 3, $\dots$, 3^{m-1} work. The heaviest token is only used if $w > \frac{3^{m-1}-1}{2}$, and the balancing can be achieved up to $\frac{3^m-1}{2}$. Hence if we replace this token by one of weight $3^{m-1}-1$, we can still balance any weight up to $\frac{3^m-1}{2} - 1$. It follows that the minimal set of tokens is not unique.

Consider now the remaining case where $n = \frac{3^m-1}{2}$. We claim that $a_i = 3^{i-1}$ for $1 \le i \le m$ is the only minimal set of tokens that works. Since $\sum_{i=1}^{m} x_i a_i$, where $x_i = -1$, 0 or 1, can represent any of the $2n+1$ integers w from $-n$ to n, and $2n+1 = 3^m$, there are no duplicate representations.

Now
$$0 \le w + \sum_{i=1}^{m} 3^{i-1} \le 3^m - 1,$$
and it has a unique representation $\sum_{i=1}^{m} y_i a_i$ where $y_i = 0$, 1 or 2. For $i = 1$, the smallest positive integer not yet represented

is 1. Hence we must have $a_1 = 1$. Suppose $a_i = 3^{i-1}$ for $1 \le i \le j$. Since

$$\sum_{i=1}^{j} y_1 a_i = \sum_{i=1}^{j} y_i 3^{i-1}$$

are the base 3 representations of $0, 1, \ldots, 3^j - 1$, we must have $a_{j+1} = 3^j$. The desired conclusion follows from mathematical induction.

Olympiad Paper I

1. We have

$$\begin{aligned}
&\frac{a+b-2R-2r}{2R} \\
= \ & \sin A + \sin B - 1 - 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\
= \ & 2\sin\frac{B+A}{2}\cos\frac{B-A}{2} - 1 \\
&+2\left(\cos\frac{B+A}{2} - \cos\frac{B-A}{2}\right)\sin\frac{C}{2} \\
= \ & 2\cos\frac{B-A}{2}\left(\sin\frac{\pi - C}{2} - \sin\frac{C}{2}\right) - 1 \\
&+2\cos\frac{\pi - C}{2}\sin\frac{C}{2} \\
= \ & 2\cos\frac{B-A}{2}\left(\cos\frac{C}{2} - \sin\frac{C}{2}\right) - \left(\cos^2\frac{C}{2} - \sin^2\frac{C}{2}\right) \\
= \ & \left(\cos\frac{C}{2} - \sin\frac{C}{2}\right)\left(2\cos\frac{B-A}{2} - \cos\frac{C}{2} - \sin\frac{C}{2}\right).
\end{aligned}$$

Since $0 \le B - A < B \le C$ and $0 \le B - A < B + A$, we have

$$\cos\frac{B-A}{2} > \cos\frac{C}{2}$$

and

$$\cos\frac{B-A}{2} > \cos\frac{B+A}{2} = \sin\frac{C}{2}.$$

It follows that the second factor in the last displayed expression is always positive.

(a) If $a + b - 2R - 2r > 0$, then $\cos\frac{C}{2} > \sin\frac{C}{2}$ or $\angle C < 90°$.

(b) If $a + b - 2R - 2r = 0$, then $\cos\frac{C}{2} = \sin\frac{C}{2}$ or $\angle C = 90°$.

(c) If $a + b - 2R - 2r < 0$, then $\cos\frac{C}{2} < \sin\frac{C}{2}$ or $\angle C > 90°$.

2. Solution 1

We may define $a_0 = 1$. Then $a_1 = 0 = a_0 - 1$. We prove by induction on n that $a_n = na_{n-1} + (-1)^n$ for all $n \ge 1$. We have

$$\begin{aligned}
a_{n+1} &= \frac{n+1}{2}a_n + \frac{n(n+1)}{2}a_{n-1} + (-1)^{n+1}\left(1 - \frac{n+1}{2}\right) \\
&= \frac{n+1}{2}a_n + \frac{n+1}{2}(a_n - (-1)^n) \\
&\quad +(-1)^{n+1}\left(1 - \frac{n+1}{2}\right) \\
&= (n+1)a_n + (-1)^{n+1}.
\end{aligned}$$

Let

$$S_n = \sum_{i=0}^{n}(i+1)\binom{n}{i}a_{n-i}.$$

Then the desired expression is equal to $S_n - (n+1)$. Note that $S_1 = 2$ while $S_2 = 4 = 2S_1$. We now prove by induction on n that $S_n = nS_{n-1}$ for all $n \ge 2$. We have

$$\begin{aligned}
S_{n+1} &= \sum_{i=0}^{n+1}(i+1)\binom{n+1}{i}a_{n+1-i} \\
&= \sum_{i=0}^{n+1}(i+1)\binom{n+1}{i}((n+1-i)a_{n-i} + (-1)^{n+1-i}) \\
&= (n+1)\sum_{i=0}^{n}(i+1)\binom{n}{i}a_{n-i} \\
&\quad +\sum_{i=0}^{n+1}(i+1)\binom{n+1}{i}(-1)^{n+1-i} \\
&= (n+1)S_n + \sum_{i=0}^{n+1}\binom{n+1}{i}(-1)^{n+1-i} \\
&\quad +(n+1)\sum_{i=1}^{n+1}\binom{n}{i-1}(-1)^{n+1-i} \\
&= (n+1)S_n + (1 + (-1))^{n+1} \\
&\quad +(n+1)\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i} \\
&= (n+1)S_n + (n+1)(1 + (-1))^n \\
&= (n+1)S_n.
\end{aligned}$$

It follows that $S_n = 2n!$ and that the desired expression is equal to $2n! - (n+1)$.

Solution 2

In a permutations of $\{1, 2, \ldots, n\}$, i is said to be a fixed point if it is in the i-th place. The derangement number d_n counts the number of permutations without fixed points. We have $d_0 = 1$ and $d_1 = 0$.

Suppose $n \geq 2$, There are $n-1$ places where 1 can be. Suppose it is in the k-th place for some $k > 1$. If k is in the first place, then the remaining $n-2$ numbers can be deranged in d_{n-2} ways.

Suppose k is not in the first place. We can pretend that it is 1. Apart from the real 1 staying put in the k-th place, the remaining $n-1$ numbers can be deranged in d_{n-1} ways.

It follows that

$$d_n = (n-1)(d_{n-1} + d_{n-2}).$$

This may be rewritten as

$$\begin{aligned} d_n - nd_{n-1} &= -(d_{n-1} - (n-1)d_{n-2}) \\ &= d_{n-2} - (n-2)d_{n-3} \\ &= \cdots \\ &= (-1)^{n-1}(d_1 - d_0) \\ &= (-1)^n. \end{aligned}$$

In the Solution 1, we have shown that the sequence $\{a_n\}$ satisfies the same recurrence relation and has the same initial values. It follows that $a_n = d_n$ for all n. We claim that

$$d_n + \binom{n}{1}d_{n-1} + \binom{n}{2}d_{n-2} + \cdots + \binom{n}{n-2}d_2 + \binom{n}{n-1}d_1$$

is equal to $n! - 1$. This is because the first term d_n counts all permutations of the set $\{1, 2, \ldots, n\}$ with 0 fixed points, the second term $\binom{n}{1}d_{n-1}$ counts all those with 1 fixed point, and so on. Since there are $n!$ permutations over all and the only one not counted is the one with all n points fixed, we have justified the claim. Similarly,

$$\begin{aligned} &\binom{n}{1}d_{n-1} + 2\binom{n}{2}d_{n-2} + \cdots + (n-1)\binom{n}{n-1}d_1 \\ = \;& n\left(d_{n-1} + \binom{n-1}{1}d_{n-2} + \cdots + \binom{n-1}{n-2}d_1\right) \\ = \;& n((n-1)! - 1). \end{aligned}$$

It follows that the desired expression is equal to

$$n! - 1 + n((n-1)! - 1) = 2n! - (n+1).$$

3. We first prove that the number of participants is at least $\frac{a_k}{2} + 3$. Consider a participant X who has played a_k games. If X is in only one pair, then there are a_k other pairs who have played this pair. Since each participant is in at most two pairs, there are at least a_k participants in those a_k pairs, yielding a total of at least $a_k + 2 > \frac{a_k}{2} + 3$. If X is in two pairs, then there are a_k other pairs who have played either of these two pairs, so that one of them has played $\frac{a_k}{2}$ pairs. It follows that the total number of participants is at least $\frac{a_k}{2} + 3$.

We now give a construction that $\frac{a_k}{2} + 3$ participants are sufficient. We use induction on k. For $k = 1$, divide the $\frac{a_1}{2} + 3$ participants into sets of three, and any two of the three form a pair. Pairs in different sets play each other. The number of games each participant plays is $2\frac{a_1}{2} = a_1$ as desired.

For $k = 2$, divide the $\frac{a_2}{2} + 3$ participants into two subsets, with $\frac{a_1}{2}$ in S and the other $\frac{a_2 - a_1}{2} + 3$ in T. Each of S and T is divided into sets of three, and any two of the three form a pair. Pairs in different sets play each other unless both are in T. The number of games each participant in T plays is $2\frac{a_1}{2} = a_1$ while the number of games each participant in S plays is $2(\frac{a_1}{2} - 3) + 2(\frac{a_2 - a_1}{2} + 3) = a_2$ as desired.

Assume that such a tournament exists for some $k - 1$ with $k \geq 2$. Consider a tournament with $\frac{a_{k+1}}{2} + 3$ players. Divide them into three subsets, with $\frac{a_1}{2}$ in S, $\frac{a_k - a_1}{2} + 3$ in T and $\frac{a_{k+1} - a_k}{2}$ in U. Each of S, T and U is divided into sets of three, and any two of the three form a pair. Pairs in S play all pairs not in the same set.

By the induction hypothesis, there exists a mini-tournament within T for the set $\{a_2 - a_1, a_3 - a_1, \ldots, a_k - a_1\}$. The number of games each participant in U plays is $2\frac{a_1}{2} = a_1$. The number of games played by a participant in T who has played $a_i - a_1$ games in the mini-tournament within T is $2\frac{a_1}{2} + (a_i - a_1) = a_i$ for $i = 2, \ldots, k$. The number of games each participant in S plays is

$$2\left(\frac{a_1}{2} - 3\right) + 2\left(\frac{a_k - a_1}{2} + 3\right) + 2\left(\frac{a_{k+1} - a_k}{2}\right) = a_{k+1}.$$

This completes the inductive argument.

Olympiad Paper II

1. Solution 1

Note that the sequence $\{b_1, b_2, \ldots, b_n\}$ is non-descending and we have $b_{k+1} = b_{k+2} = \cdots = b_n = n$ for some k. Hence we must have $b_1 = b_2 = \cdots = b_k = m$ for some $m < n$. Clearly, $a_1 = m$, $a_{k+1} = n$ and $a_i < m$ for $2 \le i \le k$. It follows that $k \le m$. For fixed m and k, $\langle a_2, a_3, \ldots, a_k\rangle$ is permutation of $k-1$ of the elements in $\{1, 2, \ldots, m-1\}$. These elements can be chosen in $\binom{m-1}{k-1}$ ways and then permuted in $(k-1)!$ ways.

On the other hand, $\langle a_{k+2}, a_{k+3}, \ldots, a_n\rangle$ is a permutation of the remaining elements, and there are $(n-k-1)!$ such permutations. It follows that the total number of permutations $\langle a_1, a_2, \ldots, a_n\rangle$ with the desired property is

$$\begin{aligned}
&\sum_{m=1}^{n-1}\sum_{k=1}^{m}\frac{(m-1)!(n-k-1)!}{(m-k)!}\\
&= \sum_{m=1}^{n-1}(m-1)!(n-m-1)!\sum_{k=1}^{m}\binom{n-k-1}{n-m-1}\\
&= \sum_{m=1}^{n-1}(m-1)!(n-m-1)!\binom{n-1}{n-m}\\
&= (n-1)!\sum_{m=1}^{n-1}\frac{1}{n-m}.
\end{aligned}$$

Similarly, the total value of the first term of these permutations is

$$\begin{aligned}
&\sum_{m=1}^{n-1}\sum_{k=1}^{m}\frac{m!(n-k-1)!}{(m-k)!}\\
&= \sum_{m=1}^{n-1}m!(n-m-1)!\binom{n-1}{n-m}\\
&= (n-1)!\sum_{m=1}^{n-1}\frac{m}{n-m}.
\end{aligned}$$

It follows that the desired average is

$$\sum_{m=1}^{n-1}\frac{m}{n-m}\left(\sum_{m=1}^{n-1}\frac{1}{n-m}\right)^{-1} = n-(n-1)\left(\sum_{m=1}^{n-1}\frac{1}{n-m}\right)^{-1}.$$

Solution 2

We have $a_1 = m$ where m can be any value less than n. We can then choose $\binom{n-1}{n-m}$ positions for the numbers $m+1$, $m+2$, $\ldots$, n. Among these numbers, n must come first, while the remaining ones can be permuted in $(m-1)!$ ways. The numbers 1, 2, $k-1$, $k+1$, $\ldots$, $m-1$ fill in the remaining positions, in $(n-m-1)!$ ways. Thus the number of permutations with the desired property and $a_1 = m$ is given by

$$(m-1)!(n-m-1)!\binom{n-1}{n-m}$$

as in Solution 1. We can continue as before.

2. Denote

$$\frac{1}{2^k}(n_1-1)(n_2-1)\cdots(n_k-1)$$

by m. Since $n = 2^m - 1$ is odd, each n_i, $1 \le i \le k$, is odd and hence at least 5, so that

$$\left(\frac{n_i-1}{2}\right)^3 \ge 4\cdot\frac{n_i-1}{2} > n_i.$$

Suppose $m \ge 10$. It is easy to show then that $2^m - 1 > m^3$. Hence

$$\begin{aligned}
n &\ge \left(\frac{1}{2^k}\left(\frac{n_1-1}{2}\right)\left(\frac{n_2-1}{2}\right)\cdots\left(\frac{n_k-1}{2}\right)\right)^3\\
&> n_1n_2\cdots n_k,
\end{aligned}$$

which is a contradiction.

It follows that $m \le 9$, and it is routine to check that only when $m = 3$ do we have a number $n = 7$ with the desired property.

3. We first prove that $n \ge 25$. Divide the 2000 students into $4^4 = 256$ groups according to how they answer the first four questions. Since $7 \times 256 = 1792 < 2000$, at least one group consists of at least 8 students. Take 8 of them aside. Since $1792 < 1984$, we can take aside two more sets of 8 students, each set from one of the groups.

Among any 4 of these 24 students, 2 of them must be from the same group and can answer differently at most one question. We now give an example to show that we can have $n = 25$. Let the groups be as before, and let the multiple choices be 0, 1, 2 and 3.

We discard 6 groups at random and put 8 of the 2000 students in each of the remaining groups.

Within each group, we insist that the last question be answered in the same way, so that the total of the five answers is a multiple of 4. Among any 25 students, there will always be 4 who are from different groups, and they must answer at least one question differently.

However, if they answer exactly one question differently, at least one of the totals of their answers will not be a multiple of 4. It follows that the desired minimum value is $n = 25$.

2000/01

Paper I.

Section 1. Questions with Multiple Choices.

1. Since $\sqrt{x-2} \leq 0$, we have $x - 2 = 0$ so that $A = \{2\}$. Since $10^{2^2-2} = 100 = 10^2$, $2 \in B$. Hence $A \cap \overline{B} = \emptyset$.

2. Let k be an arbitrary integer. From $\sin\alpha > 0 > \cos\alpha$, we have $2k\pi + \frac{\pi}{2} < \alpha < 2k\pi + \pi$ or

$$\frac{2k\pi}{3} + \frac{\pi}{6} < \frac{\alpha}{3} < \frac{2k\pi}{3} + \frac{\pi}{3}.$$

When $k = 3t$, this becomes

$$2t\pi + \frac{\pi}{6} < \frac{\alpha}{3} < 2t\pi + \frac{\pi}{3}.$$

When $k = 3t + 1$, we have

$$2t\pi + \frac{5\pi}{6} < \frac{\alpha}{3} < 2t\pi + \pi.$$

Finally,

$$2t\pi + \frac{3\pi}{2} < \frac{\alpha}{3} < 2t\pi + \frac{5\pi}{3}$$

when $k = 3t + 2$. From $\sin\frac{\alpha}{3} > \cos\frac{\alpha}{3}$, we have

$$2k\pi + \frac{\pi}{4} < \frac{\alpha}{3} < 2k\pi + \frac{5\pi}{4}.$$

Hence the range of $\frac{\alpha}{3}$ is

$$(2k\pi + \frac{\pi}{4}, 2k\pi + \frac{\pi}{3}) \cup (2k\pi + \frac{5\pi}{6}, 2k\pi + \pi).$$

3. Let B be above the x-axis. Then the line AB has slope $\frac{\sqrt{3}}{3}$ and its equation is

$$y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}.$$

Substituting this into $x^2 - y^2 = 1$, we have

$$0 = 2x^2 - 2x - 4 = 2(x+1)(x-2).$$

Since B is to the right of the y-axis, its coordinates are $(2, \sqrt{3})$. Those of C are $(2, -\sqrt{3})$ and the area of triangle ABC is

$$(2 - (-1))\sqrt{3} = 3\sqrt{3}.$$

4. We have $pq = a^2$, $2b = p + c$ and $2c = q + b$, so that $b = \frac{2p+q}{3}$ and $c = \frac{p+2q}{3}$. Now

$$\begin{aligned} bc &= \left(\frac{p+p+q}{3}\right)\left(\frac{p+q+q}{3}\right) \\ &> \left(\sqrt[3]{p^2q}\right)\left(\sqrt[3]{pq^2}\right) \\ &= pq \\ &= a^2. \end{aligned}$$

The inequality is strict since $p \neq q$. The discriminant of the quadratic equation is $4a^2 - 4bc < 0$, and neither root is real.

5. Rewrite the equation of the line as $25x - 15y + 12 = 0$. Then the distance from a lattice point (u, v) to this line is

$$\frac{|25u - 15v + 12|}{\sqrt{25^2 + (-15)^2}} = \frac{5|5u - 3v + 2| + 2}{5\sqrt{34}}.$$

Hence the numerator is at least 2, and this minimum value is attained when $u = v = -1$. Hence the minimum distance is $\frac{2}{5\sqrt{34}} = \frac{\sqrt{34}}{85}$.

6. We have

$$(x-1)(x-\omega)\cdots(x-\omega^9) = x^{10} - 1.$$

On the other hand,

$$(x-1)(x-\omega^2)\cdots(x-\omega^8) = x^5 - 1.$$

Hence

$$(x-\omega)(x-\omega^3)\cdots(x-\omega^9) = x^5 + 1.$$

Dividing both sides by $x - \omega^5 = x + 1$, we have

$$(x-\omega)(x-\omega^3)(x-\omega^7)(x-\omega^9) = x^4 - x^3 + x^2 - x + 1.$$

Section 2. Questions requiring Answers Only:

1. Note that $2000° = 11\pi + \frac{\pi}{9}$. Hence

$$\sin 2000° = \sin(-\frac{\pi}{9}).$$

Since $-\frac{\pi}{2} < -\frac{\pi}{9} < \frac{\pi}{2}$,

$$\arcsin(\sin 2000°) = -\frac{\pi}{9}.$$

2. From the Binomial Theorem, $a_n = \binom{n}{2}3^{n-2}$. It follows that

$$\frac{3^n}{a_n} = 18\left(\frac{1}{n-1} - \frac{1}{n}\right).$$

Now

$$\begin{aligned} &\frac{3^2}{a_2} + \frac{3^3}{a_3} + \cdots + \frac{3^n}{a_n} \\ = \ & 18\left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n}\right) \\ = \ & 18\left(1 - \frac{1}{n}\right). \end{aligned}$$

As n tends to infinity, the sum tends to 18.

3. Let the common ratio be r. Then

$$\begin{aligned} r &= \frac{a + \log_4 3}{a + \log_2 3} \\ &= \frac{a + \log_8 3}{a + \log_4 3} \\ &= \frac{\log_4 3 - \log_8 3}{\log_2 3 - \log_4 3} \\ &= \frac{\frac{1}{2}\log_2 3 - \frac{1}{3}\log_2 3}{\log_2 3 - \frac{1}{2}\log_2 3} \\ &= \frac{1}{3}. \end{aligned}$$

4. The coordinates of F are $(-c, 0)$ where $c^2 = a^2 - b^2$. From $\frac{c}{a} = \frac{\sqrt{5}-1}{2}$, we have $2c + a = \sqrt{5}a$ which leads to $c^2 + ac - a^2 = 0$. Hence

$$\begin{aligned} AF^2 &= (c+a)^2 \\ &= 3a^2 - c^2 \\ &= a^2 + 2b^2 + c^2 \\ &= (a^2 + b^2) + (b^2 + c^2) \\ &= AB^2 + BF^2. \end{aligned}$$

It follows that $\angle ABF = 90°$.

5. Let G be the centre of triangle BCD. Then $BG = \frac{\sqrt{3}}{3}a$. Hence

$$AG = \sqrt{AB^2 - BG^2} = \frac{\sqrt{6}}{3}a.$$

Let O be the centre of the sphere. Then

$$(AG - OA)^2 = OG^2 = OB^2 - BG^2.$$

Since $OA = OB$, we have

$$OA = \frac{AG^2 - BG^2}{2AG} = \frac{\sqrt{6}}{4}a.$$

Let E be the midpoint of AB. Then

$$OE = \sqrt{OA^2 - AE^2} = \frac{\sqrt{2}}{4}a$$

is the radius of the sphere. Hence its volume is

$$\frac{4\pi}{3}\left(\frac{\sqrt{2}}{4}a\right)^3 = \frac{\sqrt{2}\pi}{24}a^3.$$

6. If only two different digits are used, they can be chosen in $\binom{4}{2} = 6$ ways. The smaller serves as the first and third digits while the other serves as the second and fourth digits. If three different digits are used, they can be chosen in $\binom{4}{3} = 4$ ways.

The smallest goes first. If it also goes third, the other two can be permuted in 2 ways. If the smallest digit is not repeated, then one of the other two goes third while the other goes second and fourth. Thus the number of choices in this case is 4(2+2)=16.

Finally, suppose all four digits are used. The smallest goes first while the other three can be permuted in $3! = 6$ ways. Hence the total number of choices is 6+16+6=28.

Section 3. Questions requiring Full Solutions:

1. By the AM-GM Inequality, we have

$$\begin{aligned}\frac{(n+32)S_{n+1}}{S_n} &= \frac{(n+32)(n+2)}{n} \\ &= n + 34 + \frac{64}{n} \\ &\geq 34 + 2\sqrt{64} \\ &= 50.\end{aligned}$$

Hence

$$\frac{S_n}{(n+32)S_{n+1}} \leq \frac{1}{50}.$$

For $n = 8$, we have

$$\frac{8}{(8+32)(8+2)} = \frac{1}{50}.$$

Thus this is indeed the maximum value.

2. Let

$$f(x) = \frac{1}{2}(13 - x^2).$$

Suppose $0 \leq a < b$. Then $f(x)$ is decreasing. Hence its maximum value is $f(a) = 2b$ and its minimum value is $f(b) = 2a$. Subtracting $\frac{1}{2}(13 - a^2) = 2b$ from $\frac{1}{2}(13 - b^2) = 2a$, we have

$$\frac{1}{2}(a-b)(a+b) = 2(a-b)$$

so that

$$a+b=4.$$

From $\frac{1}{2}(13 - a^2) = 2(4 - a)$, we have $(a-1)(a-3) = 0$. Hence $a = 1$ and $b = 3$. Suppose $a < 0 < b$. Then $f(x)$ is increasing on $(a, 0)$ and decreasing on $(0, b)$. Hence its maximum value is $f(0) = \frac{13}{2} = 2b$, so that $b = \frac{13}{4}$. Its minimum value is either $f(b) = 2a$ or $f(a) = 2a$. However,

$$f\left(\frac{13}{4}\right) = \frac{1}{2}\left(13 - \frac{169}{16}\right) = \frac{39}{32} \neq 2a.$$

Thus it is $2a = f(a) = \frac{1}{2}(13 - a^2)$ so that $a^2 + 4a - 13 = 0$. Hence $a = -2 - \sqrt{7}$, the positive root $-2 + \sqrt{7}$ being rejected. Suppose $a < b \leq 0$. Then $f(x)$ is increasing. Hence its minimum value is

$$2a = f(a) = \frac{1}{2}(13 - a^2),$$

so that $a^2 + 4a + 13 = 0$. There are no real roots. In summary, the desired values are $(a, b) = (1, 3)$ or $(-2 - \sqrt{7}, \frac{13}{4})$.

3. Such a parallelogram must be a rhombus whose centre is at the origin O. Suppose it exists with P at $(a, 0)$. Then the opposite vertex is at $(-a, 0)$ and the other two vertices are at $(0, \pm b)$.

The equation of the side of the rhombus in the first quadrant is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{or} \quad bx + ay - ab = 0.$$

Since it is tangent to the unit circle, we have

$$\frac{ab}{\sqrt{a^2+b^2}}=1 \quad \text{or} \quad \frac{1}{a^2}+\frac{1}{b^2}=1.$$

Conversely, suppose this condition holds. Let P be represented in the complex plane by $r_1(\cos\theta+i\sin\theta)$. Then an adjacent vertex Q is represented by

$$r_2\left(\cos\left(\theta+\frac{\pi}{2}\right)+i\sin\left(\theta+\frac{\pi}{2}\right)\right)=r_2(\sin\theta+i\cos\theta)$$

for some r_2. Then

$$\begin{aligned}\frac{1}{OP^2}+\frac{1}{OQ^2} &= \frac{1}{r_1^2}+\frac{1}{r_2^2}\\ &= \frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}+\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}\\ &= \frac{1}{a^2}+\frac{1}{b^2}\\ &= 1.\end{aligned}$$

In the right triangle POQ, the distance h from O to PQ satisfies

$$\frac{1}{h^2}=\frac{1}{OP^2}+\frac{1}{OQ^2}=1.$$

Hence $h=1$ and PQ is tangent to the unit circle. By symmetry, so are the other sides of the rhombus.

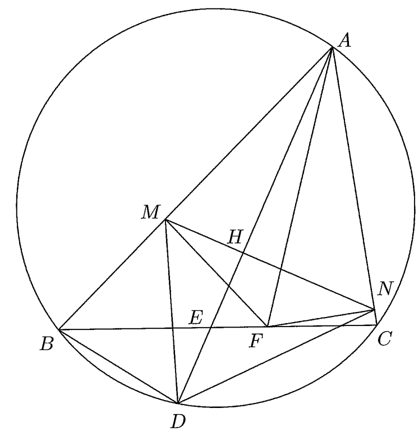
Paper II

1. Since $\angle AMF=90^\circ=\angle ANF$, $AMFN$ is a cyclic quadrilateral. Hence $\angle AMN=\angle AFN$ and we have

$$\begin{aligned}\angle AHM &= 180^\circ-\angle AMN-\angle MAE\\ &= 180^\circ-\angle AFN-\angle FAN\\ &= \angle ANF=90^\circ,\end{aligned}$$

where H is the point of intersection of AD and MN. Hence the area of $AMDN$ is equal to $\frac{1}{2}AD\cdot MN$. Now AF is a diameter of the circumcircle of triangle MAN.

By the Extended Law of Sines, $MN=AF\sin CAB$. Since the area of ABC is equal to $\frac{1}{2}AB\cdot AC\sin CAB$, we only need $AD\cdot AF=AB\cdot AC$. We have $\angle ADB=\angle ACF$ and $\angle BAD=\angle FAC$, so that triangles BAD and FAC are similar.



The desired result follows from

$$\frac{AB}{AD}=\frac{AF}{AC}.$$

2. We have

$$\begin{aligned}a_1 &= 7a_0+6b_0-3=4,\\ b_1 &= 8a_0+7b_0-4=4\\ \text{and}\quad a_2 &= 7a_1+6b_1-3=49.\end{aligned}$$

Substituting

$$b_n=\frac{1}{6}(a_{n+1}-7a_n+3)$$

into the other recurrence relation, we have

$$a_{n+2}-14a_{n+1}+a_n=6.$$

Subtracting from this

$$a_{n+1}-14a_n+a_{n-1}=6,$$

we have

$$a_{n+2}-15a_{n+1}+15a_n-a_{n-1}=0.$$

The characteristic equation is

$$x^3 - 15x^2 + 15x - 1 = (x-1)(x^2 - 14x + 1) = 0$$

with characteristic roots 1 and $7 \pm 4\sqrt{3}$. Hence

$$a_n = c_1 + c_2(7+4\sqrt{3})^n + c_3(7-4\sqrt{3})^n.$$

Using the initial values, we have

$$\begin{aligned} 1 &= c_1 + c_2 + c_3, & (1)\\ 4 &= c_1 + c_2(7-4\sqrt{3}) + c_3(7-4\sqrt{3}), & (2)\\ 49 &= c_1 + c_2(97+56\sqrt{3}) + c_3(97-56\sqrt{3}). & (3) \end{aligned}$$

Subtracting (1) from each of (2) and (3), we obtain

$$\begin{aligned} 3 &= c_2(6+4\sqrt{3}) + c_3(6-4\sqrt{3}), & (4)\\ 6 &= c_2(12+7\sqrt{3}) + c_3(12-7\sqrt{3}). & (5) \end{aligned}$$

Subtracting $6+4\sqrt{3}$ times (5) from $12+7\sqrt{3}$ times (4), we have $-3\sqrt{3} = -12\sqrt{3}c_3$ so that $c_3 = \frac{1}{4}$. From (4), we have $c_2 = \frac{1}{4}$ and from (1), we have $c_1 = \frac{1}{2}$. It follows that

$$\begin{aligned} a_n &= \frac{1}{2} + \frac{1}{4}(7+4\sqrt{3})^n + \frac{1}{4}(7-4\sqrt{3})^n\\ &= \frac{1}{4}(2+\sqrt{3})^{2n} + \frac{1}{2} + \frac{1}{4}(2-\sqrt{3})^{2n}\\ &= \frac{1}{4}((2+\sqrt{3})^n + (2-\sqrt{3})^n)^2. \end{aligned}$$

Expanding the binomials inside the bracket in the last expression, all the irrational terms cancel out and the final value is an even integer. Hence a_n is the square of an integer for all $n \geq 0$.

3. Clearly, we must have $n \geq 5$. If $n = 5$ and every two friends have a phone conversation exactly once, then the total number of phone conversations among any 3 of them is 3. Hence the scenario is possible with $n = 5$.

We now prove that this is the only possible value. For $1 \leq i, j \leq n$, let $c_{i,j}$ be the number of phone conversations between the i-th and the j-th friend. Then $c_{i,i} = 0$ and $c_{i,j} = 0$ or 1.

Let

$$t_i = c_{i,1} + c_{i,2} + \cdots + c_{i,n}$$

for $1 \leq i \leq n$, and we may assume that $t_1 \geq t_2 \geq \cdots \geq t_n$. The total number of phone conversations is

$$T = \frac{1}{2}(t_1 + t_2 + \cdots + t_n).$$

For any i and j,

$$T - t_i - t_j + c_{i,j} = 3^m$$

for some constant $m \geq 1$. For $2 \leq k \leq n-1$, we have

$$\begin{aligned} t_1 - t_n &= (t_1 + t_k) - (t_n - t_k)\\ &= (T - 3^m + c_{1,k}) - (T - 3^m + c_{n,k})\\ &= c_{1,k} - c_{n,k}\\ &\leq 1. \end{aligned}$$

If $t_1 - t_n = 1$, then $c_{1,k} = 1$ and $c_{n,k} = 0$ for $2 \leq k \leq n-1$. Hence $t_1 \geq n-2$ while $t_n \leq 1$. However, $t_1 - t_n \geq n-2-1 \geq 2$, which is a contradiction.

It follows that t_k has a constant value t for $1 \leq k \leq n$. From

$$c_{i,j} = T - 2t - 3^m,$$

$c_{i,j}$ also has a constant value c for $1 \leq i,j \leq n$, and we must have $c = 1$ so that $t = n-1$ and

$$T = \frac{n(n-1)}{2}.$$

Now

$$1 = \frac{n(n-1)}{2} - 2(n-1) - 3^m$$

simplifies to

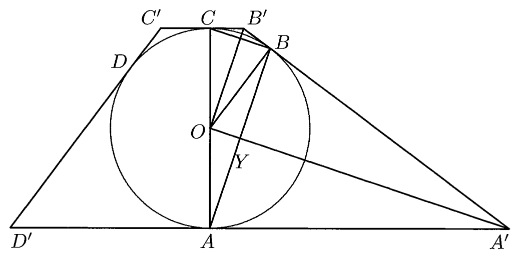
$$(n-2)(n-3) = 2 \cdot 3^m.$$

Since $n-2$ and $n-3$ are relatively prime, one of them is equal to 2 and the other 3^m. If $n-2 = 2$, then $n-3 = 1$ and $m = 0$. This is not permitted. Hence $n-3 = 2$, $n-2 = 3$ and $m = 1$.

This completes the proof that $n = 5$ is the only possible value.

Olympiad Paper I

1. Let O be the centre of the unit circle. If we cut up $A'B'C'D'$ along the radii OA, OB, OC and OD, we can rearrange the four pieces in any order without affecting the result of the problem. Thus we may assume that $AB = a$ while $BC = \sqrt{4-a^2}$. Let AX be a diameter of the circle.



Then $C = X$ since

$$BX^2 = AX^2 - AB^2 = 4 - a^2 = BC^2.$$

Let AB cut OA' at Y. Then $OY = \frac{BC}{2}$. Since triangles $AA'Y$ and OBY are similar, we have

$$AA' = \frac{AY \cdot OB}{OY} = \frac{a}{\sqrt{4-a^2}}.$$

Now

$$AA' \cdot CB' = BA' \cdot BB' = OB^2 = 1.$$

Hence

$$CB' = \frac{\sqrt{4-a^2}}{a}.$$

The area of $AA'B'C$ is

$$AA' + CB' = \frac{a}{\sqrt{4-a^2}} + \frac{\sqrt{4-a^2}}{a} = \frac{4}{a\sqrt{4-a^2}}$$

while the area of ABC is

$$\frac{AB \cdot BC}{2} = \frac{a\sqrt{4-a^2}}{2}.$$

Consider now $AD'C'C$ and ADC. The area of the former is given by $AD' + CC'$. Since $AD' \cdot CC' = 1$, its value increases as D moves away from the midpoint of the semicircular AC not passing through B. On the other hand, the area of ADC, considering AC as its base, decreases as D moves away from the midpoint of this arc.

(a) The minimum ratio of the areas of $A'B'C'D'$ and $ABCD$ occurs when $AD = CD$, and we have

$$\frac{\frac{4}{a\sqrt{4-a^2}} + 2}{\frac{a\sqrt{4-a^2}}{2} + 1} = \frac{4}{a\sqrt{4-a^2}}.$$

(b) Since we must have $\min\{AD, CD\} \geq BC$, the maximum ratio occurs when $AD = BC$ or $CD = BC$, and we have

$$\frac{\frac{4}{a\sqrt{4-a^2}}}{\frac{a\sqrt{4-a^2}}{2}} = \frac{8}{a^2(4-a^2)}.$$

2. The smallest power of 2 greater than 2001 is

$$2048 = 2001 + 47 = 2000 + 48 = \cdots = 1025 + 1023.$$

Leaving out 1024 for now, the next largest number not accounted for is 46. The smallest power of 2 greater than 46 is

$$64 = 46 + 18 = 45 + 19 = \cdots = 33 + 31.$$

Leaving out 32 for now, we have $32 = 17 + 15$. Leaving out 16 for now, we have

$$16 = 14 + 2 = 13 + 3 = \ldots = 9 + 7.$$

Only 1 and 8 are left. Thus we have divided the first 2001 positive integers into 998 pairs plus the five single elements 1, 8, 16, 32 and 1024. In any subset with 999 elements, if it contains any of these five, then it will have two identical elements whose sum is a power of 2.

Otherwise, by the Pigeonhole Principle, it must contain both elements of one of the 998 pairs. Hence it will have two distinct elements whose sum is a power of 2. To prove that 999 is indeed the minimum, we construct a set of 998 elements no two of which add up to a power of 2. We simply take the larger element from each of the above 998 pairs. In other words, we take

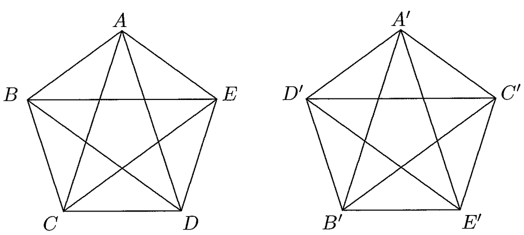
$$\{9, 10, \ldots, 14, 17, 33, 34, \ldots, 46, 1025, 1026, \ldots, 2001\}.$$

For any two elements in this set, if both are at least 1025, their sum is greater than 2048 but less than 2096. If only one is at least

1025, their sum is greater than 1024 but less than 2048. If neither is, their sum is greater than 16 and less than 128, and it is easy to verify that it cannot be 32 or 64.

3. Identify each blue jay by its initial position J and denote its new position by J'. Consider first a regular $2n$-gon for $n \geq 2$. Let A and B be two blue jays which are diametrically opposite. If A' and B' are still diametrically opposite, then any third blue jay C will work since $\angle ACB = 90° = \angle A'C'B'$.

Otherwise, let C be the blue jay such that C' is diametrically opposite to A'. Then $\angle ACB = 90° = \angle A'B'C'$. Note that the result is trivially true for an equilateral triangle, while the following example shows that it is false for a regular pentagon.



Consider now a regular $2n+1$-gon for $n \geq 3$. Clearly there are no right triangles. The number of obtuse triangles with a particular diagonal as the longest side is equal to the number of vertices between the endpoints of this diagonal, going the shorter way.

Since there are $2n+1$ diagonals of each length, the total number of obtuse triangles is

$$(2n+1)(1+2+\cdots+(n-1)) = \frac{1}{2}(n-1)n(2n+1).$$

The total number of triangles is

$$\binom{2n+1}{3} = \frac{1}{3}(2n-1)n(2n+1).$$

Since

$$\frac{\frac{1}{2}(n-1)}{\frac{1}{3}(2n-1)} = \frac{1}{2} + \frac{n-2}{4n-2} > \frac{1}{2}$$

for $n \geq 3$, there are more obtuse triangles than acute ones. By the Pigeonhole Principle, there exist three blue jays whose initial and final positions both determine obtuse triangles.

Olympiad Paper II

1. We may assume that $a < b$. Note that $a+b \equiv 2 \pmod 3$. Suppose $a \equiv 0 \pmod 3$ and $b \equiv 2 \pmod 3$. Then $a = 3$ and $c \not\equiv 0 \pmod 3$. If $c \equiv 1 \pmod 3$, then $a+b+c \equiv 0 \pmod 3$ is composite since it is distinct from 3. If $c \equiv 2 \pmod 3$, then $a+b-c \equiv 0 \pmod 3$ is also composite.

Hence we must have $a \equiv b \equiv 1 \pmod 3$. If $c \equiv 1 \pmod 3$, then $a+b+c \equiv 0 \pmod 3$. Since $a+b+c$ is the largest of the 7 distinct primes, it cannot be equal to 3. If $c \equiv 0 \pmod 3$, then $c = 3$ and $c+a-b \equiv 0 \pmod 3$ is composite. Hence we must have $c \equiv 2 \pmod 3$.

Now $a+b-c \equiv 0 \pmod 3$ is prime. Hence it is equal to 3, and is the smallest of the 7 distinct primes. We have $c = 800 - 3 = 797$ and the desired difference is

$$(a+b+c) - (a+b-c) = 2c = 1594.$$

Note that $a = 7$ and $b = 793$ yield the distinct primes 3, 7, 11, 793, 797, 1571 and 1597.

2. Label the points 0 to 23 in cyclic order and arrange the labels in a 3×8 array as shown below.

0	3	6	9	12	15	18	21
8	11	14	17	20	23	2	5
16	19	22	1	4	7	10	13

Two adjacent labels in the same row (the first and the last labels being considered adjacent) differ by 3 while two adjacent labels in the same column (the top and the bottom labels being considered adjacent) differ by 8.

Thus the problem is equivalent to choosing 8 mutually non-adjacent labels. Since we can choose at most one from each column, we have to take exactly one from each. In general, let x_n be the number of valid choices from a $3 \times n$ array where $n \geq 2$. There are 3 choices in the first column and 2 in each subsequent column.

However, it may happen that the labels in the first and the last columns are adjacent. In this case, omission of the last column yields a valid choice for the $3 \times (n-1)$ array. It follows that

$x_n + x_{n-1} = 3 \cdot 2^{n-1}$ for $n \geq 3$, with $x_2 = 6$. Hence

$$\begin{aligned} x_8 &= (x_8 + x_7) - (x_7 + x_6) + (x_6 + x_5) - (x_5 + x_4) \\ &\quad +(x_4 + x_3) - (x_3 + x_2) + x_2 \\ &= 3(2^7 - 2^6 + 2^5 - 2^4 + 2^3 - 2^2 + 2) \\ &= 258. \end{aligned}$$

3. (a) Let

$$k = \frac{4002m - m^2 + n^2}{2n}.$$

Then

$$n(2k - n) = m(4002 - m).$$

If either m or n is odd, all four factors are odd. It follows that $m \equiv n \pmod 2$. Now

$$\begin{aligned} k &= \frac{4002m - 2mn + n^2 - m^2 + 2mn}{2n} \\ &\leq \frac{4002m - 2mn + 4002(n - m)}{2n} \\ &= 2001 - m. \end{aligned}$$

Hence

$$2k - n < 2k \leq 4002 - 2m < 4002 - m,$$

so that $n > m$. Now

$$\begin{aligned} 2mn &\leq 4002(n - m) - (n^2 - m^2) \\ &= (n - m)(4002 - n - m). \end{aligned}$$

It follows that $4002 - n - m > 0$, so that the given expression

$$E(m, n) = \frac{m(4002 - m - n)}{n}$$

is positive. Also,

$$\begin{aligned} E(m, n) &= \frac{4002m - m^2 + n^2 - n^2 - mn}{n} \\ &= 2k - n - m \\ &\equiv 0 \pmod 2. \end{aligned}$$

Hence $E(m, n) \geq 2$.

We have

$$E(2, n) = \frac{8000}{n} - 2,$$

and the minimum value is attained at $E(2, 2000) = 2$. It is easy to verify that $(m, n) = (2, 2000)$ satisfies the hypothesis.

(b) Recall that $n > m$ and $m \equiv n \pmod 2$. Consider first

$$\begin{aligned} E(n - 2, n) &= \frac{4002(n - 2) - (n - 2)^2 - n(n - 2)}{n} \\ &= 4008 - 2\left(n + \frac{4004}{n}\right). \end{aligned}$$

To maximize the value of $E(n - 2, n)$, we want n and $\frac{4004}{n}$ to be as close to each other as possible. Since n must be a factor of $4004 = 52 \times 77$, we can take $n = 52$.

It is easy to verify that $(m, n) = (50, 52)$ satisfies the hypothesis while $(m, n) = (75, 77)$ does not. We claim that the maximum value occurs at $E(50, 52) = 3750$. In other words, $E(m, n) < 3750$ for all $n \geq m + 4$. In such cases, we have

$$\begin{aligned} E(m, n) &= \frac{(4002 - m)m}{n} - m \\ &\leq \frac{(4002 - m)m}{m + 4} - m \\ &= 3998 - 2\left(m + 4 + \frac{16024}{m + 4}\right) \\ &\leq 3998 - 2\sqrt{16024} \\ &\leq 3998 - 2(126) \\ &= 3746. \end{aligned}$$

This justifies the claim.

GENERAL REFERENCES

1. Andreescu T and Feng Z, *101 Problems in Algebra: From the Training of the USA IMO Team*, AMT Publishing, Canberra, 2001.
2. Artino RA, Gaglione AM and Shell N, *The Contest Problem Book IV*, Mathematical Association of America, 1983.
3. Atkins WJ, *Problem Solving via the AMC*, AMT Publishing, Canberra, 1992.
4. Atkins WJ, Edwards JD, King DJ, O'Halloran PJ and Taylor PJ, *Australian Mathematics Competition, Book 1, 1978-1984*, AMT Publishing, Canberra, 1986 and 2000.
5. Atkins WJ, Munro JEM and Taylor PJ, *Australian Mathematics Competition, Book 3, 1992-1998*, AMT Publishing, Canberra, 1998.
6. Barbeau EJ, *Polynomials*, Springer-Verlag, New York, 1989.
7. Beckenbach EF and Bellman R, *An Introduction to Inequalities*, Mathematical Association of America, 1961.
8. Berzsenyi G and Maurer SB, *The Contest Problem Book V*, Mathematical Association of America, 1996.
9. Burns JC, *Seeking Solutions*, AMT Publishing, Canberra, 2000.
10. Chinn WG and Steenrod NE, *First Concepts of Topology*, Mathematical Association of America, 1966.
11. Coxeter HSM, *Introduction to Geometry*, John Wiley & Sons, New York, 1961 and 1969.
12. Coxeter HSM and Greitzer SL, *Geometry Revisited*, Mathematical Association of America, 1967.
13. Fomin D and Kirichenko A, *Leningrad Mathematical Olympiads*, Mathpro Press, Westford, Massachusetts, 1994.
14. Greitzer SL, *International Mathematical Olympiads 1959-1977*, Mathematical Association of America, 1978.
15. Grossman I and Magnus W, *Groups and Their Graphs*, Mathematical Association of America, 1964.

16. Henry JB, Dowsey J, Edwards AR, Mottershead LJ, Nakos A, Vardaro G, *Challenge! 1991-1995*, AMT Publishing, 1997.

17. Holton D, *Let's Solve Some Math Problems*, Canadian Mathematics Competition, Waterloo, Ontario, 1993.

18. Honsberger R, *From Erdös to Kiev*, Mathematical Association of America, 1996.

19. Honsberger R, *Ingenuity in Mathematics*, Mathematical Association of America, 1970.

20. Honsberger R, *Mathematical Gems I*, Mathematical Association of America, 1973.

21. Honsberger R, *Mathematical Gems II*, Mathematical Association of America, 1976.

22. Honsberger R, *Mathematical Gems III*, Mathematical Association of America, 1985.

23. Honsberger R, *Mathematical Morsels*, Mathematical Association of America, 1978.

24. Honsberger R, *Mathematical Plums*, Mathematical Association of America, 1989.

25. Honsberger R, *Mathematical Chestnuts from Around the World*, Mathematical Association of America, 2001.

26. Kazarinoff ND, *Geometric Inequalities*, New Mathematical Association of America, 1961.

27. Klamkin MS, *International Mathematical Olympiads 1978-1985*, Mathematical Association of America, 1986.

28. Klamkin MS, *USA Mathematical Olympiads*, 1972-1986, Mathematical Association of America, 1988.

29. Kusczma ME, *144 Problems of the Austrian-Polish Mathematics Competition 1978-1993*, The Academic Distribution Center, Freeland, Maryland, 1994.

30. Kuczma ME and Windisbacher E, *Polish and Austrian Mathematical Olympiads 1981-1995: Selected Problems with Multiple Solutions*, AMT Publishing, Canberra, 1998.

31. Larson LC, *Problem Solving Through Problems*, Springer-Verlag, New York, 1983.

32. Lausch H and Bosch Giral C, *Asian Pacific Mathematics Olympiad, 1989-2000*, AMT Publishing, Canberra, 1994.

33. Lausch H and Taylor PJ, *Australian Mathematical Olympiads 1979-1995*, AMT Publishing, Canberra, 1997.

34. Liu A, *Chinese Mathematics Competitions and Olympiads 1981-1993*, AMT Publishing, Canberra, 1997.

35. Liu A, *Hungarian Problem Book III: Based on the Eötvos Competitions, 1929-1943*, Mathematical Association of America, 2001.

36. Niven Ivan, *Mathematics of Choice - How to count without counting*, Mathematics Association of America, 1965.

37. O'Halloran PJ, Pollard GH and Taylor PJ, *Australian Mathematics Competition, Book 2, 1985-1991*, AMT Publishing, Canberra, 1992 and 1998.

38. Ore O, *Graphs and Their Uses*, Mathematical Association of America, 1963, revised and updated by R Wilson 1990.

39. Ore O, *Invitation to Number Theory*, Mathematical Association of America, 1967.

40. Plank AW and Williams NH, *Mathematical Toolchest*, AMT Publishing, Canberra, 1992 and 1996.

41. Rabinowitz S, *Index to Mathematical Problems 1980-1984*, Mathpro Press, Westford, Masachusetts, 1992.

42. Rapaport E, *Hungarian Problem Book I*, Mathematical Association of America, 1963.

43. Rapaport E, *Hungarian Problem Book II*, Mathematical Association of America, 1963.

44. Salkind CT, *The Contest Problem Book*, Random House, 1961.

45. Salkind CT, *The Contest Problem Book II*, Random House, 1966.

46. Salkind CT and Earl JM, *The Contest Problem Book III*, Random House, 1973.

47. Schneider LJ, *The Contest Problem Book VI*, Mathematical Association of America, 1998.

48. Sharygin IF, *Problems in Plane Geometry*, Mir, Moscow, 1988.

49. Sharygin IF, *Problems in Solid Geometry*, Mir, Moscow, 1986.

50. Slinko AM, *USSR Mathematical Olympiads 1989-1992*, AMT Publishing, Canberra, 1997.

51. Tabov JB and Taylor PJ, *Methods of Problem Solving, Book 1*, AMT Publishing, Canberra, 1996.

52. Taylor PJ, *International Mathematics Tournament of Towns, Problems and Solutions, 1980-1984*, AMT Publishing, Canberra, 1993.

53. Taylor PJ, *International Mathematics Tournament of Towns, Problems and Solutions, 1984-1989*, AMT Publishing, Canberra, 1993.

54. Taylor PJ, *International Mathematics Tournament of Towns, Problems and Solutions, 1989-1993*, AMT Publishing, Canberra, 1994.

55. Taylor PJ and Storozhev A, *International Mathematics Tournament of Towns, Problems and Solutions, 1993-1997*, AMT Publishing, Canberra, 1998.

56. Yaglom IM, *Geometric Transformations*, Mathematical Association of America, 1962.

57. Yaglom IM, *Geometric Transformations II*, Mathematical Association of America, 1968.

58. Yaglom IM, *Geometric Transformations III*, Mathematical Association of America, 1973.

Andy Liu earned his Doctorate in Mathematics and Diploma in Elementary Education simultaneously (1976) from the University of Alberta where he is now a Professor of Mathematical Sciences. His research interests are in combinatorics, geometry, mathematics education and mathematics recreations. Since 1981, he has been running the Saturday Mathematical Activities, Recreations and Tutorials program for students in the 12 to 15 age group. He is on the editorial board of the *Journal of High School Mathematics* in China, *Mathematics and Informatics Quarterly* in Bulgaria and *Crux Mathematicorum* in Canada, as well as the series of books of which this is one volume.

From 1981 to 1984, Andy was the deputy leader of the United States IMO team (under Murray Klamkin). In 1990, he was an honorary member of the IMO organizing Committee in China. In 1994, he was a member of the IMO Academic Committee in Hong Kong. In 1995, he was chairman of the IMO Problem Selection Committee. He is a vice-president of the International Mathematics Tournament of the Towns. He is serving or has served on the problem committees of various national and local competitions. In 1996, he received the David Hilbert International Award from the World Federation of National Mathematics Competitions.

THE AUSTRALIAN MATHEMATICS TRUST

ENRICHMENT SERIES

ISBN 187642016-2

9 781876 420161