Ph.D Qalifying Examination
Complex Analysis Fall 2001

Do all six problems, each in a separate blue book. All problems have equal weight.

1. Let Ω be the domain $|z| < 2$ and let $[0,1] \subset \Omega$ be the line segment between 0 and 1.

(a) Suppose $f : \Omega \to \mathbb{C}$ is a continuous function that is analytic over $\Omega \setminus [0,1]$. Show that f is analytic over Ω.

(b) Show that there are bounded analytic functions $f : \Omega \setminus [0,1] \to \mathbb{C}$ that can not be extended to analytic functions over Ω.

2. Let $f(z)$ be an analytic function defined over $|z| < 1$ that maps $|z| < 1$ one-one and onto $\mathbb{C} \setminus (-\infty, -1/4]$. Find the most general form of $f(z)$.

3. Establish the following identity

$$\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right)\frac{1}{t} = 2\sum_{n=1}^{\infty} \frac{1}{t^2 + 4n^2\pi^2}.$$

(Hint: Consider the difference of the terms on the left and the right hand side.)

4. Let f and $g_0 \neq 0, g_2, \cdots, g_n$ be analytic functions over the punctured disk $D^* = \{0 < |z| < 1\}$. Suppose $g_0, \cdots, g_n$ all have at most poles at $z = 0$. Suppose further that f satisfies the identity

$$g_0 f^n + g_1 f^{n-1} + \cdots + g_n = 0.$$

Show that f has at most a pole at $z = 0$.

5. Let B be the square $\{|x| \leq 1, |y| \leq 1\}$ and let T be the boundary of B, in the xy-plane. Let $f : T \to \mathbb{R}$ be a piecewise smooth function defined over T. Show that there is a sequence of real polynomials $p_n(x,y)$ so that p_n converges uniformly to f on T.
(Hint: The Runge's approximation theorem might be helpful.)

6. Let $f(z)$ be an analytic function defined over the disk $|z| < 2$. Show that

$$\int_0^1 f(x)dx = \frac{1}{2\pi i}\oint_{|z|=1} f(z)\log z\, dz.$$

Here the path integral is taken counterclockwise.