Real Analysis Exam: Part I (Autumn 2000)

Do all five problems.

1. Let $A, B \subseteq \mathbf{R}$ be Lebesgue measurable subsets with finite positive measure.

   (a) Show that the convolution $\chi_A * \chi_B$ of their characteristic functions is a continuous function that is not identically zero.

   (b) Show that the set $A + B = \{a + b : \ a \in A, \ b \in B\}$ contains a nonempty open interval.

2. Suppose $\{f_n\}_{n=1}^{\infty}$ is a norm-bounded sequence of functions in $L^2(0,1)$ that converges in measure to a function f.

   (a) Show that $f \in L^2(0,1)$ and $\|f\|_2 \leq \liminf \|f_n\|_2$.

   (b) Show that $\|f_n\|_2$ converges to $\|f\|_2$ if and only if $\|f_n - f\|_2 \to 0$.

3.

   (a) Let $g(t)$ be a continuous function in $L^2(\mathbf{R})$. Assume that $g(t) \neq 0$ for almost every $t \in \mathbf{R}$. Show that the linear span of the functions $\{g(t)e^{itx} : \ x \in \mathbf{R}\}$ is dense in $L^2(\mathbf{R})$. (Hint: Hahn-Banach and Fourier Inversion.)

   (b) Suppose $f \in L^2(\mathbf{R})$ and let $\hat{f}$ denote its L^2 Fourier transform. Assume that $\hat{f}(t) \neq 0$ for almost every $t \in \mathbf{R}$. Show that the linear span of the set of functions $\{f_y(x) = f(x+y) : \ y \in \mathbf{R}\}$ is dense in $L^2(\mathbf{R})$.

4. Show that there is no sequence of positive continuous functions $\{f_n\}$ such that $\{f_n(x)\}$ is bounded for each irrational x, and unbounded for each rational x.

5.

   (a) Let A be a closed bounded subset of a Hilbert space H. Show there is a unique smallest ball in H containing the set A. The center of this ball is called the circumcenter of A. Show that the circumcenter of A lies in A if A is a convex set.

   (b) Let $\{A_i\}$ be a nested decreasing sequence of nonempty closed bounded convex sets in the Hilbert space H. Show that the sequence $\{p_i\}$ of circumcenters of the sets A_i is a Cauchy sequence in H. Show that the limit of this sequence lies in the intersection of the A_i.

   (c) Construct an example of a Banach space X containing a nested decreasing sequence of closed bounded convex sets A_i such that $\cap_{i=1}^{\infty} A_i = \phi$. (Hint: You could try $X = c_0$, the space of sequences of real numbers converging to zero, with the maximum norm.)

Real Analysis Exam: Part II (Autumn 2000)

Do all five problems.

1. Let (X, ρ) be a metric space, and let $C_1(X)$ denote the space of continuous real valued functions f on X with the property that

$$\sup\{|f(x) - \rho(x, x_0)| : x \in X\} < \infty$$

for some chosen $x_0 \in X$. For $f, g \in C_1(X)$, define $\sigma(f, g) = \sup_X |f - g|$.

(a) Show that $(C_1(X), \sigma)$ is a complete metric space which is independent of the choice of x_0.

(b) Define a map $\iota : X \to C_1(X)$ by $\iota(x) = \rho_x$ where ρ_x denotes the function $\rho_x(y) = \rho(x, y)$. Show that ι is an isometric embedding of X into $C_1(X)$.

2.

(a) Let $E_n \subseteq (0, 1)$, $n = 1, \ldots, N$, be measurable sets such that $\sum_{n=1}^{N} m(E_n) > N - 1$. Show that $m(\cap_{n=1}^{N} E_n) > 0$.

(b) Let $E \subseteq (0, 1)$ be a measurable set with $m(E) > 1 - 1/N$. Let $\hat{E}$ denote the union of E with all of its integer translates $E + k,\ k \in \mathbf{Z}$. Given any collection of points $x_1, \ldots, x_N \in \mathbf{R}$, show that there is a point $x \in (0, 1)$ such that each $x - x_n \in \hat{E}$ for $n = 1, \ldots, N$.

3. Let $\{f_k\}_{k=1}^{\infty}$ be an orthogonal set of functions in $L^2(0, 1)$ with $|f_k(x)| \leq M$ for all k and almost every $x \in (0, 1)$. For $n = 1, 2, \ldots$ let $\sigma_n = \frac{1}{n} \sum_{k=1}^{n} f_k$.

(a) Show that $\sum_{k=1}^{\infty} \|\sigma_{k^2}\|_2^2 < \infty$.

(b) Show that $\sigma_n(x) \to 0$ for almost every $x \in (0, 1)$.

4. Suppose that $F \subseteq \mathbf{R}$ is a closed set without interior points. Construct a strictly increasing C^1 function f such that $f'(x) = 0$ if and only $x \in F$.

5.

(a) Find an algebra of continuous complex-valued functions on the unit circle that separates points and contains the constants, but is not dense in the Banach space of continuous complex-valued functions with the usual maximum norm. Justify your answer. (Here an algebra means a complex vector space which is closed under multiplication.)

(b) Let X be a compact Hausdorff space, and let $C_{\mathbf{C}}(X)$ denote the Banach space of continuous complex-valued functions on X. Let A be a complex subalgebra of $C_{\mathbf{C}}(X)$ that contains the constants, separates points, and is closed under complex conjugation (i.e. $\bar{f} \in A$ whenever $f \in A$). Show that A is dense. (You may assume the Stone-Weierstrass Theorem for real-valued functions.)