Real Analysis Exam: Part I (Spring 2000)

Do all five problems.

1. Let μ be a finite measure on $\mathbb{R}$. Prove that for every $x \in \mathbb{R}$,

$$|\mu(\{x\})| \leq \limsup_{\xi \to \infty} |\hat{\mu}(\xi)|.$$

Here $\hat{\mu}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi}\, d\mu(x)$ denotes the Fourier transform of μ.

2. Let $f : S^1 \to \mathbb{R}$ be a Hölder continuous function on S^1 with Hölder exponent α. Thus

$$\sup_{x,y \in S^1,\, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

Prove that for some constant $C > 0$ depending on (the Hölder norm of) f but independent of n, the Fourier coefficient $\hat{f}(n) = \frac{1}{2\pi}\int_{S^1} f(x) e^{-inx}\, dx$ satisfies $|\hat{f}(n)| \leq C/|n|^\alpha$.

3(a). Let E be a Banach space and E^* be its dual. Assume that E^* is uniformly convex. Prove that for every $f \in E$, there is one and only one g in the weak* unit ball in E^* such that $\|f\|_E = \langle f, g \rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing between E and E^*.

(b). Suppose that f and g are two real-valued functions in $L^3(I)$, where $I = [a,b]$, and that

$$\|f\|_3 = \|g\|_3 = \int f(x)^2 g(x)\, dx = 1.$$

Prove that $g(x) = |f(x)|$ for almost every $x \in I$. You may do this either directly or else using part (a).

4. Let μ be a positive Borel measure supported in a compact set $E \subset \mathbb{R}$, with μ not identically zero. Assume that there are constants $a > 0$ and $C > 0$ such that for all intervals I, $\mu(I) \leq C|I|^a$.

(a) Define $K_b(x) = |x|^{-b}$ for $b > 0$. Prove that $\mu * K_b$ is well-defined and continuous for all $b < a$.

(b) Give an explicit positive lower bound for the Hausdorff dimension of E. (Recall that the Hausdorff dimension of E is defined to be the supremum of values α' such that the Hausdorff α'-dimensional measure of E is infinite.)

5. Let T be a unitary operator on a Hilbert space H such that its spectrum $\sigma(T)$ is a countable set. Prove that there exists a sequence $\{n_j\} \subset \mathbb{N}$ such that $T^{n_j} \to I$ (the identity operator) in the strong operator norm topology.

(Hint: prove that there exists a sequence of exponentials $\{e^{in_j t}\}$ such that $e^{in_j t_k} \to 1$ when $e^{it_k} \in \sigma(T)$. You may use Dirichlet's theorem that for any finite set $\{t_k\}_{k=1}^N$ and for every $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $|e^{int_k} - 1| < \epsilon$ for $k = 1, \ldots, N$.)

Real Analysis Exam: Part II (Spring 2000)

Do all five problems.

1. Let $f \in C^2(\mathbb{R})$ be a function such that $M_k = \sup_{x\in\mathbb{R}} |f^{(k)}(x)| < \infty$ for $k = 0, 1, 2$.
   (a) Prove that $M_1 \leq 2\sqrt{M_0 M_2}$. (Hint: use the second order Taylor formula for $f(x+t)$ about $t = 0$.)
   (b) Show that the only functions for which equality is attained in part (a) are the constant functions.

2. Let m be Lebesgue outer measure on $\mathbb{R}$ and let μ be any outer measure on $\mathbb{R}$. Suppose that all Borel sets in $\mathbb{R}$ are μ-measurable and, furthermore, that for every set $A \subset \mathbb{R}$, $\mu(A)$ is the infimum over all open sets $U \supset A$ of $\mu(U)$. If

$$\limsup_{\rho \to 0} \frac{\mu([x-\rho, x+\rho])}{2\rho} \geq 1$$

for all $x \in \mathbb{R}$, prove that $\mu(A) \geq m(A)$ for every subset $A \subset \mathbb{R}$.

3. Let $\mathcal{H} = \oplus \mathcal{H}_j$ be an orthonormal decomposition of the Hilbert space $\mathcal{H}$ into finite-dimensional subspaces $\mathcal{H}_j$, and let $\{c_j\}$ be a sequence of positive numbers. The *generalized cube* determined by these data is the set

$$\mathcal{Q} = \left\{ v \in \mathcal{H} : v = \sum_{j=1}^{\infty} v_j \text{ with } v_j \in \mathcal{H}_j \text{ and } \|v_j\| \leq c_j \right\}.$$

   (a) Prove that the condition $\sum_{j=1}^{\infty} c_j^2 < \infty$ is necessary and sufficient for the compactness of $\mathcal{Q}$.
   (b) Prove that every compact set $E \subset \mathcal{H}$ is contained in some compact generalized cube $\mathcal{Q}$.

4. Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of measurable subsets of a measure space (X, μ). Suppose that

$$\sum_{i=1}^{\infty} \mu(A_i) < \infty.$$

Let $Z = \{x \in X : x \in A_i \text{ for infinitely many } i\}$. Prove that $\mu(Z) = 0$.

5. Assume that $f \in \mathcal{L}^1(\mathbb{R})$ and that

$$\left| \int \phi''(x) f(x)\, dx \right| \leq 3\|\phi\|_{\infty}$$

for every $\phi \in \mathcal{C}_0^{\infty}(\mathbb{R})$. Prove that there is a Lipschitz continuous function g such that $f(x) = g(x)$ for almost every x.