

Người làm: Mai Xuân

Zalo: Mai Xuân - số đt zalo: 0834831388

Email: xuanmai5656@gmail.com

CD3: CÁC BÀI TOÁN VỀ LŨY THỪA

Dạng 1. So sánh hai lũy thừa

Dạng 2. So sánh biểu thức lũy thừa với một số (so sánh hai biểu thức lũy thừa)

Câu 1. (HSG 7 huyện Hưng Hà 2, trường TH và THCS Kỳ Đồng 2022 - 2023)

Cho $B = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2^{99}}$. So sánh B với 50

Lời giải

$$\begin{aligned} B &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4} \right) + \dots + \left(\frac{1}{2^{98} + 1} + \dots + \frac{1}{2^{99}} \right) \\ B &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{2^2} \right) + \dots + \left(\frac{1}{2^{98} + 1} + \dots + \frac{1}{2^{99}} \right) \quad (\text{Có } 98 \text{ nhóm trong } B) \\ B &> 1 + \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{2^2} \right) + \dots + \left(\frac{1}{2^{99}} + \dots + \frac{1}{2^{99}} \right) \quad (\text{Có } 98 \text{ nhóm trong ngoặc}) \\ B &> 1 + \frac{1}{2} + 2 \cdot \frac{1}{2^2} + \dots + 2^{98} \cdot \frac{1}{2^{99}} = 1 + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} = 1 + \frac{99}{2} = \frac{101}{2} > \frac{100}{2} = 50 \\ \text{Vậy } B &> 50 \end{aligned}$$

Câu 2. (HSG 7 huyện Hưng Hà, tỉnh, trường Thái Hưng 2022 - 2023)

Cho $D = \left(\frac{1}{2^2} - 1 \right) \left(\frac{1}{3^2} - 1 \right) \left(\frac{1}{4^2} - 1 \right) \dots \left(\frac{1}{100^2} - 1 \right)$
So sánh D với $-\frac{1}{2}$.

Lời giải

$$\begin{aligned} \text{Vì } D &\text{ là tích của } 99 \text{ số âm nên:} \\ -D &= \left(1 - \frac{1}{4} \right) \left(1 - \frac{1}{9} \right) \left(1 - \frac{1}{16} \right) \dots \left(1 - \frac{1}{100^2} \right) \\ -D &= \frac{3}{2^2} \cdot \frac{8}{3^2} \cdot \frac{15}{4^2} \dots \frac{9999}{100^2} = \frac{1.3}{2^2} \cdot \frac{2.4}{3^2} \cdot \frac{3.5}{4^2} \dots \frac{99.101}{100^2} \\ -D &= \frac{1.2.3.4.5 \dots 98.99}{2.3.4.5 \dots 99.100} \cdot \frac{3.4.5 \dots 100.101}{2.3.4 \dots 99.100} \\ -D &= \frac{1}{100} \cdot \frac{101}{2} = \frac{101}{200} > \frac{100}{200} \quad \text{hay } -D > \frac{1}{2} \\ D &< -\frac{1}{2} \\ \text{Vậy} \end{aligned}$$

Câu 3. (HSG 7 huyện Quang Sơn, 2022 - 2023)

$A = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \dots + \frac{2018}{3^{2018}}$. So sánh A và $\frac{3}{4}$.

Lời giải

$$A = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \dots + \frac{2018}{3^{2018}}$$

$$3A = 1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \dots + \frac{2018}{3^{2017}}$$

$$3A - A = \left(1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \dots + \frac{2018}{3^{2017}} \right) - \left(\frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \dots + \frac{2018}{3^{2018}} \right)$$

$$2A = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{2017}} - \frac{2018}{3^{2018}}$$

$$2A < 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{2017}} \quad (1)$$

$$B = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{2017}}$$

Đặt

$$3B = 3 + 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{2016}}$$

Suy ra:

$$3B - B = \left(3 + 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{2016}} \right) - \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{2017}} \right)$$

$$2B = 3 - \frac{1}{3^{2017}} \Rightarrow B = \frac{3}{2} - \frac{1}{2 \cdot 3^{2017}} < \frac{3}{2} \quad (2)$$

Từ (1) và (2) ta suy ra $2A < \frac{3}{2}$ Hay $A < \frac{3}{4}$.

$$A < \frac{3}{4}$$

Vậy

Câu 4. (HSG 7 huyện Hưng Hà, trường Trần Thái Tông 2022 - 2023)

Cho $M = \frac{2022^{2024} - 2021^{2024}}{2022^{2024} + 2021^{2024}}; N = \frac{2022^{2023} - 2021^{2023}}{2022^{2023} + 2021^{2023}}$. Hãy so sánh M và N

Lời giải

Ta có: $2022 > 2021 \Rightarrow \frac{2022}{2021} > 1$

Mà $2024 > 2023 \Rightarrow \left(\frac{2022}{2021} \right)^{2024} > \left(\frac{2022}{2021} \right)^{2023}$

$$\Rightarrow 1 + \left(\frac{2022}{2021} \right)^{2024} > 1 + \left(\frac{2022}{2021} \right)^{2023}$$

$$\Rightarrow \frac{1}{1 + \left(\frac{2022}{2021} \right)^{2024}} < \frac{1}{1 + \left(\frac{2022}{2021} \right)^{2023}}$$

$$\Rightarrow \frac{2021^{2024}}{2022^{2024} + 2021^{2024}} < \frac{2021^{2023}}{2022^{2023} + 2021^{2023}}$$

$$\Rightarrow \frac{2 \cdot 2021^{2024}}{2022^{2024} + 2021^{2024}} < \frac{2 \cdot 2021^{2023}}{2022^{2023} + 2021^{2023}}$$

$$\Rightarrow 1 - \frac{2 \cdot 2021^{2024}}{2022^{2024} + 2021^{2024}} > 1 - \frac{2 \cdot 2021^{2023}}{2022^{2023} + 2021^{2023}}$$

$$\Rightarrow \frac{2022^{2024} - 2021^{2024}}{2022^{2024} + 2021^{2024}} > \frac{2022^{2023} - 2021^{2023}}{2022^{2023} + 2021^{2023}}$$

Vậy $M > N$

Câu 5. (HSG 7 huyện Hưng Hà, trường Thái Phương 2022 - 2023)

So sánh A với 1 biết:

$$A = \frac{1}{2} \left(1 + \frac{1}{1.3} \right) \cdot \left(1 + \frac{1}{2.4} \right) \cdot \left(1 + \frac{1}{3.5} \right) \cdots \left(1 + \frac{1}{2021.2023} \right)$$

Lời giải

$$A = \frac{1}{2} \left(1 + \frac{1}{1.3} \right) \cdot \left(1 + \frac{1}{2.4} \right) \cdot \left(1 + \frac{1}{3.5} \right) \cdots \left(1 + \frac{1}{2021.2023} \right)$$

$$A = \frac{1}{2} \cdot \frac{3+1}{1.3} \cdot \frac{2.4+1}{2.4} \cdots \frac{2021.2023+1}{2021.2023}$$

$$A = \frac{1}{2} \cdot \frac{2.2}{1.3} \cdot \frac{3.3}{2.4} \cdot \frac{4.4}{3.5} \cdots \frac{2022.2022}{2021.2023}$$

$$A = \frac{1}{2} \cdot \frac{2.3.4 \cdots 2022}{1.2.3 \cdots 2021} \cdot \frac{2.3.4 \cdots 2022}{3.4.5 \cdots 2023}$$

$$A = \frac{1}{2} \cdot \frac{2022}{1} \cdot \frac{2}{2023}$$

$$A = \frac{2022}{2023} < \frac{2023}{2023} = 1$$

Vậy $A < 1$

Câu 6. (HSG 7 huyện Thái Thụy, 2022 - 2023)

Cho $B = \frac{1}{11} + \frac{1}{11^2} + \cdots + \frac{1}{11^{100}}$. So sánh B với $\frac{1}{10}$.

Lời giải

$$B = \frac{1}{11} + \frac{1}{11^2} + \cdots + \frac{1}{11^{100}}$$

$$11B = 1 + \frac{1}{11} + \cdots + \frac{1}{11^{99}}$$

$$11B - B = 1 - \frac{1}{11^{100}}$$

$$\Rightarrow 10B = 1 - \frac{1}{11^{100}} < 1$$

$$\Rightarrow 10B < 1$$

$$B < \frac{1}{10}$$

Vậy

Dạng 3. Một số bài toán khác

Câu 1. (HSG 7 huyện Hưng Hà 2022 - 2023)

a) $\frac{1.3+2}{2^2} + \frac{2.4+2}{3^2} + \frac{3.5+2}{4^2} + \cdots + \frac{2021.2023+2}{2022^2} + \frac{2022.2024+2}{2023^2} < 2023$

b) Cho $P = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{99}{100}$. Chứng minh rằng $P < \frac{1}{10}$

Lời giải

$$\text{Đặt } A = \frac{1.3+2}{2^2} + \frac{2.4+2}{3^2} + \frac{3.5+2}{4^2} + \dots + \frac{2021.2023+2}{2022^2} + \frac{2022.2024+2}{2023^2} \quad (1)$$

Ta thấy tổng A có 2022 số hạng

$$A = \frac{1.3+2}{2^2} + \frac{2.4+2}{3^2} + \frac{3.5+2}{4^2} + \dots + \frac{2021.2023+2}{2022^2} + \frac{2022.2024+2}{2023^2}$$

$$A = \left(1 + \frac{1}{2^2}\right) + \left(1 + \frac{1}{3^2}\right) + \left(1 + \frac{1}{4^2}\right) + \dots + \left(1 + \frac{1}{2022^2}\right) + \left(1 + \frac{1}{2023^2}\right)$$

$$A = \underbrace{\left(1 + 1 + 1 + \dots + 1\right)}_{2022 \text{ s/h}} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{2023^2}\right)$$

$$A = 2022 + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{2022^2} + \frac{1}{2023^2}\right) \quad (2)$$

Ta có:

$$\left\{ \begin{array}{l} \frac{1}{2^2} < \frac{1}{1.2} \\ \frac{1}{3^2} < \frac{1}{2.3} \\ \frac{1}{4^2} < \frac{1}{3.4} \\ \dots\dots\dots \\ \frac{1}{2023^2} < \frac{1}{2022.2023} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \frac{1}{2^2} < 1 - \frac{1}{2} \\ \frac{1}{3^2} < \frac{1}{2} - \frac{1}{3} \\ \frac{1}{4^2} < \frac{1}{3} - \frac{1}{4} \\ \dots\dots\dots \\ \frac{1}{2023^2} < \frac{1}{2022} - \frac{1}{2023} \end{array} \right.$$

$$\Rightarrow \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{2023^2} = 1 - \frac{1}{2023} < 1$$

$$\Rightarrow \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{2023^2} < 1$$

$$A = 2022 + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{2023^2}\right) < 2022 + 1 = 2023 \quad (3)$$

Do đó

Từ (1), (2), (3)

$$\text{Chúng tỏ: } \frac{1.3+2}{2^2} + \frac{2.4+2}{3^2} + \frac{3.5+2}{4^2} + \dots + \frac{2021.2023+2}{2022^2} + \frac{2022.2024+2}{2023^2} < 2023$$

$$\frac{n}{n+1} < \frac{n+1}{n+2} \quad (n \in \mathbb{N}^*)$$

b) Ta đi chứng minh bài toán phụ sau:

Thật vậy: với mọi $n \in \mathbb{N}^*$ ta có: $n^2 + n < n^2 + n + n + 1$

$$(n + 1) < (n + 1)^2$$

$$\Rightarrow \frac{n}{n+1} < \frac{n+1}{n+2}$$

$$n(n + 2) < (n + 1)(n + 1)$$

(Nhân chéo chứng minh)

Cho n các giá trị từ 1 đến 99 ta có:

$$\begin{aligned} \frac{1}{2} < \frac{2}{3}; \frac{3}{4} < \frac{5}{6}; \frac{5}{6} < \frac{6}{7}; \dots; \frac{99}{100} < \frac{100}{101} \\ \Rightarrow \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \dots \frac{99}{100} < \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \dots \frac{100}{101} \\ \Rightarrow \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \dots \frac{99}{100} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \dots \frac{99}{100} < \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \dots \frac{100}{101} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \dots \frac{99}{100} \\ \Rightarrow \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{99}{100} \right)^2 < \frac{1}{101} < \frac{1}{100} \\ \text{Vậy } P < \frac{1}{10} \end{aligned}$$

Câu 2. (HSG 7 huyện Quang Sơn, 2022 - 2023)

Chứng minh rằng
$$\frac{1}{4048} < \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \dots \frac{2021}{2022} \cdot \frac{2023}{2024} \right)^2 < \frac{1}{2025}.$$

Lời giải

$$\frac{1}{4048} < \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \dots \frac{2021}{2022} \cdot \frac{2023}{2024} \right)^2 < \frac{1}{2025}.$$

Với mọi $n \geq 1$ ta có:

$$\frac{n^2}{(n+1)^2} = \frac{n^2}{n^2 + 2n + 1} < \frac{n^2}{n^2 + 2n} = \frac{n}{n+2}.$$

Từ đó suy ra:

$$\left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \dots \frac{2021}{2022} \cdot \frac{2023}{2024} \right)^2 < \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \dots \frac{2021}{2023} \cdot \frac{2013}{2025} = \frac{1}{2025} \quad (1)$$

Với mọi $n \geq 2$ ta có:

$$\frac{n^2}{(n+1)^2} > \frac{n^2 - 1}{(n+1)^2} = \frac{(n-1)(n+1)}{(n+1)^2} = \frac{n-1}{n+1}$$

Từ đó suy ra:

$$\left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \dots \frac{2021}{2022} \cdot \frac{2023}{2024} \right)^2 > \frac{1}{4} \cdot \frac{2}{4} \cdot \frac{4}{6} \dots \frac{2020}{2022} \cdot \frac{2022}{2024} = \frac{1}{4048} \quad (2)$$

Từ (1) và (2) $\Rightarrow \frac{1}{4048} < \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \dots \frac{2021}{2022} \cdot \frac{2023}{2024} \right)^2 < \frac{1}{2025}$ (đpcm)

Câu 3. (HSG 7 huyện Tri Thức, tỉnh Đồng Nai, 2022 - 2023)

Chứng minh:
$$\frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots + \frac{1}{100^2} < \frac{1}{8}$$

Lời giải

Ta có:

$$\begin{aligned} \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots + \frac{1}{100^2} &= \frac{1}{2^2} \left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots + \frac{1}{50^2} \right) \\ &< \frac{1}{2^2} \left(\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \dots + \frac{1}{49 \cdot 50} \right) = \frac{1}{2^2} \left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \dots + \frac{1}{49} - \frac{1}{50} \right) \end{aligned}$$

$$= \frac{1}{4} \cdot \left(\frac{1}{2} - \frac{1}{50} \right) = \frac{1}{8} - \frac{1}{200} < \frac{1}{8} \quad (\text{đpcm})$$

$$\text{Vậy } \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \dots + \frac{1}{100^2} < \frac{1}{8}$$

Câu 4. (HSG 7 huyện Thường Xuân, 2022 - 2023)

Cho dãy số $a_1, a_2, a_3, \dots, a_n$ được xác định như sau: $a_1 = 1$; $a_2 = 1 + \frac{1}{2}$; $a_3 = 1 + \frac{1}{2} + \frac{1}{3}$; ...;
 $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$. Chứng minh rằng: $\frac{1}{a_1^2} + \frac{1}{2a_2^2} + \frac{1}{3a_3^2} + \dots + \frac{1}{na_n^2} < 2$, với mọi số tự nhiên $n > 1$.

Lời giải

Với mọi $\forall k \geq 2$, ta có: $\frac{1}{k \cdot a_k^2} < \frac{1}{k \cdot a_{k-1} \cdot a_k}$ (vì $a_k > a_{k-1}$).

$$\text{Lại có: } \frac{1}{a_{k-1}} - \frac{1}{a_k} = \frac{a_k - a_{k-1}}{a_{k-1} \cdot a_k} = \frac{1}{k \cdot a_{k-1} \cdot a_k}$$

$$\text{Suy ra: } \frac{1}{k \cdot a_k^2} < \frac{1}{a_{k-1}} - \frac{1}{a_k}$$

Cho $k = 2; 3; \dots; n$ ta có:

$$\frac{1}{2a_2^2} < \frac{1}{a_1} - \frac{1}{a_2}; \quad \frac{1}{3a_3^2} < \frac{1}{a_2} - \frac{1}{a_3}; \dots; \quad \frac{1}{na_n^2} < \frac{1}{a_{n-1}} - \frac{1}{a_n}$$

Cộng theo vế ta được:

$$\frac{1}{2a_2^2} + \dots + \frac{1}{na_n^2} < \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_2} - \frac{1}{a_3} + \dots + \frac{1}{a_{n-1}} - \frac{1}{a_n} = \frac{1}{a_1} - \frac{1}{a_n} < \frac{1}{a_1} = 1$$

$$\Rightarrow \frac{1}{a_1^2} + \frac{1}{2a_2^2} + \dots + \frac{1}{na_n^2} < 1 + 1 = 2$$

$$\text{Vậy } \frac{1}{a_1^2} + \frac{1}{2a_2^2} + \frac{1}{3a_3^2} + \dots + \frac{1}{na_n^2} < 2, \text{ với mọi số tự nhiên } n > 1.$$

Câu 5. (HSG 7 huyện Tiên Du, tỉnh Bắc Ninh, 2022 - 2023)

$$1. \text{ Cho } A = \frac{1}{7^2} - \frac{1}{7^4} + \frac{1}{7^6} - \frac{1}{7^8} + \dots + \frac{1}{7^{98}} - \frac{1}{7^{100}}. \text{ Chứng minh rằng } A < \frac{1}{50}.$$

$$2. \text{ Cho } A = \frac{1}{7} + \frac{2}{7^2} + \frac{3}{7^3} + \frac{4}{7^4} + \dots + \frac{99}{7^{99}} + \frac{100}{7^{100}}. \text{ Chứng minh rằng } A < \frac{7}{36}.$$

Lời giải

$$1) \text{ Ta có } A = \frac{1}{7^2} - \frac{1}{7^4} + \frac{1}{7^6} - \frac{1}{7^8} + \dots + \frac{1}{7^{98}} - \frac{1}{7^{100}}$$

$$7^2 A = 7^2 \left(\frac{1}{7^2} - \frac{1}{7^4} + \frac{1}{7^6} - \frac{1}{7^8} + \dots + \frac{1}{7^{98}} - \frac{1}{7^{100}} \right)$$

$$49A = 1 - \frac{1}{7^2} + \frac{1}{7^4} - \frac{1}{7^6} + \frac{1}{7^8} - \dots + \frac{1}{7^{96}} - \frac{1}{7^{98}}$$

$$\Rightarrow 49A + A = \left(1 - \frac{1}{7^2} + \frac{1}{7^4} - \frac{1}{7^6} + \frac{1}{7^8} - \dots + \frac{1}{7^{96}} - \frac{1}{7^{98}}\right) + \left(\frac{1}{7^2} - \frac{1}{7^4} + \frac{1}{7^6} - \frac{1}{7^8} + \dots + \frac{1}{7^{98}} - \frac{1}{7^{100}}\right)$$

$$50A = 1 - \frac{1}{7^{100}} < 1$$

$$\Rightarrow A < \frac{1}{50}$$

$$\text{Vậy } A < \frac{1}{50}.$$

$$A = \frac{1}{7} + \frac{2}{7^2} + \frac{3}{7^3} + \frac{4}{7^4} + \dots + \frac{99}{7^{99}} + \frac{100}{7^{100}}$$

2) Ta có

$$7A = 1 + \frac{2}{7^1} + \frac{3}{7^2} + \frac{4}{7^3} + \dots + \frac{99}{7^{98}} + \frac{100}{7^{99}}$$

$$\Rightarrow 7A - A = \left(1 + \frac{2}{7^1} + \frac{3}{7^2} + \frac{4}{7^3} + \dots + \frac{99}{7^{98}} + \frac{100}{7^{99}}\right) - \left(\frac{1}{7} + \frac{2}{7^2} + \frac{3}{7^3} + \frac{4}{7^4} + \dots + \frac{99}{7^{99}} + \frac{100}{7^{100}}\right)$$

$$\Rightarrow 6A = 1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4} + \dots + \frac{1}{7^{99}} - \frac{100}{7^{100}}$$

$$B = 1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4} + \dots + \frac{1}{7^{99}}$$

Đặt

$$\Rightarrow 7B = 7 + 1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4} + \dots + \frac{1}{7^{98}}$$

$$\Rightarrow 7B - B = \left(7 + 1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4} + \dots + \frac{1}{7^{98}}\right) - \left(1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4} + \dots + \frac{1}{7^{99}}\right)$$

$$6B = 7 - \frac{1}{7^{99}} < 7$$

$$\Rightarrow B < \frac{7}{6}$$

Câu 6. (HSG 7 huyện Nga Sơn, tỉnh Thanh Hóa, 2022 - 2023)

Cho $A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2^{2023} - 1}$. Chứng minh rằng: $A > \frac{2023}{2}$.

Lời giải

Ta có

$$A = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{2022} + 1} + \dots + \frac{1}{2^{2023}}\right) - \frac{1}{2^{2023}}$$

$$A > 1 + \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3}\right) + \dots + \left(\frac{1}{2^{2022}} + \dots + \frac{1}{2^{2023}}\right) - \frac{1}{2^{2023}}$$

$$A > 1 + \frac{1}{2} + 2 \cdot \frac{1}{2^2} + 2^2 \cdot \frac{1}{2^3} + \dots + 2^{2022} \cdot \frac{1}{2^{2023}} - \frac{1}{2^{2023}}$$

$$A > 1 + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} - \frac{1}{2^{2006}} = 1 + 2016 \cdot \frac{1}{2} - \frac{1}{2^{2016}} = \frac{2023}{2} + \left(1 - \frac{1}{2^{2023}}\right) > \frac{2023}{2}$$

$$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2^{2023} - 1} > \frac{2023}{2}$$

Vậy

Câu 7. (HSG 7 huyện Hưng Hà 2, tỉnh Thanh Hóa, 2022 - 2023)

a) Chứng minh rằng: $A = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2} < 1$

b) Chứng minh rằng $\frac{2n \cdot (2n - 1) \cdot \dots \cdot (n + 1)}{n \cdot (n - 1) \cdot \dots \cdot 2 \cdot 1} \geq \frac{4^n}{n + 1}, \forall n \in \mathbb{N}^*$

Lời giải

a) Chứng minh rằng: $A = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2} < 1$

Ta có : $A = \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \dots + \frac{1}{n \cdot n} < \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{(n - 1) \cdot n} = 1 - \frac{1}{n} < 1$

b) Chứng minh rằng $\frac{2n \cdot (2n - 1) \cdot \dots \cdot (n + 1)}{n \cdot (n - 1) \cdot \dots \cdot 2 \cdot 1} \geq \frac{4^n}{n + 1}, \forall n \in \mathbb{N}^*$

Với $n = 1$, ta có: $2 \leq 2$ nên đúng.

Giả sử đúng với $n = k$, tức là $\frac{4^k}{k + 1} \leq \frac{2k \cdot (2k - 1) \cdot \dots \cdot (k + 1)}{k \cdot (k - 1) \cdot \dots \cdot 2 \cdot 1}$

Ta đi chứng minh đúng khi $n = k + 1$, tức là

$$\frac{4^{k+1}}{k+2} \leq \frac{(2k+2) \cdot (2k+1) \cdot 2k \cdot \dots \cdot (k+2)}{(k+1) \cdot k \cdot (k-1) \cdot \dots \cdot 2 \cdot 1}$$

Thật vậy:

$$\frac{4^k}{k+1} \cdot \frac{4(k+1)}{k+2} \leq \frac{2k \cdot (2k-1) \cdot \dots \cdot (k+1)}{k \cdot (k-1) \cdot \dots \cdot 2 \cdot 1} \cdot \frac{4(k+1)}{k+2}$$

$$\frac{4^{k+1}}{k+2} \leq \frac{2k \cdot (2k-1) \cdot \dots \cdot (k+1)}{k \cdot (k-1) \cdot \dots \cdot 2 \cdot 1} \cdot \frac{4(k+1)}{k+2}$$

Ta có thể chứng minh :

$$VT \leq \frac{2k \cdot (2k-1) \cdot \dots \cdot (k+1)}{k \cdot (k-1) \cdot \dots \cdot 2 \cdot 1} \cdot \frac{(2k+1)(2k+2)}{(k+1)^2} = VP$$

$$\frac{2n \cdot (2n - 1) \cdot \dots \cdot (n + 1)}{n \cdot (n - 1) \cdot \dots \cdot 2 \cdot 1} \geq \frac{4^n}{n + 1}, \forall n \in \mathbb{N}^*$$

Vậy

Câu 8. (HSG 7 huyện Hưng Hà, 2022 - 2023)

$$B = \frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \dots + \frac{2n - 1}{(n - 1)^2 \cdot n^2}$$

Cho (với n là số tự nhiên lớn hơn 1).

Chứng minh $B < 1$

Lời giải

$$B = \frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \dots + \frac{2n-1}{(n-1)^2 \cdot n^2}$$

Cho (với n là số tự nhiên lớn hơn 1).

$$B = \frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \dots + \frac{2n-1}{(n-1)^2 \cdot n^2}$$

$$B = \frac{2^2 - 1^2}{1^2 \cdot 2^2} + \frac{3^2 - 2^2}{2^2 \cdot 3^2} + \frac{4^2 - 3^2}{3^2 \cdot 4^2} + \dots + \frac{n^2 - (n-1)^2}{(n-1)^2 \cdot n^2}$$

$$B = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots + \frac{1}{(n-1)^2} - \frac{1}{n^2}$$

$$B = 1 - \frac{1}{n^2}$$

$$\text{Vì } n \text{ là số tự nhiên lớn hơn 1 nên } \frac{1}{n^2} > 0 \Rightarrow 1 - \frac{1}{n^2} < 1$$

$$\text{Vậy } B > 1$$

Câu 9. (HSG 7 huyện Hưng Hà, tỉnh, trường Lê Quý Đôn 2022 - 2023)

$$\text{Chứng minh rằng: } \frac{1}{7^2} - \frac{1}{7^4} + \dots + \frac{1}{7^{4n-2}} - \frac{1}{7^{4n}} + \dots + \frac{1}{7^{98}} - \frac{1}{7^{100}} < \frac{1}{50}$$

Lời giải

$$\text{Đặt } A = \frac{1}{7^2} - \frac{1}{7^4} + \dots + \frac{1}{7^{4n-2}} - \frac{1}{7^{4n}} + \dots + \frac{1}{7^{98}} - \frac{1}{7^{100}}$$

Ta có:

$$7^2 A = 7^2 \left(\frac{1}{7^2} - \frac{1}{7^4} + \dots + \frac{1}{7^{4n-2}} - \frac{1}{7^{4n}} + \dots + \frac{1}{7^{98}} - \frac{1}{7^{100}} \right)$$

$$\Rightarrow 49A = 1 - \frac{1}{7^2} + \dots + \frac{1}{7^{4n-4}} - \frac{1}{7^{4n-2}} + \dots + \frac{1}{7^{96}} - \frac{1}{7^{98}}$$

$$49A + A = \left(1 - \frac{1}{7^2} + \dots + \frac{1}{7^{96}} - \frac{1}{7^{98}} \right) + \left(\frac{1}{7^2} - \frac{1}{7^4} + \dots + \frac{1}{7^{98}} - \frac{1}{7^{100}} \right)$$

$$\Rightarrow 50A = 1 - \frac{1}{7^{100}} < 1$$

$$\Rightarrow A < \frac{1}{50}$$

Câu 10. (HSG 7 huyện Hưng Hà, tỉnh, trường Lê Nam Đế 2022 - 2023)

$$A = \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \dots + \frac{2}{2007^2} < \frac{1003}{2008}$$

Chứng minh rằng:

Lời giải

Tổng A không có thừa số chung, do vậy ta đánh giá từng số hạng

$$\frac{2}{3^2} = \frac{2}{9} < \frac{2}{8} = \frac{2}{2 \cdot 4}$$

$$\frac{2}{5^2} = \frac{2}{25} < \frac{2}{24} = \frac{2}{4.6}$$

$$\frac{2}{7^2} = \frac{2}{49} < \frac{2}{48} = \frac{2}{6.8}$$

.....

$$\frac{2}{2007^2} < \frac{2}{2006.2008}$$

$$A < \frac{2}{2.4} + \frac{2}{4.6} + \frac{2}{6.8} + \dots + \frac{2}{2006.2008} = \frac{1}{2} - \frac{1}{2008} = \frac{1003}{2008}$$

Câu 11. (HSG 7 huyện Hưng Hà, tỉnh, trường Thái Phương 2022 - 2023)

Cho $S_1 = 1 + \frac{1}{5}; S_2 = 1 + \frac{1}{5} + \frac{1}{5^2}; S_3 = 1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3}; \dots; S_n = 1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots + \frac{1}{5^n}$

$(n \in \mathbb{N}^*)$. Chứng minh rằng: $\frac{1}{5S_1^2} + \frac{1}{5^2S_2^2} + \frac{1}{5^3S_3^2} + \dots + \frac{1}{5^nS_n^2} < \frac{1}{4}$

Lời giải

+) Ta có

$$S_n^2 > S_{n-1}^2 > \dots > S_2^2 > S_1^2 > 1$$

$$\Rightarrow \frac{1}{5S_1^2} + \frac{1}{5^2S_2^2} + \frac{1}{5^3S_3^2} + \dots + \frac{1}{5^nS_n^2} < \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots + \frac{1}{5^n}$$

$$A = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots + \frac{1}{5^n}$$

$$\Rightarrow 5A = 1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots + \frac{1}{5^{n-1}}$$

$$\Rightarrow 4A = 1 - \frac{1}{5^n} < 1$$

Vậy $A < \frac{1}{4}$

Câu 12. (HSG 7 huyện Hưng Hà, tỉnh, trường2022 - 2023)

Chứng minh rằng $\frac{1}{6} < \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \dots + \frac{1}{100^2} < \frac{1}{4}$

Lời giải

Đặt: $A = \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \dots + \frac{1}{100^2}$

Ta có:

$$A < \frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \dots + \frac{1}{99.100}$$

$$= \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{1}{99} - \frac{1}{100} = \frac{1}{4} - \frac{1}{100} < \frac{1}{4}$$

$$A > \frac{1}{5.6} + \frac{1}{6.7} + \dots + \frac{1}{99.100} + \frac{1}{100.101} = \frac{1}{5} - \frac{1}{101} > \frac{1}{6}$$

Vậy $\frac{1}{6} < \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \dots + \frac{1}{100^2} < \frac{1}{4}$ (đpcm)

Tài liệu được chia sẻ bởi Website VnTeach.Com

<https://www.vnteach.com>