Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before March 15, 2009

- 5044: *Proposed by Kenneth Korbin, New York, NY.*

  Let N be a positive integer and let

  $$\begin{cases} x = 9N^2 + 24N + 14 \text{ and} \\ y = 9(N+1)^2 + 24(N+1) + 14. \end{cases}$$

  Express the value of y in terms of x, and express the value of x in terms of y.

- 5045: *Proposed by Kenneth Korbin, New York, NY.*

  Given convex cyclic hexagon ABCDEF with sides

  $$\begin{array}{rcl} \overline{AB} & = & \overline{BC} = 85 \\ \overline{CD} & = & \overline{DE} = 104, \text{ and} \\ \overline{EF} & = & \overline{FA} = 140. \end{array}$$

  Find the area of $\triangle BDF$ and the perimeter of $\triangle ACE$.

- 5046: *Proposed by R.M. Welukar of Nashik, India and K.S. Bhanu, and M.N. Deshpande of Nagpur, India.*

  Let $4n$ successive Lucas numbers $L_k, L_{k+1}, \cdots, L_{k+4n-1}$ be arranged in a $2 \times 2n$ matrix as shown below:

  $$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & 2n \end{array}$$
  $$\begin{pmatrix} L_k & L_{k+3} & L_{k+4} & L_{k+7} & \cdots & L_{k+4n-1} \\ \\ L_{k+1} & L_{k+2} & L_{k+5} & L_{k+6} & \cdots & L_{k+4n-2} \end{pmatrix}$$

  Show that the sum of the elements of the first and second row denoted by R_1 and R_2 respectively can be expressed as

  $$R_1 = 2F_{2n}L_{2n+k}$$

  $$R_2 = F_{2n}L_{2n+k+1}$$

  where $\{L_n, n \geq 1\}$ denotes the Lucas sequence with $L_1 = 1, L_2 = 3$ and $L_{i+2} = L_i + L_{i+1}$ for $i \geq 1$ and $\{F_n, n \geq 1\}$ denotes the Fibonacci sequence,
  $F_1 = 1, F_2 = 1, F_{n+2} = F_n + F_{n+1}$.

- 5047: *Proposed by David C. Wilson, Winston-Salem, N.C.*

  Find a procedure for continuing the following pattern:

$$\begin{aligned}
S(n,0) &= \sum_{k=0}^{n}\binom{n}{k} = 2^n \\
S(n,1) &= \sum_{k=0}^{n}\binom{n}{k}k = 2^{n-1}n \\
S(n,2) &= \sum_{k=0}^{n}\binom{n}{k}k^2 = 2^{n-2}n(n+1) \\
S(n,3) &= \sum_{k=0}^{n}\binom{n}{k}k^3 = 2^{n-3}n^2(n+3) \\
&\vdots
\end{aligned}$$

- 5048: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.*

  Let $a, b, c,$ be positive real numbers. Prove that

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \ \geq \ \frac{54}{(a+b+c)^2}\frac{(abc)^3}{\sqrt{(ab)^4+(bc)^4+(ca)^4}}.$$

- 5049: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find a function $f:\Re \rightarrow \Re$ such that

$$2f(x)+f(-x)=\begin{cases} -x^3-3, & x \leq 1, \\ 3-7x^3, & x > 1. \end{cases}$$

Solutions

- 5026: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with coordinates $A(-3,0),\ B(12,0),\ C(4,15)$, and $D(0,4)$. Point P has coordinates $(x,3)$. Find the value of x if

$$\text{area } \triangle \text{PAD} + \text{area } \triangle \text{PBC} = \text{area } \triangle \text{PAB} + \text{area } \triangle \text{PCD}. \qquad (1)$$

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

$$(1) \quad \Leftrightarrow \quad \frac{1}{2}\left| det\begin{pmatrix} x & 3 & 1 \\ -3 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right| + \frac{1}{2}\left| det\begin{pmatrix} x & 3 & 1 \\ 12 & 0 & 1 \\ 4 & 15 & 1 \end{pmatrix}\right|$$

$$+\frac{1}{2}\left|det\begin{pmatrix} x & 3 & 1\\ -3 & 0 & 1\\ 12 & 0 & 1\end{pmatrix}\right| + \frac{1}{2}\left|det\begin{pmatrix} x & 3 & 1\\ 4 & 15 & 1\\ 0 & 4 & 1\end{pmatrix}\right|$$

$$\Leftrightarrow \quad |-4x-3| \ + \ |156-15x| \ = \ 45 \ + \ |11x+4|. \qquad (2)$$

If $x \le \dfrac{-3}{4}$, then (2) $\Leftrightarrow -4x-3-15x+156 = 45-11x-4 \Leftrightarrow x = 14$, impossible.
If $\dfrac{-3}{4} < x \le \dfrac{-4}{11}$, then (2) $\Leftrightarrow 4x+3-15x+156 = 45-11x-4 \Leftrightarrow x = 159 = 41$, impossible.
If $\dfrac{-4}{11} < x \le \dfrac{52}{5}$, then (2) $\Leftrightarrow 4x+3-15x+156 = 45+11x+4 \Leftrightarrow x = 5$.
If $x > \dfrac{52}{5}$, then (2) $\Leftrightarrow 4x+3+15x-156 = 45+11x+4 \Leftrightarrow x = \dfrac{101}{4}$.
Thus, there are two possible values of $x:\ x = 5$ and $\dfrac{101}{4}$.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Mark Cassell (student, St. George's School), Spokane, WA; Grant Evans (student, St. George's School), Spokane, WA; John Hawkins and David Stone (jointly), Statesboro, GA; Peter E. Liley, Lafayette, IN; Paul M. Harms, North Newton, KS; Charles, McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; Britton Stamper, (student, St. George's School), Spokane, WA; Vu Tran (student, Texas A&M University), College Station, TX, and the proposer.

- 5027: *Proposed by Kenneth Korbin, New York, NY.*

Find the x and y intercepts of

$$y = x^7 + x^6 + x^4 + x^3 + 1.$$

Solution by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.

The point $(0,1)$ is trivial. To find the x intercept we decompose
$x^7+x^6+x^4+x^3+1 = (x^4+x^3+x^2+x+1)(x^3-x+1)$ and the value we are looking for is given by $x^3-x+1=0$ since

$$x^4+x^3+x^2+x+1 = (x^2 - x\frac{-1+\sqrt{5}}{2}+1)(x^2 - x\frac{-1-\sqrt{5}}{2}+1) \neq 0.$$

Applying the formula for solving cubic equations, the only real root of $x^3-x+1=0$ is

$$\left(-\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{1}{27}}\right)^{1/3} + \left(-\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{1}{27}}\right)^{1/3} = \left(-\frac{1}{2}+\sqrt{\frac{69}{18}}\right)^{1/3} + \left(-\frac{1}{2}-\sqrt{\frac{69}{18}}\right)^{1/3}$$

whose approximate value is $-1.3247\ldots$

Also solved by Brian D. Beasley, Clinton, SC; Mark Cassell and Britton Stamper (jointly, students at St. George's School), Spokane, WA; Michael

Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA, and the propser.

- 5028: *Proposed by Michael Brozinsky, Central Islip, NY .*

If the ratio of the area of the square inscribed in an isosceles triangle with one side on the base to the area of the triangle uniquely determine the base angles, find the base angles.

Solution 1 by Brian D. Beasley, Clinton, SC.

Let θ be the measure of each base angle in the triangle, and let y be the length of each side opposite a base angle. Let x be the side length of the inscribed square. We first consider the right triangle formed with θ as an angle and x as a leg, denoting its hypotenuse by z. Then $x = z\sin\theta$. Next, we consider the isosceles triangle formed with the top of the inscribed square as its base; taking the right half of the top of the square as a leg, we form another right triangle with angle θ and hypotenuse $y - z$. Then $\frac{1}{2}x = (y-z)\cos\theta$, so $y = x(\csc\theta + \frac{1}{2}\sec\theta)$. Denoting the area of the square by S and the area of the original triangle by T, we have

$$\frac{T}{S} = \frac{\frac{1}{2}y^2\sin(\pi - 2\theta)}{x^2} = \frac{1}{2}\sin(2\theta)\left(\csc\theta + \frac{1}{2}\sec\theta\right)^2 = \frac{1}{4}\tan\theta + \cot\theta + 1.$$

Let $f(\theta) = \frac{1}{4}\tan\theta + \cot\theta + 1$ for $0 < \theta < \pi/2$. Then it is straightforward to verify that

$$\lim_{\theta\to 0^+} f(\theta) = \lim_{\theta\to\frac{\pi}{2}^-} f(\theta) = \infty$$

and that f attains an absolute minimum value of 2 at $\theta = \arctan(2)$. Hence the ratio T/S (and thus S/T) is uniquely determined when $\theta = \arctan(2) \approx 63.435°$.

Solution 2 by J. W. Wilson, Athens, GA.

With no loss of generality, let the base of the isosceles triangle b be a fixed value and vary the height h of the triangle. Then if $f(h)$ is a function giving the ratio for the compared areas, in order for it to uniquely determine the base angles, there must be either a minimum or maximum value of the function. Let $f(h)$ represent the ratio of the area of the triangle to the area of the square.

It is generally known (and easy to show) that side s of an inscribed square on base b of a triangle is on-half of the harmonic mean of the base b and the altitude h to that base. Thus

$$\begin{aligned} s &= \frac{hb}{h+b}. \text{ So,} \\ f(h) &= \frac{bh}{2s^2}. \text{ Substituting and simplifying this gives :} \\ f(h) &= \frac{h^2+2bh+b^2}{2bh}. \end{aligned}$$

For $h > 0$ it can be shown, by using the arithmetic mean−geometric mean inequality, that this function has a minimum value of 2 when $h = b$.

$$f(h) = \frac{h^2+2bh+b^2}{2bh}$$

$$= \frac{h + 2b + \dfrac{b^2}{h}}{2b}.$$

Since b is fixed, and using the arithmetic mean−geometric mean inequaltiy, we may write:

$$h + \frac{b^2}{h} \geq 2\sqrt{h\frac{b^2}{h}} = 2b, \text{ with equality holding if, and only if,}$$

$$h = \frac{b^2}{h}.$$

Therefore $f(h)$ reaches a maximum if, and only if, $h = b$. This means the base angles can be uniquely determined when the altitude and the base are the same length. Thus, by considering the right triangle formed by the altitude and the base, the base angle would be arctan 2.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; John Hawkins and David Stone (jointly; two solutions), Statesboro, GA; Peter E. Liley, Lafayette, IN; Kenneth Korbin, New York, NY; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA, and the proposer.

- 5029: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let $x > 1$ be a non-integer number. Prove that

$$\left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right) > \frac{9}{2},$$

where $[x]$ and $\{x\}$ represents the entire and fractional part of x.

Solution by John Hawkins and David Stone, Statesboro, GA.

We improve the lower bound by verifying the more accurate inequality

$$\# \quad \left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right) > \frac{16}{3}.$$

In fact, $\frac{16}{3}$ is a sharp lower bound for $\left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right)$ for x in the interval $(1, 2)$, while this expression becomes much larger for larger x.

For convenience, we let

$$f(x) = \left(\frac{x + \{x\}}{[x]} - \frac{[x]}{x + \{x\}}\right) + \left(\frac{x + [x]}{\{x\}} - \frac{\{x\}}{x + [x]}\right).$$

The function f, defined for $x > 1$, x not an integer, has a"repetitive" behavior. Its graph has a vertical asymptote at each positive integer. On each interval $(n, n+1)$, $f(x)$ decreases(strictly) from infinity to a specific limit, h_n (which we will specify), then repeats the behavior on the next interval, but does not drop down as far, because $h_n < h_{n+1}$ (so $f(x)$ never comes close to $h_1 = \frac{16}{3}$ again.)

We verify these statements by fixing n and examining the behavior on $f(x)$ on the interval $(n, n+1)$. In this case, we let $x = n + t$, where $0 < t < 1$; therefore, $[x] = n$ and $|x| = t$. Thus

$$\begin{aligned} f(x) &= \left(\frac{n+t+t}{n} - \frac{n}{n+t+t}\right) + \left(\frac{n+t+n}{t} - \frac{t}{n+t+n}\right) \\ &= \frac{n+2t}{n} - \frac{n}{n+2t} + \frac{2n+t}{t} - \frac{t}{2n+t}. \end{aligned}$$

We handle the above claims in order:

(1) $\lim_{t\to 0^+} f(x) = \lim_{t\to 0^+} \frac{n+2t}{n} - \frac{n}{n+2t} + \frac{2n+t}{t} - \frac{t}{2n+t} = +\infty.$

(2) Because $f(x)$ has been expressed in terms of t, say

$$g(t) = \frac{n+2t}{n} - \frac{n}{n+2t} + \frac{2n+t}{t} - \frac{t}{2n+t},$$

we can show that $g(t)$ is decreasing by showing its derivative is negative.

We compute the derivative with respect to t:

$$g'(t) = \frac{2}{n} + \frac{2n}{(2t+n)^2} - \frac{2n}{t^2} - \frac{2n}{(t+2n)^2}.$$

Basically, this is negative because of the dominant term $\frac{-2n}{t^2}$, but we can make this more precise:

$$\begin{aligned} g'(t) &< 0 \\ \Leftrightarrow\ & \frac{2}{n} + \frac{2n}{(2t+n)^2} - \frac{2n}{t^2} - \frac{2n}{(t+2n)^2} < 0 \\ \Leftrightarrow\ & \frac{1}{n} + \frac{n}{(2t+n)^2} < \frac{n}{t^2} + \frac{n}{(t+2n)^2} \\ \Leftrightarrow\ & \frac{(2t+n)^2 + n^2}{n(2t+n)^2} < \frac{n(t+2n)^2 + nt^2}{t^2(t+2n)^2} \\ \Leftrightarrow\ & t^2(t+2n)^2\left[(2t+n)^2 + n^2\right] < n(2t+n)^2\left[n(t+2n)^2 + nt^2\right] \\ \Leftrightarrow\ & t^2(t+2n)^2\left[(2t^2 + 2tn + n^2\right] < n^2(2t+n)^2\left[t^2 + 2tn + 2n^2\right] \\ \Leftrightarrow\ & 2t^6 + 10t^5n + 17t^4n^2 + 12t^3n^3 + 4t^2n^4 < 2n^6 + 10n^5t + 17n^4t^2 + 12n^3t^3 + 4n^2t^4 \\ \Leftrightarrow\ & 0 < 2\left(n^6 - t^6\right) + 10nt\left(n^4 - t^4\right) + 17n^2t^2\left(n^2 - t^2\right) - 4n^2t^2\left(n^2 - t^2\right) \\ \Leftrightarrow\ & 0 < 2\left(n^6 - t^6\right) + 10nt\left(n^4 - t^4\right) + 13n^2t^2\left(n^2 - t^2\right), \end{aligned}$$

and this last inequality is true because $0 < t < 1 < n$.

(3) Finally, we compute the lower bound at the right-hand endpoint:

$$\lim_{t\to 1^-} f(x) = \lim_{t\to 1^-}\left[\frac{n+2t}{n} - \frac{n}{n+2} + \frac{2n+t}{t} - \frac{t}{2n+t}\right]$$

$$
\begin{aligned}
&= \frac{n+2}{n} - \frac{n}{n+2} + \frac{2n+1}{1} - \frac{1}{2n+1} \\
&= 2n+1 - \frac{1}{2n+1} + \frac{4(n+1)}{n(n+2)}.
\end{aligned}
$$

Thus, we see that $h_n = 2n+1 - \dfrac{1}{2n+1} + \dfrac{4(n+1)}{n(n+2)} \approx 2n+1$, so the intervals' lower bounds increase linearly with n.

Note that $h_1 = 3 + \dfrac{7}{3} = \dfrac{16}{3}$, so $f(x) > \dfrac{16}{3}$ for $1 < x < 2$. So inequality (#) has been verified.

As stated above, the lower bound on x then grows, for instance,

$h_2 = 5 + \dfrac{13}{10} = \dfrac{63}{10}$, so $f(x) > \dfrac{63}{10}$ for $2 < x < 3$,

and

$h_3 = 7 + \dfrac{97}{105} = \dfrac{832}{105}$, so $f(x) > \dfrac{832}{105}$ for $3 < x < 4$.

Comment: The inequality # is sharp in the sense that no value larger than $\dfrac{16}{3}$ can be used. That is, by (3) above, we know that values of x very close to 2 produce values of $f(x)$ just above and arbitrarily close to $\dfrac{16}{3}$. We can see this precisely:

$$
\begin{aligned}
f\left(2 - \frac{1}{m}\right) &= f\left(1 + \frac{m-1}{m}\right) \\
&= \left(\frac{\dfrac{2m-1}{m} + \dfrac{m-1}{m}}{1} - \frac{1}{\dfrac{2m-1}{m} + \dfrac{m-1}{m}} \right) + \left(\frac{\dfrac{2m-1}{m} + 1}{\dfrac{m-1}{m}} - \frac{\dfrac{m-1}{m}}{\dfrac{2m-1}{m} + 1} \right) \\
&= \frac{3m-2}{m} - \frac{m}{3m-2} + \frac{3m-1}{m-1} - \frac{m-1}{3m-1} \\
&= \frac{16}{3} + \frac{2}{3}\left[\frac{3}{m(m-1)} - \frac{1}{3(m-1)(3m-2)} \right].
\end{aligned}
$$

(John and David accompanied their above solution with a graph generated by Maple showing how the lower bounds increase from $\dfrac{16}{3}$ for various values of x.)

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy; Vu Tran (student, Texas A&M University), College Station, TX; Boris Rays, Chesapeake, VA, and the proposer.

- 5030: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $A_1, A_2, \cdots, A_n \in M_2(\mathbf{C}), (n \geq 2)$, be the solutions of the equation $X^n = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$.

Prove that $\sum_{k=1}^{n} Tr(A_k) = 0.$

Solution by John Hawkins and David Stone, Statesboro, GA.

The involvement of the Trace function is a red herring. Actually, for $A_1, A_2, A_3, \ldots, A_n$ as specified in the problem, we have $\sum_{k=1}^{n} A_k = 0$. Therefore, since Tr is linear, $\sum_{k=1}^{n} Tr(A_k) = Tr\left(\sum_{k=1}^{n}(A_k\right) = Tr(0) = 0$. In fact $\sum_{k=1}^{n} Tr(A_k) = 0$ for any linear transformation $T : M_2(C) \longrightarrow W$ to any complex vector space W.

Here is our argument. For convenience, let $B = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$. Note that $B^2 = 5B$. Thus $B^3 = BB^2 = B5B = 5B^2 = 5^2B$. Inductively, $B^k = 5^{k-1}B$ for $k \geq 1$.

Therefore, $B = \frac{1}{5^{n-1}}B^n = \left[\frac{1}{5^{(n-1)/n}}B\right]^n$, so $\mathrm{A}_1 = \frac{1}{5^{(n-1)/n}}B$ is an n^{th} root of B:

$$A_1^n = \left[\frac{1}{5^{(n-1)/n}}B\right]^n = \frac{1}{5^{n-1}}B^n = \frac{1}{5^{n-1}}5^{n-1}B = B.$$

Now let $\xi = e^{2\pi i/n}$ be the primitive n^{th} root of unity. Then

$$0 = \xi^n - 1 = (\xi - 1)(\xi^{n-1} + \xi^{n-2} + \xi^{n-3} + \cdots + \xi + 1),$$

so,

$$(\#) \qquad (\xi^{n-1} + \xi^{n-2} + \xi^{n-3} + \cdots + \xi + 1) = 0.$$

With $A_1 = \frac{1}{5^{(n-1)/n}}B$ as above, let $A_k = \xi^{k-1}A_1$ for $k = 2, 3, \ldots, n$. These n distinct matrices are the n^{th} roots of B, namely:

$$A_k^n = [\xi^{k-1}A_1]^n = \xi^{(k-1)n}A_1^n = (\xi^n)^{k-1}A_1^n = 1^{k-1}A_1^n = A_1^n = B.$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{n} A_k &= \sum_{k=1}^{n} \xi^{k-1}A_1 = \left(\sum_{k=1}^{n} \xi^{k-1}\right)A_1 \\ &= 0 \cdot A_1 \text{ by } (\#) \\ &= 0. \end{aligned}$$

Comment 1: Implicit in the problem statement is that the given matrix equation has exactly n solutions. This is true for this particular matrix B. But it is not true in general. Gantmacher ("Matrix Theory", page 233) gives an example of a 3×3 matrix with infinitely many square roots: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Comment 2: The result would be true for B any 2×2 matrix having determinant zero but trace non-zero. In that case, we would have $B^2 = Tr(B)B$ and we use $A_1 = \dfrac{1}{Tr(B)^{(n-1)/n}} B$.

Comment 3: More generally, let V be a vector space over C and $c_1, c_2, \ldots, c_n$ be complex scalars whose sum is zero. Also let A be any vector in V and let $A_k = c_k A$ for $k = 1, 2, \cdots, n$. Then

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} c_k A = \Big(\sum_{k=1}^{n} c_k\Big) A = 0 \cdot A = 0.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- 5031: *Ovidiu Furdui, Toledo, OH.*

Let x be a real number. Find the sum

$$\sum_{n=1}^{\infty} (-1)^{n-1} n \Big(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!} \Big).$$

Solution 1 by Paul M. Harms, North Newton, KS.

We know that $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} + \cdots$.

The expression

$(-1)^{n-1} n \Big(e^x - 1 - x - \dfrac{x^2}{2!} - \cdots - \dfrac{x^n}{n!} \Big) = (-1)^{n-1} n \Big(\dfrac{x^{n+1}}{(n+1)!} + \dfrac{x^{n+2}}{(n+2)!} + \cdots \Big)$.

So the sum $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n \Big(\frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \cdots \Big)$ equals

$$\begin{aligned}
& \Big(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\Big) - 2\Big(\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\Big) + 3\Big(\frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\Big) - 4\Big(\frac{x^5}{5!} + \cdots\Big) + \cdots \\
= \; & \frac{(1)x^2}{2!} + \frac{(1-2)x^3}{2!} + \frac{(1-2+3)x^4}{4!} + \frac{(1-2+3-4)x^5}{5!} + \cdots \\
= \; & \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{2x^4}{4!} - \frac{2x^5}{5!} + \frac{3x^6}{6!} - \frac{3x^7}{7!} + \frac{4x^8}{8!} - \frac{4x^9}{9!} \cdots
\end{aligned}$$

We need to find the sum of this alternating series..

We have

$$\begin{aligned}
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\cdots \\
\frac{x}{2} \sinh x &= \frac{1}{2} x^2 + \frac{\frac{2}{4} x^4}{3!} + \frac{\frac{3}{6} x^6}{5!} + \frac{\frac{4}{8} x^8}{7!} + \cdots \\
&= \frac{x}{2!} + \frac{2x^4}{4!} + \frac{3x^6}{6!} + \frac{4x^8}{8!} + \cdots.
\end{aligned}$$

The positive terms of the alternating series sum to $\dfrac{x}{2} \sinh x$. Each negative term of the alternating series is an antiderivative of the previous term except for the minus sign. The general anitderivative of $\frac{x}{2}\sinh x$ is $\frac{1}{2}\Big[x\cosh x - \sinh x\Big] + C$. Using Taylor series we can show that $\frac{-1}{2}\Big[x\cosh x - \sinh x\Big]$ equals the sum of the negative terms of the alternating series. The sum in the problem is

$$\frac{x}{2}\sinh x - \frac{1}{2}\Big[x\cosh x - \sinh x\Big] = \frac{x+1}{2}\sinh x - \frac{x}{2}\cosh x.$$

Solution 2 by N. J. Kuenzi, Oshkosh, WI.

Let

$$F(x) = \sum_{n=1}^{\infty}(-1)^{n-1}n\Big(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\Big).$$

Differentiation yields

$$\begin{aligned}
F'(x) &= \sum_{n=1}^{\infty}\Big((-1)^{n-1}n(e^x - 1 - x - \cdots - \frac{x^{n-1}}{(n-1)!}\Big) \\
&= \sum_{n=1}^{\infty}\Big((-1)^{n-1}n(e^x - 1 - x - \cdots - \frac{x^{n-1}}{(n-1)!} - \frac{x^n}{n!} + \frac{x^n}{n!}\Big) \\
&= F(x) + \sum_{n=1}^{\infty}(-1)^{n-1}n\frac{x^n}{n!} \\
&= F(x) + x\Big(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^m\frac{x^m}{m!} + \cdots\Big) \\
&= F(x) + xe^{-x}.
\end{aligned}$$

Solving the differential equation

$$\begin{aligned}
F'(x) &= F(x) + xe^{-x} \text{ with initial conditions F(0) = 0 yields} \\
F(x) &= \frac{e^x - (1+2x)e^{-x}}{4}.
\end{aligned}$$

Also solved by Charles Diminnie and Andrew Siefker (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
April 15, 2009

- 5050: *Proposed by Kenneth Korbin, New York, NY.*

  Given $\triangle ABC$ with integer-length sides, and with $\angle A = 120^o$, and with $(a, b, c) = 1$. Find the lengths of b and c if side $a = 19$, and if $a = 19^2$, and if $a = 19^4$.

- 5051: *Proposed by Kenneth Korbin, New York, NY.*

  Find four pairs of positive integers (x, y) such that $\dfrac{(x-y)^2}{x+y} = 8$ with $x < y$.

  Find a formula for obtaining additional pairs of these integers.

- 5052: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain.*

  If $a \geq 0$, evaluate:

  $$\int_0^{+\infty} arctg \frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2}\,\frac{dx}{1+x^2}.$$

- 5053: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b and c be the sides, r the in-radius, and R the circum-radius of $\triangle ABC$. Prove or disprove that

  $$\frac{(a+b-c)(b+c-a)(c+a-b)}{a+b+c} \leq 2rR.$$

- 5054: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let x, y, z be positive numbers such that $xyz = 1$. Prove that

  $$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2} \geq 1.$$

- 5055: *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

  Let α be a positive real number. Find the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k^{\alpha}}.$$

Solutions

- 5032: *Proposed by Kenneth Korbin, New York, NY.*

Given positive acute angles A, B, C such that

$$\tan A\cdot\tan B+\tan B\cdot\tan C+\tan C\cdot\tan A=1.$$

Find the value of

$$\frac{\sin A}{\cos B\cdot\cos C}+\frac{\sin B}{\cos A\cdot\cos C}+\frac{\sin C}{\cos A\cdot\cos B}.$$

Solution 1 by Brian D. Beasley, Clinton, SC.

Since A, B, and C are positive acute angles with

$$\begin{aligned} 1 &= \frac{\sin A\sin B\cos C+\cos A\sin B\sin C+\sin A\cos B\sin C}{\cos A\cos B\cos C}\\ &= \frac{\cos A\cos B\cos C-\cos(A+B+C)}{\cos A\cos B\cos C}, \end{aligned}$$

we have $\cos(A+B+C)=0$ and thus $A+B+C=90°$. Then

$$\frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos A\cos C}+\frac{\sin C}{\cos A\cos B}=\frac{\sin A\cos A+\sin B\cos B+\sin C\cos C}{\cos A\cos B\cos C}.$$

Letting N be the numerator of this latter fraction, we obtain

$$\begin{aligned} N &= \sin A\cos A+\sin B\cos B+\cos(A+B)\sin(A+B)\\ &= \sin A\cos A+\sin B\cos B+(\cos A\cos B-\sin A\sin B)(\sin A\cos B+\cos A\sin B)\\ &= \sin A\cos A(1+\cos^2 B-\sin^2 B)+\sin B\cos B(1+\cos^2 A-\sin^2 A)\\ &= \sin A\cos A(2\cos^2 B)+\sin B\cos B(2\cos^2 A)\\ &= 2\cos A\cos B(\sin A\cos B+\cos A\sin B)\\ &= 2\cos A\cos B\sin(A+B)\\ &= 2\cos A\cos B\cos C. \end{aligned}$$

Hence the desired value is 2.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

The condition $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$ is equivalent to $\cot A+\cot B+\cot C=\cot A\cot B\cot C$. Since it is well known that

$$\cos(A+B+C)=-\sin A\sin B\sin C\Big(\cot A+\cot B+\cot C-\cot A\cot B\cot C\Big),$$

so $\cos(A+B+C)=0$ and $A+B+C=\frac{\pi}{2}$. Hence,

$$\sin 2A+\sin 2B+\sin 2C \;=\; 2\sin(A+B)\cos(A-B)+2\sin C\cos C$$

$$
\begin{aligned}
&= 2\cos C\big(\cos(A-B)+\cos(A+B)\big) \\
&= 4\cos A\cos B\cos C.
\end{aligned}
$$

If follows that

$$\frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos A\cos C}+\frac{\sin C}{\cos A\cos B}=\frac{\sin 2A+\sin 2B+\sin 2C}{2\cos A\cos B\cos C}=2.$$

Solution 3 by Boris Rays, Chesapeake, VA.

$\tan A\tan B+\tan B\tan C+\tan C\tan A=1$ implies,

$$
\begin{aligned}
\tan B(\tan A+\tan C) &= 1-\tan A\tan C \\
\frac{\tan A+\tan C}{1-\tan A\tan C} &= \frac{1}{\tan B} \\
\tan(A+C) &= \cot B=\tan(90^o-B).
\end{aligned}
$$

Similarly, we obtain:

$$
\begin{aligned}
\tan(B+C) &= \frac{1}{\tan A}=\cot A=\tan(90^o-A) \\
\tan(A+B) &= \frac{1}{\tan C}=\cot C=\tan(90^o-C), \text{which implies} \\
A &= 90^o-(B+C) \\
B &= 90^o-(A+C) \\
C &= 90^o-(A+B).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos A\cos C}+\frac{\sin C}{\cos A\cos B} \\
&= \frac{\sin(90^o-(B+C))}{\cos B\cos C}+\frac{\sin(90^o-(A+C))}{\cos A\cos C}+\frac{\sin(90^o-(A+B))}{\cos A\cos B} \\
&= \frac{\cos(B+C)}{\cos B\cos C}+\frac{\cos(A+C)}{\cos A\cos C}+\frac{\cos(A+B)}{\cos A\cos B} \\
&= \frac{\cos B\cos C-\sin B\sin C}{\cos B\cos C}+\frac{\cos A\cos C-\sin A\sin C}{\cos A\cos C}+\frac{\cos A\cos B-\sin A\sin B}{\cos A\cos B} \\
&= \Big(1-\tan B\tan C\Big)+\Big(1-\tan A\tan C\Big)+\Big(1-\tan A\tan B\Big) \\
&= 1+1+1-(\tan A\tan B+\tan B\tan C+\tan A\tan C) \\
&= 3-1=2.
\end{aligned}
$$

Also solved by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M.

Harms, North Newton, KS; John Hawkins, and David Stone (jointly), Statesboro, GA; Valmir Krasniqi, Prishtin, Kosovo; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; David C.Wilson, Winston-Salem, NC, and the proposer.

- 5033: *Proposed by Kenneth Korbin, New York, NY.*

Given quadrilateral $ABCD$ with coordinates $A(-3,0), B(12,0), C(4,15)$, and $D(0,4)$. Point P is on side $\overline{AB}$ and point Q is on side $\overline{CD}$. Find the coordinates of P and Q if area $\triangle PCD$ = area $\triangle QAB = \frac{1}{2}$area quadrilateral $ABCD$. (1)

Solution by Bruno Salgueiro Fanego, Viveiro, Spain.

P is on side $\overline{AB} : y = 0 \Rightarrow P(p,0)$.

Q is on side $\overline{CD} : y = \dfrac{11}{4}x + 4 \Rightarrow Q(4q,\ 11q+4)$.

Area quadrilateral $ABCD$=area $\triangle ABD$+area $\triangle BCD$, so

$$(1) \quad \Leftrightarrow \quad \frac{1}{2}\left|\det\begin{pmatrix} p & 0 & 1 \\ 4 & 15 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right| = \frac{1}{2}\left|\det\begin{pmatrix} 4q & 11q+4 & 1 \\ -3 & 0 & 1 \\ 12 & 0 & 1 \end{pmatrix}\right|$$

$$= \frac{1}{2}\left|\det\begin{pmatrix} -3 & 0 & 1 \\ 12 & 0 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right| + \frac{1}{2}\left|\det\begin{pmatrix} 12 & 0 & 1 \\ 4 & 15 & 1 \\ 0 & 4 & 1 \end{pmatrix}\right|$$

$$\Leftrightarrow \quad \left|11p+16\right| = 30+74 = 15\left|11q+4\right| \Leftrightarrow 11p+16 = \pm 104 = 15(11q+4)$$

$$\Leftrightarrow \quad P_1(8,0) \text{ or } \mathrm{P}_2(-120/11,0) \text{ and } \mathrm{Q}_1(16/15,104/15) \text{ or } \mathrm{Q}_2(-656/165,-104/15).$$

Observations by Ken Korbin. The following four points are on a straight line: midpoint of $\overline{AC}$, midpoint of $\overline{BD}$, P_1, and Q_1. Moreover, the midpoint of $\overline{P_1P_2}$ = the midpoint of $\overline{Q_1, Q_2}$ = the intersection point of lines AB and CD.

Also solved by Brian D. Beasley, Clinton, SC; Michael N. Fried, Kibbutz Revivim, Israel; John Hawkins and David Stone (jointly), Statesboro, GA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David C.Wilson, Winston-Salem, NC, and the proposer.

- 5034: *Proposed by Roger Izard, Dallas, TX.*

In rectangle $MDCB$, $MB \perp MD$. F is the midpoint of BC, and points N, E and G lie on line segments DC, DM and MB respectively, such that $NC = GB$. Let the area of quadrilateral $MGFC$ be A_1 and let the area of quadrilateral $MGFE$ be A_2. Determine the area of quadrilateral $EDNF$ in terms of A_1 and A_2.

Solution by Paul M. Harms, North Newton, KS.

Put the rectangle $MDCB$ on a coordinate system. Assume all nonzero coordinates are positive with coordinates

$$M(0,0), B(0,b), C(c,b), D(c,0) \text{ and } E(e,0), F(c/2,b), G(0,g), N(c,g).$$

The coordinates satisfy $e < c$ and $g < b$. The area A_1 of the quadrilateral $MGFC =$ the area of $\triangle MGF+$ area of $\triangle MFC$. Then

$$A_1 = \frac{1}{2}g(c/2) + \frac{1}{2}(c/2)b = \frac{1}{2}(c/2)(b+g).$$

The area A_2 of the quadrilateral $MGFE =$ area of $\triangle MGF+$ area of $\triangle MEF$. Then

$$A_2 = \frac{1}{2}g(c/2) + \frac{1}{2}eb.$$

The area of the quadrilateral $EDNF =$ area of $\triangle EFD+$ area of $\triangle FDN$. The area of the quadrilateral $EDNF$ is then

$$\begin{aligned} &= \frac{1}{2}(c-e)b + \frac{1}{2}g(c/2) \\ &= 2(\frac{1}{2})(c/2)b - \frac{1}{2}eb + \frac{1}{2}g(c/2) \\ &= 2A_1 - A_2. \end{aligned}$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; John Hawkins and David Stone (jointly), Statesboro,GA; Kenneth Korbin, New York, NY; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA, and the proposer.

- 5035: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be positive numbers. Prove that

$$(a^a b^b c^c)^2 (a^{-(b+c)} + b^{-(c+a)} + c^{-(a+b)})^3 \geq 27.$$

Solution 1 by David E. Manes, Oneonta, NY.

Note that the inequality is equivalent to

$$\frac{3}{a^{\frac{1}{b+c}} + b^{\frac{1}{c+a}} + c^{\frac{1}{b+c}}} \leq \sqrt[3]{a^{2a}b^{2b}c^{2c}}.$$

Since the problem is symmetrical in the variables a, b, and c, we can assume $a \geq b \geq c$. Therefore, $\ln a \geq \ln b \geq \ln c$. By the Rearrangement Inequality

$$\begin{aligned} a\ln a + b\ln b + c\ln c &\geq b\ln a + c\ln b + a\ln c \text{ and} \\ a\ln a + b\ln b + c\ln c &\geq c\ln a + a\ln b + b\ln c. \end{aligned}$$

Adding the two inequalities yields

$$2a\ln a + 2b\ln b + 2c\ln c \geq (b+c)\ln a + (c+a)\ln b + (a+b)\ln c.$$

Therefore,

$$\begin{aligned} \ln\left(a^{2a}b^{2b}c^{2c}\right) &\geq \ln\left(a^{b+c}b^{c+a}c^{a+b}\right) \text{ or} \\ a^{2a}b^{2b}c^{2c} &\geq a^{b+c}b^{c+a}c^{a+b} \text{ and so} \\ \sqrt[3]{a^{2a}b^{2b}c^{2c}} &\geq \sqrt[3]{a^{b+c}b^{c+a}c^{a+b}}. \end{aligned}$$

By the Harmonic-Geometric Mean Inequality

$$\frac{3}{a^{\frac{1}{b+c}} + b^{\frac{1}{c+a}} + c^{\frac{1}{b+c}}} \leq \sqrt[3]{a^{b+c}b^{c+a}c^{a+b}} \leq \sqrt[3]{a^{2a}b^{2b}c^{2c}}.$$

Solution 2 by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.

Taking the logarithm we obtain,

$$2\sum_{\text{cyc}} \ln a + 3\ln\left(\sum_{\text{cyc}} a^{-(b+c)}\right) \geq 3\ln 3.$$

The concavity of the logarithm yields,

$$2\sum_{\text{cyc}} \ln a + 3\left(\ln 3 - \sum_{\text{cyc}}(b+c)\ln a\right) \geq 3\ln 3.$$

Defining $s = a + b + c$ gives,

$$\sum_{\text{cyc}}(3a - s)\ln a \geq 0.$$

Since the second derivative of the function $f(x) = (3x - s)\ln x$ is positive for any x and s, ($f''(x) = 3/x + s/x^2$) it follows that,

$$\sum_{\text{cyc}}(3a - s)\ln a \geq \sum_{\text{cyc}}(3a - s)\ln a\Big|_{a=s/3} = 0.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Boris Rays, Chesapeake, VA, and the proposer.

- 5036: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Find all triples (x, y, z) of nonnegative numbers such that

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ 3^x + 3^y + 3^z = 5 \end{cases}$$

Solution 1 by John Hawkins and David Stone, Statesboro, GA.

We are looking for all first octant points of intersection of the unit sphere with the surface $3^x + 3^y + 3^z = 5$. Clearly, the intercept points $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ are solutions. We claim there no other solutions.

Consider the traces of our two surfaces in the xy-plane: the unit circle and the curve give by $3^x + 3^y = 4$. Our only concern is in the first quadrant, where we have a unit quarter circle and the curve $y = \dfrac{\ln(4 - 3^x)}{\ln 3}$. The two curves meet on the coordinate axes; otherwise graphing software shows that the logarithmic curve lies *inside* the quarter circle.

By the symmetry of the variables, we have the same behavior when we look at the traces in the xz- and yz-planes. That is, at our boundaries of concern, the exponential surface starts inside the sphere. By implicit differentiation of $3^x + 3^y + 3^z = 5$, we have the partial derivatives $\dfrac{\partial z}{\partial x} = -\dfrac{3^x}{3^z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{3^y}{3^z}$, which are both negative for nonnegative x, y and z. Therefore, the exponential surface *descends* from a trace inside the sphere to a trace which lies within the sphere. So the two surfaces have no points of intersection within the interior of the first octant.

Solution 2 by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Such triples are $(x, y, z) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$. We note that the first equation implies that $x, y, z \in [0, 1]$. On the other hand, using Bernoulli's inequality we obtain that

$$\begin{cases} 3^x = (1+2)^x \leq 1 + 2x \\ 3^y = (1+2)^y \leq 1 + 2y \\ 3^z = (1+2)^z \leq 1 + 2, \end{cases}$$

and hence, $5 = 3^x + 3^y + 3^z \leq 3 + 2(x + y + z)$. It follows that $1 \leq x + y + z$. This implies that $x^2 + y^2 + z^2 \leq x + y + z$, and hence, $x(1-x) + y(1-y) + z(1-z) \leq 0$. Since the left hand side of the preceding inequality is nonnegative we obtain that $x(1-x) = y(1-y) = z(1-z) = 0$ from which it follows that x, y, z are either 0 or 1. This combined with the first equation of the system shows that exactly one of x, y, and z is 1 and the other two are 0, and the problem is solved.

Solution 3 by the proposer.

By inspection we see that $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ are solutions of the given system. We claim that they are the only solutions of the system. In fact, for all $t \in [0, 1]$ the function $f(t) = 3^t$ is greater than or equal to the function $g(t) = 2t^2 + 1$, as can be easily proven, for instance, by drawing their graphs when $0 \leq t \leq 1$.

Since $x^2 + y^2 + z^2 = 1$, then $x \in [0, 1], y \in [0, 1]$ and $z \in [0, 1]$. Therefore

$$\begin{aligned} 3^x &\geq 2x^2 + 1, \\ 3^y &\geq 2y^2 + 1, \\ 3^z &\geq 2z^2 + 1. \end{aligned}$$

Adding up the preceding expressions yields

$$3^x + 3^y + 3^z \geq 2(x^2 + y^2 + z^2) + 3 \geq 5$$

and we are done

Also solved by Charles McCracken, Dayton, OH; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy, and Boris Rays, Chesapeake,VA.

- **5037:** *Ovidiu Furdui, Campia Turzii, Cluj, Romania*

Let k, p be natural numbers. Prove that

$$1^k + 3^k + 5^k + \cdots + (2n+1)^k = (1 + 3 + \cdots + (2n+1))^p$$

for all $n \geq 1$ if and only if $k = p = 1$.

Solution 1 by Carl Libis, Kingston, RI.

Since $\Big(1 + 3 + \cdots + (2n+1)\Big)^p = \Big[(n+1)^2\Big]^p = (n+1)^{2p}$, it is clear that $\Big(1 + 3 + \cdots + (2n+1)\Big)^p$ is a monic polynomial of degree $2p$.

Let $S_k^{2n+1} = \sum\limits_{i=1}^{2n+1} i^k$. Then

$$S_k^{2n+1} = \sum_{i=1}^{n+1}(2i-1)^k + \sum_{i=1}^{n}(2i)^k = \sum_{i=0}^{n}(2i+1)^k + 2^k \sum_{i=1}^{n} i^k = \sum_{i=0}^{n}(2i+1)^k + 2^k S_k^n.$$

Then $\sum\limits_{i=0}^{n}(2i+1)^k = S_k^{2n+1} - 2^k S_k^n$. It is well known for sums of powers of integers S_k^n, that the leading term of S_k^n is $\dfrac{n^{k+1}}{k+1}$. Thus the leading term of $1^k + 3^k + 5^k + \cdots + (2n+1)^k$ is

$$\frac{(2n+1)^{k+1}}{k+1} - \frac{2^k n^{k+1}}{k+1} = \frac{2^{k+1}n^{k+1} - 2^k n^{k+1}}{k+1} = \frac{2^k n^{k+1}}{k+1}.$$

This is monic if, and only if, $k = 1$. When $k = 1$ we have that

$$\sum_{i=0}^{n}(2i+1) = S_1^{2n+1} - 2S_1^n = \frac{(2n+1)(2n+2)}{2} - 2\frac{n(n+1)}{2} = (n+1)^2.$$

For k, p natural numbers we have that $1^k + 3^k + 5^k + \cdots + (2n+1)^k = \Big(1 + 3 + \cdots + (2n+1)\Big)^p$ for all $n \geq 1$ if, and only if, $k = p = 1$.

Solution 2 by Kee-Wai Lau, Hong Kong, China.

If $k = p = 1$, the equality $1^k + 3^k + 5^k + \cdots + (2n+1)^k = (1 + 3 + \cdots + (2n+1))^p$ is trivial. Now suppose that the equality holds for all $n \geq 1$. By putting $n = 1, 2$, we obtain $1 + 3^k = 4^p$ and $1 + 3^k + 5^k = 9^p$. Hence

$$\begin{aligned} 3^k &= 4^p - 1 \text{ and} \\ 5^k &= 9^p - 4^p. \end{aligned}$$

Eliminating k from the last two equations, we obtain $9^p = 4^p + (4^p - 1)^{(\ln 5/\ln 3)}$. Hence,

$$\begin{aligned} 9^p &< 2\Big(4^{p(\ln 5/\ln 3)}\Big) \\ p\ln 9 &< \ln 2 + \frac{p(\ln 4)(\ln 5)}{\ln 3}, \text{and} \end{aligned}$$

$$p \quad < \quad \frac{(\ln 2)(\ln 3)}{(\ln 3)(\ln 9) - (\ln 4)(\ln 5)} = 4.16\cdots.$$

Thus $p = 1, 2, 3, 4$. But it is easy to check that only the case $p = 1$ and $k = 1$ admits solutions in the natural numbers for the equation $1 + 3^k = 4^p$, and this completes the solution.

Solution 3 by Paul M. Harms, North Newton, KS.

Clearly if $k = p = 1$, the equation holds for all appropriate integers n. For the *only if* part of the statement consider the contrapostive statement:

If $p \neq 1$ or $k \neq 1$, then for some $n \geq 1$ the equation does not hold.

Consider $n = 1$. Then the equation in the problem is $1^k + 3^k = (1 + 3)^p = 4^p$. If $k = 1$ with $p > 1$, then $4 < 4^p$ so the equation does not hold.

If $k > 1$ with $p = 1$, then $1^k + 3^k > 4$ so the equation does not hold.

Now consider both $p > 1$ and $k > 1$ using the equation in the form
$3^k = 4^p - 1^k = (2^p - 1)(2^p + 1)$.

If $p > 1$, then $2^p - 1 > 1$ and $2^p + 1 > 1$. Also, the expressions $2^p - 1$ and $2^p + 1$ are 2 units apart so that if 3 is a factor of one of these expressions then 3 is not a factor of the other expression. Since both expressions are greater than one, if 3 is a factor of one of the expressions, then the other expression has a prime number other than 3 as a factor. Thus $(2^p - 1)(2^p + 1)$ has a prime number other than 3 as a factor and cannot be equal to 3^k, a product of just the prime number 3. Thus the equation does not hold when both $p > 1$ and $k > 1$.

Solution 4 by John Hawkins and David Stone, Statesboro, GA.

Denote $1^k + 3^k + 5^k + \cdots + (2n + 1)^k = (1 + 3 + \cdots + (2n + 1))^p$ by (#). The condition requesting *all* $n \geq 1$ is overkill. Actually, we can prove the following are equivalent:

(a) condition (#) holds for all $n \geq 1$,

(b) condition (#) holds for all $n = 1$,

(c) $k = p = 1$.

Clearly, (a)$\Rightarrow$ (b).

Also (c) $\Rightarrow$ (a), for if $k = p = 1$, then (#) becomes the identity

$$1 + 3 + 5 + \cdots + (2n + 1) = (1 + 3 + \cdots + (2n + 1)).$$

Finally, we prove that (b) $\Rightarrow$ (c). Assuming the truth of (#) for $n = 1$ tells us that
$3^k = 4^p - 1$.

If $k = 1$, we immediately conclude that $p = 1$ and we are finished.

Arguing by contradiction, suppose $k \geq 2$, so 3^k is actually a multiple of 9. Thus
$4^p \equiv 1 (\text{mod } 9)$. Now consider the powers of 4 modulo 9:

$$4^0 \equiv 1(\text{mod } 9)$$
$$4^1 \equiv 4(\text{mod } 9)$$

$$4^2 \equiv 7 (\text{mod } 9)$$
$$4^3 \equiv 1 (\text{mod } 9)$$

That is, 4 has order 3(mod 9), so $4^p \equiv 1 (\text{mod } 9)$ if and only if p is a multiple of 3. Based upon some numerical testing, we consider 4^p modulo 7: $4^p = 4^{3t} \equiv 64^t \equiv 1^t \equiv 1 (\text{mod } 7)$. That is, 7 divides $4^p - 1$, so $4^p - 1$ cannot be a power of 3. We have reached a contradiction.

Solution 5 by the proposer.

One implication is easy to prove. To prove the other implication we note that

$$1+3+\cdots+(2n+1) = \sum_{k=1}^{n+1}(2k-1) = 2\sum_{k=1}^{n+1} k-(n+1) = (n+1)(n+2)-(n+1) = (n+1)^2.$$

It follows that

$$1^k + 3^k + 5^k + \cdots + (2n+1)^k = (n+1)^{2p}.$$

We multiply the preceding relation by $2/(2n+1)^{k+1}$ and we get that

$$\frac{2}{2n+1}\left(\left(\frac{1}{2n+1}\right)^k + \left(\frac{3}{2n+1}\right)^k + \cdots + \left(\frac{2n+1}{2n+1}\right)^k\right) = 2\frac{(n+1)^{2p}}{(2n+1)^{k+1}}. \qquad (1)$$

Letting $n \to \infty$ in (1) we get that

$$\int_0^1 x^k dx = \frac{1}{k+1} = \lim_{n\to\infty} 2\frac{(n+1)^{2p}}{(2n+1)^{k+1}}.$$

It follows that $2p = k+1$ and that $\frac{1}{k+1} = \frac{1}{2^k}$. However, the equation $k+1 = 2^k$ has a unique positive solution namely $k = 1$. This can be proved by applying Bernouli's inequality as follows

$$2^k = (1+1)^k \geq 1 + k \cdot 1 = k+1,$$

with equality if and only if $k = 1$. Thus, $k = p = 1$ and the problem is solved.

Also solved by Boris Rays, Chesapeake, VA.

Late Solutions

Late solutions were received from **David C. Wilson of Winston-Salem, NC** to problems 5026, 5027, and 5028.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before May 15, 2009

- 5056: *Proposed by Kenneth Korbin, New York, NY.*

  A convex pentagon with integer length sides is inscribed in a circle with diameter $d = 1105$. Find the area of the pentagon if its longest side is 561.

- 5057: *Proposed by David C. Wilson, Winston-Salem, N.C.*

  We know that $1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ where $-1 < x < 1$.

  Find formulas for $\sum_{k=1}^{\infty} kx^k$, $\sum_{k=0}^{\infty} k^2x^k$, $\sum_{k=0}^{\infty} k^3x^k$, $\sum_{k=0}^{\infty} k^4x^k$, and $\sum_{k=0}^{\infty} k^5x^k$.

- 5058: *Proposed by Juan-Bosco Romero Márquez, Avila, Spain.*

  If p, r, a, A are the semi-perimeter, inradius, side, and angle respectively of an acute triangle, show that
  $$r + a \leq p \leq \frac{p}{\sin A} \leq \frac{p}{\tan\dfrac{A}{2}},$$
  with equality holding if, and only if, $A = 90^o$.

- 5059: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Prove that for all triangles ABC
  $$\sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right) \leq \frac{6\sqrt{3}+1}{8}.$$

- 5060: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Show that there exists $c \in (0, \pi/2)$ such that
  $$\int_0^c \sqrt{\sin x}\, dx + c\sqrt{\cos c} = \int_c^{\pi/2} \sqrt{\cos x}\, dx + (\pi/2 - c)\sqrt{\sin c}$$

- 5061: *Michael P. Abramson, NSA, Ft. Meade, MD.*

  Let $a_1, a_2, \ldots, a_n$ be a sequence of positive integers. Prove that

$$\sum_{i_m=1}^{n} \sum_{i_{m-1}=1}^{i_m} \cdots \sum_{i_1=1}^{i_2} a_{i_1} = \sum_{i=1}^{n} \binom{n-i+m-1}{m-1} a_i.$$

Solutions

- 5038: *Proposed by Kenneth Korbin, New York, NY.*

  Given the equations

$$\begin{cases} \sqrt{1+\sqrt{1-x}} \quad - 5 \cdot \sqrt{1-\sqrt{1-x}} = 4 \cdot \sqrt[4]{x} \ \text{ and} \\ 4 \cdot \sqrt{1+\sqrt{1-y}} \quad - 5 \cdot \sqrt{1-\sqrt{1-y}} = \sqrt[4]{y}. \end{cases}$$

Find the positive values of x and y.

Solution by Brian D. Beasley, Clinton, SC.

(a) To find x, we square and simplify to obtain

$$13(1-\sqrt{x}) = 12\sqrt{1-x}.$$

Squaring again and factoring produces

$$(313\sqrt{x}-25)(\sqrt{x}-1) = 0;$$

we note that $x = 1$ fails in the original equation but $x = (25/313)^2 = 625/97969$ works.

(b) To find y, we square and simplify to obtain

$$41(1-\sqrt{y}) = 9\sqrt{1-y}.$$

Squaring again and factoring produces

$$(881\sqrt{y}-800)(\sqrt{y}-1) = 0;$$

we note that $y = 1$ fails in the original equation but $y = (800/881)^2 = 640000/776161$ works.

Addendum. The two given equations, along with the equation in Problem 5024 (see 108(5), May 2008), generalize to

$$a\sqrt{1+\sqrt{1-x}} \ - \ b\sqrt{1-\sqrt{1-x}} = c\sqrt[4]{x},$$

where a, b, and c are positive real numbers with $c = b - a$. Then the solution is

$$x = \left(\frac{2a^2b^2}{a^4+b^4}\right)^2.$$

Also solved by Scott H. Brown, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Grant Evans (student, Saint George's School), Spokane, WA; Bruno Salgueiro

Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Valmir Krasniqi, Prishtinë, Kosova; David E. Manes, Oneonta, NY; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; Armend Sh. Shabani, Republic of Kosova; John Hawkins and David Stone (jointly), Statesboro, GA, and the proposer.

- 5039: *Proposed by Kenneth Korbin, New York, NY.*

  Let d be equal to the product of the first N prime numbers which are congruent to 1(mod4). That is

  $$d = 5 \cdot 13 \cdot 17 \cdot 29 \cdots P_N.$$

  A convex polygon with integer length sides is inscribed in a circle with diameter d. Prove or disprove that the maximum possible number of sides of the polygon is the N^{th} term of the sequence $t = (4, 8, 20, 32, 80, \cdots, t_N, \cdots)$ where $t_N = 4t_{N-2}$ for $N > 3$.

  Examples: If $N = 1$, then $d = 5$, and the maximum polygon has 4 sides $(3, 3, 4, 4)$. If $N = 2$, then $d = 5 \cdot 13 = 65$ and the maximum polygon has 8 sides $(16, 16, 25, 25, 25, 25, 33, 33)$.

  Editor's comment: In correspondence with Ken about this problem he wrote that he has been unable to prove the formula for $N > 5$; so it remains technically a conjecture.

  Another note: No solutions to this problem were received, but Ken Korbin made observations with hope that they will encourage readers to either prove the problem or find a counter-example to it. Following are Ken's observations on this problem.

  In the following examples, let x be the length of the side, and let F be the frequency. Note that $\sqrt{d^2 - x^2}$ is always an integer.

  Examples:

- If $N = 1,\ d = 5$

x	F
3	2
4	2
	4

$$\sum F = 4 = t_1$$

- If $N = 2,\ d = 5 \cdot 13 = 65$

x	F
16	2
25	4
33	2
	8

$$\sum F = 8 = t_2$$

- If $N = 3,\ d = 5 \cdot 13 \cdot 17 = 1105$

x	F
47	2
105	4
169	8
264	6
	20

$$\sum F = 20 = t_3$$

- If $N = 4,\ d = 5 \cdot 13 \cdot 17 \cdot 29 = 32045$

x	F
716	6
1363	2
3045	10
3955	6
4901	8
	32

$$\sum F = 32 = t_4$$

- If $N = 5,\ d = 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37 = 1185665$

x	F	Fx
12415	10	124150
26492	20	529840
49959	24	1199016
50431	12	605172
70200	6	421200
105444	8	843552
	80	3722930

$$\begin{aligned} \sum Fx &= P = Perimeter = 3722930. \\ \sum F &= 80 = t_5. \\ Circumference &= C = 1185665\pi \\ P/C &\approx 0.9994782 \end{aligned}$$

- For each value of N, there is essentially exactly one maximum polygon. For example, if $N = 2$, then the maximum polygon has sides $(16, 16, 25, 25, 25, 25, 33, 33)$ in some order.
- The sides of the maximum polygon have exactly $N + 1$ different values. For example, if $N = 2$, then the three values are $(16, 25, 33)$.
- If $N > 1$, then each side of the maximum polygon is equal to the length of the shorter leg of a Pythagorean triangle with hypotenuse d. For example, if $N = 2$, then the three Pythagorean triples are $(16, 63, 65), (25, 60, 65)$, and $(33, 56, 65)$.
- Formula: $\sum F \cdot \arcsin\left(\frac{x}{d}\right) = 180^0$. For example, if $N = 2,\ d = 65$,

$$\sum F \cdot \arcsin\left(\frac{x}{d}\right) = 2 \cdot \arcsin\left(\frac{16}{65}\right) + 4 \cdot \arcsin\left(\frac{25}{65}\right) + 2 \cdot \arcsin\left(\frac{33}{65}\right) = 180^0.$$

- It will be interesting to see if the above comments hold for $N > 5$.

  The circle from this problem has yielded many interesting results. For example, the number of different inscribed integer-sided rectangles is $\dfrac{3^n - 1}{2}$.

  If the formula for SSM problem 4996 has the '+' changed to a '−', then you have the number of different inscribed integer-sided trapezoids, and the circle for $n = 3$ yields the pentagon for SSM problem 5056 listed above.

- 5040: *Proposed by John Nord, Spokane, WA.*

  Two circles of equal radii overlap to form a lens. Find the distance between the centers if the area in circle A that is not covered by circle B is $\dfrac{1}{3}\left(2\pi + 3\sqrt{3}\right)r^2$.

  Editor' s comment: A mistake was made in stating this problem and as such, there is no solution to it. But Colin Hill, Bruno Salgueriro, and John Hawkins and David Stone caught the mistake and showed how the problem could be altered to yield the solution that John had intended.

  Solution by Bruno Salgueiro Fanego, Viveiron, Spain.

  Two circles A and B of equal radii r overlap to form a lens. Find the distance bewteen the centers if the area of $A \cup B$ that is not covered by the lens $A \cap B$ is $\dfrac{1}{3}\left(2\pi + 3\sqrt{3}\right)r^2$.
  We will show that the distance d between the centers of A and B is r.
  Being A and B of equal radii r, the area in circle A that is not covered by circle B is the same as the area in circle B that is not covered by circle A, so if $C = A \cap B$ is the lens, we have $area(A - C) = area(B - C)$ (by symmetry with respect to C) and $area(A - C) + area(B - C) = area(A \cup B) - area(C) = \dfrac{1}{3}\left(2\pi + 3\sqrt{3}\right)r^2$ (by hypothesis), and hence $area(A - C) = \dfrac{1}{6}\left(2\pi + 3\sqrt{3}\right)r^2$.
  But

$$\begin{aligned}
\frac{1}{6}\left(2\pi + 3\sqrt{3}\right)r^2 + area(C) &= area(A - C) + area(C) \\
&= area(A) = \pi r^2 \Rightarrow \\
area(C) &= \frac{1}{6}\left(4\pi - 3\sqrt{3}\right)r^2.
\end{aligned}$$

  Being the area of a circular segment the difference between the areas of the circular sector and the correspondng triangle, if α is the measure in radians of the central angle of the circular sector with center in the center of A or B we have

$$\begin{aligned}
\frac{1}{2}area(C) = area(sector) &= area(sector) - area(triangle) \\
&= \frac{1}{2}\alpha r^2 - \frac{1}{2}(\sin\alpha)r^2 \\
&= \frac{1}{2}(\alpha - \sin\alpha)r^2,
\end{aligned}$$

  so $area(C) = (\alpha - \sin\alpha)r^2$.

  Hence $\dfrac{1}{6}\left(4\pi - 3\sqrt{3}\right)r^2 = (\alpha - \sin\alpha)r^2$, that is, $\alpha - \sin\alpha = \dfrac{1}{6}\left(4\pi - 3\sqrt{3}\right)$.

The function $f(x) = x = \sin x, 0 < x < 2\pi$, is strictly increasing, so there is a unique solution to the equation $f(\alpha) = \frac{1}{6}(4\pi - 3\sqrt{3})$. It is easy to show that $\alpha = \frac{2\pi}{3}$ satisfies this equation, so this is the measure of the central angle.
The equaltiy $\cos\left(\frac{\alpha}{2}\right) = \frac{d/2}{r}$ implies that $d = 2r\cos\left(\frac{\pi}{3}\right) = r$.

Also solved by Colin Hill of Spokane, WA and by John Hawkins and David Stone of Stateboro, GA.

- 5041: *Proposed by Michael Brozinsky, Central Islip, NY .*

Quadrilateral $ABCD$ (with diagonals $AC = d_1$ and $BD = d_2$ and sides $AB = s_1, BC = s_2, CD = s_3$, and $DA = s_4$) is inscribed in a circle. Show that:

$$d_1^2 + d_2^2 + d_1 d_2 > \frac{s_1^2 + s_2^2 + s_3^2 + s_4^2}{2}.$$

Solution by Kenneth Korbin, New York, NY

Let $d_1 \le d_2$ and let $R = \frac{(S_1 + S_3)^2 + (S_2 + S_4)^2}{(d_1 + d_2)^2}$.
It can be shown that $1 \le R < 2$.
If quadrilateral $ABCD$ is a rectangle, then $d_1 = d_2$ and $R = 1$.
If $\frac{d_1}{d_2} \longrightarrow 0^+$ then $R \longrightarrow 2^-$.
By the Theorem of Ptolemy, $d_1 d_2 = S_1 S_3 + S_2 S_4$.

$$\begin{aligned} 2 &> R \\ &= \frac{(S_1 + S_3)^2 + (S_2 + S_4)^2}{(d_1 + d_2)^2} \\ &= \frac{S_1^2 + S_3^2 + S_2^2 + S_4^2 + 2\Big[S_1 S_3 + S_2 S_4\Big]}{d_1^2 + d_2^2 + 2d_1 d_2} \\ &= \frac{S_1^2 + S_2^2 + S_3^2 + S_4^2 + 2d_1 d_2}{d_1^2 + d_2^2 + 2d_1 d_2} \end{aligned}$$

Therefore,

$$\begin{aligned} 2\Big[d_1^2 + d_2^2 + 2d_1 d_2\Big] &> S_1^2 + S_2^2 + S_3^2 + S_4^2 + 2d_1 d_2 \\ d_1^2 + d_2^2 + d_1 d_2 &> \frac{S_1^2 + S_2^2 + S_3^2 + S_4^2}{2}. \end{aligned}$$

Also solved by the proposer.

- 5042: *Proposed by Miquel Grau-Sánchez and José Luis Díaz-Barrero Barcelona, Spain.*

Let $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ $(a_k \neq 0)$ and $B(z) = z^{n+1} + \sum_{k=0}^{n} b_k z^k$ $(b_k \neq 0)$ be two prime polynomials with roots $z_1, z_2, \ldots, z_n$ and $w_1, w_2, \ldots, w_{n+1}$ respectively. Prove that

$$\frac{A(w_1)A(w_2)\ldots A(w_{n+1})}{B(z_1)B(z_2)\ldots B(z_n)}$$

is an integer and determine its value.

Solution by John Hawkins and David Stone, Statesboro, GA.

In factored form, we have
$A(z) = (z - z_1)(z - z_2)(z - z_3) \cdots (z - z_n)$ and
$B(z) = (z - w_1)(z - w_2)(z - w_3) \cdots (z - w_{n+1})$,
so,
$A(w_i) = (w_i - z_1)(w_i - z_2)(w_i - z_3) \cdots (w_i - z_n) = \prod_{j=1}^{n} \left(w_i - z_j \right)$ for each $i = 1, 2, \cdots, n+1$,
and
$B(z_j) = (z_j - w_1)(z_j - w_2)(z_j - w_3) \cdots (z_j - w_{n+1}) = \prod_{i=1}^{n+1} \left(z_j - w_i \right)$ for each $j = 1, 2, \cdots, n$.
If it were the case that some w_i equals some z_j, then $A(z)$ and $B(z)$ would have a common factor $z - w_i = z - z_j$, contradicting the given condition. Thus each term $w_i - z_j \neq 0$.
Therefore,

$$\frac{A(w_1)A(w_2)A(w_3) \cdots A(w_{n+1})}{B(z_1)B(z_2) \cdots B(z_n)} = \frac{\prod_{j=1}^{n+1} A(w_i)}{\prod_{j=1}^{n} B(z_j)} = \frac{\prod_{i=1}^{n+1} \prod_{j=1}^{n} (w_i - z_j)}{\prod_{j=1}^{n} \prod_{i=1}^{n+1} (z_j - w_i)} = (-1)^{n(n+1)} = 1,$$

by repeated cancellation.

Comment: It is implicit in the problem statement that none of the roots of $A(z)$ are roots of $B(z)$, else we would have division by zero. If the reverse were true, that is if some w_i were a root of $A(z)$, we would have the quotient (Q) equal to zero. This is not bad, but it muddies the water a bit. The co-primality condition eliminates any such concerns: for polynomials over the compex field, co-primality is equivalent to having no common roots. So none of the terms in (Q), numerator or denominator, are zero.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Armend Sh. Shabani, Republic of Kosova, and the proposers.

- 5043: *Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Solve the following diophantine equation in positive integers k, m, and n

$$k \cdot n! \cdot m! + m! + n! = (m + n)!.$$

Solution 1 by R. P. Sealy, Sackville, New Brunswick, Canada.

The only solution is $m = n = 2$, $k = 5$.
Write

$$\begin{aligned} k &= \frac{(m+n)! - m! - n!}{m!n!} \\ &= \frac{(m+n)!}{m!n!} - \frac{m!}{m!n!} - \frac{n!}{m!n!} \\ &= \binom{m+n}{n} - \frac{1}{n!} - \frac{1}{m!}. \end{aligned}$$

Therefore k is an integer if and only if $\frac{1}{n!} + \frac{1}{m!}$ is an integer.
The solution $m = n = 1$ implies $k = 0$.
The only other solution is $m = n = 2$, $k = 5$.

Solution 2 by Paul M. Harms, North Newton, KS.

Suppose $n < m$. Dividing $k \cdot n! \cdot m! + m! + n! = (m+n)!$ by $m!$ gives

$$k(n!) + 1 + \frac{n!}{m!} = \frac{(m+n)!}{m!}.$$

Note that every term is an integer except the term $\frac{n!}{m!}$. This term is a positive fraction less than one. Thus the equation cannot be true.

If $m < n$, the same type of argument shows that the equation cannot be true. If there are any solutions, $n = m$.

Consider $n = m = 1$. Then the equation becomes $k + 2 = 2$ or $k = 0$. Since k must be positive, this is not a solution.

Now consider $n = m = 2$. The equation becomes $4k + 4 = 24$ or $k = 5$. In this case $k = 5, n = m = 2$ represents a solution.

Now consider $n = m > 2$. The equation can be written as $k(n!)^2 + 2(n!) = (2n)!$ or

$$k(n!) + 2 = 2n(2n-1)(2n-2)\cdots(n+1).$$

Divide both sides of the previous equation by 2 to obtain

$$kn(n-1)(n-2)\cdots(3) + 1 = n(2n-1)(2n-2)\cdots(n+1).$$

When $n > 2$ we see that two of the three terms have a factor of the prime number 3,while the third does not have 3 as a factor. Thus the equation cannot be true.

The only solution is $k = 5, n = m = 2$.

Also solved by Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Bruno Salgueiro Fanego, Viveiro, Spain; John Hawkins and David Stone, Statesboro, GA; N. J. Kuenzi, Oshkosh, WI; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; Armend Sh. Shabani, Republic of Kosova, and the proposer.

Late and Misplaced Solutions

From time to time I notice solutions on my computer that seem to have been mailed to me days before their due-dates, but somehow I missed them. So in the future, if I do not acknowledge receipt of your solutions within a week or so after you have mailed them, please send me note inquiring if they have arrived.

Solutions to 5022 and to 5024 were received from **Patrick Farrell of Andover, MA,** and to 5027 from **Pat Costello of Richmond, KY.** Mea culpa, once again.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before June 15, 2009

- 5062: *Proposed by Kenneth Korbin, New York, NY.*

  Find the sides and the angles of convex cyclic quadrilateral ABCD if $\overline{AB} = \overline{BC} = \overline{CD} = \overline{AD} - 2 = \overline{AC} - 2$.

- 5063: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

  Euclid's inscribed polygon is a constructible polygon inscribed in a circle whose consecutive central angle degree measures form a positive integral arithmetic sequence with a non-zero difference.

  a) Does Euclid's inscribed n-gon exist for any prime n greater than 5?

  b) Does Euclid's n-gon exist for all composite numbers n greater than 2?

- 5064: *Proposed by Michael Brozinsky, Central Islip, NY.*

  The Lemoine point of a triangle is that point inside the triangle whose distances to the three sides are proportional to those sides. Find the maximum value that the constant of proportionality, say λ, can attain.

- 5065: *Mihály Bencze, Brasov, Romania.*

  Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers. Prove that

$$1) \qquad \sum_{i,j=1}^{n} |(i-j)(x_i - x_j)| = \frac{n}{2} \sum_{i,j=1}^{n} |x_i - x_j|.$$

$$2) \qquad \sum_{i,j=1}^{n} (i-j)^2 = \frac{n^2(n^2-1)}{6}.$$

- 5066: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b, and c be the sides of an acute-angled triangle ABC. Let $abc = 1$. Let H be the orthocenter, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB respectively. Prove or disprove that

$$3(a+b)(b+c)(c+a) \geq 32(d_a+d_b+d_c)^2.$$

- 5067: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be complex numbers such that $a+b+c=0$. Prove that

$$\max\{|a|,|b|,|c|\} \leq \frac{\sqrt{3}}{2}\sqrt{|a|^2+|b|^2+|c|^2}.$$

Solutions

- 5044: *Proposed by Kenneth Korbin, New York, NY.*

  Let N be a positive integer and let

$$\begin{cases} x = 9N^2+24N+14 \text{ and} \\ y = 9(N+1)^2+24(N+1)+14. \end{cases}$$

Express the value of y in terms of x, and express the value of x in terms of y.

Solution by Armend Sh. Shabani, Republic of Kosova.

One easily verifies that

$$y - x = 18N + 33. \qquad (1)$$

From $9N^2+24N+14-x=0$ one obtains $N_{1,2} = \dfrac{-4 \pm \sqrt{2+x}}{3}$, and since N is a positive integer we have

$$N = \frac{-4+\sqrt{2+x}}{3}. \qquad (2)$$

Substituting (2) into (1) gives:

$$y = x + 9 + 6\sqrt{2+x}. \qquad (3)$$

From $9(N+1)^2+24(N+1)+14-y=0$ one obtains $N_{1,2} = \dfrac{-7 \pm \sqrt{2+y}}{3}$, and since N is a positive integer we have

$$N = \frac{-7+\sqrt{2+y}}{3}. \qquad (4)$$

Substituting (4) into (1) gives:

$$x = y + 9 - 6\sqrt{2+y}. \qquad (5)$$

Relations (3) and (5) are the solutions to the problem.

Comments: **1. Paul M. Harms** mentioned that the equations for x in terms of y, as well as for y in terms of x, are valid for integers satisfying the x, y and N equations in the problem. The minimum x and y values occur when $N=1$ and are $x=47$ and $y=98$. **2. David Stone and John Hawkins** observed that in addition to $(47, 98)$, other integer lattice points on the curve of $y = 9 + x + 6\sqrt{2 + x}$ in the first quadrant are $(4, 98), (98, 167), (167, 254), (254, 359)$, and $(23, 62)$.

Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Bruno Salgueiro Fanego, Viveiro, Spain; Michael C. Faleski, University Center, MI; Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; José Hernández Santiago (student UTM), Oaxaca, México; David Stone and John Hawkins (jointly), Statesboro, GA; David C.Wilson, Winston-Salem, NC, and the proposer.

- **5045:** *Proposed by Kenneth Korbin, New York, NY.*

Given convex cyclic hexagon ABCDEF with sides

$$\begin{aligned} \overline{AB} &= \overline{BC} = 85 \\ \overline{CD} &= \overline{DE} = 104, \text{ and} \\ \overline{EF} &= \overline{FA} = 140. \end{aligned}$$

Find the area of $\triangle BDF$ and the perimeter of $\triangle ACE$.

Solution by Kee-Wai Lau, Hong Kong, China.

We show that the area of $\triangle BDF$ iis 15390 and the perimeter of $\triangle ACE$ is $\dfrac{123120}{221}$.

Let $\angle AFE = 2\alpha, \angle EDC = 2\beta$, and $\angle$CBA$=2\gamma$ so that

$$\angle ACE = \pi - 2\alpha,\ \angle CAE = \pi - 2\beta, \text{ and } \angle AEC = \pi - 2\gamma.$$

Since $\angle ACE + \angle CAE + \angle AEC = \pi$, so

$$\begin{aligned} \alpha + \beta + \gamma &= \pi \\ \cos\alpha + \cos\beta + \cos\gamma &= 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} + 1 \text{ or} \\ (\cos\alpha + \cos\beta + \cos\gamma - 1)^2 &= 2(1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma). \qquad (1) \end{aligned}$$

Denote the radius of the circumcircle by R. Applying the Sine Formula to $\triangle ACE$, we have

$$R = \frac{\overline{AE}}{2\sin 2\alpha} = \frac{\overline{EC}}{2\sin 2\beta} = \frac{\overline{CA}}{2\sin 2\gamma}.$$

By considering triangles AFE, EDC, and CBA respectively, we obtain

$$\overline{AE} = 280\sin\alpha,\ \overline{EC} = 208\sin\beta,\ \overline{CA} = 170\sin\gamma.$$

It follows that $\cos\alpha = \dfrac{70}{R}$, $\cos\beta = \dfrac{52}{R}$, and $\cos\gamma = \dfrac{85}{2R}$. Substituting into (1) and simplifying, we obtain

$$\begin{aligned} 4R^3 - 37641R - 1237600 &= 0 \text{ or} \\ \Big(2R - 221\Big)\Big(2R^2 + 221R + 5600\Big) &= 0. \end{aligned}$$

Hence,

$$R = \frac{221}{2}, \ \cos\alpha = \frac{140}{221}, \ \sin\alpha = \frac{171}{221}$$
$$\cos\beta = \frac{104}{221}, \ \sin\beta = \frac{195}{221}$$
$$\cos\gamma = \frac{85}{221}, \ \sin\gamma = \frac{204}{221},$$

and our result for the perimeter of $\triangle ACE$.

It is easy to check that $\angle BFD = \alpha, \angle FDB = \beta, \angle DBF = \gamma$ so that $\angle BAF = \pi - \beta, \angle DEF = \pi - \gamma$.

Applying the cosine formula to $\triangle BAF$ and $\triangle DEF$ respectively, we obtain $BF = 195$ and $DF = 204$.

It follows, as claimed, that the area of

$$\triangle BDF = \frac{1}{2}\left(\overline{BF}\right)\left(\overline{DF}\right)\sin\angle BFD = \frac{1}{2}(195)(204)\frac{171}{221} = 15390.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5046:** *Proposed by R.M. Welukar of Nashik, India and K.S. Bhanu, and M.N. Deshpande of Nagpur, India.*

  Let $4n$ successive Lucas numbers $L_k, L_{k+1}, \cdots, L_{k+4n-1}$ be arranged in a $2 \times 2n$ matrix as shown below:

$$\begin{matrix} 1 & 2 & 3 & 4 & \cdots & 2n \end{matrix}$$
$$\begin{pmatrix} L_k & L_{k+3} & L_{k+4} & L_{k+7} & \cdots & L_{k+4n-1} \\ \\ L_{k+1} & L_{k+2} & L_{k+5} & L_{k+6} & \cdots & L_{k+4n-2} \end{pmatrix}$$

Show that the sum of the elements of the first and second row denoted by R_1 and R_2 respectively can be expressed as

$$R_1 = 2F_{2n}L_{2n+k}$$

$$R_2 = F_{2n}L_{2n+k+1}$$

where $\{L_n, n \geq 1\}$ denotes the Lucas sequence with $L_1 = 1, L_2 = 3$ and $L_{i+2} = L_i + L_{i+1}$ for $i \geq 1$ and $\{F_n, n \geq 1\}$ denotes the Fibonacci sequence, $F_1 = 1, F_2 = 1, F_{n+2} = F_n + F_{n+1}$.

Solution by Angel Plaza and Sergio Falcon, Las Palmas, Gran Canaria, Spain.

$R_1 = L_k + L_{k+3} + L_{k+4} + L_{k+7} + \cdots + L_{k+4n-2} + L_{k+4n-1}$, and since $L_i = F_{i-1} + F_{i+1}$, we have:

$$\begin{aligned} R_1 &= F_{k-1} + F_{k+1} + F_{k+2} + F_{k+4} + F_{k+3} + F_{k+5} + \cdots + F_{k+4n-2} + F_{k+4n} \\ &= F_{k-1} + \sum_{j=1}^{4n} F_{k+j} - F_{k+4n-1} \\ &= F_{k-1} - F_{k+4n-1} + \sum_{j=0}^{4n+k} F_j - \sum_{j=0}^{k} F_j \end{aligned}$$

And since $\sum_{j=0}^{m} F_j = F_{m+2} - 1$ we have:

$$R_1 = F_{k-1} - F_{k+4n-1} + F_{k+4n+2} - 1 - F_{k+2} + 1 = 2F_{k+4n} - 2F_k$$

where in the last equation it has been used that $F_{i+2} - F_i = F_{i+1} + F_i - F_{i-1} = 2F_i$. Now, using the relation $L_n F_m = F_{n+m} - (-1)^m F_{n-m}$ (S. Vajda, Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Dover Press (2008)) in the form $L_{2n+k} F_{2n} = F_{4n+k} - (-1)^{2n} F_{2n+k-2n}$ it is deduced $R_1 = 2F_{2n} L_{2n+k}$.
In order to prove the fomula for R_2 note that

$$R_1 + R_2 = \sum_{j=0}^{4n-1} L_{k+j} = \sum_{j=0}^{4n+k-1} L_j - \sum_{j=0}^{k-1} L_j$$

As before, $\sum_{j=0}^{4n+k-1} L_j = F_{k+4n} + F_{k+4n+2}$, while $\sum_{j=0}^{k-1} L_j = F_k + F_{k+2}$, so

$$\begin{aligned} R_1 + R_2 &= F_{k+4n} - F_k + F_{k+4n+2} - F_{k+2} \\ &= L_{2n+k} F_{2n} + L_{2n+k+2} F_{2n} \end{aligned}$$

And therefore,

$$R_2 = F_{2n} \left(L_{2n+k+2} - L_{2n+k} \right) = F_{2n} L_{2n+k+1}$$

Also solved by Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA, and the proposers.)

- 5047: *Proposed by David C. Wilson, Winston-Salem, N.C.*

  Find a procedure for continuing the following pattern:

$$\begin{aligned} S(n,0) &= \sum_{k=0}^{n} \binom{n}{k} = 2^n \\ S(n,1) &= \sum_{k=0}^{n} \binom{n}{k} k = 2^{n-1} n \\ S(n,2) &= \sum_{k=0}^{n} \binom{n}{k} k^2 = 2^{n-2} n(n+1) \end{aligned}$$

$$S(n,3) = \sum_{k=0}^{n}\binom{n}{k}k^3 = 2^{n-3}n^2(n+3)$$
$$\vdots$$

Solution by David E. Manes, Oneonta, NY.

Let $f(x) = (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$. For $m \geq 0$,

$S(n,m) = \left(x\dfrac{d}{dx}\right)^m (f(x))\Big|_{x=1}$, where $\left(x\dfrac{d}{dx}\right)^m$ is the procedure $x\dfrac{d}{dx}$ iterated m times and then evaluating the resulting function at $x=1$. For example,

$$\begin{aligned}
S(n,0) = f(1) &= \sum_{k=0}^{n}\binom{n}{k} = 2^n. \text{ Then}\\
x\frac{d}{dx}(f(x)) &= x\frac{d}{dx}(1+x)^n = x\frac{d}{dx}\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right) \text{ implies}\\
nx(1+x)^{n-1} &= \sum_{k=0}^{n}\binom{n}{k}k\cdot x^k. \text{ If } x=1, \text{then}\\
\sum_{k=0}^{n}\binom{n}{k}k &= n\cdot 2^{n-1} = S(n,1).
\end{aligned}$$

For the value of $S(n,2)$ note that if

$$\begin{aligned}
x\frac{d}{dx}\left[nx(1+x)^{n-1}\right] &= x\frac{d}{dx}\left[\sum_{k=0}^{n}\binom{n}{k}kx^k\right], \text{ then}\\
nx(nx+1)(1+x)^{n-2} &= \sum_{k=0}^{n}\binom{n}{k}k^2x^k. \text{ If } x=1, \text{ then}\\
n(n+1)2^{n-2} &= \sum_{k=0}^{n}\binom{n}{k}k^2 = S(n,2)
\end{aligned}$$

Similarly,

$$\begin{aligned}
S(n,3) &= \sum_{k=0}^{n}\binom{n}{k}k^3 = 2^{n-3}\cdot n^2(n+3) \text{ and}\\
S(n,4) &= \sum_{k=0}^{n}\binom{n}{k}k^4 = 2^{n-4}\cdot n(n+1)(n^2+5n-2.)
\end{aligned}$$

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro GA, and the proposer.

- 5048: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy.*

  Let $a, b, c,$ be positive real numbers. Prove that

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \geq \frac{54}{(a+b+c)^2}\frac{(abc)^3}{\sqrt{(ab)^4+(bc)^4+(ca)^4}}.$$

Solution1 by Boris Rays, Chesapeake, VA.

Rewrite the inequality into the form:

$$\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2}\cdot\Big(a+b+c\Big)^2\cdot\sqrt{(ab)^4+(bc)^4+(ca)^4} \geq 54(abc)^3 \qquad (1)$$

We will use the Arithmetic-Geometric Mean Inequality (e.g., $x+y+z \geq 3\sqrt[3]{xyz}$ and $u+v \geq 2\sqrt{uv}$) for each of the three factors on the left side of (1).

$$\begin{aligned}
\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} &\geq \sqrt{3\sqrt[3]{c^2(a^2+b^2)^2\cdot b^2(c^2+a^2)^2\cdot a^2(b^2+c^2)^2}}\\
&\geq \sqrt{3\sqrt[3]{(abc)^2(a^2+b^2)^2(c^2+a^2)^2(b^2+c^2)^2}}\\
&\geq \sqrt{3\sqrt[3]{(abc)^2(4a^2b^2)(4c^2a^2)(4b^2c^2)}}\\
&= \sqrt{3(abc)^{2/3}\sqrt[3]{4^3a^4b^4c^4}}\\
&= \sqrt{3(abc)^{2/3}4(abc)^{4/3}}\\
&= \sqrt{3\cdot 2^2(abc)^2}\\
&= 2\sqrt{3}(abc) \qquad (2)
\end{aligned}$$

Also, since $(a+b+c) \geq 3\sqrt[3]{abc}$, we have

$$(a+b+c)^2 \geq 3^2\left(\sqrt[3]{abc}\right)^2 = 3^2(abc)^{2/3} \qquad (3)$$

$$\begin{aligned}
\sqrt{(ab)^4+(bc)^4+(ca)^4} &\geq \sqrt{3\sqrt[3]{(ab)^4(bc)^4(ca)^4}}\\
&= \sqrt{3\sqrt[3]{a^8b^8c^8}}\\
&= \sqrt{3(abc)^{8/3}}\\
&= \sqrt{3}(abc)^{4/3} \qquad (4)
\end{aligned}$$

Combining (2), (3), and (4) we obtain:

$$
\begin{aligned}
\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \quad &\cdot \quad \left(a+b+c\right)^2 \cdot \sqrt{(ab)^4+(bc)^4+(ca)^4} \\
&\geq \; 2\sqrt{3}(abc)\cdot 3^2(abc)^{2/3}\sqrt{3}(abc)^{4/3} \\
&= \; 2\cdot 3^3(abc)^{1+2/3+4/3} \\
&= \; 54(abc)^3.
\end{aligned}
$$

Hence, we have shown that (1) is true, with equality holding if $a=b=c$.

Solution 2 by José Luis Díaz-Barrero, Barcelona, Spain.

The inequality claimed is equivalent to

$$
\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2}\,\sqrt{(ab)^4+(bc)^4+(ca)^4} \geq \frac{54(abc)^3}{(a+b+c)^2}
$$

Applying Cauchy's inequality to the vectors $\vec{u}=(c(a^2+b^2),b(c^2+a^2),a(b^2+c^2))$ and $\vec{v}=(a^2b^2,c^2a^2,b^2c^2)$ yields

$$
\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2}\,\sqrt{(ab)^4+(bc)^4+(ca)^4}
$$

$$
\geq abc(ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2))
$$

So, it will be suffice to prove that

$$
(ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2))(a+b+c)^2 \geq 54a^2b^2c^2 \tag{1}
$$

Taking into account GM-AM-QM inequalities, we have

$$
ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2) \geq 2(a^2b^2+b^2c^2+c^2a^2) \geq 6abc\sqrt[3]{abc}
$$

and

$$
(a+b+c)^2 \geq 9\sqrt[3]{a^2b^2c^2}
$$

Multiplying up the preceding inequalities (1) follows and the proof is complete

Solution 3 by Kee-Wai Lau, Hong Kong, China.

By homogeneity, we may assume without loss of generality that $abc=1$. We have

$$
\begin{aligned}
&\sqrt{c^2(a^2+b^2)^2+b^2(c^2+a^2)^2+a^2(b^2+c^2)^2} \\
&= \sqrt{\left(\frac{a^2+b^2}{ab}\right)^2+\left(\frac{c^2+a^2}{ca}\right)^2+\left(\frac{b^2+c^2}{bc}\right)^2} \\
&= \sqrt{\left(\frac{a^2-b^2}{ab}\right)^2+\left(\frac{c^2-a^2}{ca}\right)^2+\left(\frac{b^2-c^2}{bc}\right)^2+12}
\end{aligned}
$$

$$\geq \quad 2\sqrt{3}.$$

By the arithmetic-geometric mean inequality, we have $(a+b=c)^2 \geq 9(abc)^{2/3} = 9$ and $\sqrt{(ab)^4+(bc)^4+(ca)^4} \geq \sqrt{3}(abc)^{4/3} = \sqrt{3}$. The inequality of the problem now follows immediately.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ovidiu Furdui, Campia Turzii, Cluj, Romania; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Armend Sh. Shabani, Republic of Kosova, and the proposer.

5049: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Find a function $f : \Re \to \Re$ such that

$$2f(x) + f(-x) = \begin{cases} -x^3 - 3, & x \leq 1, \\ 3 - 7x^3, & x > 1. \end{cases}$$

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX .

If $x > 1$, then

$$2f(x) + f(-x) = 3 - 7x^3. \qquad (1)$$

Also, since $-x < -1$, we have

$$2f(-x) + f(x) = -(-x)^3 - 3 = x^3 - 3. \qquad (2)$$

By (1) and (2), $f(x) = 3 - 5x^3$ and $f(-x) = -3 + 3x^3$ when $x > 1$. Further, $f(-x) = -3 + 3x^3$ when $x > 1$ implies that $f(x) = -3 + 3(-x)^3 = -3 - 3x^3$ when $x < -1$.

Finally, when $-1 \leq x \leq 1$, we get $-1 \leq -x \leq 1$ also, and hence,

$$\begin{aligned} 2f(x) + f(-x) &= -x^3 - 3, & (3) \\ 2f(-x) + f(x) &= -(-x)^3 - 3 = x^3 - 3. & (4) \end{aligned}$$

As above, (3) and (4) imply that $f(x) = -x^3 - 1$ when $-1 \leq x \leq 1$.

Therefore, $f(x)$ must be of the form

$$f(x) = \begin{cases} -3 - 3x^3 & \text{if} \quad x < -1, \\ -1 - x^3 & \text{if} \quad -1 \leq x \leq 1, \\ 3 - 5x^3 & \text{if} \quad x > 1. \end{cases} \qquad (5)$$

With some perseverance, this can be condensed to

$$f(x) = \left|x^3 + 1\right| - 2\left|x^3 - 1\right| - 4x^3$$

for all $x \in \Re$. Since it is straightforward to check that this function satisfies the given conditions of the problem, this completes the solution.

Also solved by Brian D. Beasely, Clinton, SC; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David C. Wilson, Winston-Salem, NC, and the proposer.

Late Solutions

Late solutions were received from **Pat Costello of Richmond, KY** to problem 5027; **Patrick Farrell of Andover, MA** to 5022 and 5024, and from **David C. Wilson of Winston-Salem, NC** to 5038.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent by e-mail to eisenbt@013.net. Solutions to previously stated problems can be seen at http://ssmj.tamu.edu

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Solutions to the problems stated in this issue should be posted before October 15, 2009

- 5068: *Proposed by Kenneth Korbin, New York, NY*

  Find the value of
  $$\sqrt{1+2009\sqrt{1+2010\sqrt{1+2011\sqrt{1+\cdots}}}}.$$

- 5069: *Proposed by Kenneth Korbin, New York, NY*

  Four circles having radii $\frac{1}{14}$, $\frac{1}{15}$, $\frac{1}{x}$ and $\frac{1}{y}$ respectively, are placed so that each of the circles is tangent to the other three circles. Find positive integers x and y with $15 < x < y < 300$.

- 5070: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

  Find all real solutions to the system
  $$\left.\begin{array}{c} 9(x_1^2+x_2^2-x_3^2) = 6x_3-1, \\ 9(x_2^2+x_3^2-x_4^2) = 6x_4-1, \\ \cdots\cdots\cdot \\ 9(x_n^2+x_1^2-x_2^2) = 6x_2-1. \end{array}\right\}$$

- 5071: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let h_a, h_b, h_c be the altitudes of $\triangle ABC$ with semi-perimeter s, in-radius r and circum-radius R, respectively. Prove that
  $$\frac{1}{4}\left(\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c}\right) \le \frac{R^2}{r}\left(\sin^2 A+\sin^2 B+\sin^2 C\right).$$

- 5072: *Proposed by Panagiote Ligouras, Alberobello, Italy*

  Let a, b and c be the sides, l_a, l_b, l_c the bisectors, m_a, m_b, m_c the medians, and h_a, h_b, h_c the heights of $\triangle ABC$. Prove or disprove that

  a) $$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c} \ge \frac{4}{3}\left(m_a\cdot l_a+l_b\cdot h_b+h_c\cdot m_c\right)$$

b) $3\sum_{cyc}\frac{(-a+b+c)^3}{a} \geq 2\sum_{cyc}[m_a(l_a+h_a)]$.

- 5073: *Proposed by Ovidiu Furdui, Campia-Turzii, Cluj, Romania*

  Let $m > -1$ be a real number. Evaluate

  $$\int_0^1 \{\ln x\} x^m dx,$$

  where $\{a\} = a - [a]$ denotes the fractional part of a.

Solutions

- 5050: *Proposed by Kenneth Korbin, New York, NY*

  Given $\triangle ABC$ with integer-length sides, and with $\angle A = 120^o$, and with $(a, b, c) = 1$.
  Find the lengths of b and c if side $a = 19$, and if $a = 19^2$, and if $a = 19^4$.

Solution 1 by Paul M. Harms, North Newton, KS

Using the law of cosines we have $a^2 = b^2 + c^2 - 2bc\cos 120^o = b^2 + c^2 + bc$.

When $a = 19$ we have $19^2 = 361 = b^2 + c^2 + bc$. The result $b = 5, c = 16$ with $a = 19$ satisfies the problem.

Some books indicate that the Diophantine equation $a^2 = b^2 + c^2 + bc$ has solutions of the form

$$b = u^2 - v^2,\ c = 2uv + v^2,\ \text{and}\ a = u^2 + v^2 + uv.$$

For the above $u = 3, v = 2$ and $a = 19 = 3^2 + 2^2 + 2(3)$.

Let $a_1^2 = b_1^2 + c_1^2 + b_1c_1$ be another Diophantine equation which has solutions of the form $b_1 = u_1^2 - v_1^2$, $c_1 = 2u_1v_1 + v^2$, and $a_1 = u_1 + v_1^2 + u_1v_1$. Let u_1 be the largest and v_1 be the smallest of the numbers $\{b, c\}$. If $b = c$, the Diophantine equation becomes $a_1^2 = 3b_1^2$ which has no integer solutions. Suppose $c > b$. (If $b > c$, a procedure similar to that below can be used).

Let $u_1 = c$ and $v_1 = b$. Then $b_1 = c^2 - b^2$ and $c_1 = 2cb + b^2$. The expression $b_1^2 + c_1^2 + b_1c_1 = (c^2 - b^2)^2 + (2cb + b^2)^2 + (c^2 - b^2)(2cb + b^2) = (c^2 + b^2 + bc)^2 = (a^2)^2 = a^4 = a_1^2$. In this case $a_1 = a^2$.

Now start with the above solution where $a = 19, u = 3, v = 2, b = 5$, and $c = 16$. For $a = 19^2$, let $u = 16$ and $v = 5$. Then we have the solution $b = 231^2, c = 185$ where $\mathrm{a}^2 = 19^4 = 231 + 185^2 + 231(185)$.

For $a = 19^4$, let $u = 231$ and $v = 185$. Then $b = 19136, c = 119695$ and $a^2 = 19^8 = 19136^2 + 119695^2 + 19136(119695)$. Since 19 is not a factor of the b and c solutions above, $(a, b, c) = 1$.

The solutions I have found are $(19, 5, 16)$, $(19^2, 231, 185)$, and $(19^4, 19136, 119695)$.

Solution 2 by Bruno Salguerio Fanego, Viveiro, Spain

If $\triangle ABC$ is such a triangle, by the cosine theorem $a^2 = b^2 + c^2 - 2bc\cos A$, that is

$$c^2 + bc + b^2 - a^2 = 0,\ c = \frac{-b \pm \sqrt{4a^2 - 3b^2}}{2}\ \text{and}\ 4a^2 - 3b^2$$

must be positive integers and the latter a perfect square, with $(a, b, c) = 1$.

When $a = 19,\ 0 < b \le 2 \cdot 19/\sqrt{3}\ \Rightarrow\ 0 < b \le 21;\ \ 4 \cdot 19^2 - 3b^2$ is a positive perfect square for $b \in \{2^4, 5\}$ so $c \in \{5, 2^4\}$, and $(a, b, c) = 1$.

When $a = 19^2,\ 0 < b \le 2 \cdot 19^2/\sqrt{3}\ \Rightarrow\ 0 < b \le 416;\ 4 \cdot 19^4 - 3b^2$ is a positive perfect square that is not a multiple of 19 for $b \in \{3 \cdot 7 \cdot 11,\ 5 \cdot 37\}$, so $c \in \{5 \cdot 37,\ 3 \cdot 7 \cdot 11\}$, and $(a.b.c) = 1$.

When $a = 19^4,\ 0 < b \le 2 \cdot 19^4/\sqrt{3}\ \Rightarrow\ 0 < b \le 150481;\ 4 \cdot 19^8 - 3b^2$ is a positive perfect square that is not a multiple of 19 for $b \in \{5 \cdot 37 \cdot 647,\ 2^6 \cdot 13 \cdot 23\}$. So $c \in \{2^6 \cdot 13 \cdot 23,\ 5 \cdot 37 \cdot 647\}$, and $(a, b, c) = 1$.

And reciprocally, the triangular inequalities are verified by $a = 19, 16, 5$, by $a = 19^2, 231, 185$, and by $a = 19^4,\ 119695, 19136$, so there is a $\triangle ABC$ with sides $a,\ b$ and c with these integer lengths, and with $\angle A = 120^o$, and $(a, b, c) = 1$.

Thus, if $a = 19$, then $\{b, c\} = \{5, 16\}$; if $a = 19^2$, then $\{b, c\} = \{185, 231\}$, and if $a = 19^4$, then $\{b, c\} = \{19136, 119695\}$.

Note: When $a = 19^2, 4 \cdot 19^4 - 3b^2$ is a perfect square for $b \in \{2^4 \cdot 19,\ \ 3 \cdot 7 \cdot 11,\ \ 5 \cdot 37,\ \ 5 \cdot 19\}$.

When $a = 19^4, 4 \cdot 19^8 - 3b^2$ is a perfect square for
$b \in \{5 \cdot 37 \cdot 647,\ \ 2^4 \cdot 19^3,\ \ 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 19,\ \ 3 \cdot 7 \cdot 11 \cdot 19^2,\ \ 5 \cdot 19^2 \cdot 37,\ \ 17 \cdot 19 \cdot 163,\ \ 5 \cdot 19^3,\ \ 2^6 \cdot 13 \cdot 23\}$.

Also solved by John Hawkins and David Stone (jointly), Statesboro, GA; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David C.Wilson, Winston-Salem, NC, and the proposer.

- 5051: *Proposed by Kenneth Korbin, New York, NY*

Find four pairs of positive integers (x, y) such that $\dfrac{(x-y)^2}{x+y} = 8$ with $x < y$.

Find a formula for obtaining additional pairs of these integers.

Solution 1 by Charles McCracken, Dayton, OH

The given equation can be solved for y in term of x by expanding the numerator and multiplying by the denominator to get

$$x^2 - 2xy + y^2 = 8((x+y) \Longrightarrow y^2 - (2x+8)y + (x^2 - 8x) = 0.$$

Solving this by the quadratic formula yields $y = x + 4 + 4\sqrt{x+1}$.

Since the problem calls for integers we choose values of x that will make $x + 1$ a square. Specifically

$$\begin{aligned} x &= 3, 8, 15, 24, 35, \cdots \text{or} \\ x &= k^2 + 2k,\ k \ge 1 \end{aligned}$$

The first four pairs are $(3, 15),\ (8, 24),\ (15, 35),\ (24, 48)$.

In general, $x = k^2 + 2k$ and $y = k^2 + 6k + 8,\ k \ge 1$.

Solution 2 by Armend Sh. Shabani, Republic of Kosova

The pairs are $(3, 15), (8, 24), (15, 35), (24, 48)$. In order to find a formula for additional pairs we write the given relation $(y-x)^2 = 8(x+y)$ in its equivalent form $y - x = 2\sqrt{2(x+y)}$.

From this it is clear that $x+y$ should be of the form $2s^2$, and this gives the system of equations:

$$\begin{cases} x+y=2s^2 \\ y-x=4s \end{cases}$$

The solutions to this system are $x=s^2-2s,\ y=s^2+2s$, and since the solutions should be positive, we choose $s\geq 3$.

Solution 3 by Boris Rays, Brooklyn, NY

Let

$$\begin{cases} x+y=a \\ y-x=b \end{cases}$$

Since $x<y$ and a and b are positive integers, it follows that $b^2=8a$ and that b=$2\sqrt{2a}$. Since b is a positive integer we may choose values of a so that $2a$ is a perfect square. Specifically, let $a=2^{2n-1}$, where $n=1,2,3,\cdots$. Therefore, $2a=2\cdot 2^{2n-1}=2^{2n}=(2^n)^2$, where $n=1,2,3,\cdots$. Similarly, $b=2^{n+1}\ n=1,2,3,\cdots$.
Substituting these values of a and of b into the original system gives:

$$\begin{aligned} x &= \frac{2^{2n-1}-2^{n+1}}{2}=2^n(2^{n-2}-1) \\ y &= \frac{2^{2n-1}+2^{n+1}}{2}=2^n(2^{n-2}+1) \end{aligned}$$

and since we want $x,y>0$ we choose $n=3,4,5,\cdots$. The ordered triplets

$$(n,x,y):(3,8,24),(4,48,80),(5,224,288),(6,960,1088).$$

satisfy the problem. It can also be easily shown that our general values of x and y also satisfy the original equation.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Pat Costello, Richmond, KY; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Jahangeer Kholdi (with John Viands and Tyler Winn (students),Western Branch High School, Chesapeake, VA), Portsmouth, VA; Tuan Le (student, Fairmont, High School), Anaheim, CA; David E. Manes, Oneonta, NY; Melfried Olson, Honolulu, HI; Jaquan Outlaw (student, Heritage High School) Newport News, VA and Robert H. Anderson (jointly), Chesapeake, VA; Boris Rays, Brooklyn, NY; Vicki Schell, Pensacola, FL; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5052: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain*

  If $a\geq 0$, evaluate:

$$\int_0^{+\infty} arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2}\ \frac{dx}{1+x^2}.$$

Solution by Kee-Wai Lau, Hong Kong, China

Denote the integral by I. We show that

$$I = \begin{cases} \frac{\pi}{4} arctg \frac{2a}{1-a^2}, & 0 \leq a < 1; \\ \\ \frac{\pi^2}{8}, & a = 1; \\ \\ \frac{\pi}{4}\Big(\pi - arctg\frac{2a}{a^2-1} - 4arctg\frac{\sqrt{a^4+a^2-1}-a}{1+a^2}\Big), & a > 1. \end{cases} \quad (1)$$

Let $J = \int_0^{+\infty} \frac{2a(ax^2+2x+a)arctg(x)}{(1+x^2)\Big((a^2+1)x^2+4ax+a^2+1\Big)}dx$. Integrating by parts, we see that for $0 \leq a < 1$,

$$\begin{aligned} I &= \int_0^{+\infty} arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} d(arctg(x)) \\ &= \Big[arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} arctg(x)\Big]_0^{+\infty} \\ &- \int_0^{+\infty} arctg(x) d\Big(arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2}\Big) \\ &= J. \end{aligned}$$

For $a \geq 1$, let $r_a = \frac{\sqrt{a^4+a^2-1}-a}{1+a^2}$ be the non-negative root of the quadratic equation $(1+a^2)x^2+2ax+1-a^2=0$ so that

$$\begin{aligned} I &= \Big[arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} arctg(x)\Big]_0^{r_a} \\ &+ \Big[arctg\frac{2a(1+ax)}{x^2(1+a^2)+2ax+1-a^2} arctg(x)\Big]_{r_a}^{+\infty} + J \\ &= -\pi arctg(r_a) + J. \end{aligned}$$

By substituting $x = \frac{1}{y}$ and making use of the fact that $arctg(1/y) = \frac{\pi}{2} - arctg(y)$ we obtain

$$J = 2a \int_0^{+\infty} \frac{(ay^2+2y+a)arctg(1/y)}{(1+y^2)\Big((a^2+1)y^2+4ay+a^2+1\Big)}dy$$

$$= \quad 2a\Big(\frac{\pi}{2}\int_0^{+\infty}\frac{(ay^2+2y+a)}{(1+y^2)\Big((a^2+1)y^2+4ay+a^2+1\Big)}dy\Big) - J$$

so that $J = \dfrac{\pi a}{2}\displaystyle\int_0^{+\infty}\frac{(ay^2+2y+a)}{(1+y^2)\Big((a^2+1)y^2+4ay+a^2+1\Big)}dy$. Resolving into partial fractions we obtain

$$J = \frac{\pi}{4}\Big(\int_0^{+\infty}\frac{dy}{1+y^2} + (a^2-1)\int_0^{+\infty}\frac{dy}{(1+a^2)y^2+4ay+1+a^2}\Big).$$

Clearly, $J = \dfrac{\pi^2}{8}$ for $a = 1$. For $p > 0,\ pr > q^2$, we have the well know result

$$\int_0^{+\infty}\frac{dy}{py^2+2qy+r} = \frac{1}{\sqrt{pr-q^2}}arctg\frac{q}{\sqrt{pr-q^2}},$$

so that for $a \geq 0,\ a \neq 1$

$$J = \frac{\pi}{4}\Big(\frac{\pi}{2} + \frac{a^2-1}{|a^2-1|}arctg\frac{2a}{|a^2-1|}\Big).$$

Hence (1) follows and this completes the solution.

Also solved by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy, and the proposer.

- 5053: *Proposed by Panagiote Ligouras, Alberobello, Italy*

Let a, b and c be the sides, r the in-radius, and R the circum-radius of $\triangle ABC$. Prove or disprove that

$$\frac{(a+b-c)(b+c-a)(c+a-b)}{a+b+c} \leq 2rR.$$

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Roger Zarnowski (jointly), San Angelo, TX

The given inequality is essentially the same as Padoa's Inequality which states that

$$abc \geq (a+b-c)(b+c-a)(c+a-b),$$

with equality if and only if $a = b = c$. We will prove this using the approach presented in [1]. Let $x = \dfrac{a+b-c}{2}$, $y = \dfrac{b+c-a}{2}$, and $z = \dfrac{c+a-b}{2}$. Then, $x, y, z > 0$ by the Triangle Inequality and $a = x+z$, $b = x+y$, $c = y+z$. By the Arithmetic - Geometric Mean Inequality,

$$\begin{aligned} abc &= (x+z)(x+y)(y+z) \\ &\geq (2\sqrt{xz})(2\sqrt{xy})(2\sqrt{yz}) \\ &= (2x)(2y)(2z) \\ &= (a+b-c)(b+c-a)(c+a-b), \end{aligned}$$

with equality if and only if $x = y = z$, i.e., if and only if $a = b = c$.

If $A = Area(\triangle ABC)$ and $s = \dfrac{a+b+c}{2}$, then

$$R = \frac{abc}{4A} \quad \text{and} \quad A = rs = r\left(\frac{a+b+c}{2}\right),$$

which imply that $2rR = \dfrac{abc}{a+b+c}$. Hence, the problem reduces to Padoa's Inequality.

Reference:

[1] R. B. Nelsen, Proof Without Words: Padoa's Inequality, **Mathematics Magazine** 79 (2006) 53.

Also solved by Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Tuan Le (student, Fairmont High School), Anaheim, CA; David E. Manes, Oneonta, NY; Manh Dung Nguyen (student, Special High School for Gifted Students), HUS, Vietnam; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.)

- 5054: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let x, y, z be positive numbers such that $xyz = 1$. Prove that

$$\frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2} \geq 1.$$

Solution 1 by Ovidiu Furdui, Campia Turzii, Cluj, Romania

First we note that if a and b are two positive numbers then the following inequality holds

$$\frac{a^2-ab+b^2}{a^2+ab+b^2} \geq \frac{1}{3} \qquad (1).$$

Let

$$S = \frac{x^3}{x^2+xy+y^2} + \frac{y^3}{y^2+yz+z^2} + \frac{z^3}{z^2+zx+x^2}.$$

We have,

$$\begin{aligned} S &= \frac{x^3-y^3+y^3}{x^2+xy+y^2} + \frac{y^3-z^3+z^3}{y^2+yz+z^2} + \frac{z^3-x^3+x^3}{z^2+zx+x^2} \\ &= (x-y) + \frac{y^3}{x^2+xy+y^2} + (y-z) + \frac{z^3}{y^2+yz+z^2} + (z-x) + \frac{x^3}{z^2+zx+x^2} \\ &= \frac{y^3}{x^2+xy+y^2} + \frac{z^3}{y^2+yz+z^2} + \frac{x^3}{z^2+zx+x^2}. \end{aligned}$$

It follows, based on (1), that

$$\begin{aligned} S &= \frac{1}{2}(S+S) \\ &= \frac{1}{2}\left(\frac{x^3+y^3}{x^2+xy+y^2} + \frac{y^3+z^3}{y^2+yz+z^2} + \frac{z^3+x^3}{z^2+zx+x^2}\right) \\ &= \frac{1}{2}\left((x+y)\frac{x^2-xy+y^2}{x^2+xy+y^2} + (y+z)\frac{y^2-yz+z^2}{y^2+yz+z^2} + (z+x)\frac{z^2-xz+x^2}{z^2+zx+x^2}\right) \end{aligned}$$

$$\geq \frac{1}{2}\left(\frac{x+y}{3}+\frac{y+z}{3}+\frac{z+x}{3}\right)$$
$$= \frac{x+y+z}{3} \geq \sqrt[3]{xyz}=1, \text{ and the problem is solved.}$$

Solution 2 by Manh Dung Nguyen (student, Special High School for Gifted Students) HUS, Vietnam

Firstly, we have,

$$\sum \frac{x^3-y^3}{(x^2+xy+y^2)} = \sum \frac{(x-y)(x^2+xy+y^2)}{(x^2+xy+y^2)} = \sum (x-y) = 0.$$

Hence,

$$\sum \frac{x^3}{x^2+xy+y^2} = \sum \frac{y^3}{x^2+xy+y^2}.$$

So it suffices to show that,

$$\sum \frac{x^3+y^3}{x^2+xy+y^2} \geq 2.$$

On the other hand,

$$3(x^2-xy+y^2)-(x^2+xy+y^2)=2(x-y)^2 \geq 0.$$

Thus,

$$\sum \frac{x^3+y^3}{x^2+xy+y^2} = \sum \frac{(x+y)(x^2-xy+y^2)}{x^2+xy+y^2} = \sum \frac{x+y}{3} = \frac{2(x+y+z)}{3}.$$

By the AM-GM Inequality, we have,

$$x+y+z \geq 3\sqrt[3]{xyz}=3,$$

so we are done.
Equality hold if and only if $x=y=z=1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

It can be checked readily that,

$$\frac{x^3}{x^2+xy+y^2} = \frac{(2x-y)}{3} + \frac{(x+y)(x-y)^2}{3(x^2+xy+y^2)} \geq \frac{(2x-y)}{3}.$$

Similarly, $\dfrac{y^3}{y^2+yz+z^2} \geq \dfrac{(2y-z)}{3}$, $\dfrac{z^3}{z^2+zx+x^2} \geq \dfrac{(2z-x)}{3}$.

Hence by the arithmetic mean-geometric mean inequality, we have:

$$\frac{x^3}{x^2+xy+y^2}+\frac{y^3}{y^2+yz+z^2}+\frac{z^3}{z^2+zx+x^2}$$

$$\geq \frac{x+y+z}{3}$$

$$\geq \sqrt[3]{xyz}$$

$$= 1.$$

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly), San Angelo, TX; Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Tuan Le (student, Fairmont High School), Anaheim, CA; Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Boris Rays, Brooklyn, NY; Armend Sh. Shabani, Republic of Kosova, and the proposer.

- 5055: *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania*

Let α be a positive real number. Find the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}}.$$

Solution 1 by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy

Answer:

$$\text{The limit is} \begin{cases} 0, & \text{if } \alpha > 1; \\ 1, & \text{if } 0 < \alpha < 1; \\ \ln 2, & \text{if } \alpha = 1. \end{cases}$$

Proof: Let $\alpha > 1$.

Writing $k^{\alpha} = \sum_{i=1}^{N} \frac{k^{\alpha}}{N}$, by the AGM we have

$$\frac{1}{n+k^{\alpha}} = \frac{1}{\frac{n}{2}+\frac{n}{2}+\frac{k^{\alpha}}{N}+\ldots+\frac{k^{\alpha}}{N}} \leq \frac{1}{\frac{n}{2}+\left(\frac{n}{2}\frac{k^{\alpha N}}{N^N}\right)^{\frac{1}{N+1}}}$$

$$= \frac{1}{\frac{n}{2}+\frac{n^{\frac{1}{N+1}}k^{\frac{\alpha N}{N+1}}}{2^{\frac{1}{N+1}}N^{\frac{N}{N+1}}}} \leq \frac{1}{n^{\frac{1}{N+1}}\left(\frac{1}{2}+\frac{k^{\frac{\alpha N}{N+1}}}{2^{\frac{1}{N+1}}N^{\frac{N}{N+1}}}\right)}$$

and we observe that $\alpha N/(N+1) > 1$ if $N > 1/(\alpha-1)$. Thus we write

$$0 < \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} \leq n^{-1/(N+1)} \sum_{k=1}^{\infty} \frac{1}{\left(\frac{1}{2}+\frac{k^{\frac{\alpha N}{N+1}}}{2^{\frac{1}{N+1}}N^{\frac{N}{N+1}}}\right)}$$

The series converges and the limit is zero.

Let $\alpha < 1$. Trivially we have $\sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} \leq \sum_{k=1}^{n} \frac{1}{n} = 1$.

Moreover,

$$\sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} \geq \sum_{k=1}^{n} \frac{1}{n}\frac{1}{1+\frac{k^{\alpha}}{n}} \geq \sum_{k=1}^{n} \frac{1}{n}(1-\frac{k^{\alpha}}{n}) = 1 - \sum_{k=1}^{n} \frac{k^{\alpha}}{n^2} \geq 1 - \frac{n^{1+\alpha}}{n^2},$$

$1 \geq (1-x^2)$ has been used. By comparison the limit equals one since

$$1 \leq \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \leq 1 - \frac{n^{1+\alpha}}{n^2}$$

The last step is $\alpha = 1$. We need the well known equality $H_n \approx \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)$ and then

$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=n+1}^{2n} (H_{2n} - H_n) = \ln(2n) - \ln n + o(1) \to \ln 2$$

The proof is complete.

Solution 2 by David Stone and John Hawkins, Statesboro, GA

Below we show that for $0 < \alpha < 1$, the limit is 1; for $\alpha = 1$, the limit is $\ln 2$; and for $\alpha > 1$, the limit is 0.

For $\alpha = 1$ we get

$$\int_0^1 \frac{1}{1+u} du \geq \sum_{k=1}^{n} \frac{1}{n+k} \geq \int_{1/n}^{(n+1)/n} \frac{1}{1+u} du.$$

Since $\frac{1}{2} \leq \frac{1}{1+u} \leq 1$, we know that the limit exists as n approaches infinity and is given by

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} = \int_0^1 \frac{1}{1+u} du = \ln(1+u)\Big|_0^1 = \ln 2 - \ln 1 = \ln 2.$$

Next suppose $\alpha < 1$. Then

$$0 < k^\alpha \leq n^\alpha \text{ for } 1 \leq k \leq n, \text{ so}$$

$$\begin{aligned}
n < n + k^\alpha &\leq n + n^\alpha \quad \text{and} \\
\frac{1}{n+n^n} &\leq \frac{1}{n+k^\alpha} < \frac{1}{n}. \quad \text{Thus,} \\
\sum_{k=1}^{n} \frac{1}{n+n^\alpha} &\leq \sum_{k=1}^{n} \frac{1}{n+k^\alpha} < \sum_{k=1}^{n} \frac{1}{n} = 1, \quad \text{or} \\
\frac{n}{n+n^\alpha} &\leq \sum_{k=1}^{n} \frac{1}{n+k^\alpha} < 1. \quad \text{Hence,} \\
\lim_{n\to\infty} \frac{n}{n+n^\alpha} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \leq 1 \\
\lim_{n\to\infty} \frac{1}{1+\alpha n^{\alpha-1}} &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \leq 1. \quad \text{But,} \\
\lim_{n\to\infty} \frac{1}{1+\alpha n^{\alpha-1}} &= 1, \text{ since } \alpha - 1 < 0. \text{ Therefore,} \\
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} &= 1.
\end{aligned}$$

Finally, suppose $\alpha > 1$.

We note that $\dfrac{1}{n+k^\alpha}$ is a decreasing function of k and as a result we can write

$$0 \le \sum_{k=1}^{\infty} \frac{1}{n+k^\alpha} \le \int_0^n \frac{1}{n+k^\alpha} dk = \frac{1}{n}\int_0^1 \frac{1}{1+\dfrac{k^n}{n^{\alpha/\alpha}}} dk.$$

Using the substitution $u = \dfrac{k}{u^{1/\alpha}}$ with $du = \dfrac{1}{n^{1/\alpha}} dk$, the above becomes,

$$\begin{aligned}
0 \le \sum_{k=1}^{n} \frac{1}{n+k^\alpha} &\le \frac{n^{1/\alpha}}{n}\int_0^{n^{(n-1)/n}} \frac{1}{1+u^\alpha} du = \frac{1}{n^{(\alpha-1)/\alpha}}\int_0^{n^{(n-1)/\alpha}} \frac{1}{1+u^\alpha} du \\
&\le \frac{1}{n^{(\alpha-1)/\alpha}}\int_0^{n} \frac{1}{1+u^\alpha} du \\
&\le \frac{1}{n^{(\alpha-1)}\alpha}\int_0^{1} \frac{1}{1+u^\alpha} du + \frac{1}{n^{(\alpha-1)/\alpha}}\int_1^{n} \frac{1}{1+u^\alpha} du \\
&\le \frac{1}{n^{(\alpha-1)/\alpha}}(1) + \frac{1}{n^{(\alpha-1)/\alpha}}\int_1^{n} \frac{1}{1+u} du \\
&= \frac{1}{n^{(\alpha-1)/\alpha}}(1) + \frac{1}{n^{(\alpha-1)/\alpha}}(1)\Big[\ln(1+n) - \ln 2\Big].
\end{aligned}$$

That is,

$$0 \le \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} \le \lim_{n\to\infty} \frac{1}{n^{(\alpha-1)/\alpha}} + \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{2}\right)}{n^{(\alpha-1)/\alpha}}.$$

Using L'Hospital's rule repeatedly we get,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^{(\alpha-1)/\alpha}} + \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{2}\right)}{n^{(\alpha-1)/\alpha}} &= 0 + \lim_{n\to\infty} \frac{\frac{2}{n+1}}{\left(\frac{\alpha-1}{\alpha}\right)n^{-1/\alpha}} \\
&= \lim_{n\to\infty} \frac{2\alpha n^{1/\alpha}}{(\alpha-1)(n+1)} \\
&= \lim_{n\to\infty} \frac{2}{(\alpha-1)(n)^{1-1/\alpha}} \\
&= 0.
\end{aligned}$$

Thus, $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} = 0$ for $\alpha > 1$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha} = \begin{cases} 1, & 0<\alpha<1; \\ \ln 2, & \alpha = 1; \\ 0, & \alpha > 1. \end{cases}$

For $0 < \alpha < 1$, we have

$$\frac{1}{1+n^{\alpha-1}} = \sum_{k=1}^{n} \frac{1}{n+n^{\alpha}} \le \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} < \sum_{k=1}^{n} \frac{1}{n} = 1 \text{ and so } \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = 1.$$

For $\alpha = 1$ we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{1}{(1+k/n)} = \int_0^1 \frac{dx}{1+x} = \ln 2.$$

For $\alpha > 1$, let t be any real number satisfying $\frac{1}{\alpha} < t < 1$ and let $m = \lfloor n^t \rfloor$.

We have

$$0 < \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = \sum_{k=1}^{m} \frac{1}{n+k^{\alpha}} + \sum_{k=m+1}^{n} \frac{1}{n+k^{\alpha}} < \frac{m}{n} + \frac{n-m}{(m+1)^{\alpha}} \le \frac{1}{n^{1-t}} + \frac{1}{n^{\alpha t-1}},$$

which tends to 0 as n tends to infinity. It follows that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^{\alpha}} = 0.$

This completes the solution.

Also solved by Valmir Krasniqi, Prishtina, Kosova, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before December 15, 2009

- 5074: *Proposed by Kenneth Korbin, New York, NY*

  Solve in the reals:

$$\sqrt{25+9x+30\sqrt{x}}-\sqrt{16+9x+30\sqrt{x-1}}=\frac{3}{x\sqrt{x}}.$$

- 5075: *Proposed by Kenneth Korbin, New York, NY*

  An isosceles trapezoid is such that the length of its diagonal is equal to the sum of the lengths of the bases. The length of each side of this trapezoid is of the form $a+b\sqrt{3}$ where a and b are positive integers.

  Find the dimensions of this trapezoid if its perimeter is $31+16\sqrt{3}$.

- 5076: *Proposed by M.N. Deshpande, Nagpur, India*

  Let a, b, and m be positive integers and let F_n satisfy the recursive relationship

$$F_{n+2}=mF_{n+1}+F_n, \text{ with } F_0=a, F_1=b, n\geq 0.$$

  Furthermore, let $a_n=F_n^2+F_{n+1}^2, n\geq 0$. Show that for every a, b, m, and n,

$$a_{n+2}=(m^2+2)a_{n+1}-a_n.$$

- 5077: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

  Find all triplets (x, y, z) of real numbers such that

$$\left.\begin{array}{r} xy(x+y-z)=3,\\ yz(y+z-x)=1,\\ zx(z+x-y)=1. \end{array}\right\}$$

- 5078: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy*

  Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{\sqrt{b}(b+c)}+\frac{b}{\sqrt{c}(a+c)}+\frac{c}{\sqrt{a}(a+b)}\geq\frac{3}{2}\frac{1}{\sqrt{ab+ac+cb}}.$$

- 5079: *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $x \in (0,1)$ be a real number. Study the convergence of the series

$$\sum_{n=1}^{\infty} x^{\sin\frac{1}{1}+\sin\frac{1}{2}+\cdots+\sin\frac{1}{n}}.$$

Solutions

- 5056: *Proposed by Kenneth Korbin, New York, NY*

  A convex pentagon with integer length sides is inscribed in a circle with diameter $d = 1105$. Find the area of the pentagon if its longest side is 561.

  Solution by proposer

  The answer is 25284.

  The sides are $561, 169, 264, 105$, and 47 (in any order).

  Check: $\arcsin\left(\frac{561}{d}\right) = \arcsin\left(\frac{169}{d}\right) + \arcsin\left(\frac{264}{d}\right) + \arcsin\left(\frac{105}{d}\right) + \arcsin\left(\frac{47}{d}\right)$.

  Let $\overline{AB} = 561, \overline{BC} = 105, \overline{CD} = 47, \overline{DE} = 169, \overline{EA} = 264$. Then Diag $\overline{AC} = 468$.

  Check: $\arcsin\left(\frac{468}{d}\right) = \arcsin\left(\frac{47}{d}\right) + \arcsin\left(\frac{169}{d}\right) + \arcsin\left(\frac{264}{d}\right)$.

  Area $\triangle ABC = \sqrt{567 \cdot 99 \cdot 462 \cdot 6} = 12474$.

  Diag $\overline{AD} = 425$.

  Check: $\arcsin\left(\frac{425}{d}\right) = \arcsin\left(\frac{169}{d}\right) + \arcsin\left(\frac{264}{d}\right)$.

  Area $\triangle ACD = \sqrt{470 \cdot 45 \cdot 423 \cdot 2} = 4230$, and
  Area $\triangle ADE = \sqrt{429 \cdot 260 \cdot 165 \cdot 4} = 8580$.

  Area pentagon $= 12474 + 4230 + 8580 = 25284$.

  Editor's comments: Several solutions to this problem were received each claiming, at least initially, that the problem was impossible. I sent these individuals Ken's proof and some responded with an analysis of their errors. **Brian Beasley of Clinton, SC** responded as follows:

  "My assumption was that the inscribed pentagon was large enough to contain the center of the circle, so that I could subdivide the pentagon into five isosceles triangles, each with two radii as sides along with one side of the pentagon. But this pentagon is very small compared to the circle; it does not contain the center of the circle, and the ratio of its area to the area of the circle is only bout 2.64%. Attached is a rough diagram with two attempts to draw such an inscribed pentagon."

  "This has been a fascinating exercise! I found a Wolfram site and a Monthly paper with results about cyclic pentagons: <http://mathworld.wolfram.com/CyclicPentagon.html> and Areas of Polygons Inscribed in a Circle, by D. Robbins, American Mathematical Monthly, 102(6), 1995, 523-530."

"I salute Ken for creating this problem and for finding the arcsine identities to make it work."

David Stone and John Hawkins of Statesboro GA wrote: "Using MATLAB, we found the following four cyclic pentagons which have a side of length 561 and can be inscribed in a circle of diameter 1105. The first one has longest side 561, as required by the problem."

561	264	169	105	47	Area = 25284
817	663	663	561	520	Area = 705276
817	744	576	561	520	Area = 699984
817	744	663	561	425	Area = 692340

- 5057: *Proposed by David C. Wilson, Winston-Salem, N.C.*

We know that $1 + x + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k = \dfrac{1}{1-x}$ where $-1 < x < 1$.

Find formulas for $\sum_{k=1}^{\infty} kx^k$, $\sum_{k=0}^{\infty} k^2x^k$, $\sum_{k=0}^{\infty} k^3x^k$, $\sum_{k=0}^{\infty} k^4x^k$, and $\sum_{k=0}^{\infty} k^5x^k$.

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX

By differentiating the geometric series when $|x| < 1$,

$$\begin{aligned}\sum_{k=1}^{\infty} x^k &= \frac{1}{1-x}\\ &\Rightarrow \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}\\ &\Rightarrow \sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2} \qquad (1)\end{aligned}$$

Similarly, by differentiating (1),

$$\begin{aligned}\sum_{k=1}^{\infty} k^2x^{k-1} &= \frac{1+x}{(1-x)^3}\\ &\Rightarrow \sum_{k=1}^{\infty} k^2x^k = \frac{x(1+x)}{(1-x)^3}.\end{aligned}$$

Continuing this technique, it can be shown that

$$\begin{aligned}\sum_{k=1}^{\infty} k^3x^k &= \frac{x(x^2+4x+1)}{(1-x)^4}\\ \sum_{k=1}^{\infty} k^4x^k &= \frac{x(x^3+11x^2+11x+1)}{(1-x)^5}\\ \sum_{k=1}^{\infty} k^5x^k &= \frac{x(x^4+26x^3+66x^2+26x+1)}{(1-x)^6}\end{aligned}$$

Solution 2 by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy

The sums are respectively:

$$\frac{x}{(1-x)^2},\quad \frac{x(x+1)}{(1-x)^3},\quad \frac{x(x^2+4x+1)}{(1-x)^4},$$

$$\frac{x(x^3+11x^2+11x+1)}{(1-x)^5},\quad \frac{x(x^4+26x^3+66x^2+26x+1)}{(1-x)^6}$$

One might invoke standard theorems about the differentiability of convergent power series, but we propose the following proof which we believe is attributed to Euler.
We define
$S_p(x) \doteq \sum_{k=1}^{\infty} k^p x^k$, $p = 1, \ldots, 5$ and employ $\sum_{k=1}^{\infty} x^k = \left(\sum_{k=0}^{\infty} x^k\right) - 1 = \frac{1}{1-x} - 1 = \frac{x}{1-x}$.
To compute $\sum_{k=0}^{\infty} x^k - 1 = \frac{1}{1-x}$ we proceed as follows:

$$P \doteq \sum_{k=0}^{\infty} x^k = 1 + x(1 + x + x^2 + \ldots) = 1 + xP \quad \Longrightarrow \quad P = \frac{1}{1-x}.$$

$\mathbf{S_1(x)}$:

$$\sum_{k=1}^{\infty} kx^k = \sum_{k=2}^{\infty}(k-1)x^k + \sum_{k=0}^{\infty} x^k - 1 = x\sum_{n=1}^{\infty} nx^n + \frac{1}{1-x} - 1 \text{ or}$$

$$(1-x)\sum_{k=1}^{\infty} kx^k = \frac{x}{1-x} \quad \Longrightarrow \quad \sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}.$$

$\mathbf{S_2(x)}$:

$$\sum_{k=1}^{\infty} k^2x^k = \sum_{k=2}^{\infty}(k-1)^2x^k + 2\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k \text{ or}$$

$$\begin{aligned} \sum_{k=1}^{\infty} k^2x^k - x\sum_{n=1}^{\infty} n^2x^n &= 2\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k \\ &= \frac{2x}{(1-x)^2} - \frac{x}{(1-x)} \Longrightarrow S_2(x) = \frac{x(x+1)}{(1-x)^3}. \end{aligned}$$

$\mathbf{S_3(x)}$:

$$\begin{aligned} \sum_{k=1}^{\infty} k^3x^k &= \sum_{k=2}^{\infty}(k-1)^3x^k + 3\sum_{k=1}^{\infty} k^2x^k - 3\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k \\ &= x\sum_{k=1}^{\infty} k^3x^k + 3\sum_{k=1}^{\infty} k^2x^k - 3\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k \text{ or} \end{aligned}$$

$$(1-x)\sum_{k=1}^{\infty} k^3x^k = 3S_2(x) - 3S_1(x) + \frac{x}{1-x} \Longrightarrow S_3(x) = \frac{x(x^2+4x+1)}{(1-x)^4}.$$

$\mathbf{S_4(x)}$:

$$\begin{aligned}\sum_{k=1}^{\infty} k^4x^k &= \sum_{k=2}^{\infty}(k-1)^4x^k + 4\sum_{k=1}^{\infty} k^3x^k - 6\sum_{k=1}^{\infty} k^2x^k + 4\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k\\ &= x\sum_{k=1}^{\infty} k^4x^k + 4\sum_{k=1}^{\infty} k^3x^k - 6\sum_{k=1}^{\infty} k^2x^k + 4\sum_{k=1}^{\infty} kx^k - \sum_{k=1}^{\infty} x^k \text{ or}\end{aligned}$$

$$(1-x)\sum_{k=1}^{\infty} k^4x^k = 4S_3(x) - 6S_2(x) + 4S_1(x) - \frac{x}{1-x} \Longrightarrow S_4(x) = \frac{x(x^3+11x^2+11x+1)}{(1-x)^5.}$$

$\mathbf{S_5(x)}$:

$$\begin{aligned}\sum_{k=1}^{\infty} k^5x^k &= \sum_{k=2}^{\infty}(k-1)^5x^k + 5\sum_{k=1}^{\infty} k^4x^k - 10\sum_{k=1}^{\infty} k^3x^k + 10\sum_{k=1}^{\infty} k^2x^k - 5\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k\\ &= x\sum_{k=1}^{\infty} k^5x^k + 5S_4(x) - 10\sum_{k=1}^{\infty} k^3x^k + 10\sum_{k=1}^{\infty} k^2x^k - 5\sum_{k=1}^{\infty} kx^k + \sum_{k=1}^{\infty} x^k \text{ or}\end{aligned}$$

$$\begin{aligned}(1-x)\sum_{k=1}^{\infty} k^5x^k &= 5S_4(x) - 10S_3(x) + 10S_2(x) - 5S_1(x) + \frac{x}{1-x}\\ &\Longrightarrow S_5(x) = \frac{x(x^4+26x^3+66x^2+26x+1)}{(1-x)^6}.\end{aligned}$$

Also solved by Matei Alexianu (student, St. George's School), Spokane,WA; Brian D. Beasley, Clinton, SC; Sully Blake (student, St. George's School), Spokane,WA; Michael Brozinsky, Central Islip, NY; Mark Cassell (student, St. George's School), Spokane,WA; Richard Caulkins (student, St. George's School), Spokane,WA; Pat Costello, Richmond, KY; Michael C. Faleski, University Center, MI; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Nguyen Pham and Quynh Anh (jointly; students, Belarusian State University), Belarus; Boris Rays, Brooklyn, NY, and the proposer.

- 5058: *Proposed by Juan-Bosco Romero Márquez, Avila, Spain.*

  If p, r, a, A are the semi-perimeter, inradius, side, and angle respectively of an acute triangle, show that

  $$r + a \le p \le \frac{p}{\sin A} \le \frac{p}{\tan\dfrac{A}{2}},$$

  with equality holding if, and only if, $A = 90^o$.

  Solution by Manh Dung Nguyen,(student, Special High School for Gifted Students) HUS, Vietnam

1) $\mathbf{r+a \le p}$:

$\tan\frac{A}{2} \le 1$ for all $A \in (0, \pi/2]$, so by the well known formula $\tan\frac{A}{2} = \frac{(p-b)(p-c)}{p(p-a)}$ we have $(p-b)(p-c) \le p(p-a)$. Letting S be the area of $\triangle ABC$ and using Heron's formula,

$$\begin{aligned} S^2 = p^2r^2 &= p(p-a)(p-b)(p-c) \le p^2(p-a)^2. \text{ Thus} \\ r &\le p-a \text{ or } r+a \le p. \end{aligned}$$

2) $\mathbf{p \le \frac{p}{\sin A}}$:

We have $\sin A \le 1$ for all $A \in (0, \pi)$, so $p \le \frac{p}{\sin A}$.

3) $\mathbf{\frac{p}{\sin A} \le \frac{p}{\tan\frac{A}{2}}}$:

For $A \in (0, \pi/2]$ we have

$$\sin A - \tan\frac{A}{2} = \sin\frac{A}{2}\left(2\cos\frac{A}{2} - \frac{1}{\cos\frac{A}{2}}\right) = \frac{\sin\frac{A}{2}\cos A}{\cos\frac{A}{2}} \ge 0. \text{ Hence}$$

$$\frac{p}{\sin A} \le \frac{p}{\tan\frac{A}{2}}.$$

Equality holds if and only if $A = 90^0$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Scott H. Brown, Montgomery, AL; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY, and the proposer.

- 5059: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Prove that for all triangles ABC

$$\sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right) \le \frac{6\sqrt{3}+1}{8}.$$

Editor's comment: Many readers noted that the inequality as stated in the problem is incorrect. It should have been $\frac{3(2\sqrt{3}+1)}{2}$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

We need the following inequalities

$$\sin(A)+\sin(B)+\sin(C) \ge \sin(2A)+\sin(2B)+\sin(2C) \qquad (1)$$

$$\sin(A)+\sin(B)+\sin(A) \le \frac{3\sqrt{3}}{2} \qquad (2)$$

$$\sin(\frac{A}{2}) + \sin(\frac{B}{2}) + \sin(\frac{C}{2}) \quad \le \quad \frac{3}{2} \qquad (3)$$

Inequalities (1), (2), (3) appear respectively as inequalities 2.4, 2.2(1),and 2.9 in Geometric Inequalities by O. Bottema, R.Z. Dordevic, R.R. Janic, D.S. Mitrinovic, and P.M. Vasic, (Groningen), 1969.

It follows from (1),(2),(3) that

$$\sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right) \le \frac{3(2\sqrt{3}+1)}{2}.$$

Solution 2 by John Hawkins and David Stone, Statesboro, GA

We treat this as a Lagrange Multiplier Problem: let

$$f(A,B,C) = \sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right).$$

We wish to find the maximum value of this function of three variables, subject to the constraint $g(A,B,C) : A+B+C = \pi$. That is, (A,B,C) lies in the closed, bounded, triangular region in the first octant with vertices on the coordinate axes: $(\pi,0,0),(0,\pi,0),(0,0,\pi)$.

By taking partial derivatives with respect to the variables A, B, and C and setting $\nabla f(A,B,C) = \lambda \nabla g(A,B,C)$ or $\Big\langle f_A, f_B, f_C \Big\rangle = \lambda \Big\langle g_A, g_B, g_C \Big\rangle = \lambda\langle 1,1,1\rangle$, we are lead to the system

$$\begin{cases} 2\cos(2A) + \cos(A) + \frac{1}{2}\cos\left(\frac{A}{2}\right) = \lambda \\ 2\cos(2B) + \cos(B) + \frac{1}{2}\cos\left(\frac{B}{2}\right) = \lambda \\ 2\cos(2C) + \cos(C) + \frac{1}{2}\cos\left(\frac{C}{2}\right) = \lambda \end{cases}$$

It is clear that one solution is to let $A = B = C$. We claim there are no others in our domain.

To show this, we investigate the fuction $h(\theta) = 2\cos(2\theta) + \cos(\theta) + \dfrac{1}{2}\cos\left(\dfrac{\theta}{2}\right)$ on the interval $0 \le \theta \le \pi$. Finding a solution to our system is equivalent to finding values A, B and C such that $h(A) = h(B) = h(C) = \lambda$.

We determine that $h(0) = 3.5$; then the function h decreases, passing through height 1 at (0.802,1), reaching a minimum at $(1.72, -1.73)$, then rising to height 1 at π. No horizontal line crosses the graph three times, so we cannot find distinct A, B and C with $h(A) = h(B) = h(C)$. In fact, because the function is decreasing from 0 to 1.72, and increasing from 1.72 to π, any horizontal line crossing the graph more than once must do so after $\theta = 0.802$. That is all of A, B and C would have to be greater than 0.802, and at least one of them greater than 1.72. Because $0.802 + 0.802 + 1.72 = 3.324 > \pi$, this violates the condition that $A + B + C = \pi$.

Thus the maximum value occurs when $A = B = C = \frac{\pi}{3}$:

$$f\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = 3\sin\left(\frac{2\pi}{3}\right) + 3\sin\left(\frac{\pi}{3}\right) + 3\sin\left(\frac{\pi}{6}\right) = 6\frac{\sqrt{3}}{2} + \frac{3}{2} = \frac{6\sqrt{3}+3}{2}.$$

This method tells us that the only point on the plane $A + B + C = \pi$ (in the first octant) where the function f achieves a maximum value is the point we just found. We must check the boundaries for a minimum.

Note that $f(\pi, 0, 0) = 1 = f(0, \pi, 0) = f(0, 0, \pi)$. That is f achieves the lower bound 1 at the vertices of our triangular region.

We also consider the behavior of the function f along the edges of this region. For instance, in the AB-plane where $C = 0$, we have $A + B = \pi$. Then
$f(A, \pi - A, 0) = 2\sin A + \sin\left(\frac{A}{2}\right) + \cos\left(\frac{A}{2}\right)$, which has value 1 (of course) at the endpoints $A = 0$ and $A = \pi$, and climbs to a local maximum value of $2 + \sqrt{2}$ when $A = \frac{\pi}{2}$. This value is less than $f\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$.

There is identical behavior along the other two edges.

In summary, the function f achieves an absolute maximum of $\frac{6\sqrt{3}+3}{2}$ at the interior point $A = B = C = \frac{\pi}{3}$, and f achieves its absolute minimum of 1 at the vertices.

However, for a non-degenerate triangle ABC

$$1 < \sin(2A)+\sin(2B)+\sin(2C)+\sin(A)+\sin(B)+\sin(C)+\sin\left(\frac{A}{2}\right)+\sin\left(\frac{B}{2}\right)+\sin\left(\frac{C}{2}\right) \le \frac{6\sqrt{3}+3}{2},$$

and the lower bound is never actually achieved.

Solution 3 by Tom Leong, Scranton, PA

This inequality follows from summing the three known inequalities labeled (1), (2), and (3) below. Both $\sin x$ and $\sin\frac{x}{2}$ are concave down on $(0, \pi)$. Applying the AM-GM inequality followed by Jensen's inequality gives

$$\sin A \sin B \sin C \le \left(\frac{\sin A + \sin B + \sin C}{3}\right)^3 \le \sin^3\left(\frac{A+B+C}{3}\right) = \frac{3\sqrt{3}}{8} \qquad (1)$$

and

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \left(\frac{\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2}}{3}\right)^3 \le \sin^3\left(\frac{A+B+C}{6}\right) = \frac{1}{8}. \qquad (2)$$

For the third inequality, we use the AM-GM inequality along with the identity

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$$

and (1):

$$\sin 2A \sin 2B \sin 2C \quad \le \quad \left(\frac{\sin 2A + \sin 2B + \sin 2C}{3}\right)^3 = \left(\frac{4\sin A \sin B \sin C}{3}\right)^3$$

$$\leq \left(\frac{4}{3}\cdot\frac{3\sqrt{3}}{8}\right)^3 = \frac{3\sqrt{3}}{8}. \tag{3}$$

Equality occurs if and only if $A = B = C = \pi/3$ as it does in every inequality used above.

Also solved by Brian D. Beasley, Clinton, SC; Scott H. Brown, Montgomery, AL; Michael Brozinsky, Central Islip, NY; Elsie Campbell, Dionne Bailey, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Manh Dung Nguyen (student, Special High School for Gifted Students) HUS, Vietnam; Boris Rays, Brooklyn, NY, and the proposer.

- 5060: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Show that there exists $c \in (0, \pi/2)$ such that

$$\int_0^c \sqrt{\sin x}\, dx + c\sqrt{\cos c} = \int_c^{\pi/2} \sqrt{\cos x}\, dx + (\pi/2 - c)\sqrt{\sin c}$$

Solution 1 by Paul M. Harms, North Newton, KS

Let

$$f(x) = \int_0^x \sqrt{\sin t}\, dt + x\sqrt{\cos x} - \int_x^{\pi/2} \sqrt{\cos t}\, dt - (\frac{\pi}{2} - x)\sqrt{\sin x} \text{ where } x \in [0, \pi/2].$$

For $x \in [0, \pi/2]$, each term of $f(x)$ is continuous including the integrals of continuous functions. Then $f(x)$ is continuous for $x \in [0, \pi/2]$. For any $x \in [0, \pi/2]$, the two integrals of nonnegative functions are positive except when the lower limit equals the upper limit. We have

$$f(0) = -\int_0^{\pi/2} \sqrt{\cos t}\, dt < 0 \text{ and } f(\pi/2) = \int_0^{\pi/2} \sqrt{\sin t}\, dt > 0.$$

Since $f(x)$ is continuous for $x \in [0, \pi/2]$, $f(0) < 0$ and $f(\pi/2) > 0$, there is at least one $c \in (0, \pi/2)$ such that

$$f(c) = 0 = \int_0^c \sqrt{\sin t}\, dt + c\sqrt{\cos c} - \int_c^{\pi/2} \sqrt{\cos t} - (\pi/2 - c)\sqrt{\sin c}.$$

This last equation is equivalent to the equation in the problem.

Solution 2 by Michael C. Faleski, University Center, MI

The given equation will hold if the integrals and their constants of integration are the same on each side of the equality.
For the integral $\int_0^c \sqrt{\sin x} dx$ we substitute $x = \frac{\pi}{2} - y$ to obtain

$$\int_0^c \sqrt{\sin x} dx = \int_{\pi/2}^{\pi/2-c} \sqrt{\sin\left(\frac{\pi}{2} - y\right)}(-dy) = \int_{\pi/2-c}^{\pi/2} \sqrt{\cos y} dy.$$

We substitute this into the original statement of the problem and equate the integrals on each side of the equation.

$$\int_{\pi/2-c}^{\pi/2} \sqrt{\cos y}dy = \int_{c}^{\pi/2} \sqrt{\cos y}dy$$

For equality to hold the lower limits of integration must be the same; that is,
$\frac{\pi}{2} - c = c \Longrightarrow c = \frac{\pi}{4}$

We now check the constants of integration on each side of the equality when $c = \frac{\pi}{4}$, and we see that they are equal.

$$\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^{1/2} = \frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)^{1/2}$$

Hence, the value of $c = \frac{\pi}{4}$ satisfies the original equation.

Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie, and Andrew Siefker (jointly), San Angelo, TX; Brian D. Beasley, Clinton, SC; Bruno Salgueiro Fanego, Viveiro, Spain; Ovidiu Furdui, Cluj, Romania; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Nguyen Pham and Quynh Anh (jointly; students, Belarusian State University), Belarus; Angel Plaza, Las Palmas, Spain; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy; David Stone and John Hawkins (jointly) Statesboro, GA , and the proposer.

- 5061: *Michael P. Abramson, NSA, Ft. Meade, MD.*

Let $a_1, a_2, \ldots, a_n$ be a sequence of positive integers. Prove that

$$\sum_{i_m=1}^{n} \sum_{i_{m-1}=1}^{i_m} \cdots \sum_{i_1=1}^{i_2} a_{i_1} = \sum_{i=1}^{n} \binom{n-i+m-1}{m-1} a_i.$$

Solution by Tom Leong, Scranton, PA

We treat the a's as variables; they don't necessarily have to be integers. Fix an i, $1 \le i \le n$, and imagine completely expanding all the sums on the lefthand side. We wish to show that, in this expansion, the number of times that the term a_i appears is $\binom{n-i+m-1}{m-1}$. Now each term in this expansion corresponds to some m-tuple of indices in the set

$$I = \{(i_1, i_2, \ldots, i_m) : 1 \le i_1 \le i_2 \le \cdots \le i_m \le n\}.$$

We want to count the number of elements of I of the form $(i, i_2, \ldots, i_m)$. Equivalently, using the one-to-one correspondence between I and

$$J = \{(j_1, j_2, \ldots, j_m) : 1 \le j_1 < j_2 < \cdots < j_m \le n+m-1\}$$

given by

$$(i_1, i_2, \ldots, i_m) \leftrightarrow (j_1, j_2, \ldots, j_m) = (i_1, i_2+1, i_3+2 \ldots, i_m+m-1),$$

we wish to count the number elements of J of the form $(i, j_2, \ldots, j_m)$. This number is simply the number of $(m-1)$-element subsets of $\{i+1, i+2, \ldots, n+m-1\}$ which is just $\binom{n-i+m-1}{m-1}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before January 15, 2010

- 5080: *Proposed by Kenneth Korbin, New York, NY*

  If p is a prime number congruent to 1 (mod4), then there are positive integers a, b, c, such that
  $$\arcsin\left(\frac{a}{p^3}\right) + \arcsin\left(\frac{b}{p^3}\right) + \arcsin\left(\frac{c}{p^3}\right) = 90^o.$$
  Find a, b, and c if $p = 37$ and if $p = 41$, with $a < b < c$.

- 5081: *Proposed by Kenneth Korbin, New York, NY*

  Find the dimensions of equilateral triangle ABC if it has an interior point P such that $\overline{PA} = 5, \overline{PB} = 12$, and $\overline{PC} = 13$.

- 5082: *Proposed by David C. Wilson, Winston-Salem, NC*

  Generalize and prove:
  $$\begin{aligned}
  \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} &= 1 - \frac{1}{n+1} \\
  \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} &= \frac{1}{4} - \frac{1}{2(n+1)(n+2)} \\
  \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)} &= \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)} \\
  \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)(n+4)} &= \frac{1}{96} - \frac{1}{4(n+1)(n+2)(n+3)(n+4)}
  \end{aligned}$$

- 5083: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let $\alpha > 0$ be a real number and let $f : [-\alpha, \alpha] \to \Re$ be a continuous function two times derivable in $(-\alpha, \alpha)$ such that $f(0) = 0$ and f'' is bounded in $(-\alpha, \alpha)$. Prove that the sequence $\{x_n\}_{n\geq 1}$ defined by
  $$x_n = \begin{cases} \displaystyle\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right), & n > \dfrac{1}{\alpha}; \\ 0, & n \leq \dfrac{1}{\alpha} \end{cases}$$
  is convergent and determine its limit.

- 5084: *Charles McCracken, Dayton, OH*

  A natural number is called a "repdigit" if all of its digits are alike.
  Prove that regardless of positive integral base b, no natural number with two or more digits when raised to a positive integral power will produce a repdigit.

- 5085: *Proposed by Valmir Krasniqi, (student, Mathematics Department,) University of Prishtinë, Kosova*

  Suppose that $a_k,\ (1 \le k \le n)$ are positive real numbers. Let $e_{j,k} = (n-1)$ if $j = k$ and $e_{j,k} = (n-2)$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise.

  Prove that
  $$\prod_{j=1}^{n}\sum_{k=1}^{n} e_{j,k}a_k^2 \ge \prod_{j=1}^{n}\left(\sum_{k=1}^{n} d_{j,k}a_k\right)^2.$$

Solutions

- 5062: *Proposed by Kenneth Korbin, New York, NY.*

  Find the sides and the angles of convex cyclic quadrilateral ABCD if
  $\overline{AB} = \overline{BC} = \overline{CD} = \overline{AD} - 2 = \overline{AC} - 2$.

  Solution 1 by David E. Manes, Oneonta, NY

  Let $x = \overline{AB} = \overline{BC} = \overline{CD}$ and let $y = \overline{BD}$. Then $\overline{AD} = \overline{AC} = x + 2$.

  Let $\alpha = \angle CAB,\ \beta = \angle ABD$, and $\gamma = \angle DBC$. Finally, in quadrilateral $ABCD$, we denote the angle at vertex A by $\angle A$ and similarly for the other three vertices. Then $\overline{AB} = \overline{BC}$ implies $\alpha = \angle BCA$. Since angles inscribed in the same arc are congruent, it follows that

  $$\begin{aligned}\alpha &= \angle CAB = \angle CDA,\\ \alpha &= \angle BCA = \angle BDA,\\ \beta &= \angle ABD = \angle ACD, \text{and}\\ \gamma &= \angle DBC = \angle DAC\end{aligned}$$

  Therefore,

  $$\angle A = \alpha + \gamma,\ \angle B = \beta + \gamma,\ \angle C = \alpha + \beta \text{ and } \angle D = 2\alpha = \beta \text{ since } \overline{AC} = \overline{AD}.$$

  From Ptolemy's Theorem, one obtains

  $$\begin{aligned}\overline{AC}\cdot\overline{BD} &= \overline{AB}\cdot\overline{CD} + \overline{AD}\cdot\overline{BC} \text{ or}\\ (x+2)y &= x^2 + x(x+2)\\ y &= \frac{2x(x+1)}{x+2}.\end{aligned}$$

  In triangles ACD and BCD, the law of cosines implies $\cos\gamma = \dfrac{2(x+2)^2 - x^2}{2(x+2)^2}$ and $\cos\gamma = \dfrac{y}{2x} = \dfrac{x+1}{x+2}$ respectively. Setting the two values equal yields the quadratic equation $x^2 - 2x - 4 = 0$ with positive solution $x = 1 + \sqrt{5}$. Hence,

  $$\overline{AB} = \overline{BC} = \overline{CD} = 1 + \sqrt{5} \text{ and } \overline{AD} = 3 + \sqrt{5}.$$

Moreover, note that

$$\begin{aligned}\cos\gamma &= \frac{x+1}{x+2} = \frac{2+\sqrt{5}}{3+\sqrt{5}} = \frac{1+\sqrt{5}}{4} \text{ implies that}\\ \gamma &= \arccos\left(\frac{1+\sqrt{5}}{4}\right) = 36^0\end{aligned}$$

In $\triangle ACD, \gamma+\beta+2\alpha = 180^o$ or $\gamma+2\beta = 180^0$ so that $\beta = \dfrac{180^0-36^0}{2} = 72^0$ and $\alpha = \beta/2 = 36^0$.

Therefore,

$$\angle A = \alpha+\gamma = 72^0 = 2\alpha = \angle D \text{ and}$$

$$\angle B = \beta+\gamma = 108^0 = \alpha+\beta = \angle C.$$

Solution 2 by Brian D. Beasley, Clinton, SC

We let $a = \overline{AB}$, $b = \overline{BC}$, $c = \overline{CD}$, $d = \overline{AD}$, $p = \overline{BD}$ and $q = \overline{AC}$. Then $a = b = c = d-2 = q-2$. According to the Wolfram MathWorld web site [1], for a cyclic quadrilateral, we have

$$pq = ac+bd \text{ (Ptolemy'sTheorem)} \quad \text{and} \quad q = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}.$$

Thus $a+2 = \sqrt{2a^2+2a}$, so the only positive value of a is $a = 1+\sqrt{5}$. Hence $a = b = c = 1+\sqrt{5}$ and $d = p = q = 3+\sqrt{5}$. Using the Law of Cosines, it is straightforward to verify that $\angle ABC = \angle BCD = 108°$ and $\angle CDA = \angle DAB = 72°$.

[1] Weisstein, Eric W. "Cyclic Quadrilateral." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/CyclicQuadrilateral.html

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

We show that the sides are $1+\sqrt{5}, 1+\sqrt{5}, 1+\sqrt{5}, 3+\sqrt{5}$ and the angles are $108^0,\ 72^0,\ 72^0,\ 108^0$.

Let $\alpha = \overline{AB} = \overline{BC} = \overline{CD} = \overline{AD}-2 = \overline{AC}-2, \beta = \angle CBA$ and R the circumradius of $ABCD$.

By solution 1 of SSM problem 4961,

$$R = \frac{1}{4}\sqrt{\frac{[aa+a(a+2)][a(a+2)+aa][aa+a(a+2)]}{(2a+1-a)(2a+1-a)(2a+1-a)[2a+1-(a+2)]}} = \frac{a}{2}\sqrt{\frac{2a}{a-1}}.$$

From this and the generalized sine theorem in $\triangle ABC$,

$$\frac{a}{2R} = \sin\left(\frac{180^0-\beta}{2}\right) \Longrightarrow \cos\left(\frac{\beta}{2}\right) = \sqrt{\frac{a-1}{2a}}.$$

By the law of cosines in $\triangle ABC$,

$$\cos\beta = \frac{a^2+a^2-(a+2)^2}{2a^2} \Longrightarrow \cos\left(\frac{\beta}{2}\right) = \sqrt{\frac{1+\cos\beta}{2}} = \frac{\sqrt{3a^2-4a-4}}{2a}.$$

Hence,

$$\sqrt{\frac{a-1}{2a}} = \frac{\sqrt{3a^2-4a-4}}{2a} \Longrightarrow a^2-2a-4=0 \Longrightarrow a = 1+\sqrt{5} = 2\phi,$$

so the sides are

$$\overline{AB} = \overline{BC} = \overline{CD} = 1+\sqrt{5} \text{ and } \overline{AD} = a+2 = 3+\sqrt{5}.$$

Then $\beta = 2\arccos\sqrt{\frac{\sqrt{5}}{2(1+\sqrt{5})}} = 108^0$, so the angles are $\angle CBA = 108^0,\ \angle DCB = \angle CBA = 108^0,\ \angle ADC = 180^0 - 108^0 = 72^0$ and $\angle BAD = 72^0$.

Also solved by Michael Brozinsky, Central Islip, NY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5063: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

Euclid's inscribed polygon is a constructible polygon inscribed in a circle whose consecutive central angle degree measures form a positive integral arithmetic sequence with a non-zero difference.

a) Does Euclid's inscribed n-gon exist for any prime n greater than 5?

b) Does Euclid's n-gon exist for all composite numbers n greater than 2?

Solution by Joseph Lupton, Jacob Erb, David Ebert, and Daniel Kasper, students at Taylor University, Upland, IN

a) For an inscribed polygon to fit this description, there has to be an arithmetic sequence of positive integers where the number of terms in the sequence is equal to the number of sides of the polygon and the terms sum to 360. So if the first term is f and the constant difference between the terms is d, the sum of the terms is

$$f \cdot n + \frac{n(n-1)}{2}d = 360.$$

Thus, $f \cdot n + \frac{n(n-1)}{2}d = 360 \Longrightarrow n \Big| 360$. That is, n is a prime number greater than five and $n \Big| 2^3 \cdot 3^3 \cdot 5$. But there is no prime number greater than five that divides 360. So there is no Euclidean polygon that can be inscribed in a circle whose consecutive central angle degree measures form a positive integral arithmetic sequence with a non-zero difference.

b) Euclid's inscribed n-gon does not exist for all composite numbers greater than two. Obviously, if n gets too large, then the terms $\frac{n(n-1)}{2}d$ will be greater than 360 even if $d=1$ which is the minimal d allowed. There is no Eculidean inscribed n-gon for $n=21$. If there were, the the sum of central angles would be $f \cdot n + n \cdot d \cdot \frac{n-1}{2}$ implies that 21 divides 360. Similarly, there is no 14-gon for if there were, it would imply that 7 divides 360.

- **Comments and elaborations by David Stone and John Hawkins, Statesboro GA**

We note that this problem previously appeared as part of Problem 4708 in this journal in March, 1999; however the solution was not published. Also, a Google search on the internet turned up a paper by the proposer in the Bulletin of the Malaysian Mathematical Sciences Society in which the answer to both questions is presented as being "no". {See "The Euclidean Inscribed Polygon" (Bull. Malaysian Math Sc. Soc (Second series) 27 (2004), 45-52).}

David and John solved the problem and then elaborated on it by considering the possibility that the inscribed polygon many not enclose the center of the circle. And it is here that things start to get interesting.

(In the case where the inscribed polygon does not include the center of the circle, and letting a be the first term in the arithmetic sequence and d the common difference, they noted that the largest central angle must be the sum of the previous $n-1$ central angles, and they proceeded as follows:)

$$\begin{aligned} a+(n-1)d = S_{n-1} &= \frac{n-1}{2}\Big(2a+(n-2)d\Big) \text{ or} \\ 2a+2(n-1)d &= 2a(n-1)+(n-1)(n-2)d \text{ or} \\ 2a(n-2) &= -(n-1)(n-4)d. \end{aligned}$$

For $n=3$, this happens exactly when $a=d$; although $n=3$ is of no concern for the stated problem, we shall return to this case later.

For $n \geq 4$, this condition is never satisfied because the left-hand side is positive and the right-hand side ≤ 0.

David and John then determined all Euclidean inscribed n-gons as follows:

The cited paper by the poser points out that 3^0 is the smallest constructible angle of positive integral degree. In fact, it is well known that an angle is constructible if, and only if, its degree measure is an integral multiple of 3^0. This implies that a and d must both be multiples of 3. We wish to find all solutions of the Diophantine equation
(1) $\quad n(2a+(n-1)d) = 2^4 \cdot 3^2 \cdot 5$, where a and d are multiples of 3.

Letting $a = 3A$ and $d = 3D$, the above equation becomes
(2) $\quad n\Big(2A+(n-1)D\Big) = 2^4 \cdot 3 \cdot 5 = 240$, so n must be a divisor of 240.

Moreover, the cofactor $2A+(n-1)D$ is bounded below. That is

$$\begin{aligned} 2A+(n-1)D &\geq 2+(n-1) = n+1. \text{ So} \\ \frac{240}{n} &= 2A+(n-1)D \geq 1, \text{ and} \\ n(n+1) &\leq 240. \end{aligned}$$

These conditions allow only $n = 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,$ and 15.

First we show that $n = 12$ fails. For in this case (2) becomes

$$\begin{aligned} 12(2A+11D) = 240, \text{or} \\ 2A+11D = 20, \end{aligned}$$

and this linear Diophantine equation has no positive solutions.

All other possible values of n do produce corresponding Euclidean $n-$gons.

The case $n = 3$ is perhaps the most interesting. There are twenty triangles inscribed in semi-circle: $(3A,\ 6A,\ 9A)$ for $A = 1, 2, \ldots 20$, each having $a = d$, and nineteen more triangles which properly enclose the center of the circle: $(3t, 120, 240 - 3t)$, for $t = 21,\ 22, \ldots,\ 39$, each with $d = 120 - a$.

We consider in detail the case $n = 4$, in which case Equation (2) becomes $4(2A + 3D) = 2^4 \cdot 3 \cdot 5$, or $2A + 3D = 60$. The solution of this Diophantine equation is given by

$$\begin{cases} A = 3t \\ D = 20 - 2t \end{cases}$$

where the integer parameter t satisfies $0 < t < 10$.

We exhibit the results in tabular form, with all angles in degrees:

t	A	$a = 3A$	D	$d = 3D$	*Central angles of inscribed quarilateral*
1	3	9	18	54	9, 63, 117, 171
2	6	18	16	48	18, 66, 114, 162
3	9	27	14	42	27, 69, 111 153
4	12	36	12	36	36, 72, 108, 144
5	15	45	10	30	45, 75, 105, 135
6	18	54	8	24	54, 78, 102, 126
7	21	63	6	18	63, 81, 99, 117
8	24	72	4	12	72, 84, 96, 108
9	27	81	2	6	81, 87, 93, 99

That is, the central angles are $(9t,\ 60 + 3t,\ 120 - 3t,\ 180 - 9t)$ for $t = 1, 2, \ldots, 9$. Thus we have nine Euclidean inscribed quadrilaterals.

Similarly for $n = 5$, we have eleven Euclidean inscribed pentagons, with central angles $(6t,\ 36 + 3t,\ 72,\ 108 - 3t,\ 144 - 6t)$ for $t = 1,\ 2, \ldots,\ 11$.

Similarly for $n = 6$, we have three Euclidean inscribed hexagons, with central angles $(45,\ 51,\ 57,\ 63,\ 75), (30,\ 42,\ 54,\ 66,\ 78,\ 90)$ and $(15,\ 33,\ 52,\ 69,\ 105)$.

For $n = 8$, we have two Euclidean inscribed octagons with central angles $(24,\ 30,\ 36,\ 42,\ 48,\ 54,\ 60,\ 66)$ and $(3,\ 15,\ 27,\ 39,\ 51,\ 63,\ 75,\ 87)$.

For $n = 10$, we have one Euclidean inscribed decagon, with central angles $(9,\ 15,\ 21,\ 27,\ 33,\ 39,\ 45,\ 51,\ 57,\ 63)$.

For $n = 15$, we have one Euclidean inscribed 15-gon with central angles $(3,\ 6,\ 9,\ 12,\ 15,\ 18,\ 21,\ 24\ 27,\ 30,\ 33,\ 36,\ 39,\ 42,\ 45)$.

There is a grand total of 66 Euclidean inscribed n-gons!

A final note: If $n(n+1)$ divides 240, then $a = d = 3\dfrac{240}{n(n+1)} = \dfrac{720}{n(n+1)}$ produces a Euclidean inscribed $n-$gon.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Boris Rays, Brooklyn, NY, and the proposer.

- 5064: *Proposed by Michael Brozinsky, Central Islip, NY.*

The Lemoine point of a triangle is that point inside the triangle whose distances to the three sides are proportional to those sides. Find the maximum value that the constant of proportionality, say λ, can attain.

Solution 1 by David E. Manes, Oneonta, NY

The maximum value of λ is $\sqrt{3}/6$ and is attained when the triangle is equilateral.

Given the triangle ABC let $[ABC]$ denote its area. The distance from the Lemoine point to the three sides are in the ratio $\lambda a,\ \lambda b,\ \lambda c$ where $\lambda = \dfrac{2[ABC]}{a^2+b^2+c^2}$ and a, b, c denote the length of the sides $BC,\ CA$ and AB respectively. Let $\alpha = \angle BAC, \beta = \angle CBA$, and $\gamma = \angle ACB$. Then

$$[ABC] = \frac{1}{2}bc\cdot\sin\alpha = \frac{1}{2}ac\cdot\sin\beta = \frac{1}{2}ab\cdot\sin\gamma.$$

Therefore,

$$a^2+b^2+c^2 \geq ab+bc+ca = [ABC]\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}\right).$$

The function $f(x) = \dfrac{1}{\sin x}$ is convex on the interval $(0, \pi)$. Jensen's inequality then implies

$$f(\alpha)+f(\beta)+f(\gamma) \geq 3f\left(\frac{\alpha+\beta+\gamma}{3}\right) = 3f\left(\frac{\pi}{3}\right) = \frac{3}{\sin\left(\frac{\pi}{3}\right)} = 2\sqrt{3}$$

with equality if and only if $\alpha=\beta=\gamma=\pi/3$. Therefore, $a^2+b^2+c^2 \geq 4\sqrt{3}\cdot[ABC]$ so that

$$\lambda = \frac{2[ABC]}{a^2+b^2+c^2} \leq \frac{2[ABC]}{4\sqrt{3}\cdot[ABC]} = \frac{\sqrt{3}}{6}$$

with equality if and only if the triangle ABC is equilateral.

Solution 2 by John Nord, Spokane, WA

Without loss of generality we can denote the coordinates of $\triangle ABC$ as $A(0,0), B(1,0), C(b,c)$, the coordinates of the Lemoine point L as (x_1, y_1), the constant of proportionality from L to the sides as λ, the coordinates on AB of the foot of the perpendicular from L to AB as $D(x_1, 0)$, the coordinates on BC of the foot of the perpendicular from L to BC as $E(x_2, y_2)$ and the coordinates on AC of the foot of the perpendicular from L to AC as $F(x_3, y_3)$.

The distance from L to AB equals $LD = \lambda\cdot 1$.
The distance from L to BC equals $LE = \lambda\cdot\sqrt{(1-b)^2+c^2}$ and
The distance from L to AC equals $LF = \lambda\cdot\sqrt{b^2+c^2}$.

The coordinates of E can be found by finding the intersection of LE and BC. That is, by solving:

$$\begin{cases} y = \dfrac{c}{b-1}x + \dfrac{c}{1-b}, \text{ and} \\ y = \dfrac{1-b}{c}x + y_1 + \dfrac{b-1}{c}x_1. \end{cases}$$

And the coordinates of F can be found by finding the intersection of LF and AC. That is, by solving,

$$\begin{cases} y = \dfrac{c}{b}x \text{ and} \\ y = \dfrac{-b}{c}x + y_1 + \dfrac{b}{c}x_1. \end{cases}$$

Once we have computed (x_2, y_2) and (x_3, y_3) in terms of b, c, x_1 and λ, we apply the distance relationships above. This results in:

$$x_1 = \frac{b + b^2 + c^2}{2(1 - b + b^2 + c^2)} \qquad y_1 = \lambda = \frac{c}{2(1 - b + b^2 + c^2)}.$$

The maximum value of λ is obtained by solving the system of partial derivatives

$$\begin{cases} \dfrac{\partial \lambda}{\partial b} = 0 \\ \dfrac{\partial \lambda}{\partial c} = 0. \end{cases}$$

This yields: $c = \dfrac{\sqrt{3}}{2}$ and $b = \dfrac{1}{2}$. Substituting these values into y_1 above gives $\lambda = \dfrac{\sqrt{3}}{6}$ as the maximum value of the constant of proportionality.

Solution 3 by Charles Mc Cracken, Dayton, OH

The Lemoine point is also the intersection of the symmedians.

The medians of a triangle divide the triangle in two equal areas.

The medians intersect at the centroid, G.

Any point other than G is closer than G to one side of the triangle.

In $\triangle ABC$ let a denote the side (and its length) opposite $\angle A$, b the side opposite $\angle B$, and c the side opposite $\angle C$. Let L denote the Lemoine point.

If the distance from L to side a is λa, then λa less the distance from G to a we call γa.

Similarly for sides b and c.

For $\lambda = \gamma$, L must coincide with G.

This will happen when the medians and symmedians coincide.

This occurs when the triangle is equiangular $(60^0 - 60^0 - 60^0)$ and hence equilateral $(a = b = c)$.

In that case, $\lambda = \dfrac{\sqrt{3}}{6} \equiv 0.289$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain, John Hawkins and David Stone (jointly), Statesboro, GA; Kee-Wai Lau, Hong Kong, China; Tom Leong, Scranton, PA, and the proposer.

- 5065: *Mihály Bencze, Brasov, Romania.*

Let n be a positive integer and let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers. Prove that

$$1) \qquad \sum_{i,j=1}^{n} |(i-j)(x_i - x_j)| = \frac{n}{2} \sum_{i,j=1}^{n} |x_i - x_j|.$$

2) $$\sum_{i,j=1}^{n}(i-j)^2 = \frac{n^2(n^2-1)}{6}.$$

Solution 1 by Paul M. Harms, North Newton, KS

1) Both summations in part 1) have the same terms for $i > j$ that they have for $i < j$ and have 0 for $i = j$. Equality will be shown for $i > j$.

Each row below is the left summation of part 1) of the problem for $i > j$ and for a fixed j starting with $j = 1$.

$$\begin{array}{l} 1(x_2-x_1)+2(x_3-x_1)+\ldots+(n-1)(x_n-x_1) \\ 1(x_3-x_2)+2(x_4-x_2)+\ldots+(n-2)(x_n-x_2) \\ \qquad\qquad\qquad \vdots \\ 1(x_{n-1}-x_{n-2})+2(x_n-x_{n-2}) \\ 1(x_n-x_{n-1}) \end{array}$$

The coefficient of x_1 is $(-1)[1+2+\ldots+(n-1)] = \dfrac{-(n-1)n}{2}$. Note that the coefficient of x_n (looking at the diagonal from lower left to upper right is
$1+2+\ldots+(n-1) = \dfrac{(n-1)n}{2}$.

The coefficient of x_2 is $(-1)[1+2+\ldots+(n-2)]+1 = \dfrac{-(n-2)(n-1)}{2}+1$, where the one is the coefficient of x_2 in row 1.

The coefficient of x_{n-1} is the negative of the coefficient of x_2.

The coefficient of x_r where r is a positive integer less than $\dfrac{n+1}{2}$ is

$$\begin{aligned} (-1)[1+2+\ldots(n-r)]+0+1+\ldots(r-1) &= \frac{(-1)(n-r)(n-r+1)}{2}+\frac{(r-1)r}{2} \\ &= \frac{(-1)n(n-2r+1)}{2} \\ &= (-1)\frac{n}{2}[(n-r)+(1-r)]. \end{aligned}$$

The coefficients of x_r andx_{n+1-r} are the negatives of each other.

If we write out the right summation of part 1) for $i > j$, we can obtain a triangular form like that above except that each coefficient of the difference of the $x's$ is 1. Using the form just explained, the coefficient of x_1 is $(-1)(n-1)$ and the coefficient of x_n along the diagonal is $(n-1)$.

The coefficient of x_2 is $(-1)(n-2)+1$ where the $(+1)$ is the coefficient of x_2 in row 1.

For x_r, where r is a positive integer less than $\dfrac{n+1}{2}$, the coefficient is
$(-1)(n-r)+(r-1)$ where $(r-1)$ comes from the x_r having coefficients of one in each of the first $(r-1)$ rows. The coefficient of x_r on the right side of the inequaity of part 1) is then $\dfrac{n}{2}(-1)[(n-r)+(1-r)]$ which is the same as the left side of the inequality.

Also, the coefficients of x_r and x_{n+1-r} are negative of each other.

2) To show part 2), first consider the summation of each of the three terms $i^2, j^2, -2ij$. For each j, the summation of i^2 from $i=1$ to n is $1^2+2^2+\ldots n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Then the summation of i^2 where both i and j go from 1 to n is $\dfrac{n(n+1)(2n+1)}{6}$. The summation of j^2 is the same value.

The summation of ij is

$$\begin{aligned} 1(1+2+\ldots+n)+2(1+2+\ldots+n)+\ldots+n(1+2+\ldots+n) &= (1+2+\ldots+n)^2 \\ &= \frac{n^2(n+1)^2}{2^2} \end{aligned}$$

The total summation of the left side of part 2) is

$$\begin{aligned} \frac{2n^2(n+1)(2n+1)}{6} - \frac{2n^2(n+1)^2}{2^2} &= n^2(n+1)\left[\frac{2n+1}{3} - \frac{n+1}{2}\right] \\ &= \frac{n^2(n+1)(n-1)}{6}. \end{aligned}$$

Solution 2 by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy

We begin with 1). The result is achieved by a double induction. For $n=1$ there is nothing to say. Let's suppose that 1) holds for any $1 \le n \le m$. For $n=m+1$ the equality reads as

$$\begin{aligned} &\sum_{i,j=1}^{m+1} |i-j|\,|x_i-x_j| = \\ &\sum_{i,j=1}^{m} |i-j|\,|x_i-x_j| + \sum_{i=1}^{m+1} |i-m-1|\,|x_i-x_{m+1}| + \sum_{j=1}^{m+1} |m+1-j|\,|x_{m+1}-x_j| = \\ &\frac{m}{2}\sum_{i,j=1}^{m} |x_i-x_j| + 2\sum_{i=1}^{m+1} |i-m-1|\,(x_{m+1}-x_i). \end{aligned}$$

(in the second passage the induction hypotheses has been used) and we need it equal to

$$\frac{m+1}{2}\sum_{i,j=1}^{m+1} |x_i-x_j| = \frac{m}{2}\sum_{i,j=1}^{m} |x_i-x_j| + \frac{1}{2}\sum_{i,j=1}^{m} |x_i-x_j| + (m+1)\sum_{i=1}^{m} |x_i-x_{m+1}|.$$

Comparing the two quantities we have to prove

$$2\sum_{i=1}^{m+1}(m+1-i)(x_{m+1}-x_i) = \frac{1}{2}\sum_{i,j=1}^{m} |x_i-x_j| + (m+1)\sum_{i=1}^{m} |x_i-x_{m+1}|$$

or

$$\sum_{i=1}^{m}(x_{m+1}-x_i)(m+1-2i) = \frac{1}{2}\sum_{i,j=1}^{m} |x_i-x_j|$$

or

$$-\sum_{i=1}^{m} x_i(m+1-2i) = \frac{1}{2}\sum_{i,j=1}^{m} |x_i - x_j| \qquad \text{since} \qquad \sum_{i=1}^{m}(m+1-2i) = 0.$$

Here starts the second induction. For $m = 1$ there is nothing to do as well. Let's suppose that the equality holds true for any $1 \leq m \leq r$. For $m = r+1$ we have to prove that

$$-\sum_{i=1}^{r+1} x_i(r+2-2i) = \frac{1}{2}\sum_{i,j=1}^{r} |x_i - x_j| + \frac{1}{2}\sum_{i=1}^{r+1}(x_{r+1} - x_i) + \frac{1}{2}\sum_{i=1}^{r+1}(x_{r+1} - x_i).$$

which, by using the induction hypotheses is

$$-\sum_{i=1}^{r} x_i(r+1-2i) - \sum_{i=1}^{r} x_i + r x_{r+1} = -\sum_{i=1}^{r} x_i(r+1-2i) + \sum_{i=1}^{r+1}(x_{r+1} - x_i).$$

or

$$-\sum_{i=1}^{r} x_i + r x_{r+1} = (r+1)x_{r+1} - x_{r+1} - \sum_{i=1}^{r} x_i.$$

namely the expected result.

To prove 2) we employ 1) by calculating $\frac{n}{2}\sum_{i,j=1}^{n} |i-j|$. The symmetry of the absolute value yields

$$\frac{n}{2}\sum_{i,j=1}^{n} |i-j| = n\sum_{1\leq i<j\leq n}(j-i) = n\sum_{i=1}^{n}\sum_{j=i+1}^{n}(j-i) = n\sum_{i=1}^{n}\sum_{k=1}^{n-i} k = \frac{n}{2}\sum_{i=1}^{n}(n-i)(n-i+1).$$

The last sum is equal to $\frac{n}{2}\sum_{k=1}^{n-1} k(k+1)$.

In the last step we show that $\sum_{k=1}^{n-1} k(k+1) = \frac{n^3-n}{3}$.

For $n = 1$ both sides are 0. Let's suppose it is true for $1 \leq n \leq m-1$.

For $n = m$ we have

$$\sum_{k=1}^{m-1} k(k+1) + m(m+1) = \frac{m^3-m}{3} + m(m+1) = m(m+1)\frac{m+2}{3} = \frac{(m+1)^3-(m+1)}{3}.$$

Finally,

$$\frac{n}{2}\frac{n^3-n}{3} = n^2\frac{n^2-1}{6}$$

The proof is complete.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Michael C. Faleski, University Center, MI; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5066: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b, and c be the sides of an acute-angled triangle ABC. Let $abc = 1$. Let H be the orthocenter, and let d_a, d_b, and d_c be the distances from H to the sides BC, CA, and AB respectively. Prove or disprove that

$$3(a+b)(b+c)(c+a) \geq 32(d_a + d_b + d_c)^2.$$

Solution by Kee-Wai Lau, Hong Kong, China

We prove the inequality. First we have $(a+b)(b+c)(c+a) \geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca}) = 8$. Hence it suffices to prove that $d_a + d_b + d_c \leq \dfrac{\sqrt{3}}{2}$. Let s, r, R be respectively the semi-perimeter, in-radius and circumradius of triangle ABC. Let the foot of the perpendicular from A to BC be D and the foot of the perpendicular from B to AC be E so that $\triangle BCE \sim \triangle BHD$. Hence,

$$\begin{aligned} d_a &= \overline{DH} = \frac{(\overline{BD})(\overline{CE})}{\overline{BE}} \\ &= \frac{(c\cos B)(a\cos C)}{c\sin A} = 2R\cos B\cos C, \text{ and similarly,} \\ d_b &= 2R\cos C\cos A \text{ and } d_c = 2R\cos A\cos B. \end{aligned}$$

Therefore, by the well known equality

$$\begin{aligned} \cos A\cos B + \cos B\cos C + \cos C\cos A &= \frac{r^2+s^2-4R^2}{4R^2}, \text{ we have} \\ d_a + d_b + d_c &= \frac{r^2+s^2-rR^2}{2R}. \end{aligned}$$

And by a result of J. C. Gerretsen: Ongelijkheden in de Driehoek Nieyw Tijdschr.Wisk. 41(1953), 1-7, we have $s^2 \leq 4R^2 + 4Rr + 3r^2$. Thus

$$d_a + d_b + d_c = \frac{2r(R+r)}{R} \leq 3r,$$

which follows from L. Euler's result that $R \geq 2r$.

It remains to show that $r \leq \dfrac{1}{2\sqrt{3}}$. But this follows from the well known result that $s \geq 3\sqrt{3}r$ and the fact that $1 = abc = 4rsR \geq 4r(3\sqrt{3})r(2r) = 24\sqrt{3}r^3$.

This completes the solution.

Also solved by the proposer.

- 5067: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be complex numbers such that $a + b + c = 0$. Prove that

$$\max\{|a|, |b|, |c|\} \leq \frac{\sqrt{3}}{2}\sqrt{|a|^2+|b|^2+|c|^2}.$$

Solution by Tom Leong, Scranton, PA

Since $a+b+c=0$, $|a|$, $|b|$, and $|c|$ form the sides of a (possibly degenerate) triangle. It follows from the triangle inequality that the longest side, $\max\{|a|,|b|,|c|\}$, cannot exceed half of the perimeter, $\frac{1}{2}(|a|+|b|+|c|)$, of the triangle. Using this fact along with the Cauchy-Schwarz inequality gives the desired result:

$$\begin{aligned}
\max\{|a|,|b|,|c|\} &\leq \frac{1}{2}(|a|+|b|+|c|)\\
&= \frac{1}{2}(1\cdot|a|+1\cdot|b|+1\cdot|c|)\\
&\leq \frac{1}{2}\sqrt{1^2+1^2+1^2}\sqrt{|a|^2+|b|^2+|c|^2}\\
&= \frac{\sqrt{3}}{2}\sqrt{|a|^2+|b|^2+|c|^2}.
\end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky, Centeral Islip, NY; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Manh Dung Nguyen (student, Special High School for Gifted Students), HUS, Vietnam; Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy; Boris Rays, Brooklyn, NY; Dmitri V. Skjorshammer (student, Harvey Mudd College), Claremont, CA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before February 15, 2010

- 5086: *Proposed by Kenneth Korbin, New York, NY*

  Find the value of the sum
  $$\frac{2}{3}+\frac{8}{9}+\cdots+\frac{2N^2}{3^N}.$$

- 5087: *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers a, b, c, and d such that $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$ with $a < b < c < d$. Rationalize and simplify
  $$\frac{\sqrt{x+y}-\sqrt{x}}{\sqrt{x+y}+\sqrt{x}} \quad \text{if} \quad \begin{cases} x = bc+bd+cd, & \text{and} \\ y = ab+ac+ad. \end{cases}$$

- 5088: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b be positive integers. Prove that
  $$\frac{\varphi(ab)}{\sqrt{\varphi^2(a^2)+\varphi^2(b^2)}} \le \frac{\sqrt{2}}{2},$$
  where $\varphi(n)$ is Euler's totient function.

- 5089: *Proposed by Panagiote Ligouras, Alberobello, Italy*

  In $\triangle ABC$ let $AB = c, BC = a, CA = b, r =$ the in-radius and r_a, r_b, and $r_c =$ the ex-radii, respectively.

  Prove or disprove that
  $$\frac{(r_a-r)(r_b+r_c)}{r_ar_c+rr_b}+\frac{(r_c-r)(r_a+r_b)}{r_cr_b+rr_a}+\frac{(r_b-r)(r_c+r_a)}{r_br_a+rr_c} \ge 2\left(\frac{ab}{b^2+ca}+\frac{bc}{c^2+ab}+\frac{ca}{a^2+bc}\right).$$

- 5090: *Proposed by Mohsen Soltanifar (student), University of Saskatchewan, Canada*

  Given a prime number p and a natural number n. Calculate the number of elementary matrices $E_{n\times n}$ over the field Z_p.

- 5091: *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k, p \geq 0$ be nonnegative integers. Evaluate the integral

$$\int_{-\pi/2}^{\pi/2} \frac{\sin^{2p} x}{1+\sin^{2k+1} x + \sqrt{1+\sin^{4k+2} x}} dx.$$

Solutions

- 5068: *Proposed by Kenneth Korbin, New York, NY.*

  Find the value of

$$\sqrt{1+2009\sqrt{1+2010\sqrt{1+2011\sqrt{1+\cdots}}}}.$$

Solution by Dmitri V. Skjorshammer (student, Harvey Mudd College), Claremont, CA

To solve this, we apply Ramanujan's nested radical. Consider the identity $(x+n)^2 = x^2+2nx+n^2$, which can be rewritten as

$$x+n = \sqrt{n^2 + x((x+n)+n)}.$$

Now, the $(x+n)+n$ term has the same form as the left-hand side, so we can write it in terms of a radical:

$$x+n = \sqrt{n^2 + x\sqrt{n^2+(x+n)((x+2n)+n)}}$$

Repeating this process, ad infinitum, yields Ramanujan's nested radical:

$$x+n = \sqrt{n^2 + x\sqrt{n^2+(x+n)\sqrt{n^2+\cdots}}}$$

With $n=1$ and $x=2009$, the right-hand side becomes the expression in the problem. It follows that the value is 2010.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Pat Costello, Richmond, KY; Michael N. Fried, Kibbutz Revivim, Israel; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Nguyen Van Vinh (student, Belarusian State University), Minsk, Belarus, and the proposer.

- 5069: *Proposed by Kenneth Korbin, New York, NY.*

  Four circles having radii $\frac{1}{14}$, $\frac{1}{15}$, $\frac{1}{x}$ and $\frac{1}{y}$ respectively, are placed so that each of the circles is tangent to the other three circles. Find positive integers x and y with $15 < x < y < 300$.

Solution by Bruno Salgueiro Fanego, Viveiro, Spain

If all the circles are tangent in a point, the problem is not interesting because x and y can take on any value for which $15 < x < y < 300$. So we assume that the circles are not mutually tangent at a point.
By Descarte's circle theorem with ϵ_1, ϵ_2 and ϵ_3 being the curvature of the first three circles, the curvature ϵ_4 of the fourth circle can be obtained with Soddy's formula:

$$\begin{aligned}
\epsilon_4 &= \epsilon_1 + \epsilon_2 + \epsilon_3 \pm 2\sqrt{\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1}, \text{ that is,} \\
y &= 14 + 15 + x \pm 2\sqrt{14 \cdot 15 + 15 \cdot x + x \cdot 14} \\
y &= 29 + x + \pm 2\sqrt{210 + 29x}
\end{aligned}$$

Then, $210 + 29x$ must be a perfect square, say a^2. Since, $15 < x < 300$,

$$25^2 < 210 + 29x < 95^2, \text{ so}$$
$$26 \le a \le 94.$$

Thus,

$$29 \Big| (a^2 - 210).$$

The only integers a, $26 \le a \le 94$, which satisfy this condition are $35, 52, 64, 81$, and 93. Taking into account that $15 < x < y < 300$, we have:

$$\begin{aligned}
\text{For } a &= 35,\ x = \mathit{35} \text{ and so } y = \mathit{29} + x \pm \mathit{2a} = \mathit{134} \\
\text{For } a &= 52,\ x = 86 \text{ and } y = \mathit{219}; \\
\text{For } a &= 64,\ x = 134 \text{ and } y = \mathit{291};
\end{aligned}$$

and for $a \in \{81, 93\}$, none of the obtained values of y is valid.

Thus the only pairs of integers x and y with $15 < x < y < 300$ are

$$(x, y) \in \Big\{(35, 134), (86, 219), (134, 291)\Big\}.$$

Also solved by Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; John Hawkins and David Stone (jointly), Statesboro, GA; Antonio Ledesma Vila, Requena-Valencia, Spain, and the proposer.

- **5070:** *Proposed by Isabel Iriberri Díaz and José Luis Díaz- Barrero, Barcelona, Spain.*

Find all real solutions to the system

$$\left.\begin{aligned}
9(x_1^2 + x_2^2 - x_3^2) &= 6x_3 - 1, \\
9(x_2^2 + x_3^2 - x_4^2) &= 6x_4 - 1, \\
&\cdots\cdots\cdots \\
9(x_n^2 + x_1^2 - x_2^2) &= 6x_2 - 1.
\end{aligned}\right\}$$

Solution by Antonio Ledesma Vila, Requena -Valencia, Spain

Add all

$$\begin{aligned}
9(x_1^2+x_2^2-x_3^2) &= 6x_3-1\\
9(x_2^2+x_3^2-x_4^2) &= 6x_4-1\\
&\cdots\\
9(x_n^2+x_1^2-x_2^2) &= 6x_2-1
\end{aligned}$$

$$9\left(\sum_{i=1}^n x_i^2+\sum_{i=1}^n x_i^2-\sum_{i=1}^n x_i^2\right)=6\sum_{i=1}^n x_i-n$$

$$9\sum_{i=1}^n x_i^2=6\sum_{i=1}^n x_i-n$$

$$\sum_{i=1}^n (3x_i)^2=2\sum_{i=1}^n (3x_i)-n$$

$$\sum_{i=1}^n (3x_i)^2-2\sum_{i=1}^n (3x_i)+n=0$$

$$\sum_{i=1}^n (3x_i-1)^2=0,$$

$$x_i=\frac{1}{3}\text{ for all } i$$

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong; China; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; Boris Rays, Brooklyn, NY; Dmitri V. Skjorshammer (student, Harvey Mudd College), Claremont, CA, and the proposer.

- 5071: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let h_a, h_b, h_c be the altitudes of $\triangle ABC$ with semi-perimeter s, in-radius r and circum-radius R, respectively. Prove that

$$\frac{1}{4}\left(\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c}\right)\le\frac{R^2}{r}\left(\sin^2 A+\sin^2 B+\sin^2 C\right).$$

Solution by Charles McCracken, Dayton, OH

Multiply both sides of the inequality by 4 to obtain

$$\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c} \le \frac{(2R)^2}{r}\left[\sin^2 A+\sin^2 B+\sin^2 C\right]$$

$$\frac{s(2s-a)}{h_a}+\frac{s(2s-b)}{h_b}+\frac{s(2s-c)}{h_c} \le \frac{1}{r}\left[(2R)^2\sin^2 A+(2R)^2\sin^2 B+(2R)^2\sin^2 C\right].$$

Now $2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ so the inequality becomes

$$\frac{s(2s-a)}{h_a} + \frac{s(2s-b)}{h_b} + \frac{s(2s-c)}{h_c} \le \frac{1}{r}\Big(a^2+b^2+c^2\Big).$$

From Johnson (Roger A. Johnson, Advanced Euclidean Geometry, Dover, 2007, p. 11) we have

$$h_a = \frac{2\Delta}{a},\ h_b = \frac{2\Delta}{b},\ h_c = \frac{2\Delta}{c},\ \text{where } \Delta \text{ represents the area of the triangle.}$$

The inequality now takes the form

$$\frac{as(2s-a)}{2\Delta} + \frac{bs(2s-b)}{2\Delta} + \frac{cs(2s-c)}{2\Delta} \le \frac{1}{r}\Big(a^2+b^2+c^2\Big).$$

Since $\Delta = rs$, we now have our inequality in the form

$$\begin{aligned}
\frac{as(2s-a)}{2rs} + \frac{bs(2s-b)}{2rs} + \frac{cs(2s-c)}{2rs} &\le \frac{1}{r}\Big(a^2+b^2+c^2\Big)\\
\frac{a(2s-a)}{2} + \frac{b(2s-b)}{2} + \frac{c(2s-c)}{2} &\le \Big(a^2+b^2+c^2\Big)
\end{aligned}$$

Substituting $a+b+c$ for $2s$ we have

$$\begin{aligned}
a(b+c)+b(c+a)+c(a+b) &\le 2a^2+2b^2+2c^2\\
ab+ac+bc+ba+ca+cb &\le 2a^2+2b^2+2c^2\\
ab+bc+ca &\le a^2+b^2+c^2
\end{aligned}$$

This last inequality, $ab+bc+ca \le a^2+b^2+c^2$, can be readily proved true for any triple of positive numbers a, b, c by letting $b = a+\delta$ and $c = a+\epsilon$ with $0 < \delta < \epsilon$. Hence the original inequality holds.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5072: *Proposed by Panagiote Ligouras, Alberobello, Italy.*

  Let a, b and c be the sides, l_a, l_b, l_c the bisectors, m_a, m_b, m_c the medians, and h_a, h_b, h_c the heights of $\triangle ABC$. Prove or disprove that

  a) $\dfrac{(-a+b+c)^3}{a} + \dfrac{(a-b+c)^3}{b} + \dfrac{(a+b-c)^3}{c} \ge \dfrac{4}{3}\Big(m_a \cdot l_a + l_b \cdot h_b + h_c \cdot m_c\Big)$

  b) $3\displaystyle\sum_{cyc} \frac{(-a+b+c)^3}{a} \ge 2\sum_{cyc}[m_a(l_a+h_a)].$

Solution by proposer

We have

$$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}\geq a^2+b^2+c^2. \quad (1)$$

In fact, the equality is homogeneous and putting $a+b=c=1$ gives

$$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}\geq a^2+b^2+c^2 \Leftrightarrow \sum_{cyc}\frac{(1-2a)^3}{a}\geq\sum_{cyc}a^2. \quad (2)$$

Applying Chebyshev's Inequality gives

$$\sum_{cyc}\frac{(1-2a)^3}{a}=\sum_{cyc}\frac{1}{a}(1-2a)^3\geq\frac{1}{3}\bigg(\sum_{cyc}\frac{1}{a}\bigg)\cdot\bigg[\sum_{cyc}(1-2a)^3\bigg]. \quad (3)$$

Using the well known equalities

$$\sum x^3=\Big(\sum x\Big)^3-3(x+y)(y+z)(z+x). \quad (4)$$

$$(a+b+c)\bigg(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\bigg)\geq 3^2=9 \quad (5)$$

and applying (4), (3), and (5) we have

$$\sum_{cyc}\frac{(1-2a)^3}{a}\geq\frac{1}{3}\bigg(\sum_{cyc}\frac{1}{a}\bigg)\cdot\bigg[\sum_{cyc}(1-2a)^3\bigg]$$

$$=\frac{1}{3}\bigg(\sum_{cyc}\frac{1}{a}\bigg)\cdot\bigg[(1-2a+1-2b+1-2c)^3-3(1-2a+1-2b)(1-2b+1-2c)(1-2c+1-2a)\bigg]$$

$$=\frac{1}{3}\bigg(\sum_{cyc}\frac{1}{a}\bigg)\cdot[1-24abc]$$

$$=\frac{1}{3}\bigg(\sum_{cyc}\frac{1}{a}\bigg)\cdot\Big(\sum a\Big)-\frac{24}{3}\bigg(\sum_{cyc}ab\bigg)$$

$$\geq\frac{1}{3}\cdot 9-8\bigg(\sum_{cyc}ab\bigg)$$

$$\Leftrightarrow\sum_{cyc}\frac{(1-2a)^3}{a}\geq 3-8\bigg(\sum_{cyc}ab\bigg). \quad (6)$$

We have

$$3-8\bigg(\sum_{cyc}ab\bigg)\geq\sum_{cyc}a^2. \quad (7)$$

In fact,

$$3-8\bigg(\sum_{cyc}ab\bigg)\geq\sum_{cyc}a^2\Leftrightarrow 3-6\bigg(\sum_{cyc}ab\bigg)\geq\sum_{cyc}a^2+2\bigg(\sum_{cyc}ab\bigg)$$

$$\Leftrightarrow \quad 3-6\Big(\sum_{cyc} ab\Big) \geq \Big(\sum_{cyc} a\Big)^2 = 1 \Leftrightarrow 3-6\Big(\sum_{cyc} ab\Big) \geq 1-3$$

$$\Leftrightarrow \quad \sum_{cyc} ab \leq \frac{1}{3} = \frac{\left(\sum a\right)^2}{3}$$

$$\Leftrightarrow \quad \sum_{cyc} (a-b)^2 \geq 0, \text{ and this last statement is true.}$$

Using (6) and (7) we have

$$\sum_{cyc} \frac{(1-2a)^3}{a} \geq 3 - 8\Big(\sum_{cyc} ab\Big) \geq \sum_{cyc} a^2$$
$$\Leftrightarrow \sum_{cyc} \frac{(1-2a)^3}{a} \geq \sum_{cyc} a^2, \text{ and (1) is true.}$$

Is well known that

$$\begin{aligned} a^2+b^2 &= 2m_c^2 + \frac{1}{2}c^2 \qquad \text{(A)} \\ c^2+b^2 &= 2m_a^2 + \frac{1}{2}a^2 \qquad \text{(B)} \\ c^2+a^2 &= 2m_b^2 + \frac{1}{2}b^2 \qquad \text{(C)} \end{aligned}$$

For (A),(B), and (C)

$$\begin{aligned} m_a^2+m_b^2+m_c^2 &= \frac{3}{4}(a^2+b^2+c^2) \text{ and} \\ a^2+b^2+c^2 &= \frac{4}{3}(m_a^2+m_b^2+m_c^2) \quad (8) \end{aligned}$$

It is also well known that

$$m_a \geq l_a \geq h_a, \quad m_b \geq l_b \geq h_b, \quad m_c \geq l_c \geq h_c. \quad (9)$$

Using (9) we have

$$m_a^2 \geq m_a \cdot l_a \geq m_a \cdot h_a, \quad m_b^2 \geq m_b \cdot l_b \geq m_b \cdot h_b, \quad m_c^2 \geq m_c \cdot l_c \geq m_c \cdot h_c \quad \text{(D)}$$

$$m_a^2 \geq l_a \cdot h_a, \quad m_b^2 \geq l_b \cdot h_b, \quad m_c^2 \geq l_c \cdot h_c, \quad \text{(E)}$$

Using (8) and (D) we have

$$a^2+b^2+c^2 \geq \frac{4}{3}(m_a l_a + m_b l_b + m_c l_c). \quad (10)$$

$$a^2+b^2+c^2 \ge \frac{4}{3}(m_a h_a + m_b h_b + m_c h_c). \quad (11)$$

And using (8), (D), and (E) we have

$$a^2+b^2+c^2 \ge \frac{4}{3}(m_a l_a + l_b h_b + h_c m_c). \quad (12)$$

For part a of the problem, using (1) and (12) we have

$$\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c} \ge \frac{4}{3}\Big(m_a\cdot l_a + l_b\cdot h_b + h_c\cdot m_c\Big)$$

For part b of the problem, using (1), (10) and (11) we have

$$2\Big[\frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}\Big] \ge$$
$$\frac{4}{3}\Big(m_a\cdot l_a+m_b\cdot l_b+m_c\cdot l_c\Big)+\frac{4}{3}\Big(m_a\cdot h_a+m_b\cdot h_b+m_c\cdot h_c\Big)$$

$$\Leftrightarrow \quad \frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}$$

$$\ge \quad \frac{2}{3}\Big(m_a\cdot l_a+m_b\cdot l_b+m_c\cdot l_c+m_a\cdot h_a+m_b\cdot h_b+m_c\cdot h_c\Big)$$

$$\Leftrightarrow \quad \frac{(-a+b+c)^3}{a}+\frac{(a-b+c)^3}{b}+\frac{(a+b-c)^3}{c}$$

$$\ge \quad \frac{2}{3}\Big[m_a\cdot(l_a+h_a)+m_b\cdot(l_b+h_b)+m_c\cdot(l_c+h_c)\Big]$$

- 5073: *Proposed by Ovidiu Furdui, Campia-Turzii, Cluj, Romania.*

Let $m > -1$ be a real number. Evaluate

$$\int_0^1 \{\ln x\} x^m dx,$$

where $\{a\} = a - [a]$ denotes the fractional part of a.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

$$I_m = \int_0^1 \{\ln x\}x^m dx = \int_0^1 (\ln x - [\ln x])x^m dx = \int_0^1 (\ln x)x^m dx - \int_0^1 [\ln x]x^m dx = A - B$$

where $A = \int_0^1 (\ln x)x^m dx$ and $B = \int_0^1 [\ln x]x^m dx$. Integrating by parts $\Big(\int udv = uv - \int vdu$ with $u = \ln x$ and $dv = x^m dx\Big)$, and by using Barrow's and L'Hospital's rule we obtain,

$$\int (\ln x)x^m dx \;=\; \frac{(\ln x)x^{m+1}}{m+1} - \int \frac{x^m}{m+1}dx = \frac{(\ln x)x^{m+1}}{m+1} - \frac{x^{m+1}}{(m+1)^2}$$

$$
\begin{aligned}
\Longrightarrow A &= \left.\frac{(\ln x)x^{m+1}}{m+1} - \frac{x^{m+1}}{(m+1)^2}\right|_0^1 \\
&= \frac{(\ln 1)1^{m+1}}{m+1} - \frac{1^{m+1}}{(m+1)^2} - \left(\lim_{x\to 0^+}\frac{(\ln x)x^{m+1}}{m+1} - \frac{0^{m+1}}{(m+1)^2}\right) \\
&= \frac{-1}{(m+1)^2} - \lim_{x\to 0^+}\frac{(\ln x)}{(m+1)x^{-(m+1)}} \\
&= \frac{-1}{(m+1)^2} - \lim_{x\to 0^+}\frac{x^{-1}}{-(m+1)^2x^{-(m+2)}} \\
&= \frac{-1}{(m+1)^2} + \lim_{x\to 0^+}\frac{x^{m+1}}{(m+1)^2} \\
&= \frac{-1}{(m+1)^2}
\end{aligned}
$$

With the partition $\left\{\ldots, e^{-n},\ e^{-n+1},\ e^{-n+2}, \ldots,\ e^{-2},\ e^{-1},\ e^0 = 1\right\}$ of $(0,1]$, being $[\ln x] = -n$ for $e^{-n} \le x < e^{-n+1}$, and $\left|e^{-m-1}\right| < 1$,

$$
\begin{aligned}
B &= \int_0^1 [\ln x]x^m dx = \sum_{n=1}^{\infty}\int_{e^{-n}}^{e^{-n+1}} [\ln x]x^m dx \\
&= \sum_{n=1}^{\infty}\int_{e^{-n}}^{e^{-n+1}} (-n)x^m dx = \sum_{n=1}^{\infty}\left.\frac{-nx^{m+1}}{m+1}\right|_{e^{-n}}^{e^{-n+1}} \\
&= \sum_{n=1}^{\infty}\frac{-n\left(e^{(-n+1)(m+1)} - e^{-n(m+1)}\right)}{m+1} \\
&= \sum_{n=1}^{\infty}\frac{-n\left(e^{m+1}e^{(-n)(m+1)} - e^{-n(m+1)}\right)}{m+1} \\
&= \sum_{n=1}^{\infty}\frac{-n\left(e^{(m+1)} - 1\right)e^{-n(m+1)}}{m+1} \\
&= \sum_{n=1}^{\infty}\frac{\left(1 - e^{(m+1)}\right)n\left(e^{-m-1}\right)^n}{m+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\left(1-e^{m+1}\right)e^{-m-1}}{m+1}\sum_{n=1}^{\infty}\left(e^{-m-1}\right)^{n-1} \\
&= \frac{e^{-m-1}-1}{m+1}\sum_{n=1}^{\infty}\frac{d}{dx}x^n\Big|_{x=e^{-m-1}} \\
&= \frac{e^{-m-1}-1}{m+1}\frac{d}{dx}\sum_{n=1}^{\infty}x^n\Big|_{x=e^{-m-1}} \\
&= \frac{e^{-m-1}-1}{m+1}\frac{d}{dx}\frac{x}{1-x}\Big|_{x=e^{-m-1}} \\
&= \frac{e^{-m-1}-1}{(m+1)(1-x)^2}\Big|_{x=e^{-m-1}} \\
&= \frac{e^{-m-1}-1}{(m+1)(e^{-m-1}-1)^2} = \frac{1}{(m+1)(e^{-m-1}-1)},\ \text{so} \\
I_m &= A-B = -\frac{1}{(m+1)^2} - \frac{1}{(m+1)(e^{-m-1}-1)} \\
&= \frac{me^{m+1}+1}{(m+1)^2(e^{m+1}-1)}.
\end{aligned}
$$

Solution 2 by the proposer

The integral equals

$$\frac{e^{m+1}}{(m+1)(e^{m+1}-1)} - \frac{1}{(1+m)^2}.$$

We have, by making the substitution $\ln x = y$, that

$$
\begin{aligned}
\int_0^1 \{\ln x\}\, x^m dx &= \int_{-\infty}^{0} \{y\}\, e^{(m+1)y}dy \\
&= \sum_{k=0}^{\infty}\int_{-k-1}^{-k} \{y\}\, e^{(m+1)y}dy \\
&= \sum_{k=0}^{\infty}\int_{-k-1}^{-k} (y-(-k-1))\, e^{(m+1)y}dy \\
&= \sum_{k=0}^{\infty}\int_{-k-1}^{-k} (y+k+1))\, e^{(m+1)y}dy
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \left(\frac{y+k+1}{m+1} e^{(m+1)y} \Big|_{-k-1}^{-k} - \frac{e^{(m+1)y}}{(m+1)^2} \Big|_{-k-1}^{-k} \right) \\
&= \sum_{k=0}^{\infty} \frac{e^{-(m+1)k}}{m+1} - \frac{1}{(m+1)^2} \sum_{k=0}^{\infty} \left(e^{-(m+1)k} - e^{-(m+1)(k+1)} \right) \\
&= \frac{e^{m+1}}{(m+1)(e^{m+1}-1)} - \frac{1}{(1+m)^2},
\end{aligned}
$$

and the problem is solved.

Also solved by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; and David Stone and John Hawkins (jointly), Statesboro, GA.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before March 15, 2010

- 5092: *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with altitude h and with cevian $\overline{CD}$. A circle with radius x is inscribed in $\triangle ACD$, and a circle with radius y is inscribed in $\triangle BCD$ with $x < y$. Find the length of the cevian $\overline{CD}$ if x, y and h are positive integers with $(x, y, h) = 1$.

- 5093: *Proposed by Worapol Ratanapan (student), Montfort College, Chiang Mai, Thailand*

  $6 = 1 + 2 + 3$ is one way to partition 6, and the product of $1, 2, 3$ is 6. In this case, we call each of $1, 2, 3$ a **part** of 6.

  We denote the maximum of the product of all **parts** of natural number n as $N(n)$.

  As a result, $N(6) = 3 \times 3 = 9$, $N(10) = 2 \times 2 \times 3 \times 3 = 36$, and $N(15) = 3^5 = 243$.

  More generally, $\forall n \in N$, $N(3n) = 3^n$, $N(3n+1) = 4 \times 3^{n-1}$, and $N(3n+2) = 2 \times 3^n$.

  Now let's define $R(r)$ in the same way as $N(n)$, but each **part** of r is positive real. For instance $R(5) = 6.25$ and occurs when we write $5 = 2.5 + 2.5$

  Evaluate the following:

  $$\begin{array}{rl} i) & R(2e) \\ ii) & R(5\pi) \end{array}$$

- 5094: *Proposed by Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy*

  Let a, b, c be real positive numbers such that $a + b + c + 2 = abc$. Prove that

  $$2(a^2 + b^2 + c^2) + 2(a + b + c) \geq (a + b + c)^2.$$

- 5095: *Proposed by Zdravko F. Starc, Vršac, Serbia*

  Let F_n be the Fibonacci numbers defined by

  $$F_1 = F_2 = 1,\ F_{n+2} = F_{n+1} + F_n\ (n = 1, 2, \cdots).$$

Prove that

$$\sqrt{F_{n-2}F_{n-1}}+1 \le F_n \le \sqrt{(n-2)F_{n-2}F_{n-1}}+1 \ (n=3,4,\cdots).$$

- 5096: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+\sqrt[4]{ab^3}}+\frac{b}{c+\sqrt[4]{bc^3}}+\frac{c}{a+\sqrt[4]{ca^3}} \ge \frac{3}{2}.$$

- 5097: *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $p \ge 2$ be a natural number. Find the sum

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\lfloor \sqrt[p]{n} \rfloor},$$

where $\lfloor a \rfloor$ denotes the **floor** of a. (Example $\lfloor 2.4 \rfloor = 2$).

Solutions

- 5074: *Proposed by Kenneth Korbin, New York, NY*

  Solve in the reals:

$$\sqrt{25+9x+30\sqrt{x}}-\sqrt{16+9x+30\sqrt{x-1}}=\frac{3}{x\sqrt{x}}.$$

Solution by Antonio Ledesma Vila, Requena-Valencia, Spain

Note that the domain of definition is $x \ge 1$, and that the two radicands are perfect squares:

$$\begin{aligned} 25+9x+30\sqrt{x} &= \left(3\sqrt{x}+5\right)^2 \\ 16+9x+30\sqrt{x-1} &= \left(3\sqrt{x-1}+5\right)^2 \end{aligned}$$

So

$$\begin{aligned} \sqrt{25+9x+30\sqrt{x}}-\sqrt{16+9x+30\sqrt{x-1}} &= \frac{3}{x\sqrt{x}} \\ \sqrt{\left(3\sqrt{x}+5\right)^2}-\sqrt{\left(3\sqrt{x-1}+5\right)^2} &= \frac{3}{x\sqrt{x}} \\ \left|3\sqrt{x}+5\right|-\left|3\sqrt{x-1}+5\right| &= \frac{3}{x\sqrt{x}} \\ (3\sqrt{x}+5)-(3\sqrt{x-1}+5) &= \frac{3}{x\sqrt{x}} \end{aligned}$$

$$\begin{aligned}
\sqrt{x}-\sqrt{x-1} &= \frac{1}{x\sqrt{x}} \\
\frac{1}{\sqrt{x}-\sqrt{x-1}} &= x\sqrt{x} \\
\sqrt{x}+\sqrt{x-1} &= x\sqrt{x} \\
\sqrt{x-1} &= (x-1)\sqrt{x} \\
(x-1) &= (x-1)^2 x \\
(x-1)\Big(1-(x-1)x\Big) &= 0
\end{aligned}$$

Therefore, $x=1$ or $x^2-x-1=0 \Longrightarrow x=\dfrac{1\pm\sqrt{5}}{2}$. But $\dfrac{1-\sqrt{5}}{2}$ is an extraneous root. Hence, the only two real solutions are $x=1$ and $x=\dfrac{1+\sqrt{5}}{2}=\phi,$ the golden ratio.

Also solved by Daniel Lopez Aguayo, Puebla, Mexico; José Luis Díaz-Barrero, Barcelona, Spain; Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Katherine Janell Eyre (student, Angelo State University), San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Kibbutz Revivim, Israel; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy; David C. Wilson, Winston-Salem, NC, and the proposer.

5075: *Proposed by Kenneth Korbin, New York, NY*

An isosceles trapezoid is such that the length of its diagonal is equal to the sum of the lengths of the bases. The length of each side of this trapezoid is of the form $a+b\sqrt{3}$ where a and b are positive integers.

Find the dimensions of this trapezoid if its perimeter is $31+16\sqrt{3}$.

Solution by Michael N. Fried, Kibbutz Revivim, Israel

Let the equal sides be $s=a+b\sqrt{3}$ and the bases be $b_1=p+q\sqrt{3}$ and $b_2=u+v\sqrt{3}$. Since each of its diagonals d is the sum of the bases, we have:

$$d=b_1+b_2=(p+u)+(q+v)\sqrt{3}=y+x\sqrt{3},$$

where a,b,p,q,u,v, and accordingly, y and x are all positive integers.

We begin by making some observations.
I. Since the diagonal $d = b_1 + b_2$, we have $P = 2s + d = 31 + 16\sqrt{3}$ (1)

II. From (1), we have,

$$\begin{aligned} s = a + b\sqrt{3} &= \left(\frac{31-y}{2}\right) + \left(\frac{16-x}{2}\right)\sqrt{3} \text{ or} \\ a &= \frac{31-y}{2} \quad (2) \\ b &= \frac{16-x}{2} \quad (3) \end{aligned}$$

And since a and b are positive integers, (2) and (3) imply that *y is odd and x even.*

III. Since any isosceles trapezoid can be inscribed in a circle, we can apply Ptolemy's theorem here to obtain the equation: $d^2 - s^2 = b_1 b_2$ (4). This, together with the fact that $d = b_1 + b_2$, implies that the bases b_1 and b_2 are the solutions of the equation $\mathrm{b}^2 - db + (d^2 - s^2) = 0$. Thus:

$$\begin{aligned} b_1 &= \frac{1}{2}\left(d + \sqrt{4s^2 - 3d^2}\right) \quad (5) \\ b_2 &= \frac{1}{2}\left(d - \sqrt{4s^2 - 3d^2}\right) \quad (6) \end{aligned}$$

IV. Since $b_1 = p + q\sqrt{3}$ and $b_2 = u + v\sqrt{3}$ where $p, q, u,$ and v are integers, it follows from (5) and (6) that

$$4s^2 - 3d^2 = \left(K + L\sqrt{3}\right)^2 = K^2 + 3L^2 + 2KL\sqrt{3} \quad (7)$$

where K and L are integers.

Now, let us find bounds for d and, from those, bounds for y and x. But to start, let us find bounds for $\frac{s}{d}$.

From equation (4), we have:

$$\begin{aligned} \frac{s^2}{d^2} = 1 - \frac{b_1 b_2}{d^2} &= 1 - \frac{b_1 b_2}{(b_1 + b_2)^2} \\ &= 1 - \frac{1}{4}\left(\frac{(b_1+b_2)^2 - (b_1-b_2)^2}{(b_1+b_2)^2}\right) \\ &= \frac{3}{4} + \frac{1}{4}\left(\frac{b_1 - b_2}{b_1 + b_2}\right)^2 \end{aligned}$$

Thus,

$$\frac{3}{4} < \frac{s^2}{d^2} < 1$$

or

$$\frac{\sqrt{3}}{2} < \frac{s}{d} < 1.$$

From this, we can write,

$$1 + \sqrt{3} < \frac{2s + d}{d} < 3.$$

By (1), we can substitue $31 + 16\sqrt{3}$ for $2s + d$, thus eliminating s. With that, we obtain:

$$\frac{31 + 16\sqrt{3}}{3} < d < \frac{31 + 16\sqrt{3}}{1 + \sqrt{3}} \quad (8)$$

Replacing d by $y + x\sqrt{3}$, we can rewrite (8) as bounds for y in terms of x:

$$\frac{31 + (16 - 3x)\sqrt{3}}{3} < y < \frac{(31 - 3x) + (16 - x)\sqrt{3}}{1 + \sqrt{3}} \quad (9)$$

Since y must be a positive integer, x cannot exceed 11, otherwise y will be either negative or less than 1. Also, recalling observation II, x must be even and y must be odd. Replacing x successively by $2, 4, 6, 8,$ and 10, then, we find by (9) that the corresponding values of y will be $17, 13, 11, 7,$ and 3. From these values, in turn, we can then find a and b by equations (2) and (3). The five possibilities we have to check are summarized in the following table.

$$\left\{\begin{array}{cc} d = y + x\sqrt{3} & s = a + b\sqrt{3} \\ x = 2 \quad y = 17 & a = 7 \quad b = 7 \\ x = 4 \quad y = 13 & a = 9 \quad b = 6 \\ x = 6 \quad y = 11 & a = 10 \quad b = 5 \\ x = 8 \quad y = 7 & a = 12 \quad b = 4 \\ x = 10 \quad y = 3 & a = 14 \quad b = 3 \end{array}\right\}$$

Now, in observation IV, we found $4s^2 - 3d^2 = \left(K + L\sqrt{3}\right)^2 = K^2 + 3L^2 + 2KL\sqrt{3}$ which of course must be a positive number. This immediately eliminates the first and last possibilities, $d = 17 + 2\sqrt{3}, s = 7 + 7\sqrt{2}$, and $d = 3 + 10\sqrt{3}, s = 14 + 3\sqrt{2}$ since the rational part of $4s^2 - 3d^2$ (that is, the part not multiplying $\sqrt{3}$) is negative for these pairs.

This leaves only the second, third, and fourth possibilities. The rational parts of $4s^2 - 3d^2$ for these are, respectively, $105, 13,$ and 45. It is then easy to check that only $13 = 1^2 + 3 \times 2^2$ corresponding to $d = 11 + 6\sqrt{3}, s = 10 + 5\sqrt{3}$ can be written in the form $K^2 + 3L^2$, and the irrational part is also $4 = 2KL$.

Hence, these together with equations (5) and (6), give us our solution:

$$\begin{array}{rcl} s & = & 10 + 5\sqrt{3} \\ b_1 & = & 6 + 4\sqrt{3} \\ b_2 & = & 5 + 2\sqrt{3} \end{array}$$

Also solved by Mayer Goldberg, Beer-Sheva, Israel; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

5076: *Proposed by M.N. Deshpande, Nagpur, India*

Let $a, b,$ and m be positive integers and let F_n satisfy the recursive relationship

$$F_{n+2} = mF_{n+1} + F_n, \text{ with } F_0 = a, F_1 = b, n \geq 0.$$

Furthermore, let $a_n = F_n^2 + F_{n+1}^2, n \geq 0$. Show that for every a, b, m, and n,

$$a_{n+2} = (m^2+2)a_{n+1} - a_n.$$

Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX

From the given,

$$\begin{aligned}
a_{n+2} &= F_{n+2}^2 + F_{n+3}^2 \\
&= F_{n+2}^2 + (mF_{n+2} + F_{n+1})^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + mF_{n+1}F_{n+2} + mF_{n+1}F_{n+2} + F_{n+1}^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + mF_{n+1}F_{n+2} + mF_{n+1}(F_n + mF_{n+1}) + F_{n+1}^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + mF_{n+1}(F_{n+2} + F_n) + m^2F_{n+1}^2 + F_{n+1}^2 \\
&= F_{n+2}^2 + m^2F_{n+2}^2 + (F_{n+2} - F_n)(F_{n+2} + F_n) + m^2F_{n+1}^2 + F_{n+1}^2 \\
&= F_{n+2}^2(m^2+2) + F_{n+1}^2(m^2+1) - F_n^2 \\
&= (F_{n+2}^2 + F_{n+1}^2)(m^2+2) - (F_n^2 + F_{n+1}^2) \\
&= (m^2+2)a_{n+1} - a_n.
\end{aligned}$$

Solution 2 by G. C. Greubel, Newport News, VA

Changing the terms slightly we shall use the more familiar Fibonacci polynomial terminology. The fibonacci polynomials are given by

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x).$$

The Binet form of the Fibonacci polynomials is given by

$$F_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where

$$\begin{aligned}
\alpha &= \alpha(x) = \frac{1}{2}\left(x + \sqrt{x^2+4}\right) \\
\beta &= \beta(x) = \frac{1}{2}\left(x - \sqrt{x^2+4}\right).
\end{aligned}$$

Also, the Lucas polynomials are given by

$$L_n(x) = \alpha^n + \beta^n$$

and satisfies the recurrence relation

$$L_{n+2}(x) = xL_{n+1}(x) + L_n(x).$$

The term to be considered is

$$a_n = F_{n+1}^2(x) + F_n^2(x).$$

It can be seen that

$$F_n^2(x) = \frac{1}{x^2+4}\left(L_{2n}(x) - 2(-1)^n\right).$$

This leads to the relation

$$a_n = \frac{1}{x^2+4}\left(L_{2n+1}(x) + L_{2n}(x)\right).$$

The relation being asked to show is given by

$$a_{n+2} = \left(x^2+2\right) a_{n+1} - a_n.$$

Let $\phi_n = (x^2+2)\, a_{n+1} - a_n$ for the purpose of demonstration. With the use of the above equations we can see the following:

$$\begin{aligned}
\left(x^2+4\right)\phi_n &= \left(x^2+4\right)\left[\left(x^2+2\right)a_{n+1} - a_n\right] \\
&= \left(x^2+2\right)\left(L_{2n+3} + L_{2n+2}\right) - \left(L_{2n+1} + L_{2n}\right) \\
&= \left(x^2+2\right)\left(\left(x^2+x+1\right)L_{2n+1} + (x+1)L_{2n}\right) - \left(L_{2n+1} + L_{2n}\right) \\
&= \left(x^4+x^3+3x^2+2x+1\right)L_{2n+1} + \left(x^3+x^2+2x+1\right)L_{2n} \\
&= \left(x^3+x^2+2x+1\right)L_{2n+2} + \left(x^2+x+1\right)L_{2n+1} \\
&= \left(x^2+x+1\right)L_{2n+3} + (x+1)L_{2n+2} \\
&= xL_{2n+4} + L_{2n+4} + L_{2n+3} \\
&= L_{2n+5} + L_{2n+4}. \quad (\mathbf{1})
\end{aligned}$$

From the equation $a_n = \frac{1}{x^2+4}\left(L_{2n+1}(x) + L_{2n}(x)\right)$ we have

$$\left(x^2+4\right)a_{n+2} = L_{2n+5} + L_{2n+4}. \quad (\mathbf{2})$$

Comparing the result of (1) to that of (2) leads to $\phi_n = a_{n+2}$. Thus we have the relation

$$a_{n+2} = \left(x^2+2\right)a_{n+1} - a_n$$

and this provides the relation being sought.

Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University) Seguin, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

5077: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

Find all triplets (x, y, z) of real numbers such that

$$\left.\begin{aligned} xy(x+y-z) &= 3, \\ yz(y+z-x) &= 1, \\ zx(z+x-y) &= 1. \end{aligned}\right\}$$

Solution by Ercole Suppa, Teramo, Italy

From the second and third equation it follows that

$$yz(y+z) = zx(z+x) \qquad \Longleftrightarrow \qquad (x-y)(x+y+z) = 0.$$

If $x+y+z=0$ the first two equations yield $-2xyz = 3$ and $-xyz = 1$ which is impossible.

If $x = y$ then the system can rewritten as

$$\begin{aligned} x^2(2x-z) &= 3 \\ z^2y &= 1 \\ z^2x &= 1 \end{aligned}$$

Thus $x = \dfrac{1}{z^2}$ and

$$\begin{aligned} \frac{1}{z^4}\left(\frac{2}{z^2} - z\right) &= 3 \\ 3z^6 + z^3 - 2 &= 0 \\ (3z^3-2)(z^3+1) &= 0 \end{aligned}$$

The equation $(3z^3-2)(z^3+1) = 0$ factors into

$$\left(3^{1/3}z - 2^{1/3}\right)\left(3^{2/3}z^2 + (3^{1/3}\cdot 2^{1/3})z + 2^{2/3}\right)(z+1)(z^2-z+1) = 0.$$

Setting each factor equal to zero we see that only the first and third factors give real roots for the unknown z. So, the real roots are $z = \sqrt[3]{\dfrac{2}{3}}$ and $z = -1$. And since $x = y = \dfrac{1}{z^2}$ we see that $(1, 1, -1)$ and $\left(\sqrt[3]{\dfrac{9}{4}}, \sqrt[3]{\dfrac{9}{4}}, \sqrt[3]{\dfrac{2}{3}}\right)$ are the only real triplets (x, y, z) that satisfy the given system.

Also solved by Daniel Lopez Aguayo, Puebla, Mexico; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; M. N. Deshpande, Nagpur, India; Bruno Salgueiro Fanego, Viveiro, Spain; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY;

Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA; Antonio Ledesma Vila, Requena-Valencia, Spain, and the proposers.

5078: *Proposed by Paolo Perfetti, Mathematics Department, University "Tor Vergata," Rome, Italy*

Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{\sqrt{b}(b+c)}+\frac{b}{\sqrt{c}(a+c)}+\frac{c}{\sqrt{a}(a+b)} \geq \frac{3}{2}\frac{1}{\sqrt{ab+ac+cb}}.$$

Solution by Kee-Wai Lau, Hong Kong, China

For $x>0$, let $f(x)$ be the convex function x^{-1} so that we have

$$\begin{aligned}
& \frac{a}{\sqrt{b}(b+c)}+\frac{b}{\sqrt{c}(a+c)}+\frac{c}{\sqrt{a}(a+b)} \\
= \quad & af\Big(\sqrt{b}(b+c)\Big)+bf\Big(\sqrt{c}(a+c)\Big)+cf\Big(\sqrt{a}(a+b)\Big) \\
\geq \quad & f\Big(a\sqrt{b}(b+c)+b\sqrt{c}(a+c)+c\sqrt{a}(a+b)\Big) \\
= \quad & \frac{1}{a\sqrt{b}(b+c)+b\sqrt{c}(a+c)+c\sqrt{a}(a+b)}. \qquad (1)
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& a\sqrt{b}(b+c)+b\sqrt{c}(a+c)+c\sqrt{a}(a+b) \\
= \quad & \Big(\sqrt{ab(b+c)}\Big)\Big(\sqrt{a(b+c)}\Big)+\Big(\sqrt{bc(a+c)}\Big)\Big(\sqrt{b(a+c)}\Big)+\Big(\sqrt{ca(a+b)}\Big)\Big(\sqrt{c(a+b)}\Big) \\
\leq \quad & \Big(\sqrt{ab(b+c)+bc(a+c)+ca(a+b)}\Big)\Big(\sqrt{a(b+c)+b(a+c)+c(a+b)}\Big) \\
= \quad & \Big(\sqrt{ab^2+bc^2+ca^2+3abc}\Big)\Big(\sqrt{2(ab+bc+ca)}\Big). \qquad (2)
\end{aligned}$$

By (1) and (2), it suffices for us to show that $ab^2+bc^2+ca^2+3abc \leq \dfrac{2}{9}$. In fact,

$$ab^2+bc^2+ca^2+3abc$$

$$
\begin{aligned}
&= \left(a+b+c-\frac{2}{3}\right)(ab+bc+ca)+\frac{a+b+c}{9}-b\left(a-\frac{1}{3}\right)^2-c\left(b-\frac{1}{3}\right)^2-a\left(c-\frac{1}{3}\right)^2\\
&\leq \frac{ab+bc+ca}{3}+\frac{1}{9}\\
&= \frac{(a+b+c)^2}{9}-\frac{(a-b)^2+(b-c)^2+(c-a)^2}{18}+\frac{1}{9}\\
&\leq \frac{2}{9}.
\end{aligned}
$$

This completes the solution.

Also solved by Boris Rays, Brooklyn, NY, and the proposer.

5079: *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $x \in (0,1)$ be a real number. Study the convergence of the series

$$\sum_{n=1}^{\infty} x^{\sin\frac{1}{1}+\sin\frac{1}{2}+\cdots+\sin\frac{1}{n}}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

For positive integers n and $x\in(0,1)$, let $a_n = a_n(x) = x^{\sin\frac{1}{1}+\sin\frac{1}{2}+\cdots+\sin\frac{1}{n}}$.

Since $\sin\dfrac{1}{n+1} = \dfrac{1}{n}+O\left(\dfrac{1}{n^2}\right)$ as n tends to infinity, so

$$
\begin{aligned}
\left|\frac{a_n}{a_{n+1}}\right| &= \exp\left(\left(\sin\frac{1}{n+1}\right)\left(\ln\frac{1}{x}\right)\right)\\
&= 1+\left(\sin\frac{1}{n+1}\right)\left(\ln\frac{1}{x}\right)+\sum_{m=2}^{\infty}\frac{\left(\left(\sin\frac{1}{n+1}\right)\left(\ln\frac{1}{x}\right)\right)^m}{m!}\\
&= 1+\frac{1}{n}\ln\left(\frac{1}{x}\right)+O\left(\frac{1}{n^2}\right),
\end{aligned}
$$

where the constant implied by the last O depends at most on x. Hence, by Gauss' test, the series of the problem is convergent if $0<x<\dfrac{1}{e}$ and is divergent if $\dfrac{1}{e}\leq x<1$.

Solution 2 by David Stone and John Hawkins (jointly), Statesboro, GA

Our answer: we have convergence if $0 < x < \dfrac{1}{e}$ and divergence if $\dfrac{1}{e} \le x < 1$.

We start by looking at the sum $\displaystyle\sum_{i=1}^{n} \sin\frac{1}{k}$. Each term of the sum, $\sin\dfrac{1}{k}$, can be expanded in an alternating series $\sin\dfrac{1}{k} = \dfrac{1}{k} - \dfrac{1}{3!}\left(\dfrac{1}{k}\right)^3 + \cdots$. The error from terminating the series after the first term does not exceed the second term. Thus we have

$$\begin{aligned}
\left|\sin\frac{1}{k} - \frac{1}{k}\right| &< \frac{1}{3!}\left(\frac{1}{k}\right)^3, \text{ so}\\
-\frac{1}{6k^3} &< \sin\frac{1}{k} - \frac{1}{k} < \frac{1}{6k^3}\\
\frac{1}{k} - \frac{1}{6k^3} &< \sin\frac{1}{k} < \frac{1}{k} + \frac{1}{6k^3}. \text{ Therefore,}\\
\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} &< \sum_{k=1}^{n}\sin\frac{1}{k} < \sum_{k=1}^{n}\frac{1}{k} + \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3}.
\end{aligned}$$

The series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^3}$ is known to be convergent, say to L, which is grater than any of its partial sums.

Moreover, by looking at the graph of $y = 1/x$ we see that

$$\begin{aligned}
\frac{1}{k} &< \int_{k-1}^{k}\frac{1}{u}du = \ln k - \ln(k-1), \text{ and}\\
\frac{1}{k} &> \int_{k}^{k+1}\frac{1}{u}du = \ln(k+1) - \ln(k).
\end{aligned}$$

Using these for our bound on the partial sum of $\sin\dfrac{1}{k}$, we obtain

$$\sum_{k=1}^{n}\Big(\ln(k+1) - \ln k\Big) - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < \sum_{k=1}^{n}\frac{1}{k} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < \sum_{k=1}^{n}\sin\frac{1}{k}, \text{ so}$$

$$\ln(n+1) - \frac{1}{6}L < \sum_{k=1}^{n}\frac{1}{k} - \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < \sum_{k=1}^{n}\sin\frac{1}{k}.$$

On the other hand,

$$\sum_{k=1}^{n}\sin\frac{1}{k} < \sum_{k=1}^{n}\frac{1}{k} + \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^3} < 1 + \ln n + \frac{1}{6}L.$$

Thus we have bounds on the sine sum:

$$\ln(n+1) - \frac{1}{6}L < \sum_{i=1}^{n}\sin\frac{1}{k} < 1 + \ln n + \frac{1}{6}L.$$

We use this to investigate the convergence so the series $\sum_{n=1}^{\infty} x^{\sin\frac{1}{1}+\sin\frac{1}{2}+\cdots+\sin\frac{1}{n}}$.

Since $0 < x < 1$, we know that x^u is a decreasing function of u. Thus

$$x^{-\frac{1}{6}L+\ln(n+1)} > x^{\sum_{k=1}^{n}\sin\frac{1}{k}} > x^{\frac{1}{6}L+\ln n}$$

and we have

$$x^{\frac{1}{6}L+1}\sum_{n=1}^{t} x^{\ln n} < \sum_{n=1}^{t} x^{\sum_{k=1}^{n}\sin\frac{1}{k}} < x^{-\frac{1}{6}L}\sum_{n=1}^{t} x^{\ln(n+1)}.$$

Noting that

$$x^{\ln n} = e^{\ln(x^{\ln n})} = e^{(\ln n)(\ln x)} = e^{\ln n^{\ln x}} = n^{\ln x}$$

we can rewrite the outside sums to obtain

$$x^{\frac{1}{6}L+1}\sum_{n=1}^{t} n^{\ln x} < \sum_{n=1}^{t} x^{\sum_{k=1}^{n}\sin\frac{1}{k}} < x^{-\frac{1}{6}L}\sum_{n=1}^{t}(n+1)^{\ln x}.$$

It is well known that the series $\sum_{n=1}^{\infty} n^{\alpha}$ diverges if $\alpha \geq -1$. Hence, if $\ln x \geq -1$, the series $\sum_{n=1}^{\infty} x^{\sum_{k=1}^{n}\sin\frac{1}{k}}$ dominates the divergent series $\sum_{n=1}^{\infty} x^{\ln x}$ and thus diverges. That is, we have divergence if $1 > x \geq \frac{1}{e}$.

Likewise, it is well known that $\sum_{n=1}^{\infty}(n+1)^{\alpha}$ converges if $\alpha < -1$. So if $\ln x < -1$, the series $\sum_{n=1}^{\infty} x^{\sum_{k=1}^{n}\sin\frac{1}{k}}$ is dominated by the convergent series $\sum_{n=1}^{\infty}(n+1)^{\ln x}$ and thus converges.

That is, we have convergence if $0 < x < \frac{1}{e}$.

Also solved by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and the proposer.

Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before April 15, 2010

- 5098: *Proposed by Kenneth Korbin, New York, NY*

  Given integer-sided triangle ABC with $\angle B = 60^\circ$ and with $a < b < c$. The perimeter of the triangle is $3N^2 + 9N + 6$, where N is a positive integer. Find the sides of a triangle satisfying the above conditions.

- 5099: *Proposed by Kenneth Korbin, New York, NY*

  An equilateral triangle is inscribed in a circle with diameter d. Find the perimeter of the triangle if a chord with length $d - 1$ bisects two of its sides.

- 5100: *Proposed by Mihály Bencze, Brasov, Romania*

  Prove that
  $$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}} \binom{n}{k} \le \sqrt{\frac{n(2^{n+1} - n)2^{n-1}}{n+1}}$$

- 5101: *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India*

  An unbiased coin is tossed repeatedly until r heads are obtained. The outcomes of the tosses are written sequentially. Let R denote the total number of runs (of heads and tails) in the above experiment. Find the distribution of R.

  Illustration: if we decide to toss a coin until we get 4 heads, then one of the possibilities could be the sequence $T\ T\ H\ H\ T\ H\ T\ H$ resulting in 6 runs.

- 5102: *Proposed by Miquel Grau-Sánchez and José Luis Díaz-Barrero, Barcelona, Spain*

  Let n be a positive integer and let $a_1, a_2, \cdots, a_n$ be any real numbers. Prove that
  $$\frac{1}{1 + a_1^2 + \ldots + a_n^2} + \frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \frac{a_k F_k}{1 + a_1^2 + \ldots + a_k^2} \right)^2 \le 1,$$
  where F_k represents the k^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \ge 3, F_n = F_{n-1} + F_{n-2}$.

- 5103: *Proposed by Roger Izard, Dallas, TX*

  A number of circles of equal radius surround and are tangent to another circle. Each of the outer circles is tangent to two of the other outer circles. No two outer circles intersect in two points. The radius of the inner circle is a and the radius of each outer circle is b. If

$$a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 = 0,$$

determine the number of outer circles.

Solutions

- 5080: *Proposed by Kenneth Korbin, New York, NY*

If p is a prime number congruent to 1 (mod 4), then there are positive integers a, b, c, such that

$$\arcsin\left(\frac{a}{p^3}\right) + \arcsin\left(\frac{b}{p^3}\right) + \arcsin\left(\frac{c}{p^3}\right) = 90^\circ.$$

Find a, b, and c if $p = 37$ and if $p = 41$, with $a < b < c$.

Solution 1 by Paul M. Harms, North Newton, KS

The equation in the problem is equivalent to

$$\arcsin\left(\frac{a}{p^3}\right) + \arcsin\left(\frac{b}{p^3}\right) = 90^\circ - \arcsin\left(\frac{c}{p^3}\right).$$

Taking the cosine of both sides yields

$$\begin{aligned}\frac{(p^6 - a^2)^{1/2}(p^6 - b^2)^{1/2}}{p^6} - \frac{ab}{p^6} &= \frac{c}{p^3}.\\ (p^6 - a^2)^{1/2}(p^6 - b^2)^{1/2} - ab &= cp^3.\end{aligned}$$

Since p^3 is a factor on the right side I made some assumptions on a and b so that the left side also had p^3 as a factor.
Assume $a = p^2a_1$ and $b = pb_1$ where all numbers are positive integers. Then we have

$$c = (p^2 - a_1)^{1/2}(p^4 - b_1^2)^{1/2} - a_1b_1.$$

I then looked for perfect squares for $(p^2 - a_1^2)$ and $(p^4 - b_1^2)$.

When $p = 37, (37^2 - a_1^2) = (37 - a_1)(37 + a_1)$ and $a_1 = 12$ yields a product of the squares 25 and 49.

When $p = 37, (37^4 - b_1^2) = (37^2 - b_1)(37^2 + b_1)$.

I checked for a number b_1 where both $(37^2 - b_1)$ and$(37^2 + b_1)$ were perfect squares. The numbers b_1 which make $(37^2 - b_1)$ a square are

$$0,\ 37 + 36 = 73,\ 73 + (36 + 35) = 144,\ 144 + (35 + 34) = 213,\ \cdots.$$

When $b_1 = 840$, both factors involving b_1 are perfect squares.

When $p = 37$ a result is $a = (12)37^2 = 16428, b = 840(37) = 31080$ and $c = 27755$.

Since the problem conditions state that $a < b < c$, I will switch notation. One answer is

$$a = 16428, b = 27755, \text{and } = 31080$$

with approximate angles $18.925°,\ 33.226°$ and $37.849°$.

When $p = 41, (41 - a_1)(41 + a_1)$ is a perfect square when $a_1 = 9$ or 40. The product $(41^2 - b_1)(41^2 + b_1)$ is a perfect square when $b_1 = 720$. One answer is

$$a = 9(41^2) = 15129,\ b = 720(41) = 29520 \text{ and } c = 54280$$

with approximate angles $12.757°,\ 25,361°$, and $51.959°$.

When $a_1 = 40$ and $b_1 = 720$, c was less than zero so this did not satisfy the problem.

Solution 2 by Tom Leong, Scotrun, PA

Fermat's Two-Square Theorem implies that every prime congruent to 1 mod 4 can be represented as the sum of two distinct squares. We give a solution to the following modest generalization. Suppose the positive integer n is the sum of two distinct squares, say, $n = x^2 + y^2$ with $0 < x < y$. Then a solution to

$$\arcsin\frac{A}{n} + \arcsin\frac{B}{n^2} + \arcsin\frac{C}{n^3} = 90°$$

in positive integers A, B, C is

$$(A,B,C) = \begin{cases} (s, 2st, 2(xs+yt)(xt-ys)) & \text{if } 1 < \dfrac{y}{x} < \sqrt{3} \\ (t, t^2 - s^2, 2(xs+yt)(ys-xt)) & \text{if } \sqrt{3} < \dfrac{y}{x} < 1+\sqrt{2} \\ (s, s^2 - t^2, (xs+yt)^2 - (ys-xt)^2) & \text{if } 1+\sqrt{2} < \dfrac{y}{x} < 2+\sqrt{3} \\ (t, 2st, (ys-xt)^2 - (xs+yt)^2) & \text{if } \dfrac{y}{x} > 2+\sqrt{3} \end{cases}$$

where $s = y^2 - x^2$ and $t = 2xy$.
We can verify this as follows. Since $\arcsin(A/n) + \arcsin(B/n^2)$ and $\arcsin(C/n^3)$ are complementary,

$$\tan\left(\arcsin\frac{A}{n} + \arcsin\frac{B}{n^2}\right) = \cot\left(\arcsin\frac{C}{n^3}\right).$$

Using the angle sum formula for tangent and $\tan(\arcsin z) = z/\sqrt{1-z^2}$, this reduces to

$$\frac{A\sqrt{n^4 - B^2} + B\sqrt{n^2 - A^2}}{\sqrt{n^2 - A^2}\sqrt{n^4 - B^2} - AB} = \frac{\sqrt{n^6 - C^2}}{C}.$$

Now verifying the solutions is straightforward using the following identities

$$n = x^2 + y^2, \qquad n^2 = s^2 + t^2, \qquad n^3 = (xs+yt)^2 + (ys-xt)^2$$

and the following inequalities

$$\frac{y}{x} < \sqrt{3} \Leftrightarrow ys < xt, \qquad \frac{y}{x} < 1+\sqrt{2} \Leftrightarrow s < t, \qquad \frac{y}{x} < 2+\sqrt{3} \Leftrightarrow ys - xt < xs + yt.$$

As for the original problem, for $n = 37$, since $37 = 1^2 + 6^2$, we have
$x = 1, y = 6, s = 35, t = 12$ which gives

$$\arcsin\frac{12}{37} + \arcsin\frac{840}{37^2} + \arcsin\frac{27755}{37^3} = \arcsin\frac{16428}{37^3} + \arcsin\frac{31080}{37^3} + \arcsin\frac{27755}{37^3} = 90°.$$

For $n = 41$, since $41 = 4^2 + 5^2$, we have $x = 4, y = 5, s = 9, t = 40$ which gives

$$\arcsin\frac{9}{41} + \arcsin\frac{720}{41^2} + \arcsin\frac{54280}{41^3} = \arcsin\frac{15129}{41^3} + \arcsin\frac{29520}{41^3} + \arcsin\frac{54280}{41^3} = 90°.$$

Comment by editor: David Stone and John Hawkins of Statesboro, GA developed equations:

$$\begin{aligned} b &= \sqrt{\frac{p^3(p^3-c)}{2}} \\ a &= \frac{-bc+\sqrt{b^2c^2+p^6\left(p^6-b^2-c^2\right)}}{p^3}. \end{aligned}$$

Using Matlab they found four solutions for $p = 37$,

$$\begin{array}{lll} a = 16428 & b = 27755 & c = 31080 \\ a = 3293 & b = 32157 & c = 36963 \\ a = 7363 & b = 27188 & c = 38332 \\ a = 352 & b = 25123 & c = 43808 \end{array}$$

and two solutions for $p = 41$,

$$\begin{array}{lll} a = 15129 & b = 29520 & c = 54280 \\ a = 5005 & b = 31529 & c = 58835. \end{array}$$

Also solved by Brian D. Beasley, Clinton, SC, and the proposer.

- **5081:** *Proposed by Kenneth Korbin, New York, NY*

Find the dimensions of equilateral triangle ABC if it has an interior point P such that $\overline{PA} = 5, \overline{PB} = 12$, and $\overline{PC} = 13$.

Solution 1 by Kee-Wai Lau, Hong Kong, China

Let the length of the sides of the equilateral triangle be x. We show that $x = \sqrt{169 + 60\sqrt{3}}$.

Applying the cosine formula to triangles $APB,\ BPC$, and CPA respectively, we obtain

$$\cos\angle APB = \frac{169 - x^2}{120},\ \cos\angle BPC = \frac{313 - x^2}{312},\ \cos\angle CPA = \frac{194 - x^2}{130}.$$

Since

$$\begin{aligned} \angle APB + \angle BPC + \angle CPA &= 360° \text{ so} \\ \cos\angle CPA &= \cos(\angle APB + \angle BPC) \text{ and} \\ \sin\angle APB \sin\angle BPC &= \cos\angle APB \cos\angle BPC - \cos\angle CPA. \end{aligned}$$

Hence,

$$\left(\frac{\sqrt{338x^2 - x^4 - 14161}}{120}\right)\left(\frac{\sqrt{626x^2 - x^4 - 625}}{312}\right) = \left(\frac{169 - x^2}{120}\right)\left(\frac{313 - x^2}{312}\right) - \frac{194 - x^2}{130} \text{ or}$$

$$\sqrt{338x^2 - x^4 - 14161}\sqrt{626x^2 - x^4 - 625} \quad = \quad (169 - x^2)(313 - x^2) - 288(194 - x^2).$$

Squaring both sides and simplifying, we obtain

$$576x^6 - 194668x^4 + 10230336x^2 \quad = \quad 0 \text{ or}$$

$$576x^2(x^4 - 338x^2 + 17761) \quad = \quad 0.$$

It follows that $x = \sqrt{169 - 60\sqrt{3}},\ \sqrt{160 + 60\sqrt{3}}$. Since $\angle APB,\ \angle BPC, \angle CPA$ are not all acute, the value of $\sqrt{169 - 60\sqrt{3}}$ must be rejected.

This completes the solution.

Comments and Solutions 2 & 3 by Tom Leong, Scotrun, PA

Comments: This problem is not new and has appeared in, e.g., the 1998 Irish Mathematical Olympiad and T. Andreescu & R. Gelca, *Mathematical Olympiad Challenges*, Birkhäuser, 2000, p5. A nice elementary solution to this problem uses a rotation argument (Solution 2 below). A quick solution to a more general problem can be found using a somewhat obscure result of Euler on tripolar coordinates (Solution 3 below).

Solution 2

Rotate the figure about the point C by $60°$ so that B maps onto A. Let P' denote the image of P under this rotation. Note that triangle PCP' is equilateral since $PC = P'C$ and $\angle PCP' = 60°$. So $\angle P'PC = 60°$. Furthermore, since $PP' = 13$, triangle APP' is a 5-12-13 right triangle. Consequently,

$$\cos\angle APC = \cos(\angle APP' + 60°) = \frac{5}{13}\cdot\frac{1}{2} - \frac{12}{13}\cdot\frac{\sqrt{3}}{2} = \frac{5 - 12\sqrt{3}}{26}.$$

So by the Law of Cosines,

$$AC = \sqrt{5^2 + 13^2 - 2\cdot 5\cdot 13\cdot\frac{5 - 12\sqrt{3}}{26}} = \sqrt{169 + 60\sqrt{3}}$$

Solution 3

A generalization follows from a result of Euler on *tripolar coordinates* (see, e.g., van Lamoen, Floor and Weisstein, Eric W. "Tripolar Coordinates" From MathWorld–A Wolfram Web Resource.
`http://mathworld.wolfram.com/TripolarCoordinates.html`.) Suppose triangle ABC is equilateral with side length s, and P is a point in the plane of ABC. The triple $(x, y, z) = (PA, PB, PC)$ is the tripolar coordinates of P in reference to triangle ABC. A result of Euler implies these tripolar coordinates satisfy

$$s^4 - (x^2 + y^2 + z^2)s^2 + x^4 + y^4 + z^4 - x^2y^2 - y^2z^2 - z^2x^2 = 0$$

which gives the positive solutions

$$s = \sqrt{\frac{x^2 + y^2 + z^2 \pm \sqrt{(x^2 + y^2 + z^2)^2 - 2(x-y)^2 - 2(y-z)^2 - 2(z-x)^2}}{2}}.$$

The larger solution refers to the case where P is interior to the triangle, while the smaller solution refers to the case where P is exterior to the triangle. In the case where (x, y, z) is a Pythatgorean triple with $x^2 + y^2 = z^2$, this simplifies to the surprisingly terse

$$s = \sqrt{z^2 \pm xy\sqrt{3}}.$$

In the original problem, with $(x, y, z) = (5, 12, 13)$, we find

$$s = \sqrt{169 \pm 60\sqrt{3}}$$

with the larger solution $s = \sqrt{169 + 60\sqrt{3}}$ being the desired answer.

A conjecture by David Stone and John Hawkins, Statesboro, GA

If a, b, c form a *right* triangle with $a^2 + b^2 = c^2$, then

1. the side length of the unique equilateral triangle ABC having an *interior* point P such that $\overline{PA} = a, \overline{PB} = b$, and $\overline{PC} = c$ is $s\sqrt{c^2 + ab\sqrt{3}}$, and
2. the side length of the unique equilateral triangle with an *exterior* point P satisfying $\overline{PA} = a, \overline{PB} = b$, and $\overline{PC} = c$ is $s\sqrt{c^2 - ab\sqrt{3}}$.

Also solved by Scott H. Brown, Montgomery, AL; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Pat Costello, Richmon, KY; Paul M. Harms, North Newton, KS; Antonio Ledesma López, Requena-Valencia, Spain; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; John Nord, Spokane, WA; Boris Rays, Brooklyn, NY; Armend Sh. Shabani, Republic of Kosova; David Stone and John Hawkins, Statesboro, GA, and the proposer.

- 5082: *Proposed by David C. Wilson, Winston-Salem, NC*

Generalize and prove:

$$\begin{aligned}
\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} &= 1 - \frac{1}{n+1} \\
\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} &= \frac{1}{4} - \frac{1}{2(n+1)(n+2)} \\
\frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)} &= \frac{1}{18} - \frac{1}{3(n+1)(n+2)(n+3)} \\
\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)(n+4)} &= \frac{1}{96} - \frac{1}{4(n+1)(n+2)(n+3)(n+4)}
\end{aligned}$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

We will give two different proofs, each relies on the telescoping property.

First proof:

Our quantity may be written as $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+m)}$ where m is a positive integer.

Next we observe

$$\frac{1}{k(k+1)\cdots(k+m-1)} - \frac{1}{(k+1)\cdots(k+m)} = \frac{m}{k(k+1)\cdots(k+m)}$$

yielding, also by telescoping,

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+m)} &= \frac{1}{m}\sum_{k=1}^{n}\left(\frac{1}{k(k+1)\cdots(k+m-1)} - \frac{1}{(k+1)\cdots(k+m)}\right) \\ &= \frac{1}{m}\left(\frac{1}{m!} - \frac{1}{(n+1)\cdots(n+m)}\right)\end{aligned}$$

Second proof:

If $a_k = \dfrac{1}{k(k+1)\cdots(k+m)}$, then $\dfrac{a_{k+1}}{a_k} = \dfrac{k\cdot(k+1)\cdots(k+m)}{(k+1)(k+2)\cdots(k+m)} = \dfrac{k}{k+1+m}$ and then $ma_k = ka_k - (k+1)a_{k+1}$ and therefore

$$\begin{aligned} m\sum_{k=1}^{n} a_k = m\sum_{k=0}^{n-1} a_{k+1} &= m\sum_{k=0}^{n-1}(ka_k - (k+1)a_{k+1}) \\ &= \frac{1}{m!} - \frac{1}{(n+1)(n+2)\cdots(n+m)}\end{aligned}$$

and the result is immediate.

Solution 2 by G. C. Greubel, Newport News, VA

It can be seen that all the series in question are of the form

$$S_n^m = \sum_{k=1}^{n} \frac{(k-1)!}{(k+m)!}.$$

Making a slight change we have

$$S_n^m = \frac{1}{m!}\sum_{k=1}^{n}\frac{(k-1)!m!}{(k+m)!} = \frac{1}{m!}\sum_{k=1}^{n} B(k, m+1),$$

where $B(x,y)$ is the Beta function. By using an integral form of the Beta function, namely,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

the series becomes

$$\begin{aligned} S_n^m &= \frac{1}{m!}\sum_{k=1}^{n}\int_0^1 t^m(1-t)^{k-1}dt \\ &= \frac{1}{m!}\int_0^1 t^m(1-t)^{-1}\cdot\frac{(1-t)(1-(1-t)^n)}{t}dt \\ &= \frac{1}{m!}\int_0^1 t^{m-1}(1-(1-t)^n)dt \end{aligned}$$

$$= \frac{1}{m!}\left(\int_0^1 t^{m-1}dt - B(n+1,m)\right)$$

$$= \frac{1}{m!}\left(\frac{1}{m} - B(n+1,m)\right)$$

$$= \frac{1}{m}\left[\frac{1}{m!} - \frac{n!}{(n+m)!}\right].$$

The general result is given by

$$\sum_{k=1}^{n} \frac{(k-1)!}{(k+m)!} = \frac{1}{m}\left[\frac{1}{m!} - \frac{n!}{(n+m)!}\right].$$

As examples let $m = 1$ to obtain

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$

and when $m = 2$ the series becomes

$$\sum_{k=1}^{n} \frac{1}{n(n+1)(n+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

The other series follow with higher values of m.

Comments by Tom Leong, Scotrun, PA

This series is well-known and has appeared in the literature in several places. Some references include

1. Problem 241, *College Mathematics Journal* (Nov 1984, p448–450)
2. Problem 819, *College Mathematics Journal* (Jan 2007, p65–66)
3. K. Knopp, *Theory and Application of Infinite Series*, 2nd ed., Blackie & Son, 1951, p233
4. D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, *The USSR Olympiad Problem Book*, W.H. Freeman and Company, 1962, p30

In the first reference above, four different perspectives on this series are given.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; Kee-Wai Lau, Hong Kong, China; Antonio Ledesma López, Requena-Valencia, Spain; Tom Leong, Scotrun, PA; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; Armend Sh. Shabani, Republic of Kosova; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5083:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let $\alpha > 0$ be a real number and let $f : [-\alpha, \alpha] \to \Re$ be a continuous function two times derivable in $(-\alpha, \alpha)$ such that $f(0) = 0$ and f'' is bounded in $(-\alpha, \alpha)$. Prove that the sequence $\{x_n\}_{n\geq 1}$ defined by

$$x_n = \begin{cases} \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right), & n > \frac{1}{\alpha}; \\ 0, & n \leq \frac{1}{\alpha} \end{cases}$$

is convergent and determine its limit.

Solution 1 by Michael N. Fried, Kibbutz Revivim, Israel

Clearly, for n large enough, we will have $n > \frac{1}{\alpha}$. Therefore, we only need to show that $\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right)$ converges and to find *its* limit as $n \to \infty$.

Since $f(0) = 0$ and $f'(x)$ exist in $[0, k/n^2] \subset [0, 1/n] \subset [-\alpha, \alpha]$, there is some $\xi_k \in [0, k/n^2]$ such that $f\left(\frac{k}{n^2}\right) = f'(\xi_k)\frac{k}{n^2}$ by the mean value theorem.

Let $f'(M_n)$ =max$_k f'(\xi_k)$ and $f'(m_n)$ =min$_k f'(\xi_k)$.

Then, since $\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = \sum_{k=0}^{n} f'(\xi_k)\frac{k}{n^2}$, we have:

$$f'(m_n)\sum_{k=1}^{n}\frac{k}{n^2} \leq \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) \leq f'(M_n)\sum_{k=1}^{n}\frac{k}{n^2}, \text{ or}$$

$$f'(m_n)\left(\frac{1}{2}+\frac{1}{2n}\right) \leq \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) \leq f'(M_n)\left(\frac{1}{2}+\frac{1}{2n}\right).$$

But f' is bounded in $[-\alpha, \alpha]$ and, thus, in every subinterval of $[-\alpha, \alpha]$. Therefore, f' is continuous in every subinterval of $[-\alpha, \alpha]$. Hence,

$$\lim_{n\to\infty} f'(m_n) = \lim_{n\to\infty} f'(M_n) = f'(0), \text{ so that}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = \frac{f'(0)}{2}$$

Heuristically, we can approach the problem in a slightly different way. Keeping in mind that $f(0) = 0$, write:

$$\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = n^2\sum_{k=0}^{n}\left(\frac{k}{n}\times\frac{1}{n}\right)\frac{1}{n^2} \approx n^2\int_0^{\frac{1}{n}} f(\xi)\,d\xi.$$

The approximation become exact as $n \to \infty$ (this is the heuristic part!)
Since f' is bounded in $(0, \alpha)$ (being bounded in $(-\alpha, \alpha)$), and since $f(0) = 0$ we can write, for some $s \in (0, 1/n)$:

$$n^2\int_0^{\frac{1}{n}} f(\xi)d\xi = n^2\int_0^{\frac{1}{n}}\left(f'(0)\xi + \frac{f''(s)}{2}\xi^2\right)d\xi$$

$$= \quad n^2\left(\frac{f'(0)}{2}\frac{1}{n^2}+\frac{f''(s)}{6}\frac{1}{n^3}\right)$$

$$= \quad \frac{f'(0)}{2}+\frac{f''(s)}{6}\frac{1}{n}$$

$$= \quad \frac{f'(0)}{2} \text{ as } n\to\infty.$$

Solution 2 by Ovidiu Furdui, Cluj, Romania

The limit equals

$$\frac{f'(0)}{2}.$$

We have, since $f(0)=0$, that for all $n>\dfrac{1}{\alpha}$ one has

$$\begin{aligned} x_n=\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) &= \sum_{k=1}^{n}\left(f\left(\frac{k}{n^2}\right)-f(0)\right)\\ &= \sum_{k=1}^{n}\frac{k}{n^2}f'\left(\theta_{k,n}\right)\\ &= \sum_{k=1}^{n}\frac{k}{n^2}\left(f'(\theta_{k,n})-f'(0)\right)+\sum_{k=1}^{n}\frac{k}{n^2}f'(0)\\ &= \sum_{k=1}^{n}\frac{k}{n^2}\theta_{k,n}f''(\beta_{k,n})+\frac{f'(0)(n+1)}{2n}. \qquad (1) \end{aligned}$$

We used, in the preceding calculations, the **Mean Value Theorem** twice where $0<\beta_{k,n}<\theta_{k,n}<\dfrac{k}{n^2}$. Now,

$$\left|\sum_{k=1}^{n}\frac{k}{n^2}\theta_{k,n}f''(\beta_{k,n})\right|\le M\sum_{k=1}^{n}\frac{k}{n^2}\theta_{k,n}\le M\sum_{k=1}^{n}\frac{k^2}{n^4}=M\frac{(n+1)(2n+1)}{6n^3},$$

where $M=\sup_{x\in(-\alpha,\alpha)}|f''(x)|$. Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\theta_{k,n}f''(\beta_{k,n})=0. \qquad (2)$$

Combining (1) and (2) we get that the desired limit holds and the problem is solved.

Also solved by Dionne Bailey, Elsie Campbell and Charles Diminnie (jointly), San Angelo, TX; Tom Leong, Scotrun, PA; Paolo Perfetti, Department of Mathematics, Tor Vergata Universtiy, Rome, Italy, and the proposer.

- 5084: *Charles McCracken, Dayton, OH*

A natural number is called a "repdigit" if all of its digits are alike.
Prove that regardless of positive integral base b, no natural number with two or more digits when raised to a positive integral power will produce a repdigit.

Comments by David E. Manes, Oneonta, NY; Michael N. Fried, Kibbutz Revivim, Israel, the proposer, and the editor.

Manes: The website <http://www.research.att.com/ njas/sequences/A158235> appears to have many counterexamples to problem 5084.
Editor: Following are some examples and comments from the above site.

$$11, 20, 39, 40, 49, 78, 133, 247, 494, 543, 1086, 1218,$$

$$1651, 1729, 2172, 2289, 2715, 3097, 3258, 3458, 3801,$$

$$171, 4344, 4503, 4578, 4887, 5187, 5430, 6194, 6231.$$

(And indeed, each number listed above can be written as repdigit *in some base.* For example:)

$$\begin{aligned}
11^2 &= 11111 \text{ in base } 3 \\
20^2 &= 1111 \text{ in base } 7 \\
39^2 &= 333 \text{ in base } 22 \\
40^2 &= 4444 \text{ in base } 7 \\
49^2 &= 777 \text{ in base } 18 \\
78^2 &= (12)(12)(12) \text{ in base } 22 \\
1218^2 &= (21)(21)(21)(21) \text{ in base } 41
\end{aligned}$$

McCracken: When I wrote the problem I intended that the number and it's power be written *in the same base.*

Editor: Charles McCracken sent in a proof that was convincing to me that the statement, as he had intended it to be, was indeed correct. No natural number with two or more digits (written in base b), when raised to a positive integral power, will produce a repdigit (in base b). I showed the problem, its solution, and Manes' comment, to my colleague Michael Fried, and he finally convinced me that although the intended statement might be true, the proof was in error.

Fried: The Sloan Integer Sequence site (mentioned above) also cites a paper which among other things, refers to Catalan's conjecture, now proven, stating that the only solution to $x^k - y^n = 1$ is $3^2 - 2^3 - 9 - 8 = 1$. This is the fact one needs to show that Charles' claim is true for base 2 repdigits. For in base 2 only numbers of the form $11111\dots1$ are repdigits. These numbers are equal to $2^n - 1$. So if one of these numbers were equal to x^k, we would have $2^n - 1 = x^k$ or $2^n - x^k = 1$. But by the proven Catalan conjecture, the latter can never be satisfied.

Editor: So, dear readers, let's rephrase the problem: Prove or disprove that regardless of positive integral base b, no natural number with two or more digits when raised to a positive integral power will produce a repdigit in base b.

- 5085: *Proposed by Valmir Krasniqi, (student, Mathematics Department,) University of Prishtinë, Kosova*

Suppose that $a_k,\ (1 \le k \le n)$ are positive real numbers. Let $e_{j,k} = (n-1)$ if $j = k$ and $e_{j,k} = (n-2)$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise.

Prove that

$$\prod_{j=1}^{n}\sum_{k=1}^{n} e_{j,k}a_k^2 \ge \prod_{j=1}^{n}\left(\sum_{k=1}^{n} d_{j,k}a_k\right)^2.$$

Solution by proposer

On expanding each side and reducing, the inequality becomes

$$\prod_{k=1}^{n}\left[(n-2)S + a_k^2\right] \quad \ge \quad \prod_{k=1}^{n}(T - a_k)\,, \text{ where}$$

$$S = \sum_{k=1}^{n} a_k^2 \quad \text{and} \quad T = \sum_{k=1}^{n} a_k.$$

Since $(T - a_1)^2 \le (n-1)(S - a_1^2)$, etc., it suffices to prove that

$$\prod_{k=1}^{n}\left[(n-2)S + a_k^2\right] \ge (n-1)^n \prod_{k=1}^{n}(S - a_k)\,. \qquad (1)$$

If we now let $x_k = S - a_k^2$ where $k = 1, 2, 3, \ldots, n$ so that $S = \dfrac{x_1 + x_2 + \ldots + x_n}{n-1}$ and $a_k^2 = S - x_k$, then (1) becomes

$$\prod_{k=1}^{n}\left(S^{'} - x_k\right) \ge (n-1)^n \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n, \text{ where } S^{'} = \sum_{k=1}^{n} x_k.$$

The result now follows by applying the AM-GM inequality to each of the factors $(S^{'} - x_k)$ on the left hand side. There is equality if, and only if, all the a_k's are equal.

Also solved by Tom Leong, Scotrun, PA

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before May 15, 2010

- 5104: *Proposed by Kenneth Korbin, New York, NY*

  There are infinitely many primitive Pythagorean triangles with hypotenuse of the form $4x^4+1$ where x is a positive integer. Find the dimensions of all such triangles in which at least one of the sides has prime integer length.

- 5105: *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation
  $$x+y-\sqrt{x^2+xy+y^2}=2+\sqrt{5}$$
  if x and y are of the form $a+b\sqrt{5}$ where a and b are positive integers.

- 5106: *Proposed by Michael Brozinsky, Central Islip, NY*

  Let a, b, and c be the sides of an acute-angled triangle ABC. Let H be the orthocenter and let d_a, d_b and d_c be the distances from H to the sides BC,CA, and AB respectively.

  Show that
  $$d_a+d_b+d_c \le \frac{3}{4}D$$
  where D is the diameter of the circumcircle.

- 5107: *Proposed by Tuan Le (student, Fairmont, H.S.), Anaheim, CA*

  Let a, b, c be positive real numbers. Prove that
  $$\frac{\sqrt{a^3+b^3}}{a^2+b^2}+\frac{\sqrt{b^3+c^3}}{b^2+c^2}+\frac{\sqrt{c^3+a^3}}{c^2+a^2} \ge \frac{6(ab+bc+ac)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}$$

- 5108: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Compute
  $$\lim_{n\to\infty}\frac{1}{n}\tan\left[\sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right)\right].$$

- 5109 *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k \geq 1$ be a natural number. Find the value of

$$\lim_{n\to\infty} \frac{(k\sqrt[n]{n} - k + 1)^n}{n^k}.$$

Solutions

- 5086: *Proposed by Kenneth Korbin, New York, NY*

Find the value of the sum

$$\frac{2}{3} + \frac{8}{9} + \cdots + \frac{2N^2}{3^N}.$$

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

If $x \neq 1$, the formula for a geometric sum yields

$$\sum_{k=0}^{N} x^k = \frac{x^{N+1} - 1}{x - 1}.$$

If we differentiate and simplify, we obtain

$$\sum_{k=1}^{N} kx^{k-1} = \frac{Nx^{N+1} - (N+1)\,x^N + 1}{(x-1)^2}.$$

Next, multiply by x and differentiate again to get

$$\sum_{k=1}^{N} kx^k = \frac{Nx^{N+2} - (N+1)\,x^{N+1} + x}{(x-1)^2}$$

and

$$\sum_{k=1}^{N} k^2x^{k-1} = \frac{N^2x^{N+2} - (2N^2 + 2N - 1)\,x^{N+1} + (N+1)^2\,x^N - x - 1}{(x-1)^3}.$$

Finally, multiply by x once more to yield

$$\sum_{k=1}^{N} k^2x^k = \frac{N^2x^{N+3} - (2N^2 + 2N - 1)\,x^{N+2} + (N+1)^2\,x^{N+1} - x^2 - x}{(x-1)^3}.$$

In particular, when we substitute $x = \dfrac{1}{3}$ and simplify, the result is

$$\sum_{k=1}^{N} \frac{k^2}{3^k} = \frac{3^{N+1} - (N^2 + 3N + 3)}{2 \cdot 3^N}.$$

Therefore, the desired sum is

$$\sum_{k=1}^{N} \frac{2k^2}{3^k} = \frac{3^{N+1} - (N^2 + 3N + 3)}{3^N}.$$

Solution 2 by Ercole Suppa, Teramo, Italy

The required sum can be written as $S_N = \dfrac{2}{3^N} \cdot x_n$, where x_n denotes the sequence

$$x_n = 1^2 \cdot 3^{n-1} + 2^2 \cdot 3^{n-2} + 3^2 \cdot 3^{n-3} + \cdots + n^2 \cdot 3^0.$$

Since

$$x_{n+1} = 1^2 \cdot 3^n + 2^2 \cdot 3^{n-1} + 3^2 \cdot 3^{n-2} + \cdots + n^2 \cdot 3^1 + (n+1)^2 \cdot 3^0,$$

such a sequence satisfies the linear recurrence

$$x_{n+1} - 3x_n = (n+1)^2. \qquad (*)$$

Solving the characteristic equation $\lambda - 3 = 0$, we obtain the homogeneous solutions $x_n = A \cdot 3^n$, where A is a real parameter. To determine a particular solution, we look for a solution of the form $x_n^{(p)} = Bn^2 + Cn + D$. Substituting this into the difference equation, we have

$$\begin{aligned} B\,(n+1)^2 + C\,(n+1) + D - 3\left[Bn^2 + Cn + D\right] &= (n+1)^2 \Leftrightarrow \\ -2Bn^2 + 2\,(B - C)\,n + B + C - 2D &= n^2 + 2n + 1. \end{aligned}$$

Comparing the coefficients of n and the constant terms on the two sides of this equation, we obtain

$$B = -\frac{1}{2}, \qquad C = -\frac{3}{2}, \qquad D = -\frac{3}{2}$$

and thus

$$x_n^{(p)} = -\frac{1}{2}n^2 + -\frac{3}{2}n - \frac{3}{2}$$

The general solution of $(*)$ is simply the sum of the homogeneous and particular solutions, i.e.,

$$x_n = A \cdot 3^n - \frac{1}{2}n^2 + -\frac{3}{2}n - \frac{3}{2}$$

From the boundary condition $x_1 = 1$, the constant is determined as $\dfrac{3}{2}$.
Finally, the desired sum is

$$S_N = \frac{3^{N+1} - N^2 - 3N - 3}{3^N}$$

and we are done.

Also solved by Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Pat Costello, Richmond, KY; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Enkel Hysnelaj, Sydney, Australia & Elton Bojaxhiu, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Paolo Perfetti, Department of Mathematics, Tor Vergata Universtiy, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Taylor University Problem Solving Group, Upland, IN, and the proposer.

- **5087:** *Proposed by Kenneth Korbin, New York, NY*

  Given positive integers a, b, c, and d such that $(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2)$ with $a < b < c < d$. Rationalize and simplify

$$\frac{\sqrt{x+y}-\sqrt{x}}{\sqrt{x+y}+\sqrt{x}} \quad \text{if} \quad \begin{cases} x = bc+bd+cd, & \text{and} \\ y = ab+ac+ad. \end{cases}$$

Solution by Paul M. Harms, North Newton, KS

From the equation given in the problem we have

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd = 2\left(a^2+b^2+c^2+d^2\right).$$

From the last equation we have

$$2\left(ab+ac+ad+bc+bd+cd\right) = a^2+b^2+c^2+d^2.$$

We note that,

$$\begin{aligned} x+y &= ab+ac+ad+bc+bd+cd, \text{ then} \\ 2\left(x+y\right) &= a^2+b^2+c^2+d^2 \end{aligned}$$

From the identity in the problem,

$$\begin{aligned} 2\left(x+y\right) &= \frac{(a+b+c+d)^2}{2} \text{ or} \\ (x+y) &= \frac{(a+b+c+d)^2}{2^2} \end{aligned}$$

Also note that,

$$\begin{aligned} y &= a(b+c+d) \text{ or} \\ \frac{y}{a} &= b+c+d. \text{ Then} \\ x+y &= \frac{(a+(y/a))^2}{2^2} = \frac{(a^2+y)^2}{(2a)^2}. \end{aligned}$$

We have,

$$\begin{aligned} x &= (x+y)-y \\ &= \frac{(a^2+y)^2}{(2a)^2} - y \\ &= \frac{a^4+2a^2y+y^2-4a^2y}{4a^2} \\ &= \frac{(a^2-y)^2}{(2a)^2}. \end{aligned}$$

From $a < b < c < d$, we see that

$$\begin{aligned} a^2 - y &= a^2 - a\,(b+c+d) < 0. \text{ Thus} \\ \sqrt{\left(a^2-y\right)^2} &= y - a^2. \end{aligned}$$

Working with the expression to be simplified, we have

$$\begin{aligned} \frac{\sqrt{x+y}-\sqrt{x}}{\sqrt{x+y}+\sqrt{x}} &= \frac{\left(\sqrt{x+y}-\sqrt{x}\right)^2}{y} \\ &= \frac{\left[(a^2+y)/(2a) - (y-a^2)/(2a)\right]^2}{y} \\ &= \frac{\left(2a^2/2a\right)^2}{y} \\ &= \frac{a^2}{y} \\ &= \frac{a}{b+c+d}. \end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; G. C., Greubel, Newport News, VA; Enkel Hysnelaj, Sydney, Australia & Elton Bojaxhiu, Germany; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5088: *Proposed by Isabel Iriberri Díaz and José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b be positive integers. Prove that

$$\frac{\varphi(ab)}{\sqrt{\varphi^2(a^2)+\varphi^2(b^2)}} \le \frac{\sqrt{2}}{2},$$

where $\varphi(n)$ is Euler's totient function.

Solution by Tom Leong, Scotrun, PA

We show

$$\varphi(ab) \le \sqrt{\varphi(a^2)\varphi(b^2)} \le \sqrt{\frac{\varphi^2 a^2) + \varphi^2(b^2)}{2}}$$

which implies the desired result. The second inequality used here is simply the AM-GM Inequality. To prove the first inequality, let p_i denote the prime factors of both a and b, and let q_j denote the prime factors of a only and r_k the primes factors of b only. Then

$$\begin{aligned} \varphi(ab) &= ab\prod_i\left(1-\frac{1}{p_i}\right)\prod_j\left(1-\frac{1}{q_j}\right)\prod_k\left(1-\frac{1}{r_k}\right) \\ \varphi(a^2)\varphi(b^2) &= \left[a^2\prod_i\left(1-\frac{1}{p_i}\right)\prod_j\left(1-\frac{1}{q_j}\right)\right]\left[b^2\prod_i\left(1-\frac{1}{p_i}\right)\prod_k\left(1-\frac{1}{r_k}\right)\right] \end{aligned}$$

where we understand the empty product to be 1. Then $\varphi(ab) \leq \sqrt{\varphi(a^2)\varphi(b^2)}$ reduces to

$$\prod_j \left(1 - \frac{1}{q_j}\right) \prod_k \left(1 - \frac{1}{r_k}\right) \leq 1$$

which is obviously true.

Editor's comment: **Kee-Wai Lau of Hong Kong, China** mentioned in his solution to this problem that in the *Handbook of Number Theory I* (Section 1.2 of Chapter I by J. Sándor, D.S. Mitrinovi, and B. Crstic, Springer, 1995), the proof of $(\varphi(mn))^2 \leq \varphi(m^2)\varphi(n^2)$, for positive integers m and n is attributed to a 1940 paper by T. Popoviciu. Kee-Wai then wrote $\sqrt{\varphi^2(a^2) + \varphi^2(b^2)} \geq \sqrt{2\varphi(a^2)\varphi(b^2)} \geq \sqrt{2}\varphi(ab)$, proving the inequality.

Also solved by Brian D. Beasley, Clinton, SC; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Enkel Hysnelaj, Sydney, Australia & Elton Bojaxhiu, Germany; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy; and the proposers.

- 5089: *Proposed by Panagiote Ligouras, Alberobello, Italy*

  In $\triangle ABC$ let $AB = c, BC = a, CA = b, r =$ the in-radius and r_a, r_b, and r_c= the ex-radii, respectively.

  Prove or disprove that

$$\frac{(r_a - r)(r_b + r_c)}{r_a r_c + r r_b} + \frac{(r_c - r)(r_a + r_b)}{r_c r_b + r r_a} + \frac{(r_b - r)(r_c + r_a)}{r_b r_a + r r_c} \geq 2\left(\frac{ab}{b^2 + ca} + \frac{bc}{c^2 + ab} + \frac{ca}{a^2 + bc}\right).$$

Solution by Kee-Wai Lau, Hong Kong, China

We prove the inequality.

Let s and S be respectively the semi-perimeter and area of $\triangle ABC$. It is well known that

$$r = \frac{S}{s},\ r_a = \frac{S}{s-a},\ r_b = \frac{S}{s-b},\ r_c = \frac{S}{s-c}.$$

Using these relations, we readily simplify

$$\frac{(r_a - r)(r_b + r_c)}{r_a r_c + r r_b} \text{ to } \frac{a}{c},\ \frac{(r_c - r)(r_a + r_b)}{r_c r_b + r r_a} \text{ to } \frac{c}{b},\ \text{and } \frac{(r_b - r)(r_c + r_a)}{r_b r_a + r r_c} \text{ to } \frac{b}{a}.$$

Since $b^2 + ca \geq 2b\sqrt{ca}$, $c^2 + ab \geq 2c\sqrt{ab}$, and $a^2 + bc \geq 2a\sqrt{bc}$, so

$$2\left(\frac{ab}{b^2 + ca} + \frac{bc}{c^2 + ab} + \frac{ca}{a^2 + bc}\right) \leq \sqrt{\frac{a}{c}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}}.$$

By the Cauchy-Schwarz inequality, we have

$$\sqrt{\frac{a}{c}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}} \leq \sqrt{3\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right)},$$

and by the arithmeic mean-geometric mean inequality we have

$$3 = 3\left(\sqrt[3]{\left(\frac{a}{c}\right)\left(\frac{b}{a}\right)\left(\frac{c}{b}\right)}\right) \leq \frac{a}{c} + \frac{b}{a} + \frac{c}{b}.$$

It follows that $\sqrt{\frac{a}{c}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}} \leq \frac{a}{c} + \frac{b}{a} + \frac{c}{b}$ and this completes the solution.

Also solved by Tom Leong, Scotrun, PA; Ercole Suppa, Teramo, Italy, and the proposer.

- 5090: *Proposed by Mohsen Soltanifar (student), University of Saskatchewan, Canada*

Given a prime number p and a natural number n. Calculate the number of elementary matrices $E_{n\times n}$ over the field Z_p.

Solution by Paul M. Harms, North Newton, KS

The notation 0 and 1 will be used for the additive and multiplicative identities, respectively.

There are three types of matrices which make up the set of elementary matrices. One type is a matrix where two rows of the identity matrix are interchanged. Since there are n rows and we interchange two at a time, the number of elementary matrices of this type is $\frac{n(n-1)}{2}$, the combination of n things taken two at a time.

Another type of elementary matrix is a matrix where one of the elements along the main diagonal is replaced by an element which is not 0 or 1. There are $(p-2)$ elements which can replace a 1 on the main diagonal. The number of elementary matrices of this type is $(p-2)n$.

The third type of elementary matrix is the identity matrix where at most one position, not on the main diagonal, is replaced by a non-zero element. There are (n^2-n) positions off the main diagonal and $(p-1)$ non-zero elements. Then there are $(n^2-n)(p-1)$ different elementary matrices where a non-zero element replaces one zero element in the identity matrix. If the identity matrix is included here, the number of elementary matrices of this type is $(n^2-n)(p-1)+1$.

The total number of elementary matrices is

$$\frac{n(n-1)}{2} + (p-2)n + (n^2-n)(p-1) + 1 = n^2\left(p - \frac{1}{2}\right) - \frac{3n}{2} + 1.$$

Comment by David Stone and John Hawkins of Statesboro, GA. There doesn't seem to be any need to require that p be prime as we form and count these elementary matrices. However, if m were not prime then Z_m would not be a field and the algebraic properties would be affected. For instance, it's preferable that any elementary matrix be invertible and the appearance of non-invertible scalars would produce non-invertible elementary matrices such as $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ over Z_4.

Also solved by David E. Manes, Oneonta, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5091: *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $k, p \geq 0$ be nonnegative integers. Evaluate the integral

$$\int_{-\pi/2}^{\pi/2} \frac{\sin^{2p} x}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} dx.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We show that the integral equals $\dfrac{(2p-1)!!}{(2p)!!}\dfrac{\pi}{2}$, independent of k.

Here $(-1)!! = 0!! = 1,\ n!! = n(n-2)\dots(3)(1)$ if n is a positive odd integer and $n!! = n(n-2)\dots(4)(2)$ if n is a positive even integer.

By substituting $x = -y$, we have

$$\int_{-\pi/2}^{0} \frac{\sin^{2p} x}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} dx = \int_{0}^{\pi/2} {}_0 \frac{\sin^{2p} y}{1-\sin^{2k+1} y+\sqrt{1+\sin^{4k+2} y}} \text{ so that}$$

$$\begin{aligned}
&\int_{-\pi/2}^{\pi/2} \frac{\sin^{2p} x}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}} dx \\
= &\int_{0}^{\pi/2} \sin^{2p} x\left(\frac{1}{1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}}+\frac{1}{1-\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}}\right) dx \\
= &\ 2\int_{0}^{\pi/2} \sin^{2p} x\left(\frac{1+\sqrt{1+\sin^{4k+2} x}}{\left(1+\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}\right)\left(1-\sin^{2k+1} x+\sqrt{1+\sin^{4k+2} x}\right)}\right) dx \\
= &\int_{0}^{\pi/2} \sin^{2p} x dx.
\end{aligned}$$

The last integral is standard and its value is well known to be $\dfrac{(2p-1)!!}{(2p)!!}\dfrac{\pi}{2}$.

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata, Rome, Italy

The answer is: $\dfrac{(2p)!}{2^{2p}(p!)^2}\dfrac{\pi}{2}$ for any k.

Proof Let's substitute $\sin x = t$

$$\int_{-1}^{1} \frac{t^{2p}}{1+t^{2k+1}+\sqrt{1+t^{4k+2}}}\frac{dt}{\sqrt{1-t^2}} = \int_{-1}^{1} \frac{t^{2p}(1+t^{2k+1}-\sqrt{1+t^{4k+2}})}{2t^{2k+1}}\frac{dt}{\sqrt{1-t^2}}$$

Now

$$\int_{-1}^{1} \frac{t^{2p}}{2t^{2k+1}}\frac{dt}{\sqrt{1-t^2}} = \int_{-1}^{1} \frac{t^{2p}\sqrt{1+t^{4k+2}}}{2t^{2k+1}}\frac{dt}{\sqrt{1-t^2}} = 0$$

since the integrands are odd functions. It remains

$$\frac{1}{2}\int_{-1}^{1}\frac{t^{2p}}{\sqrt{1-t^2}}dt = \frac{1}{2}\int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx$$

after changing variable $t = \sin x$. Integrating by parts we obtain

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx &= \int_{-\pi/2}^{\pi/2}(-\cos x)'(\sin x)^{2p-1}dx \\
&= -\cos x(\sin x)^{2p-1}\Big|_{-\pi/2}^{\pi/2} + (2p-1)\int_{-\pi/2}^{\pi/2}\cos^2 x(\sin x)^{2p-2}dx \\
&= (2p-1)\int_{-\pi/2}^{\pi/2}(\sin x)^{2p-2}dx - (2p-1)\int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx
\end{aligned}$$

and if we call $I_{2p} = \int_{-\pi/2}^{\pi/2}(\sin x)^{2p}dx$, then we have $I_{2p} = \frac{2p-1}{2p}I_{2p-2}$. It results that $I_{2p} = \frac{(2p-1)!!}{(2p)!!}\pi = \frac{(2p)!}{2^{2p}(p!)^2}\pi$ and then $\frac{1}{2}\int_{-1}^{1}\frac{t^{2p}}{\sqrt{1-t^2}}dt = \frac{\pi}{2}\frac{(2p-1)!!}{(2p)!!} = \frac{(2p)!}{2^{2p}(p!)^2}\frac{\pi}{2}$

Editor's comment: The two solutions presented, $\frac{(2p-1)!!}{(2p)!!}\frac{\pi}{2}$ and $\frac{(2p)!}{2^{2p}(p!)^2}\frac{\pi}{2}$, are equivalent to one another.

Also solved by Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
June 15, 2010

- **5110:** *Proposed by Kenneth Korbin, New York, NY.*

  Given triangle ABC with an interior point P and with coordinates $A(0,0), B(6,8)$, and $C(21,0)$. The distance from point P to side $\overline{AB}$ is a, to side $\overline{BC}$ is b, and to side $\overline{CA}$ is c where $a:b:c = \overline{AB}:\overline{BC}:\overline{CA}$.

  Find the coordinates of point P.

- **5111:** *Proposed by Michael Brozinsky, Central Islip, NY.*

  In Cartesianland where immortal ants live, it is mandated that any anthill must be surrounded by a triangular fence circumscribed in a circle of unit radius. Furthermore, if the vertices of any such triangle are denoted by A, B, and C, in counterclockwise order, the anthill's center must be located at the interior point P such that $\angle PAB = \angle PBC = \angle PCA$.

  Show $\overline{PA} \cdot \overline{PB} \cdot \overline{PC} \leq 1$.

- **5112:** *Proposed by Juan-Bosco Romero Márquez, Madrid, Spain*

  Let $0 < a < b$ be real numbers with a fixed and b variable. Prove that

$$\lim_{b \to a} \int_a^b \frac{dx}{\ln \dfrac{b+x}{a+x}} = \lim_{b \to a} \int_a^b \frac{dx}{\ln \dfrac{b(a+x)}{a(b+x)}}.$$

- **5113:** *Proposed by Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy*

  Let x, y be positive real numbers. Prove that

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \leq \sqrt{xy} + \frac{x+y}{2} + \frac{\left(\dfrac{x+y}{6} - \dfrac{\sqrt{xy}}{3}\right)^2}{\dfrac{2xy}{x+y}}.$$

- **5114:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let M be a point in the plane of triangle ABC. Prove that

$$\frac{\overline{MA}^2+\overline{MB}^2+\overline{MC}^2}{\overline{AB}^2+\overline{BC}^2+\overline{CA}^2}\geq\frac{1}{3}.$$

  When does equality hold?

- **5115:** *Proposed by Mohsen Soltanifar (student, University of Saskatchewan), Saskatoon, Canada*

  Let G be a finite cyclic group. Compute the number of distinct composition series of G.

Solutions

- **5092:** *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with altitude h and with cevian $\overline{CD}$. A circle with radius x is inscribed in $\triangle ACD$, and a circle with radius y is inscribed in $\triangle BCD$ with $x<y$. Find the length of the cevian $\overline{CD}$ if x,y and h are positive integers with $(x,y,h)=1$.

  Solution by David Stone and John Hawkins (jointly), Statesboro, GA;

  We let the length of cevian $=d$. Since the altitude of the equilateral triangle is h, the length of the side $\overline{AC}$ is $\dfrac{2h}{\sqrt{3}}$. Let F be the center of the circle inscribed in $\triangle ACD$. Let $\alpha=\angle ACF=\angle FCD$. Therefore $\angle ACD=2\alpha$.

  Let E be the point where the inscribed circle in $\triangle ACD$ is tangent to side $\overline{AC}$. Since $\overline{AF}$ bisects the base angle of 60°, we know that $\triangle AEF$ is a $30^\circ-60^\circ-90^\circ$ triangle, implying that $\overline{AE}=\sqrt{3}x$. Thus the length of $\overline{CE}$ is $\overline{AC}-\overline{AE}=\dfrac{2h}{\sqrt{3}}-\sqrt{3}x=\dfrac{2h-3x}{\sqrt{3}}$.

  Applying the Law of Sines in triangle $\triangle ADC$, we have

$$\frac{\sin 2\alpha}{\overline{AD}}=\frac{\sin 60^\circ}{d}=\frac{\sin(\angle ADC)}{\overline{AC}}. \qquad (1)$$

  Because $\angle ADC=180^\circ-60^\circ-2\alpha=120^\circ-2\alpha$, we have

$$\begin{aligned}\sin\left(\angle ADC\right)&=\sin\left(120^\circ-2\alpha\right)\\&=\sin 120^\circ\cos 2\alpha-\cos 120^\circ\sin 2\alpha\\&=\frac{\sqrt{3}}{2}\cos 2\alpha+\frac{1}{2}\sin 2\alpha\\&=\frac{\sqrt{3}}{2}\left(\cos^2\alpha-\sin^2\alpha\right)+\frac{1}{2}\left(2\sin\alpha\sin\alpha\right).\end{aligned}$$

Thus from (1) we have

$$\frac{\sqrt{3}}{2d} = \frac{\left[\sqrt{3}\left(\cos^2\alpha - \sin^2\alpha\right) + \left(2\sin\alpha\sin\alpha\right)\right]\sqrt{3}}{4h}.$$

Therefore, we can solve for d in terms of h and α:

$$d = \frac{2h}{\left[\sqrt{3}\left(\cos^2\alpha - \sin^2\alpha\right) + \left(2\sin\alpha\sin\alpha\right)\right]}.$$

In the right triangle $\triangle EFC$, we have

$$\overline{FC} = \sqrt{x^2 + \left(\frac{2h-3x}{\sqrt{3}}\right)^2} = \sqrt{\frac{3x^2 + 4h^2 - 12hx + 9x^2}{3}} = \frac{2}{\sqrt{3}}\sqrt{3x^2 + h^2 - 3hx}.$$

Thus, $\sin\alpha = \dfrac{\sqrt{3}x}{2\sqrt{3x^2 + h^2 - 3hx}}$ and $\cos\alpha = \dfrac{2h-3x}{2\sqrt{3x^2 + h^2 - 3hx}}$. Therefore,

$$\begin{aligned} cos^2\alpha - \sin^2\alpha &= \frac{(2h-3x)^2}{4\left(3x^2 + h^2 - 3hx\right)} - \frac{3x^2}{4\left(3x^2 + h^2 - 3hx\right)} \\ &= \frac{4h^2 - 12hx + 6x^2}{4\left(3x^2 + h^2 - 3hx\right)} = \frac{2h^2 - 6hx + 3x^2}{2\left(3x^2 + h^2 - 3hx\right)}. \end{aligned}$$

and $2\sin\alpha\cos\alpha = \dfrac{\sqrt{3}x\left(2h-3x\right)}{2\left(3x^2 + h^2 - 3hx\right)}$.

Therefore the denominator in the expression for d becomes

$$\frac{\sqrt{3}(2h^2 - 6hx + 3x^2)}{2\left(3x^2 + h^2 - 3hx\right)} + \frac{\sqrt{3}x\left(2h-3x\right)}{2\left(3x^2 + h^2 - 3hx\right)} = \sqrt{3}\frac{2h^2 - 4hx}{2\left(3x^2 + h^2 - 3hx\right)}.$$

Thus, $d = \dfrac{2h}{\dfrac{\sqrt{3}(2h^2 - 4hx)}{2(3x^2 + h^2 - 3hx)}} = \dfrac{2(3x^2 + h^2 - 3hx)}{\sqrt{3}(h-2x)}.$

Similarly, working in $\triangle BCD$, we can show that $d = \dfrac{2(3y^2 + h^2 - 3hy)}{\sqrt{3}(h-2y)}$.

We note that x and y both satisfy the same equation when set equal to d. Thus for a given value of d, the equation should have two solutions. The smaller one can be used for x and the larger for y.

We also note that if x, h and y are integers, then d has the form $d = \dfrac{r}{\sqrt{3}}$, for r a rational number. We substitute this into the equation x:

$$d \;=\; \frac{2\left(3x^2 + h^2 - 3hx\right)}{\sqrt{3}(h-2x)} = \frac{r}{\sqrt{3}}, \text{ so}$$

$$r = \frac{2(3x^2 + h^2 - 3hx)}{h - 2x}.$$

Now we solve this for x:

$$rh - 2xr = 6x^2 + 2h^2 - 6hx$$

$$6x^2 - (6h - 2r)\,x + 2h^2 - rh = 0$$

$$x = \frac{6h - 2r \pm \sqrt{36h^2 - 24hr + 4r^2 - 48h^2 + 24hr}}{12} = \frac{3h - r \pm \sqrt{r^2 - 3h^2}}{6}.$$

Of course we would have the exact same expression for y.

Thus we take $x = \dfrac{3h - r - \sqrt{r^2 - 3h^2}}{6}$ and $y = \dfrac{3h - r - \sqrt{r^2 - 3h^2}}{6}$ and find h and r so that x and y turn out to be positive integers.

Subtracting x from y gives $y - x = \dfrac{\sqrt{r^2 - 3h^2}}{3}$. Thus we need r and h such that $\dfrac{\sqrt{r^2 - 3h^2}}{3}$ is an integer.

It must be the case that $r^2 - 3h^2 \geq 0$, which requires $0 < \sqrt{3}h \leq r$. In addition it must be true that

$$\begin{aligned}
3h - r - \sqrt{r^2 - 3h^2} &> 0 \\
9h^2 - 6hr + r^2 &> r^2 - 3h^2 \\
12h^2 - 6hr &> 0 \\
6h(2h - r) &> 0 \\
0 < \quad r &< 2h. \quad \text{Thus,} \\
\sqrt{3}h &\leq r < 2h.
\end{aligned}$$

If we restrict our attention to integer values of r, then both h and r must be divisible by 3.

For $h = 3, 6$ and 9, no integer values of r divisible by 3 satisfy $\sqrt{3}h \leq r < 2h$. So the first allowable value of h is 12. Then the condition $12\sqrt{3} \leq r < 24$ forces $r = 21$. From this we find that $x = 2$ and $y = 3$ and $d = 7\sqrt{3}$. (Note that $(2, 3, 12) = 1$.)

This is only the first solution. We programmed these constraints and let MatLab check for integer values of h and appropriate integer values of r which make x and y integers satisfying $(x, y, h) = 1$. There are many solutions:

$$\begin{pmatrix} r & y & x & y & \text{cevian} \\ & & & & \\ 21 & 12 & 2 & 3 & 7\sqrt{3} \\ & & & & \\ 78 & 45 & 9 & 10 & 26\sqrt{3} \\ & & & & \\ 111 & 60 & 5 & 18 & 37\sqrt{3} \\ & & & & \\ 114 & 63 & 7 & 18 & 38\sqrt{3} \\ & & & & \\ 129 & 72 & 9 & 20 & 43\sqrt{3} \end{pmatrix}$$

Editor's note: David and John then listed another 47 solutions. They capped their search at $h = 1000$, but stated that solutions exist for values of $h > 1000$. They ended the write-up of their solution with a formula for expressing the cevian in terms of x, y and h.

$$\begin{aligned} y - x &= \frac{\sqrt{r^2 - 3h^2}}{3} \\ 9(y-x)^2 &= r^2 - 3h^2 \\ r^2 &= 3h^2 + 9(y-x)^2 \\ r &= \sqrt{3h^2 + 9(y-x)^2} \\ \text{Length of cevian } \frac{r}{\sqrt{3}} &= \sqrt{h^2 + 3(y-x)^2}. \end{aligned}$$

Ken Korbin, the proposer of this problem, gave some insights into how such a problem can be consructed. He wrote:

Begin with any prime number P congruent to 1(mod 6). Find positive integers $[a, b]$ such that $a^2 + ab + b^2 = P^2$. Construct an equilateral triangle with side $a + b$ and with Cevian P. The Cevian will divide the base of the triangle into segments with lengths a and b. Find the altitude of the triangle and the inradii of the 2 smaller triangles. Multiply the altitude, the inradii and the Cevian P by $\sqrt{3}$ and then by their LCD. This should do it. Examples: $P = 7, [a, b] = [3, 5]$. $P = 13, [a, b] = [7, 8]$. And so on.

- **5093:** *Proposed by Worapol Ratanapan (student), Montfort College, Chiang Mai, Thailand*

$6 = 1 + 2 + 3$ is one way to partition 6, and the product of $1, 2, 3$, is 6. In this case, we call each of $1, 2, 3$ a **part** of 6.

We denote the maximum of the product of all **parts** of natural number n as $N(n)$.

As a result, $N(6) = 3 \times 3 = 9, N(10) = 2 \times 2 \times 3 \times 3 = 36$, and $N(15) = 3^5 = 243$.

More generally, $\forall n \in N, N(3n) = 3^n, N(3n+1) = 4 \times 3^{n-1}$, and $N = (3n+2) = 2 \times 3^n$.

Now let's define $R(r)$ in the same way as $N(n)$, but each **part** of r is positive real. For instance $R(5) = 6.25$ and occurs when we write $5 = 2.5 + 2.5$

Evaluate the following:

$$\begin{array}{l} i)\ \ R(2e) \\ ii)\ \ R(5\pi) \end{array}$$

Solution by Michael N. Fried, Kibbutz Revivim, Israel

Let $R(r) = \prod_i x_i$, where $\sum_i x_i = r$ and $x_i > 0$ for all i. For any given r, find the maximum of $R(r)$.

Since for any given r and n the arithmetic mean of every set $\{x_i\}$ $i = 1, 2, 3 \ldots n$ is $\frac{r}{n}$ by assumption, the geometric-arithemetic mean inequality implies that

$$R(r) = \prod_{i=1}^{n} x_i \le \left(\frac{r}{n}\right)^n.$$

Hence the maximum of $R(r)$ is a function of n. Let us then find the maximum of the function $R(x) = \left(\frac{r}{x}\right)^x$, which is the same as the maximum of

$$L(x) = \ln\left(R(x)\right) = x \ln r - x \ln x.$$

$L(x)$ indeed has a *single* maximum at $x = \frac{r}{e}$.

Let $m = \lfloor \frac{r}{e} \rfloor$ and $M = \lceil \frac{r}{e} \rceil$. Then the maximum value of $R(r)$ is

$$\max\left(\left(\frac{r}{m}\right)^m, \left(\frac{r}{M}\right)^M\right).$$

To make this concrete consider $r = 5,\ 2e$, and 5π.

For $r = 5, r/e = 1.8393\ldots$, so $\max R(5) = \max\left(5, (5/2)^2\right) = \max(5, 6.25) = 6.25$

For $r = 2e, r/e = 2$, so $\max R(2e) = e^2$.

For $r = 5\pi, r/e = 5.7786\ldots$, so $\max R(5\pi) = \max\left(\left(\frac{5\pi}{5}\right)^5, \left(\frac{5\pi}{6}\right)^6\right) = \left(\frac{5\pi}{6}\right)^6$.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Statesboro, GA; The Taylor University Problem Solving Group, Upland, IN, and the proposer.

- **5094:** *Proposed by Paolo Perfetti, Mathematics Department Tor Vergata University, Rome, Italy*

  Let a, b, c be real positive numbers such that $a + b + c + 2 = abc$. Prove that

$$2\left(a^2 + b^2 + c^2\right) + 2\left(a + b + c\right) \ge \left(a + b + c\right)^2.$$

Solution 1 by Ercole Suppa, Teramo, Italy

We will use the "magical" substitution given in "Problems from The Book" by Titu Andreescu and Gabriel Dospinescu, which is explained in the following lemma:

If a, b, c are positive real numbers such that $a+b+c+2=abc$, then there exists three real numbers $x, y, z>0$ such that

$$a=\frac{y+z}{x}, \ b=\frac{z+x}{y}, \text{ and } c=\frac{x+y}{z}. \qquad (*)$$

Proof: By means of a simple computation the condition $a+b+c+2=abc$ can be written in the following equivalent form

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=1.$$

Now if we take

$$x=\frac{1}{1+a}, \quad y=\frac{1}{1+b}, \text{ and } \mathrm{z}=\frac{1}{1+\mathrm{c}},$$

then $x+y+z=1$ and $a=\dfrac{1-x}{x}=\dfrac{y+z}{x}$. Of course, in the same way we find $b=\dfrac{z+x}{y}$ and $c=\dfrac{x+y}{z}$.

By using the substitution $(*)$, after some calculations, the given inequality rewrites as

$$\frac{z^4(x-y)^2+x^4(y-z)^2+y^4(x-z)^2+2(x^3y^3+x^3z^3+y^3z^3-3x^2y^2z^2)}{x^2y^2z^2}\geq 0,$$

which is true since

$$x^3y^3+x^3z^3+y^3z^3\geq 3x^2y^2z^2$$

by virtue of the AM-GM inequality.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

First let $x=a+b$ and $y=ab$. Hence $x\geq 2\sqrt{y}$.

From $a+b+c+2=abc$, we have $c=\dfrac{x+2}{y-1}$. Hence, $y>1$.

Noting that $x^2-2y=a^2+b^2$, it follows readily that the original inequality can be rewritten as

$$(y-2)^2x^2+2\left(y^2-3y+4\right)x-4y^3+8y^2\geq 0, \qquad (1)$$

where $y>1$ and $x\geq 2\sqrt{y}$. For $y>1$ arbitrary but fixed, we denote by $f_y(x)$, for $x\geq 2\sqrt{y}$, the function on the left-hand side of (1).

Trivially, $f_y(x)\geq 0$ for $y=2$. For $y\neq 2$ (which we henceforth assume), $f_y(\cdot)$, when extended to $\Re$, is a quadratic function (parabola) attaining its minimum at $x_0=\dfrac{-\left(y^2-3y+4\right)}{(y-2)^2}$.

Noting that $x_0<0$, it follows that

$$\min_{\{x:x\geq 2\sqrt{y}\}} f_y(x) \quad = \quad f_y(2\sqrt{y})$$

$$= \quad 4\sqrt{y}\left(y^2 - 3y + 4 - 2y^{3/2} + 4\sqrt{y}\right).$$

Thus the inequality (1) will be proved if we show that

$$\varphi(y) := y^2 - 3y + 4 - 2y^{3/2} + 4\sqrt{y} \geq 0. \qquad (2)$$

This is trivial for $1 < y < 2$ since in this case both $y^2 - 3y + 4$ and $-2y^{3/2} + 4\sqrt{y}$ are greater than 0.

For $y > 2$, it is readily seen that $\varphi''(y) > 0$. Hence, $\varphi'(y)$ is increasing for $y > 2$. Noting that $\varphi'(4) = 0$, it thus follows that $\min_{\{y>2\}} \varphi(y) = \varphi(4)$. Since $\varphi(4) = 0$, inequality (2) is proved.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Firstly, we have

$$2(a^2+b^2+c^2) + 2(a+b+c) - (a+b+c)^2 = (a+b+c)(a+b+c+2) - 4(ab+bc+ca)$$

Let $p = a+b+c,\ q = ab+bc+ca,\ r = abc$, so that $r = p+2$.

We need to show that $q \leq \dfrac{p(p+2)}{4}$ (1)

It is well known that $a,\ b$, and c are the positive real roots of the cubic equation

$$x^3 - px^2 + qx - r \quad = \quad 0 \ \text{ if, and only if,}$$

$$p^2q^2 - 4p^3r + 18pqr - 4q^3 - 27r^2 \quad \geq 0.$$

By substituting $r = p+2$ and simplifying, we reduce the last inequality to $f(q) \leq 0$, where

$$\begin{aligned} f(q) &= 4q^3 - p^2q^2 - \left(36p + 18p^2\right)q + 4p^4 + 8p^3 + 27p^2 + 108p + 108 \\ &= (q+2p+3)\left(4q^2 - (p^2+8p+12)q + 2p^3 + p^2 + 12p + 36\right). \text{ Thus} \end{aligned}$$

$$4q^2 - (p^2+8p+12)q + 2p^3 + p^2 + 12p + 36 \leq 0. \qquad (2)$$

By the arithmetic mean-geometric inequality we have
$abc = a+b+c+2 \geq 4(2abc)^{1/4}$ so that $abc \geq 8$ and $p = a+b+c \geq 6$.

From (2) we obtain $q \leq \dfrac{1}{8}\left(p^2 + 8p + 12 + \sqrt{(p+2)(p-6)^3}\right)$ and it remains to show that

$$\frac{1}{8}\left(p^2 + 8p + 12 + \sqrt{(p+2)(p-6)^3}\right) \leq \frac{p(p+2)}{4}. \qquad (3)$$

Now (3) is equivalent to $\sqrt{(p+2)(p-6^3)} \leq (p-6)(p+2)$ or, on squaring both sides and simplifying, $-8(p+2)(p-6)^2 \leq 0$.

Since the last inequality is clearly true, we see that (1) is true, and this completes the solution.

Also solved by Tom Leong, Scotrun, PA; Bruno Salgueiro Fanego, Viveiro, Spain, and the proposer.

- **5095:** *Proposed by Zdravko F. Starc, Vršac, Serbia*

  Let F_n be the Fibonacci numbers defined by

$$F_1 = F_2 = 1,\ F_{n+2} = F_{n+1} + F_n\ (n = 1, 2, \cdots).$$

  Prove that

$$\sqrt{F_{n-2}F_{n-1}} + 1 \le F_n \le \sqrt{(n-2)F_{n-2}F_{n-1}} + 1 \quad (n = 3, 4, \cdots).$$

Solution 1 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

First, using mathematical induction, we show that

$$F_n^2 = F_{n-1}F_{n+1} + (-1)^{n+1}, \text{for } n = 2, 3, \ldots \qquad (2).$$

For $n = 2$ we have:

$$F_2^2 = 1 = 1 \cdot 2 - 1 = F_1F_3 + (-1)^3.$$

Assume that (2) holds for n. We show that it is true also for $n + 1$.

$$\begin{aligned} F_nF_{n+2} + (-1)^{n+2} &= F_n\left(F_n + F_{n+1}\right) + (-1)^{n+2} \\ &= F_n^2 + F_nF_{n+1} + (-1)^{n+2} \\ &= F_{n-1}F_{n+1} + F_nF_{n+1} + (-1)^{n+1} + (-1)^{n+2} \\ &= F_{n+1}\left(F_{n-1} + F_n\right) = F_{n+1}^2. \end{aligned}$$

So (2) hold for any $n \ge 2$.

Next we show that,

$$\sqrt{F_{n-2}F_{n-1}} + 1 \le F_n, \text{ holds.}$$

By applying (2) several times we obtain:

$$\begin{aligned} F_n^2 &= F_{n-1}F_{n+1} + (-1)^{n+1} \\ &= F_{n-1}\left(F_n + F_{n-1}\right) + (-1)^{n+1} \\ &= F_{n-1}F_n + F_{n-1}^2 + (-1)^{n+1} \\ &= F_{n-1}F_n + F_{n-2}F_n + (-1)^n + (-1)^{n+1} \\ &= F_{n-1}F_n + F_{n-2}F_{n-1} + F_{n-2}^2 \\ &= 2F_{n-1}F_{n-2} + F_{n-2}F_n + F_{n-2}^2 + (-1)^{n+1} \\ &= 3F_{n-1}F_{n-2} + 2F_{n-2}^2 + (-1)^{n+1} \\ &= F_{n-1}F_{n-2} + 2F_{n-1}F_{n-2} + 2F_{n-2}^2 + (-1)^{n+1} \end{aligned}$$

$$
\begin{aligned}
&\geq F_{n-1}F_{n-2} + 2\sqrt{F_{n-1}F_{n-2}} + 1 \\
&= \left(\sqrt{F_{n-1}F_{n-2}} + 1\right)^2
\end{aligned}
$$

Taking the square root of both sides we obtain:

$$F_n \geq \sqrt{F_{n-1}F_{n-2}} + 1,$$

which is the first part of (1).

To prove the second part of (1), we proceed similarly. That is:

$$
\begin{aligned}
F_n^2 &= F_{n-1}F_{n+1} + (-1)^{n+1} \\
&= F_{n-1}\left(F_n + F_{n-1}\right) + (-1)^{n+1} \\
&= F_{n-1}F_n + F_{n-1}^2 + (-1)^{n+1} \\
&= F_{n-1}F_n + F_{n-2}F_n + (-1)^n + (-1)^{n+1} \\
&= F_{n-1}F_n + F_{n-2}F_{n-1} + F_{n-2}^2 \\
&= 2F_{n-1}F_{n-2} + F_{n-2}F_n + F_{n-2}^2 + (-1)^{n+1} \\
&= 3F_{n-1}F_{n-2} + 2F_{n-2}^2 + (-1)^{n+1} \\
&\leq 3F_{n-1}F_{n-2} + 2F_{n-1}F_{n-2} + 1 \\
&= 5F_{n-1}F_{n-2} + 1 \\
&\leq (n-2)\,F_{n-1}F_{n-2} + 1 \text{ for } n \geq 7.
\end{aligned}
$$

Taking the square root of both sides we obtain:

$$F_n \leq \sqrt{(n-2)F_{n-1}F_{n-2} + 1} \leq \sqrt{(n-2)F_{n-1}F_{n-2}} + 1, \qquad (4)$$

which proves the second part of (1) for $n \geq 7$.

On can easily show that (4) also holds for $n = 3, 4, 5,$ and 6 by checking each of these cases separately. So combining (3) and (4) we have proved that:

$$\sqrt{F_{n-2}F_{n-1}} + 1 \leq F_n \leq \sqrt{(n-2)F_{n-2}F_{n-1}} + 1 \quad (n = 3, 4, \cdots).$$

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

Given $n = 3, 4, \cdots$, we can use (because all the F_n are positive) the Geometric Mean-Arithmetic Mean Inequality applied to $F_i, i = n-1, n-2$, the facts that $F_n = F_{n-1} + F_{n-2}$ and $F_n \geq 2$ with equality if, and only if, $n = 3$, to obtain:

$$\sqrt{F_{n-2}F_{n-1}} + 1 \leq \frac{F_{n-2} + F_{n-1}}{2} + 1 = \frac{F_n}{2} + 1 \leq F_n,$$

which is the first inequality to prove, with equality if, and only if, $n = 3$.

The second inequality, if $n = 3, 4, \cdots$ can be proved using that $F_n = \sum_{i=1}^{n-2} F_i + 1$, the Quadratic Mean-Arithmetic Mean inequality applied to the positive numbers F_i, $i = 1, 2, \cdots, n-2$, and that $F_{n-2}F_{n-1} = \sum_{i=1}^{n-2} F_i^2$, because

$$F_n = \sum_{i=1}^{n-2} F_i + 1 \leq \sqrt{(n-2)\sum_{i=1}^{n-2} F_i^2} + 1 = \sqrt{(n-2)F_{n-2}F_{n-1}} + 1,$$

with equality if, and only if, $n = 3$ or $n = 4$.

Solution 3 by Shai Covo, Kiryat-Ono, Israel

The left inequality is trivial. Indeed, for any $n \geq 3$,

$$\sqrt{F_{n-2}F_{n-1}} + 1 \leq \sqrt{F_{n-1}F_{n-1}} + F_{n-2} = F_n.$$

As for the right inequality, the result is readily seen to hold for $n = 3, 4, 5, 6$. Hence, it suffices to show that for any $n \geq 7$ the following inequality holds:

$$F_n = F_{n-2} + F_{n-1} < \sqrt{5F_{n-2}F_{n-1}}.$$

With x and y playing the role of F_{n-2} and F_{n-1} ($n \geq 7$), respectively, it thus suffices to show that $x + y < \sqrt{5xy}$, subject to $x < y < 2x$ ($x \geq F_5 = 5$).

It is readily checked that, for any fixed $x > 0$ (real), the function $\phi_x(y) = \sqrt{5xy} - (x + y)$, defined for $y \in [x, 2x]$, has a global minimum at $y = 2x$, where $\phi_x(y) = (\sqrt{10} - 3)x > 0$. The result is now established.

Solution 4 by Brian D. Beasley, Clinton, SC

Let $L_n = \alpha\sqrt{\alpha F_{n-2}F_{n-1}} - 1$ and $U_n = \alpha\sqrt{\alpha F_{n-2}F_{n-1}} + 1$, where $\alpha = (1 + \sqrt{5})/2$. We prove the stronger inequalities $L_n \leq F_n \leq U_n$ for $n \geq 3$, with improved lower bound for $n \geq 5$ and improved upper bound for $n \geq 7$.

First, we note that the inequalities given in the original problem hold for $3 \leq n \leq 6$. Next, we apply induction on n, verifying that $L_3 \leq F_3 \leq U_3$ and assuming that $L_n \leq F_n \leq U_n$ for some $n \geq 3$. Then $(F_n - 1)^2 \leq \alpha^3 F_{n-2}F_{n-1} \leq (F_n + 1)^2$, which implies

$$(F_{n+1} - 1)^2 = (F_n - 1)^2 + 2F_{n-1}(F_n - 1) + F_{n-1}^2 \leq \alpha^3 F_{n-2}F_{n-1} + 2F_{n-1}(F_n - 1) + F_{n-1}^2$$

and

$$(F_{n+1} + 1)^2 = (F_n + 1)^2 + 2F_{n-1}(F_n + 1) + F_{n-1}^2 \geq \alpha^3 F_{n-2}F_{n-1} + 2F_{n-1}(F_n + 1) + F_{n-1}^2.$$

Since $\alpha^3 F_{n-1}F_n = \alpha^3 F_{n-2}F_{n-1} + \alpha^3 F_{n-1}^2$, it suffices to show that

$$2F_{n-1}(F_n - 1) + F_{n-1}^2 \leq \alpha^3 F_{n-1}^2 \leq 2F_{n-1}(F_n + 1) + F_{n-1}^2,$$

that is, $2(F_n - 1) + F_{n-1} \leq \alpha^3 F_{n-1} \leq 2(F_n + 1) + F_{n-1}$. Using the Binet formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, where $\beta = (1 - \sqrt{5})/2$, these latter inequalities are equivalent to $2\beta^{n-1} - 2 \leq 0 \leq 2\beta^{n-1} + 2$, both of which hold since $-1 < \beta < 0$. (We also used the identities $2\alpha + 1 - \alpha^3 = 0$ and $\alpha^3 - 1 - 2\beta = 2\sqrt{5}$.)

Finally, we note that U_n is smaller than the original upper bound for $n \geq 7$, since $\alpha^3+2 < 7$. Also, a quick check verifies that L_n is larger than the original lower bound for $n \geq 5$; this requires

$$(\alpha^3-1)^2(F_{n-2}F_{n-1})^2 - 8(\alpha^3+1)F_{n-2}F_{n-1} + 16 \geq 0,$$

which holds if $F_{n-2}F_{n-1} \geq 4$.

Also solved by Paul M. Harms, North Newton, KS; Tom Leong, Scotrun, PA; Boris Rays, Brooklyn NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5096:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+\sqrt[4]{ab^3}} + \frac{b}{c+\sqrt[4]{bc^3}} + \frac{c}{a+\sqrt[4]{ca^3}} \geq \frac{3}{2}.$$

Solution 1 by Ovidiu Furdui, Cluj, Romania

We have, since $\sqrt[4]{xy^3} \leq \dfrac{x+3y}{4}$, that

$$\sum_{cyclic} \frac{a}{b+\sqrt[4]{ab^3}} \geq 4\sum_{cyclic} \frac{a}{7b+a} = 4\sum_{cyclic} \frac{a^2}{7ba+a^2} \geq 4\frac{(a+b+c)^2}{\sum a^2 + 7\sum ab},$$

and hence it suffices to prove that

$$8(a+b+c)^2 \geq 3(a^2+b^2+c^2) + 21(ab+bc+ca).$$

However, the last inequality reduces to proving that

$$a^2+b^2+c^2 \geq ab+bc+ca,$$

and the problem is solved since the preceding inequality holds for all real a, b, and c.

Solution 2 by Ercole Suppa, Teramo, Italy

By the weighted AM-GM inequality we have

$$\begin{aligned} & \frac{a}{b+\sqrt[4]{ab^3}} + \frac{b}{c+\sqrt[4]{bc^3}} + \frac{c}{a+\sqrt[4]{ca^3}} \\ \geq\ & \frac{a}{b+\frac{1}{4}a+\frac{3}{4}b} + \frac{b}{c+\frac{1}{4}b+\frac{3}{4}c} + \frac{c}{a+\frac{1}{4}c+\frac{3}{4}a} \\ =\ & \frac{4a}{a+7b} + \frac{4b}{b+7c} + \frac{4c}{c+7a}. \end{aligned}$$

So it suffices to prove that

$$\frac{a}{a+7b} + \frac{b}{b+7c} + \frac{c}{c+7a} \geq \frac{3}{8}.$$

This inequality is equivalent to

$$\frac{7(13a^2b+13b^2c+13ac^2+35ab^2+35a^2c+35bc^2-144abc)}{8(a+7b)(b+7c)(c+7a)} \geq 0$$

which is true. Indeed according to the AM-GM inequality we obtain

$$13a^2b+13b^2c+13ac^2 \geq 13\cdot 3\cdot \sqrt[3]{a^3b^3c^3} \quad = \quad 39abc$$

$$35ab^2+35a^2c+35bc^2 \geq 35\cdot 3\cdot \sqrt[3]{a^3b^3c^3} \quad = \quad 105abc$$

and, summing these inequalities we obtain:

$$13a^2b+35ab^2+35a^2c+13b^2c+13ac^2+35bc^2 \geq 144abc.$$

This ends the proof. Clearly, equality occurs for $a=b=c$.

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

We start by considering the function

$$f(t) = \frac{1}{e^t + e^{\frac{3}{4}t}}$$

on $\Re$. Then for all $t \in \Re$,

$$f''(t) = \frac{16e^{2t}+23e^{\frac{7}{4}t}+9e^{\frac{3}{2}t}}{16\left(e^t+e^{\frac{3}{4}t}\right)^3} > 0,$$

and hence, $f(t)$ is strictly convex on $\Re$.

If $x = \ln\left(\frac{b}{a}\right)$, $y = \ln\left(\frac{b}{a}\right)$, and $z = \ln\left(\frac{b}{a}\right)$, then

$$x+y+z = \ln\left(\frac{b}{a}\cdot\frac{c}{b}\cdot\frac{a}{c}\right) = \ln 1 = 0.$$

By Jensen's Theorem,

$$\begin{aligned}
\frac{a}{b+\sqrt[4]{ab^3}}+\frac{b}{c+\sqrt[4]{bc^3}}+\frac{c}{a+\sqrt[4]{ca^3}} \quad &= \quad \frac{1}{\left(\frac{b}{a}\right)+\left(\frac{b}{a}\right)^{3/4}}+\frac{1}{\left(\frac{c}{b}\right)+\left(\frac{c}{b}\right)^{3/4}}+\frac{1}{\left(\frac{a}{c}\right)+\left(\frac{a}{c}\right)^{3/4}} \\
&= \quad f(x)+f(y)+f(z) \\
&\geq \quad 3f\left(\frac{x+y+z}{3}\right)
\end{aligned}$$

$$= \quad 3f(0)$$

$$= \quad \frac{3}{2}.$$

Further, equality is attained if, and only if, $x = y = z = 0$, i.e., if, and only if, $a = b = c$.

Solution 4 by Shai Covo, Kiryat-Ono, Israel

Let us first represent b and c as $b = xa$ and $c = yxa$, where x and y are arbitrary positive real numbers. By doing so, the original inequality becomes

$$\frac{1}{x + x^{3/4}} + \frac{1}{y + y^{3/4}} + \frac{yx}{1 + (yx)^{1/4}} \geq \frac{3}{2}. \qquad (1)$$

Let us denote by $f(x, y)$ the expression on the left-hand side of this inequality. Clearly, $f(x, y)$ has a global minimum at some point $(\alpha, \beta) \in (0, \infty) \times (0, \infty)$, a priori not necessarily unique. This point is, in particular, a critical point of f; that is, $f_x(\alpha, \beta) = f_y(\alpha, \beta) = 0$, where f_x and f_y denote the partial derivatives of f with respect to x and y. Calculating derivatives, the conditions $f_x(\alpha, \beta) = 0$ and $f_y(\alpha, \beta) = 0$ imply that

$$\left\{ \begin{array}{l} \dfrac{1 + \frac{3}{4}\alpha^{-1/4}}{\left(\alpha + \alpha^{3/4}\right)^2} = \dfrac{\beta\left[1 + \frac{3}{4}(\beta\alpha)^{1/4}\right]}{\left[1 + (\beta\alpha)^{1/4}\right]^2} \ \text{ and} \\ \\ \dfrac{1 + \frac{3}{4}\beta^{-1/4}}{\left(\beta + \beta^{3/4}\right)^2} = \dfrac{\alpha\left[1 + \frac{3}{4}(\beta\alpha)^{1/4}\right]}{\left[1 + (\beta\alpha)^{1/4}\right]^2} \end{array} \right. , \qquad (2)$$

respectively. From this it follows straight forwardly, that

$$\frac{1 + \frac{3}{4}\alpha^{-1/4}}{\alpha\left(1 + \alpha^{-1/4}\right)^2} = \frac{1 + \frac{3}{4}\beta^{-1/4}}{\beta\left(1 + \beta^{-1/4}\right)^2}.$$

Writing this equality as $\varphi(\alpha) = \varphi(\beta)$ and noting that φ is strictly decreasing, we conclude (by virtue of φ being one-to-one) that $\alpha = \beta$. Substituting this into (2) gives

$$\frac{1 + \frac{3}{4}\alpha^{-1/4}}{\left(\alpha + \alpha^{3/4}\right)^2} = \frac{\alpha\left(1 + \frac{3}{4}\alpha^{1/2}\right)}{\left(1 + \alpha^{1/2}\right)^2}.$$

Comparing the numerators and denominators of this equation shows that the right-hand side is greater than the left-hand side for $\alpha > 1$, while the opposite is true for $\alpha < 1$. We conclude that $\alpha = \beta = 1$. Thus f has a unique global minimum at $(x, y) = (1, 1)$, where $f(x, y) = 3/2$. The inequality (1), and hence the one stated in the problem, is thus proved.

Also solved by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Tom Leong, Scotrun, PA; Paolo Perfetti, Mathematics Department Tor Vergata University, Rome, Italy, and the proposer.

- **5097:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $p \geq 2$ be a natural number. Find the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\lfloor \sqrt[p]{n} \rfloor},$$

where $\lfloor a \rfloor$ denotes the **floor** of a. (Example $\lfloor 2.4 \rfloor = 2$).

Solution 1 by Paul M. Harms, North Newton, KS

Since the series is an alternating series it is important to check whether the number of terms with the same denominator is even or odd. It is shown below that the number of terms with the same denominator is an odd number.

Consider p=2. The series starts:

$$\frac{(-1)^1}{1} + \frac{(-1)^2}{1} + \frac{(-1)^3}{1} + \frac{(-1)^4}{1} + \ldots + \frac{(-1)^8}{2} + \frac{(-1)^9}{3} + \ldots$$

$$\begin{aligned} &= \frac{(-1)^3}{1} + \frac{(-1)^8}{2} \frac{(-1)^{15}}{3} + \ldots \\ &= -1 + \frac{1}{2} - \frac{1}{3} + \ldots . \end{aligned}$$

The terms with 1 in the denominator are from $n = 1^2$ up to (not including) $n = 2^2$, and the terms with 2 in the denominator come from $n = 2^2$ up to $n = 3^2$. The number of terms with 1 in the denominator is $2^2 - 1^2 = 3$ terms.

For $p = 2$ the number of terms with a positive integer m in the denominator is $(m+1)^2 - m^2 = 2m + 1$ terms which is an odd number of terms.

For a general positive integer p, the number of terms with a positive integer m in the denominator is $(m + 1)^p - m^p$ terms. Either $(m + 1)$ is even and m is odd or vice versa. An odd integer raised to a positive power is an odd integer, and an even integer raised to a positive power is an even integer. Then $(m + 1)^p - m^p$ is the difference of an even integer and an odd integer which is an odd integer. Since, for every positive integer p the series starts with $\dfrac{(-1)^1}{1} = -1$ and we have an odd number of terms with denominator 1, the last term with 1 in the denominator is $\dfrac{-1}{1}$ and the other terms cancel out.

The terms with denominator 2 start and end with positive terms. They all cancel out except the last term of $\dfrac{1}{2}$.

Terms with denominator 3 start and end with negative terms. For every p we have the series

$$\frac{-1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots = -\ln 2.$$

Solution 2 by The Taylor University Problem Solving Group, Upland, IN

First note that the denominators of the terms of this series will be increasing natural numbers, because $\sqrt[p]{n}$ will always be a real number greater than or equal to 1 for $n \geq 1$, meaning that its floor will be a natural number. Furthermore, for a natural number a, a^p is the smallest n for which a is the denominator, because $\lfloor \sqrt[p]{a^p} \rfloor = \lfloor a \rfloor = a$. In other words, the denominator increases by 1 each time n is a perfect *pth* power. Thus, a natural number k occurs as the denominator $(k+1)^p - k^p$ times in the series. Because multiplying a number by itself preserves parity and $k+1$ and k always have opposite parity, $(k+1)^p$ and k^p also have opposite parity, hence their difference is odd. So each denominator occurs an odd number of times. Because the numberator alternates between 1 and -1, all but the last of the terms with the same denominator will cancel each other out. This leaves an alternating harmonic series with a negative first term, which converges to $-\ln 2$.

This can be demonstrated by the fact that the alternating harmonic series with a positive first term is the Mercator series evaluated at $x = 1$, and this series is simply the opposite of that.

Incidentally, this property holds for $p = 1$ as well.

Also solved by Shai Covo, Kiryat-Ono, Israel; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Mathematics Department Tor Vergata University, Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA; Ercole Suppa, Teramo, Italy, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before September 15, 2010

- **5116:** *Proposed by Kenneth Korbin, New York, NY*

  Given square $ABCD$ with point P on side AB, and with point Q on side BC such that

  $$\frac{AP}{PB} = \frac{BQ}{QC} > 5.$$

  The cevians DP and DQ divide diagonal AC into three segments with each having integer length. Find those three lengths, if $AC = 84$.

- **5117:** *Proposed by Kenneth Korbin, New York, NY*

  Find positive acute angles A and B such that

  $$\sin A + \sin B = 2\sin A \cdot \cos B.$$

- **5118:** *Proposed by David E. Manes, Oneonta, NY*

  Find the value of

  $$\sqrt{2011 + 2007\sqrt{2012 + 2008\sqrt{2013 + 2009\sqrt{2014 + \cdots}}}}$$

- **5119:** *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain*

  Let n be a non-negative integer. Prove that

  $$2 + \frac{1}{2^{n+1}} \prod_{k=0}^{n} \csc\left(\frac{1}{F_k}\right) < F_{n+1}$$

  where F_n is the n^{th} Fermat number defined by $F_n = 2^{2^n} + 1$ for all $n \geq 0$.

- **5120:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Calculate

  $$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \log\left(\frac{2n-k}{2n+k}\right).$$

- **5121:** *Proposed by Tom Leong, Scotrun, PA*

  Let n, k and r be positive integers. It is easy to show that

  $$\sum_{n_1+n_2+\cdots+n_r=n} \binom{n_1}{k}\binom{n_2}{k}\cdots\binom{n_r}{k} = \binom{n+r-1}{kr+r-1}, \quad n_1, n_2, \cdots n_r \in N$$

  using generating functions. Give a combinatorial argument that proves this identity.

Solutions

- **5098:** *Proposed by Kenneth Korbin, New York, NY*

  Given integer-sided triangle ABC with $\angle B = 60^\circ$ and with $a < b < c$. The perimeter of the triangle is $3N^2 + 9N + 6$, where N is a positive integer. Find the sides of a triangle satisfying the above conditions.

  Solution 1 by Michael N. Fried, Kibbutz Revivim, Israel

  Since $3n^2 + 9n + 6 = 3(n+1)(n+2) = 3K$, we can rephrase the problem as follows: Find an integer-sided triangle ABC with angle $B = 60^\circ$ and $a < b < c$ whose perimeter is the same as an equilateral triangle PBQ whose side is K.

  Let us then consider a triangle derived from PBQ by lengthening PB by an integer x and shortening BQ by an integer y such that the resulting triangle still has perimeter $3K$.

  Thus, we can write the following expression:

  $$\text{Perimeter } (\triangle ABC) = (K+x)+(K-y)+\Big[(K+x)^2 + (K-y)^2 - (K+x)(K-y)\Big]^{1/2} = 3K \qquad (1)$$

  Also we must make sure that,

  $$(K+x)^2 + (K-y)^2 - (K+x)(K-y) = M^2, \text{ for some interger } M \qquad (2)$$

  Note also that $BC < CA < AB$ since $\angle BAC < 60^\circ < \angle BCA$.

  Equation (1) can be transformed into the much simpler equation,

  $$xy = K(y-x) = (n+1)(n+2)(y-x) \qquad (3)$$

  The most obvious solution of (3) is $x = n+1$ and $y = n+2$.

  Substituting these expressions into the left hand side of (2) and simplifying, we get

  $$(K+x)^2+(K-y)^2-(K+x)(K-y) = (n+1)^4+2(n+1)^3+3(n+1)^2+2(n+1)+1 \qquad (4)$$

  But the right hand side of (4) is just $\big[(n+1)^2 + (n+1) + 1\big]^2$, so that (2) is satisfied when $x = n+1$ and $y = n+2$.

  Hence, we have at least one solution:

  $$\begin{aligned} AB &= K + x = (n+1)(n+2) + (n+1) = (n+1)(n+3) \\ BC &= K - y = (n+1)(n+2) - (n+2) = n(n+2) \\ CA &= (n+1)^2 + (n+1) + 1 \end{aligned}$$

Solution 2 by David Stone and John Hawkins, Statesboro, GA

We show that the following triangle satisfies the conditions posed in the problem:

$$\begin{aligned} a &= N2 + 2N = N(N+2) \\ b &= N^2 + 3N + 3 = (N+1)(N+2) + 1 \\ c &= N^2 + 4N + 3 = (N+1)(N+3). \end{aligned}$$

But by no means does this give all acceptable triangles and we exhibit some others (and methods to produce them).

The given sum for the perimeter does have a connection to triangles: $3N^2 + 9N + 6$ is $6T_{N+1}$, the $N + 1st$ triangular number!

Since $a < b < c$ are all integers, we let m and n be positive integers such that $b = a + m$ and $c = a + m + n$.

By the Law of Cosines, $b^2 = a^2 + c^2 - 2ac\cos 60^\circ = a^2 - ac + c^2$. Replacing $b = a + m$ and $c = a + m + n$ we get

$$\begin{aligned} &(a+m)^2 = a^2 - a(a+m+n) + (a+m+n)^2 \text{ or} \\ &-am + an + n^2 + 2mn = 0. \qquad (1) \end{aligned}$$

Likewise, substituting $b = a + m$ and $c = a + m + n$ into the proscribed perimeter conditions produces

$$3a + 2m + n = 3(N+1)(N+2). \qquad (2)$$

From equation (1), we have $am = n(a + 2m + n)$; and from this we see that n must be a factor of am. There are many ways for this to happen, but the simplest possible is that $n|s$ or $n|m$.

Case I: $a = nA$.

Then

$$\begin{aligned} nAm &= n(nA + 2m + n), \\ Am &= nA + 2m + n, \text{ or} \\ (A-2)m &= n(A+1). \qquad (1a) \end{aligned}$$

The simplest possible solution to Equation (1a) is

$$\begin{aligned} n &= A - 2 \\ m &= A + 1 \end{aligned}$$

In this case, equation (2) becomes

$$\begin{aligned} 3nA + 2m + n &= 3\,(N+1)\,(N+2)\,, \\ 3(A-2)A + 2(A+1) + (A-2) &= 3(N+1)(N+2), \\ 3A^2 - 3A &= 3(N+1)(N+2), \text{ or} \\ (A-1)A &= (N+1)(N+2). \end{aligned}$$

Because $A - 1$ and A are consecutive integers, as are $N + 1$ and $N + 2$, we must have $A = N + 2$ (so $n = N$ and $m = N + 3$). It then follows that

$$a = nA = N(N+2)$$

$$\begin{aligned} b &= a+m = N(N+2)+(N+3) = N^2+3N+3 \\ c &= a+m+n = N(N+2)+(N+3)+N = N^2+4N+3. \end{aligned}$$

It is straightforward to check that such a, b, c satisfy equations (1) and (2). Here are the first few solutions:

N	a	b	c
1	3	7	8
2	8	13	15
3	15	21	24
4	24	31	35
5	35	43	48
6	48	57	63
7	63	73	80
8	80	91	99
9	99	111	120
10	120	133	143
11	143	157	168
12	168	183	195
13	195	211	224
14	224	241	255
15	225	273	288

There are more solutions to the equation $(1a) : (A-2)m = n(A+1)$. For instance, we could look for solutions with $m = d(A+1)$ and $n = d(A-2)$, with $d > 1$. In this case, equation (2) becomes $dA(A-1) = (N+1)(N+2)$ which is quadratic in A. By varying d (and using Excel) we find more solutions:

d	N	a	b	c
2	2	6	14	16
2	19	390	422	448
3	8	72	93	105
3	34	1197	1263	1320
5	4	15	35	40
5	13	175	215	240
5	98	9675	9905	10120

Note that these solutions are scalar multiples of the (fundamental?) solutions found above. Many more solutions are possible.

Case II: $n|m$, or $m = nC$.

In this case, $am = n(a+2m+n)$ becomes

$$\begin{aligned} anC &= n(a+2nC+n) \text{ or} \\ aC &= a+2nC+n \text{ or} \\ (C-1)a &= n(2C+1) \qquad (1b) \end{aligned}$$

Once again, there are many ways to find solutions to this, but no general solution valid for all values of N. We stop by giving one more: with $N = 54$ we find
$a = 231, b = 4449, c = 4560$.

Also solved by Brian D. Beasley, Clinton, SC; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; David C. Wilson, Winston-Salem, NC, and the proposer.

- **5099:** *Proposed by Kenneth Korbin, New York, NY*

  An equilateral triangle is inscribed in a circle with diameter d. Find the perimeter of the triangle if a chord with length $d-1$ bisects two of its sides.

  Solution 1 by Boris Rays, Brooklyn, NY

  Let O be the center of the inscribed equilateral triangle ABC. Let the intersection of the altitude from vertex A with side BC be F; from vertex B with side AC be H, and from vertex C with side AB be E . Since $\triangle ABC$ is equilateral, AF, BH, and CE are also the respective angle bisectors, perpendicular bisectors, and medians of the equilateral triangle, and $AH = HC = CE = FB = BE = EA$.

  Let line segment EF be extended in each direction, intersecting BH at K, and the circumscribing circle of $\triangle ABC$ at points D and G, where D is on the minor arc$\widehat{AB}$ and G is on the minor arc $\widehat{BC}$. Note that points D, E, K, F, and G lie on line segment $\overline{DG}$ and that $AO = OG$. Also note, by the givens of the problem, that

$$\begin{aligned} DG &= d-1 \text{ and} \\ AO &= BO = CO = r = \frac{d}{2}, \qquad (1) \end{aligned}$$

  where r and d are correspondingly the radius and diameter of the circumscribed circle.

$$\begin{aligned} BH &\perp AC, AH = HC, \angle BAO = \angle OAH = 30^\circ. \\ OH &= \frac{1}{2}AO = \frac{d}{4}. \\ AH &= \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{d}{4}\right)^2} = \frac{d}{4}\sqrt{3}. \\ AC &= 2AH = \frac{d}{2}\sqrt{3} \qquad (2) \end{aligned}$$

  The perimeter P of triangle $\triangle ABC$ will be

$$\begin{aligned} P &= 3 \cdot AC = \frac{3}{2}\sqrt{3}d. \qquad (3) \\ BK &= \frac{1}{2}BH = \frac{1}{2} \cdot 3 \cdot OH = \frac{3}{8}d. \\ KO &= BO - BK = \frac{d}{2} - \frac{3}{8}d = \frac{d}{8}. \\ GK &= \frac{1}{2}DG = \frac{d-1}{2}. \end{aligned}$$

  Triangle $\triangle GKO$ is a right triangle with $DG \perp BH$ and $GK \perp BO$. Therefore,

$$GO^2 = GK^2 + KO^2 \qquad (4)$$

  Substituting the values of the component parts of $\triangle GKO$ into (4),

$$GO = r = \frac{d}{2},\ GK = \frac{d-1}{2},\ KO = \frac{d}{8},$$

we obtain

$$\left(\frac{d-1}{2}\right)^2 - \left(\frac{d}{8}\right)^2 = \left(\frac{d}{2}\right)^2. \qquad (5)$$

Simplifying the last equation (5) we find that $d = 4\cdot\left(4+\sqrt{15}\right)$. Therefore,

$$\begin{aligned} AC &= \frac{4(4+\sqrt{15})}{2}\sqrt{3} = 2\left(4\sqrt{3}+3\sqrt{5}\right), \text{and} \\ P &= 3\cdot 2\left(4\sqrt{3}+3\sqrt{5}\right) = 24\sqrt{3}+18\sqrt{5}. \end{aligned}$$

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Solution 2 by Brian D. Beasley, Clinton, NC

We model the circle using $x^2+y^2=r^2$, where $r=d/2$, and place the triangle with one vertex at $(0,r)$, leaving the other two vertices in the third and fourth quadrants. Labeling the fourth quadrant vertex as (a,b), we have $b=r-\sqrt{3}a$ and thus $a=\sqrt{3}r/2$, $b=-r/2$. Then two of the midpoints of the triangle's sides are $\left(\frac{\sqrt{3}}{4}r, \frac{1}{4}r\right)$ and $\left(0, -\frac{1}{2}r\right)$. We find the endpoints of the chord through these two midpoints by substituting its equation, $y=\sqrt{3}x-r/2$, into the equation of the circle; the two x-coordinates of these endpoints are $x=sr$ and $x=tr$, where

$$s=\frac{\sqrt{3}+\sqrt{15}}{8} \quad \text{and} \quad t=\frac{\sqrt{3}-\sqrt{15}}{8}.$$

Hence the length of the chord is

$$\sqrt{(s-t)^2r^2+(\sqrt{3}(s-t))^2r^2} = d(s-t).$$

If the chord length is $d-k$, where $0<k<d$, then $d=k/(1-s+t)=4k(4+\sqrt{15})$. Thus the perimeter of the triangle is $P=3\sqrt{3}r=k(24\sqrt{3}+18\sqrt{5})$. For the given problem, since $k=1$, we obtain $P=24\sqrt{3}+18\sqrt{5}$.

Also solved by Michael Brozinsky, Central Islip, NY; Paul M. Harms, North Newton, KS; John Nord, Spokane, WA; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5100:** *Proposed by Mihály Bencze, Brasov, Romania*

Prove that

$$\sum_{k=1}^{n}\sqrt{\frac{k}{k+1}}\binom{n}{k} \le \sqrt{\frac{n(2^{n+1}-n)2^{n-1}}{n+1}}$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

We need the identities

$$\sum_{k=0}^{n}\binom{n}{k}x^k = (1+x)^n \qquad (1)$$

$$\sum_{k=0}^{n}k\binom{n}{k}x^{k-1} = n\,(1+x)^{n-1} \qquad (2)$$

$$\text{and} \quad \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1} = \frac{(1+x)^{n+1}-1}{n+1} \qquad (3)$$

Identity (1) is the well known binomial expansion, whilst identities (2) and (3) follow respectively by differentiating and integrating (1). By the Cauchy-Schwarz inequality and putting $x = 1$ in (2) and (3) we obtain

$$\begin{aligned}
\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}\binom{n}{k}} &= \sum_{k=1}^{n} \left(\sqrt{k\binom{n}{k}}\right)\left(\sqrt{\binom{n}{k}\frac{1}{k+1}}\right) \\
&\le \sqrt{\left(\sum_{k=1}^{n} k\binom{n}{k}\right)\left(\sum_{k=1}^{n}\binom{n}{k}\frac{1}{k+1}\right)} \\
&= \sqrt{(n2^{n-1})\left(\frac{2^{n+1}-1}{n+1}-1\right)} \\
&= \sqrt{\frac{n\left(2^{n+1}-n-2\right)2^{n-1}}{n+1}},
\end{aligned}$$

and the inequality of the problem follows.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

We shall prove a substantially better upper bound than the one stated in the problem. Namely, we show that

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}\binom{n}{k}} < \frac{n}{n+1}\left(2^n - \frac{1}{2}\right).$$

It is readily checked that our bound is less than the bound of $\sqrt{\dfrac{n(2^{n+1}-n)2^{n-1}}{n+1}}$ that the problem suggests; moreover, we have verified numerically that it is much tighter.

Now to the proof. The key observation is that

$$\sqrt{\frac{k}{k+1}} < 1 - \frac{1}{2(k+1)}$$

for all $k \in N$ (actually, for any real $k > 0$; its origin lies in the *mean value theorem* applied to the function $f(x) = \sqrt{x}$ and points $a = k/(k+1), b = 1$.

Thus, using the elementary identity $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ (twice), we get

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}}\binom{n}{k} < \sum_{k=1}^{n}\binom{n}{k} - \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+1}\binom{n}{k}$$

$$
\begin{aligned}
&= 2^n - 1 - \frac{1}{2(n+1)} \sum_{k=1}^{n} \binom{n+1}{k+1} \\
&= 2^n - 1 - \frac{2^{n+1} - (n+1) - 1}{2(n+1)} \\
&= \frac{n}{n+1} \left(2^n - \frac{1}{2}\right).
\end{aligned}
$$

Solution 3 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

We prove the slightly more general statement

$$\sum_{k=1}^{n} \sqrt{\frac{k}{k+1}} \binom{n}{k} \le \sqrt{\frac{(2^n-1)\left[(n-1)\,2^n+1\right]}{n+1}}. \qquad (1)$$

To show that this implies the desired inequality, we begin by letting $P(n)$ be the statement: $2^{n+1} > (n-1)^2 + 3$. $P(1)$ is obvious and if we assume $P(n)$ is true for some $n \ge 1$, then

$$
\begin{aligned}
2^{n+2} &= 2 \cdot 2^{n+1} > 2\left[(n-1)^2 + 3\right] = \left(n^2+3\right) + \left(n^2 - 4n + 5\right) \\
&= \left(n^2+3\right) + \left[(n-2)^2 + 1\right] > n^2 + 3,
\end{aligned}
$$

and $P(n+1)$ is also true. By Mathematical Induction, $P(n)$ is true for all $n \ge 1$.

Then, for $n \ge 1$,

$$
\begin{aligned}
& n\left(2^{n+1} - n\right) 2^{n-1} - (2^n - 1)\left[(n-1)\,2^n + 1\right] \\
=\ & 2^{n-1}\left[2^{n+1} - (n-1)^2 - 3\right] + 1 \\
>\ & 0
\end{aligned}
$$

and we have

$$(2^n-1)\left[(n-1)\,2^n+1\right] < n\left(2^{n+1} - n\right) 2^{n-1}. \qquad (2)$$

It follows that statement (1) implies the given inequality.

To prove statement (1), we note that since $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$, we get $\displaystyle\sum_{k=1}^{n} \frac{\binom{n}{k}}{2^n - 1} = 1$.
Because $f(x) = \sqrt{x}$ is concave down on $[0, \infty)$, Jensen's Theorem implies that

$$\sum_{k=1}^{n} \binom{n}{k} \frac{1}{2^n-1} \sqrt{\frac{k}{k+1}} \le \sqrt{\sum_{k=1}^{n} \binom{n}{k} \frac{1}{2^n-1} \frac{k}{k+1}} = \sqrt{\frac{1}{2^n-1} \sum_{k=1}^{n} \binom{n}{k} \frac{k}{k+1}},$$

and hence,

$$\sum_{k=1}^{n} \binom{n}{k} \sqrt{\frac{k}{k+1}} \le \sqrt{(2^n-1) \sum_{k=1}^{n} \binom{n}{k} \frac{k}{k+1}}. \qquad (3)$$

For $k = 1, 2, \ldots, n$,

$$\binom{n}{k}\frac{k}{k+1} = \frac{k}{n+1}\binom{n+1}{k+1}$$

and we get

$$\sum_{k=1}^{n}\binom{n}{k}\frac{k}{k+1} = \frac{1}{n+1}\sum_{k=1}^{n}k\binom{n+1}{k+1} = \frac{1}{n+1}\sum_{k=2}^{n+1}(k-1)\binom{n+1}{k}. \qquad (4)$$

Finally, the Binomial Theorem yields

$$\sum_{k=0}^{n+1}\binom{n+1}{k}x^k = (1+x)^{n+1}.$$

It follows that when $x \neq 0$,

$$\sum_{k=1}^{n+1}\binom{n+1}{k}x^{k-1} = \frac{(1+x)^{n+1}-1}{x}$$

and, by differentiating,

$$\sum_{k=2}^{n+1}(k-1)\binom{n+1}{k}x^{k-2} = \frac{x\,(n+1)\,(1+x)^n - \left[(1+x)^{n+1}-1\right]}{x^2}.$$

In particular, when $x = 1$,

$$\sum_{k=2}^{n+1}(k-1)\binom{n+1}{k} = (n+1)\,2^n - 2^{n+1} + 1 = (n-1)\,2^n + 1. \qquad (5)$$

Then, statements (3), (4), and (5) imply statement (1), which (by statement (2)) yields the desired inequality.

Also solved by G. C. Greubel, Newport News, VA, and the proposer

- **5101:** *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India*

An unbiased coin is tossed repeatedly until r heads are obtained. The outcomes of the tosses are written sequentially. Let R denote the total number of runs (of heads and tails) in the above experiment. Find the distribution of R.

Illustration: if we decide to toss a coin until we get 4 heads, then one of the possibilities could be the sequence $T\ T\ H\ H\ T\ H\ T\ H$ resulting in 6 runs.

Solution by Shai Covo, Kiryat-Ono, Israel

It is readily seen that R can be represented as

$$R = 1 + Y_1 + 2\sum_{i=2}^{r} Y_i, \qquad (1)$$

where $Y_i,\ i = 1, \ldots, r$, is a random variable equal to 1 if the i-th head follows a tail and equal to 0 otherwise. The $Y_{i's}$ are thus *independent* Bernoulli (1/2) variables, that is

$P(Y_i = 1) = P(Y_i = 0) = 1/2$. Noting that R is odd if and only if $Y_1 = 0$, and even if and only if $Y_1 = 1$, it follows straightforwardly from (1) that

$$\begin{aligned} P(R=n) &= \frac{1}{2}P\left(\sum_{i=2}^{r} Y_i = \frac{n-1}{2}\right) \text{ for } n = 1,3,\ldots,(2r-1) \text{ and} \\ P(R=n) &= \frac{1}{2}P\left(\sum_{i=2}^{r} Y_i = \frac{n-2}{2}\right) \text{ for } n = 2,4,\ldots,2r. \end{aligned} \qquad (2)$$

Finally, since $\sum_{i=2}^{r} Y_i$ has a binomial distribution with parameters $r-1$ and $\frac{1}{2}$ (defined as 0 if $r=1$), we conclude that

$$P(R=n) = \binom{r-1}{(n-1)/2}\frac{1}{2^r} \text{ for } n = 1,3,\ldots,(2r-1)$$

and

$$P(R=n) = \binom{r-1}{(n-2)/2}\frac{1}{2^r} \text{ for } n = 2,4,\ldots,2r.$$

Remark 1. More generally, if the probability of getting a head on each throw is $p \in (0,1)$, then $P(R=n)$ is given, in a shorter form, by

$$P(R=n) = \binom{r-1}{\lfloor \frac{n-1}{2} \rfloor}(1-p)^{\lfloor n/2 \rfloor}p^{r-\lfloor n/2 \rfloor}, \; n = 1,2,\ldots,2r,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. This is proved in the same way as in the unbiased case, only that now the Y_i are Bernoulli $(1-p)$ variables.

Remark 2. From (1) and the fact that $E(Y_i) = 1/2$ and $Var(Y_i) = 1/4$, we find that the expectation and variance of R are given by
$E(R) = 1 + \frac{1}{2} + 2(r-1)\frac{1}{2} = r + \frac{1}{2}$ and $Var(R) = \frac{1}{4} + 4(r-1)\frac{1}{4} = r - \frac{3}{4}$.
In the more general case of Remark 1, where $E(Y_i) = 1-p$ and $Var(Y_i) = (1-p)p$, the expectation and variance of R are given by

$$E(R) = 2(1-p)r + p \quad \text{and} \quad Var(R) = 4(1-p)pr - 3(1-p)p.$$

Also solved by David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **5102:** *Proposed by Miquel Grau-Sánchez and José Luis Díaz-Barrero, Barcelona, Spain*

Let n be a positive integer and let $a_1, a_2, \cdots, a_n$ be any real numbers. Prove that

$$\frac{1}{1+a_1^2+\ldots+a_n^2} + \frac{1}{F_nF_{n+1}}\left(\sum_{k=1}^{n}\frac{a_kF_k}{1+a_1^2+\ldots+a_k^2}\right)^2 \le 1,$$

where F_k represents the k^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3, F_n = F_{n-1} + F_{n-2}$.

Solution by Kee-Wai Lau, Hong Kong, China

By Cauchy-Schwarz's inequality and the well known identity $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$ we have

$$\begin{aligned}
&\frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \frac{a_k F_k}{1 + a_1^2 + \ldots + a_k^2} \right)^2 \\
= &\frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \left(\frac{a_k}{1 + a_1^2 + \ldots + a_k^2} \right) F_k \right)^2 \\
\leq &\frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \frac{a_k^2}{\left(1 + a_1^2 + \ldots + a_k^2\right)^2} \right) \left(\sum_{k=1}^{n} F_k^2 \right) \\
= &\sum_{k=1}^{n} \frac{a_k^2}{\left(1 + a_1^2 + \ldots + a_k^2\right)^2}
\end{aligned}$$

Hence it remains for us to show that

$$\frac{1}{1 = a_1^2 + \ldots + a_n^2} + \sum_{k=1}^{n} \frac{a_k^2}{\left(1 + a_1^2 + \ldots + a_k^2\right)^2} \leq 1. \qquad (1)$$

Denote the left hand side of (1) by $f(n)$. Since $f(1) = \dfrac{1 + 2a_1^2}{1 + 2a_1^2 + a_1^4}$, so $f(1) \leq 1$.

Now

$$\begin{aligned}
&f(m+1) - f(m) \\
= &\frac{1}{1 + a_1^2 + \ldots + a_{m+1}^2} + \frac{a_{m+1}^2}{\left(1 + a_1^2 + \ldots + a_{m+1}^2\right)^2} - \frac{1}{1 + a_1^2 + \ldots + a_m^2} \\
= &\frac{\left(1 + a_1^2 + \ldots + a_m^2\right)\left(1 + a_1^2 + \ldots + a_m^2 + 2a_{m+1}^2\right) - \left(1 + a_1^2 + \ldots + a_{m+1}^2\right)^2}{\left(1 + a_1^2 + \ldots + a_{m+1}^2\right)^2 \left(1 + a_1^2 + \ldots + a_m^2\right)} \\
= &- \frac{a_{m+1}^4}{\left(1 + a_1^2 + \ldots + a_{m+1}^2\right)^2 \left(1 + a_1^2 + \ldots + a_m^2\right)} \\
\leq &\ 0,
\end{aligned}$$

so in fact $f(n) \leq 1$ for all positive integers n. Thus (1) holds and this completes the solution.

Also solved by the proposers.

- **5103:** *Proposed by Roger Izard, Dallas, TX*

A number of circles of equal radius surround and are tangent to another circle. Each of the outer circles is tangent to two of the other outer circles. No two outer circles intersect in two points. The radius of the inner circle is a and the radius of each outer circle is b. If

$$a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 = 0,$$

determine the number of outer circles.

Solution by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie, San Angelo, TX

Let $\mathcal{C_A}$ be the inner circle centered at point A with radius a. Similarly, let $\mathcal{C_B}$ be a fixed outer circle centered at point B with radius b. Circle $\mathcal{C_B}$ is tangent to two other outer circles; let T_1 and T_2 be these points of tangency. Then,

$$\overline{BT_1} \perp \overline{AT_1} \quad \text{and} \quad \overline{BT_2} \perp \overline{AT_2}.$$

If θ is the measure of $\angle T_1AT_2$, then $0^\circ < \theta < 180^\circ$. Further, triangle T_1AB is a right triangle where

$$m\angle T_1AB = \frac{\theta}{2}, \quad T_1B = b, \quad \text{and} \quad AB = a + b$$

which yields

$$\sin\left(\frac{\theta}{2}\right) = \frac{b}{a+b}. \qquad (1)$$

The given condition $a^4 + 4a^3b - 10a^2b^2 - 28ab^3 + b^4 = 0$ implies that

$$\begin{aligned}
a^4 + 4a^3b + b^4 &= 10a^2b^2 + 28ab^3 \\
a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 &= 16a^2b^2 + 32ab^3 \\
(a+b)^4 &= 16b^2(a^2 + 2ab) \\
(a+b)^4 &= 16b^2(a^2 + 2ab + b^2) - 16b^4 \\
(a+b)^4 &= 16b^2(a+b)^2 - 16b^4 \\
1 &= \frac{16b^2(a+b)^2 - 16b^4}{(a+b)^4} \\
1 &= 16\left(\frac{b}{a+b}\right)^2 - 16\left(\frac{b}{a+b}\right)^4.
\end{aligned}$$

By equation (1) and the half-angle formula, $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$, it follows that:

$$\begin{aligned}
1 &= 16\left(\frac{1-\cos\theta}{2}\right) - 16\left(\frac{1-\cos\theta}{2}\right)^2 \\
1 &= 8(1-\cos\theta) - 4(1-\cos\theta)^2 \\
1 &= 4 - 4\cos^2\theta
\end{aligned}$$

$$\begin{aligned}\cos^2\theta &= \frac{3}{4}\\ \cos\theta &= \pm\frac{\sqrt{3}}{2}\\ \theta &= 30^\circ \text{ or } 150^\circ.\end{aligned}$$

Since the number of outer circles is $\dfrac{360^\circ}{\theta}$, then $\theta = 30^\circ$ and there must be 12 outer circles.

Comment by editor: **David Stone and John Hawkins of Statesboro, GA** observed in their solution that "the circle passing through the centers of the outer bracelet of circles has circumference almost equal, but slightly larger than, the perimeter of the regular polygon determined by these centers: $2\pi(a+b) \approx n(2b)$. Thus $n \approx \dfrac{a+b}{b}\pi$ (in fact, n must be slightly smaller than $\dfrac{a+b}{b}\pi$)."

They went on to say that since

$$\begin{aligned}a^4+4a^3b-10a^2b^2-28ab^3+b^4 &= 0,\\ \frac{a^4}{b^4}+\frac{4a^3b}{b^4}-\frac{10a^2b^2}{b^4}-\frac{28ab^3}{b^4}+\frac{b^4}{b^4} &= 0, \text{ implies}\\ x^4+4x^3-10x^2-28x+1 &= 0, \text{ where } x=\frac{a}{b}.\end{aligned}$$

Therefore, $\dfrac{a}{b} = \sqrt{6}\pm\sqrt{2}-1$, and since $n \approx \dfrac{a+b}{b}\pi$, $n = 12$. But then they went further. The equation $\sin\left(\dfrac{\pi}{n}\right) = \dfrac{b}{a+b} = \dfrac{1}{1+\dfrac{a}{b}}$, provides the link between n and the ratio $\dfrac{a}{b}$; we can solve for either:

$$n = \frac{\pi}{\sin^{-1}\left(\dfrac{1}{1+a/b}\right)} \quad\text{and}\quad \frac{a}{b} = \frac{1}{\sin(\pi/n)} - 1.$$

The problem poser cleverly embedded a nice ratio for $\dfrac{a}{b}$ in the fourth degree polynomial; nice in the sense that the n turned out to be an integer. In fact, the graph of the increasing function $y = \dfrac{\pi}{\sin^{-1}\left(\dfrac{1}{1+r}\right)}$ is continuous and increasing for the positive ratio r. Thus **any** lager value of n is uniquely attainable (given the correct choice of $r = \dfrac{a}{b}$).

Or we can reverse the process: fix the number of surrounding circles and calculate $r = \dfrac{a}{b}$.

A nice example (by letting $b = 1$): if we want to surround a circle with a bracelet of 100 unit circles, how large should it be? Answer:

$$\text{radius } = a = \frac{a}{1} = \frac{1}{\sin\dfrac{\pi}{100}} - 1 = 30.836225.$$

Also solved by Michael Brozinsky, Central Islip, NY; Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; Kenneth

Korbin, New York, NY; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA; The Taylor University Problem Solving Group, Upland, IN, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before December 15, 2010

- **5122:** *Proposed by Kenneth Korbin, New York, NY*

  Partition the first 32 non-negative integers from 0 to 31 into two sets A and B so that the sum of any two distinct integers from set A is equal to the sum of two distinct integers from set B and vice versa.

- **5123:** *Proposed by Kenneth Korbin, New York, NY*

  Given isosceles triangle ABC with $\overline{AB} = \overline{BC} = 2011$ and with cevian $\overline{BD}$. Each of the line segments $\overline{AB}$, $\overline{BD}$, and $\overline{CD}$ have positive integer length with $\overline{AD} < \overline{CD}$.

  Find the lengths of those three segments when the area of the triangle is minimum.

- **5124:** *Proposed by Michael Brozinsky, Central Islip, NY*

  If $n > 2$ show that $\sum_{i=1}^{n} \sin^2\left(\frac{2\pi i}{n}\right) = \frac{n}{2}$.

- **5125:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

  $$\frac{ab}{2(c+a)+5b} + \frac{bc}{2(a+b)+5c} + \frac{ca}{2(b+c)+5a} < \frac{11}{32}.$$

- **5126:** *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c, d be positive real numbers and $f : [a, b] \to [c, d]$ be a function such that $|f(x) - f(y)| \geq |g(x) - g(y)|$, for all $x, y \in [a, b]$, where $g : R \to R$ is a given injective function, with $g(a)$, $g(b) \in \{c, d\}$.

  Prove

  (**i**) $f(a) = c$ and $f(b) = d$, or $f(a) = d$ and $f(b) = c$.

  (**ii**) If $f(a) = g(a)$ and $f(b) = g(b)$, then $f(x) = g(x)$ for $a \leq x \leq b$.

- **5127:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $n \geq 1$ be an integer and let $T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \dfrac{x^{2k-1}}{(2k-1)!}$, denote the $(2n-1)$th Taylor polynomial of the sine function at 0. Calculate

$$\int_0^\infty \frac{T_n(x) - \sin x}{x^{2n+1}} dx.$$

Solutions

- **5104:** *Proposed by Kenneth Korbin, New York, NY*

  There are infinitely many primitive Pythagorean triangles with hypotenuse of the form $4x^4 + 1$ where x is a positive integer. Find the dimensions of all such triangles in which at least one of the sides has prime integer length.

  Solution by Brian D. Beasley, Clinton, SC

  It is well-known that a primitive Pythagorean triangle (a, b, c) satisfies $a = 2st$, $b = s^2 - t^2$, and $c = s^2 + t^2$ for integers $s > t > 0$ of opposite parity with $\gcd(s, t) = 1$. Then a is never prime. Letting x be a positive integer and taking

$$c = 4x^4 + 1 = (2x^2 + 2x + 1)(2x^2 - 2x + 1),$$

  we see that c can only be prime if $2x^2 - 2x + 1 = 1$, meaning $x = 1$. Thus $s = 2$ and $t = 1$, which produces the triangle $(4, 3, 5)$. Similarly, $b = (s + t)(s - t)$ can only be prime if $s - t = 1$, which would yield

$$4x^4 + 1 = 2t^2 + 2t + 1 \quad \text{and hence} \quad 2x^4 = t(t + 1).$$

  But this would force one of the consecutive positive integers t or $t + 1$ to be a fourth power and the other to be twice a fourth power, meaning $t = 1$. Once again, our only solution is the triangle $(4, 3, 5)$.

  Also solved by Paul M. Harms, North Newton, KS; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer

- **5105:** *Proposed by Kenneth Korbin, New York, NY*

  Solve the equation

$$x + y - \sqrt{x^2 + xy + y^2} = 2 + \sqrt{5}$$

  if x and y are of the form $a + b\sqrt{5}$ where a and b are positive integers.

  Solution by Shai Covo, Kiryat-Ono, Israel

  We let $x = a + b\sqrt{5}$ and $y = c + d\sqrt{5}$, with $a, b, c, d \in N$. Since the solution of

$$x + y - \sqrt{x^2 + xy + y^2} = 2 + \sqrt{5} \qquad (1)$$

  is symmetric in x and y, it suffices to consider the case $x \leq y$. Hence, we let $y = x\alpha$ with $\alpha \geq 1$. Substituting into (1) gives

$$(a + b\sqrt{5}) \left[(1 + \alpha) - \sqrt{1 + \alpha + \alpha^2}\right] = 2 + \sqrt{5} \qquad (2)$$

It is immediately verified by taking the derivative that the function $\varphi(\alpha) = (1+\alpha) - \sqrt{1+\alpha+\alpha^2}$ is increasing . From $\varphi(\alpha)\left[(1+\alpha)+\sqrt{1+\alpha+\alpha^2}\right] = \alpha$ it is readily seen that $\varphi(\alpha) \to \dfrac{1}{2}$ as $\alpha \to \infty$. On the other hand, $\varphi(1) = 2-\sqrt{3}$. We thus conclude from (2) that

$$4+2\sqrt{5} < a+b\sqrt{5} \le \frac{2+\sqrt{5}}{2-\sqrt{3}}.$$

We verify numerically that this leaves us with the following set of pairs (a,b):

$$\begin{array}{ccccccc} \{(1,4), & (1,5), & (1,6), & (2,3), & (2,4), & (2,5), & (2,6), \\ (3,3), & (3,4), & (3,5), & (4,3), & (4,4), & (4,5), & (5,2), \\ (5,3), & (5,4), & (6,2), & (6,3), & (6,4), & (7,1), & (7,2), \\ (7,3), & (8,1), & (8,2), & (8,3), & (9,1), & (9,2), & (9,3), \\ (10,1), & (10,2), & (11,1), & (11,2), & (12,1), & (13,1)\}. & \end{array}$$

It follows straightforwardly from (1) that

$$y = \frac{\left(4+2\sqrt{5}\right)x - 9 - 4\sqrt{5}}{x-4-2\sqrt{5}}.$$

Substituting $x = a+b\sqrt{5}$ and multiplying the numerator and denominator on the right hand side by $(a-4)-(b-2)\sqrt{5}$ gives, after some algebra,

$$y = \frac{4a^2-5a+20b-20b^2-4}{(a-4)^2-5(b-2)^2} + \frac{2a^2-4a+13b-10b^2-2}{(a-4)^2-5(b-2)^2}\sqrt{5}. \qquad (3)$$

This determines the constants c and d forming y in an obvious manner, since $a,b,c,d \in N$. In particular, we see that

$$c-2d = \frac{3a-6b}{(a-4)^2-5(b-2)^2}. \qquad (4)$$

From this, noting that $c-2d$ is an integer, it follows readily that a and b cannot be both odd; furthermore if a and b are both even, then a must be divisible by 4. This restricts the set of all possible pairs (a,b)given above to

$$\begin{array}{ccccccc} \{(1,4), & (1,6), & (2,3), & (2,5), & (3,4), & (4,3), & (4,4), \\ (4,5), & (5,2), & (5,4), & (6,3), & (7,2), & (8,1), & (8,2), \\ (8,3), & (9,2), & (10,1), & (11,2), & (12,1)\}. & & \end{array}$$

The requirement that the right-handside of (4) be an integer further restricts the set to

$$\{(2,3),\ (5,2),\ (6,3),\ (7,2)\}.$$

With these values of a and b, calculating c and d according to (3) give the following x, y pairs:

$$x = 2+3\sqrt{5}, \qquad y = 118+53\sqrt{5}$$

$$x = 5+2\sqrt{5}, \qquad y = 31+14\sqrt{5}$$

$$x = 6 + 3\sqrt{5}, \qquad y = 10 + 5\sqrt{5}$$

$$x = 7 + 2\sqrt{5}, \qquad y = 13 + 6\sqrt{5}.$$

Substituting into (1) show that these x, y pairs constitute the solution of (1) for $x \leq y$. The complete solution then follows by symmetry in x and y.

Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro,GA, and the proposer.

- **5106:** *Proposed by Michael Brozinsky, Central Islip, NY*

Let a, b, and c be the sides of an acute-angled triangle ABC. Let H be the orthocenter and let d_a, d_b and d_c be the distances from H to the sides BC, CA, and AB respectively.

Show that

$$d_a + d_b + d_c \leq \frac{3}{4}D$$

where D is the diameter of the circumcircle.

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

From the published solution of SSM problem 5066 and Gerretsen and Euler's inequality, we have that

$$d_a + d_b + d_c = \frac{r^2 + s^2 - 4R^2}{2R} \leq \frac{r^2 + 4Rr + 3r^2}{2R} = \frac{2}{R}(r + R) \leq 1\left(\frac{R}{2} + R\right) = \frac{3}{4}D,$$

with equality if and only if $\triangle ABC$ is equilateral.

Solution 2 by Ercole Suppa, Teramo, Italy

Let H_a, H_b, H_c be the feet of A, B, C onto the sides BC, CA, AB respectively and let R be the circumradius of $\triangle ABC$. We have

$$d_a = BH_a \cdot \tan(90^\circ - C) = c \cos B \cot C.$$

Hence, taking into account the extended sine law, we get

$$d_a = 2R \sin C \cos B \cot C = 2R \cos B \cos C. \qquad (1)$$

Now, by using (1) and its cyclic permutations, the given inequality rewrites as

$$\begin{aligned} 2R\cos B\cos C + 2R\cos C\cos A + 2R\cos A\cos B &\leq \frac{3}{4}\cdot 2R \\ \cos B\cos C + \cos C\cos A + \cos A\cos B &\leq \frac{3}{4} \qquad (2) \end{aligned}$$

which is true. In fact, from the well known formulas

$$\sum \cos^2 A = 1 - 2\cos A \cos B \cos C$$

and

$$0 \leq \cos A \cos B \cos C \leq \frac{1}{8},$$

each of which is valid for an acute-angled triangle, we immediately obtain

$$\sum \cos^2 A \geq \frac{3}{4}. \qquad (3)$$

Hence, by applying the known inequality

$$1 < \cos A + \cos B + \cos C \leq \frac{3}{2},$$

we obtain

$$\begin{aligned}
(\cos A + \cos B + \cos C)^2 &\leq \frac{9}{4} \Rightarrow \\
\sum \cos^2 A + 2\sum \cos B \cos C &\leq \frac{9}{4} \Rightarrow \\
2\sum \cos B \cos C \leq \frac{9}{4} - \sum \cos^2 A &\leq \frac{9}{4} - \frac{3}{4} = \frac{3}{2} \Rightarrow \\
\sum \cos B \cos C &\leq \frac{3}{4},
\end{aligned}$$

and the conclusion follows. Equality holds for $a = b = c$.

Also solved by Scott H. Brown, Montgomery, AL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China;

- **5107:** *Proposed by Tuan Le (student, Fairmont, H.S.), Anaheim, CA*

Let a, b, c be positive real numbers. Prove that

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2} + \frac{\sqrt{b^3+c^3}}{b^2+c^2} + \frac{\sqrt{c^3+a^3}}{c^2+a^2} \geq \frac{6(ab+bc+ac)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}}$$

Solution by Kee-Wai Lau, Hong Kong, China

By the Cauchy-Schwarz inequality, we have

$$a^2+b^2 \leq \sqrt{(a+b)\left(a^3+b^3\right)},\ b^2+c^2 \leq \sqrt{(b+c)\left(b^3+c^3\right)},\ c^2+a^2 \leq \sqrt{(c+a)\left(c^3+a^3\right)}.$$

Hence it suffices to show that

$$\frac{1}{\sqrt{a+b}} + \frac{1}{\sqrt{b+c}} + \frac{1}{\sqrt{c+a}} \geq \frac{6(ab+bc+ac)}{(a+b+c)\sqrt{(a+b)(b+c)(c+a)}} \quad \text{or}$$

$$\sqrt{(a+b)(b+c)} + \sqrt{(b+c)(c+a)} + \sqrt{(c+a)(a+b)} \geq \frac{6(ab+bc+ac)}{(a+b+c)}.$$

By the arithmetic mean-geometric mean-harmonic inequalities, we have

$$\begin{aligned}
&\sqrt{(a+b)(b+c)} + \sqrt{(b+c)(c+a)} + \sqrt{(c+a)(a+b)} \\
\geq\ & 3\sqrt[3]{(a+b)(b+c)(c+a)}
\end{aligned}$$

$$\geq \frac{9}{\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{c+a}}$$

$$= \frac{9(a+b)((b+c)(c+a)}{a^2+b^2+c^2+3(ab+bc+ca)}.$$

It remains to show that

$$3(a+b+c)(a+b)(b+c)(c+a) \geq 2(ab+bc+ca)\left(a^2+b^2+c^2+3\,(ab+bc+ca)\right).$$

But this follows from the fact that

$$\begin{aligned}
& 3(a+b+c)(a+b)(b+c)(c+a) - 2(ab+bc+ca)\left(a^2+b^2+c^2+3(ab+bc+ca)\right) \\
= \; & a^3b+ab^3+a^3c+ac^3+b^3c+bc^3-2a^2bc-2ab^2c-2abc^2 \\
= \; & a(b+c)(b-c)^2+b(c+a)(c-a)^2+c(a+b)(a-b)^2 \\
\geq \; & 0,
\end{aligned}$$

and this completes the solution.

Also solved by Pedro H.O. Pantoja (student, UFRN), Natal, Brazil; Paolo Perfetti, Department of Mathematics, University of Rome, Italy, and the proposer.

- **5108:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Compute

$$\lim_{n\to\infty}\frac{1}{n}\tan\left[\sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right)\right].$$

Solution 1 by Ovidiu Furdui, Cluj, Romania

The limit equals 4. A calculation shows that

$$\begin{aligned}
\arctan\left(1+\frac{2}{k(k+1)}\right) &= \arctan(1)+\arctan\frac{1}{k^2+k+1}, \\
&= \frac{\pi}{4}+\arctan\frac{1}{k}-\arctan\frac{1}{k+1}.
\end{aligned}$$

And it follows that

$$\begin{aligned}
\sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right) &= (4n+1)\frac{\pi}{4}+\arctan 1-\arctan\frac{1}{4n+2} \\
&= (4n+1)\frac{\pi}{4}+\arctan\frac{4n+1}{4n+3}.
\end{aligned}$$

Thus,

$$\tan\left[\sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right)\right] = \tan\left((4n+1)\frac{\pi}{4}+\arctan\frac{4n+1}{4n+3}\right)$$

$$= \frac{\tan((4n+1)\frac{\pi}{4}) + \frac{4n+1}{4n+3}}{1 - \tan((4n+1)\frac{\pi}{4})\frac{4n+1}{4n+3}}$$

$$= \frac{1 + \frac{4n+1}{4n+3}}{1 - \frac{4n+1}{4n+3}}$$

$$= 4n+2.$$

So the limit equals 4, and the problem is solved.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

We will show that

$$\lim_{n\to\infty} \frac{1}{n} \tan\left[\sum_{k=1}^{4n+1} \arctan\left(1 + \frac{2}{k(k+1)}\right)\right] = 4. \qquad (1)$$

From the identity $\tan(x + m\pi) = \tan(x)$, m integer, it follows that the equality (1) will be proved if we show that

$$\sum_{k=1}^{4n+1} \arctan\left(1 + \frac{2}{k(k+1)}\right) = \arctan(4n+2) + m\pi, \qquad (2)$$

for some integer m. In fact, as we will see at the end, the m in (2) is equal to n. We first prove the following lemma.

Lemma. Define a sequence $(a_k)_{k\geq 1}$ recursively by $a_1 = 2$ and, for $k \geq 2$,

$$a_k = \frac{a_{k-1} + \left(1 + \frac{2}{k(k+1)}\right)}{1 - a_{k-1}\left(1 + \frac{2}{k(k+1)}\right)}. \qquad (3)$$

If $a_{k-1} = k$, for some $k \geq 2$, then

$$a_k = -\frac{k+2}{k},\ a_{k+1} = -\frac{1}{k+2},\ a_{k+2} = \frac{k+2}{k+4},\ a_{k+3} = k+4.$$

Hence, in particular, $a_{4n+1} = 4n+2$ for all $n \geq 0$.

Proof. Suppose that $a_{k-1} = k, k \geq 2$. Substituting this into (3) gives

$$a_k = -\frac{(k^2+1)(k+2)}{(k^2+1)k} = -\frac{k+2}{k}. \qquad (4)$$

From (3) and (4) we find

$$a_{k+1} = -\frac{k^2+2k+2}{(k^2+2k+2)(k+2)} = -\frac{1}{k+2}. \qquad (5)$$

From (3) and (5) we find

$$a_{k+2} = \frac{(k^2+4k+5)(k+2)}{(k^2+4k+5)(k+4)} = \frac{k+2}{k+4}. \qquad (6)$$

Finally, from (3) and (6) we find

$$a_{k+3} = \frac{(k^2+6k+10)(k+4)}{k^2+6k+10} = k+4.$$

The lemma is thus established.

We make use of the addition formula for arctan:

$$\arctan(x)+\arctan(y) = \begin{cases} \arctan\left(\dfrac{x+y}{1-xy}\right), & \text{if } xy<1, \\ \arctan\left(\dfrac{x+y}{1-xy}\right)+\pi\text{sign}(x), & \text{if } xy>1, \end{cases} \qquad (7)$$

the case where $xy=1$ being irrelevant here. Now, let $(a_k)_{k\geq 1}$ be the sequence defined in the above lemma. If follows readily from (7) and the lemma that, for all $k \geq 2$,

$$\arctan(a_{k-1}) + \arctan\left(1+\frac{2}{k(k+1)}\right) = \arctan(a_k)+\pi\sigma_k,$$

where $\sigma_k = 1$ or 0 accordingly, as k is or is not of the form $k=4j+2, j\geq 0$ integer. From this it follows that

$$\sum_{k=1}^{l} \arctan\left(1+\frac{2}{k(k+1)}\right) = \arctan(a_l)+\pi\sum_{k=1}^{l}\sigma_k.$$

Recalling the conclusion in the lemma, it thus follows that (2) holds with $m=n$, and so we are done.

Remark: From (2), where $m=n$, and the fact that

$$\begin{aligned} \int \arctan\left(1+\frac{2}{x(x+1)}\right)dx &= \frac{1}{2}\log(x^2+1)-\frac{1}{2}\log(x^2+2x+2) \\ &+ \arctan\left(1+\frac{2}{x(x+1)}\right)+\arctan(x+1)+C, \end{aligned}$$

it follows readily the following interesting result:

$$\int_0^{4n+1} \arctan\left(1+\frac{2}{x(x+1)}\right)dx - \sum_{k=1}^{4n+1}\arctan\left(1+\frac{2}{k(k+1)}\right) \longrightarrow \frac{1}{2}\log 2, \text{as } n\to\infty.$$

Also solved by Kee-Wai, Hong Kong, China, and the proposer.

- **5109:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k \geq 1$ be a natural number. Find the value of

$$\lim_{n\to\infty}\frac{(k\sqrt[n]{n}-k+1)^n}{n^k}.$$

Solution 1 by Angel Plaza and Sergio Falcon, Las Palmas de Gran Canaria, Spain

Let

$$\begin{aligned}
x_n &= \frac{(k\sqrt[n]{n}-k+1)^n}{n^k}. \quad \text{Then,}\\
\ln x_n &= \ln(k\sqrt[n]{n}-k+1)^n - \ln n^k = n\ln(k\sqrt[n]{n}-k+1) - k\ln n\\
&= n\left(\ln(k\sqrt[n]{n}-k+1) - k\ln\sqrt[n]{n}\right)\\
&= \frac{\ln\dfrac{k\sqrt[n]{n}-k+1}{(\sqrt[n]{n})^k}}{\dfrac{1}{n}} \approx \frac{\dfrac{k\sqrt[n]{n}-k+1}{(\sqrt[n]{n})^k}-1}{\dfrac{1}{n}} = \frac{k\sqrt[n]{n}-k+1-(\sqrt[n]{n})^k}{(\sqrt[n]{n})^k\,\dfrac{1}{n}}.
\end{aligned}$$

Now, taking into account that $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$ and the equivalence of the infinitesimals $k(x-1)+1-x^k \approx \dfrac{k(k-1)}{2}(x-1)^2$ when $x\to 1$, we have

$$\begin{aligned}
\lim_{n\to\infty}\ln x_n &= \lim_{n\to\infty}\frac{\dfrac{k(k-1)}{2}\left(\sqrt[n]{n}-1\right)^2}{\dfrac{1}{n}} = \frac{k(k-1)}{2}\lim_{n\to\infty}\frac{(\ln n)^2}{\dfrac{1}{n}}\\
&= \frac{k(k-1)}{2}\lim_{n\to\infty}\frac{(\ln n)^2}{n} = 0. \quad \text{Therefore,}\\
\lim_{n\to\infty} x_n &= 1.
\end{aligned}$$

Solution 2 by Kee-Wai Lau of Hong Kong, China

As $n\to\infty$, we have $\sqrt[n]{n} = e^{\ln n/n} = 1 + \dfrac{\ln n}{n} + O\left(\dfrac{\ln^2 n}{n^2}\right)$. Since $\ln(1+x) = x + O\left(x^2\right)$ as $x\to 0$, so

$$n\ln\left(1+k\left(\sqrt[n]{n}-1\right)\right) - k\ln n = n\left(\frac{k\ln n}{n} + O\left(\frac{\ln^2 n}{n^2}\right)\right) - k\ln n = O\left(\frac{\ln^2 n}{n}\right),$$

where the constant implied by the last O depends at most on k. It follows that the limit of the problem equal 1, independent of k.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paolo Perfetti, Department of Mathematics, University of Rome, Italy, and the proposer.

Late Solution

A late solution to 5099 was received from **Charles McCracken of Dayton, OH.**

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before January 15, 2011

- **5128:** *Proposed by Kenneth Korbin, New York, NY*

  Find all positive integers less than 1000 such that the sum of the divisors of each integer is a power of two.

  For example, the sum of the divisors of 3 is 2^2, and the sum of the divisors of 7 is 2^3.

- **5129:** *Proposed by Kenneth Korbin, New York, NY*

  Given prime number c and positive integers a and b such that $a^2 + b^2 = c^2$, express in terms of a and b the lengths of the legs of the primitive Pythagorean Triangles with hypotenuses c^3 and c^5, respectively.

- **5130:** *Proposed by Michael Brozinsky, Central Islip, NY*

  In Cartesianland, where immortal ants live, calculus has not been discovered. A bride and groom start out from $A(-a, 0)$ and $B(b, 0)$ respectively where $a \neq b$ and $a > 0$ and $b > 0$ and walk at the rate of one unit per second to an altar located at the point P on line $L : y = mx$ such that the time that the first to arrive at P has to wait for the other to arrive is a maximum. Find, without calculus, the locus of P as m varies through all nonzero real numbers.

- **5131:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive real numbers. Prove that

$$\frac{a+b+3c}{3a+3b+2c} + \frac{a+3b+c}{3a+2b+3c} + \frac{3a+b+c}{2a+3b+3c} \geq \frac{15}{8}.$$

- **5132:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Find all all functions $f : C \to C$ such that $f(f(z)) = z^2$ for all $z \in C$.

- **5133:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Let $n \geq 1$ be a natural number. Calculate

$$I_n = \int_0^1 \int_0^1 (x - y)^n dx dy.$$

- **5110:** *Proposed by Kenneth Korbin, New York, NY.*

Given triangle ABC with an interior point P and with coordinates $A(0,0), B(6,8)$, and $C(21,0)$. The distance from point P to side $\overline{AB}$ is a, to side $\overline{BC}$ is b, and to side $\overline{CA}$ is c where $a:b:c = \overline{AB} : \overline{BC} : \overline{CA}$.

Find the coordinates of point P

Solution 1 by Boris Rays, Brooklyn, NY

From the given triangle we have $\overline{AB} = 10, \overline{BC} = 17$ and $\overline{CA} = 21$. Also $a:b:c = 10:17:21$.

Let $a = 10t,\ b = 17t$, and $c = 21t$, where t is real number, $t > 0$. (1)

$$\text{Area } \triangle\text{ABC} = \text{Area } \triangle\text{APB} + \text{Area } \triangle\text{BPC} + \text{Area } \triangle\text{CPA}. \qquad (2)$$

Express all of the terms in (2) by using formulas in (1).

$$\begin{aligned}\frac{1}{2}\cdot 21\cdot 8 &= \frac{1}{2}\cdot 10\cdot 10t + \frac{1}{2}\cdot 17\cdot 17t + \frac{1}{2}\cdot 21\cdot 21t \\ &= \frac{1}{2}t\left(10^2+17^2+21^2\right) = \frac{1}{2}830t\end{aligned}$$

From the above we find that $t = \dfrac{84}{415} = \dfrac{2^2\cdot 3\cdot 7}{5\cdot 83}$.

The y-coordinate of point P is c, the distance to side $\overline{CA}$.

$$y_P = c = 21t = 21\cdot\frac{84}{415} = \frac{1764}{415}.$$

Let points E and F lie on side $\overline{CA}$, where $\overline{PE} \perp \overline{CA}$ and $\overline{BF} \perp \overline{CA}$.

Hence we have $\overline{PE} = C = \dfrac{42^2}{415}$, $\overline{BF} = 8$, and $\overline{AF} = 6$.

$$\text{Area } \triangle\text{APB} + \text{Area } \triangle\text{APE} + \text{Area BPEF} = \text{Area } \triangle\text{ABF}.$$

Letting $\overline{AE} = x$ we have $\overline{EF} = 6 - x$. Therefore,

$$\frac{1}{2}\cdot 10\cdot a + \frac{1}{2}\cdot x\cdot c + \frac{1}{2}\left(\overline{PE} + \overline{BF}\right)\cdot\overline{EF} = \frac{1}{2}\overline{AF}\cdot\overline{BF}$$

$$\frac{1}{2}\cdot 100\cdot\frac{84}{415} + \frac{1}{2}\cdot x\cdot\frac{42^2}{415} + \frac{1}{2}\left(\frac{42^2}{415} + 8\right)(6-x) = \frac{1}{2}6\cdot 8.$$

From the above equation we find x.

$$x = \frac{1}{8}\left(\frac{8400 + 6(42)^2}{415}\right) = \frac{2373}{415}.$$

Hence, the coordinates of point P are $\left(\dfrac{2373}{415}, \dfrac{1764}{415}\right)$.

Solution 2 by Charles McCracken, Dayton, OH

$$\overline{AB} = 10 \qquad \overline{BC} = 17 \qquad \overline{CA} = 21$$

The equations of $\overline{AB}, \overline{BC}$ and $\overline{CA}$ are respectively,

$$4x - 3y = 0 \qquad 8x + 15y - 168 = 0 \qquad y = c$$

Then,

$$a = \frac{4x - 3y}{5} \qquad b = \frac{8x + 15y - 168}{17} \qquad c = y$$

$$\frac{\left(\dfrac{4x-3y}{5}\right)}{y} = \frac{10}{21} \qquad \frac{\left(\dfrac{8x+15y-168}{-17}\right)}{y} = \frac{17}{21}$$

$$21(4x - 3y) = 50y \qquad 21(8x + 15y - 168) = -289y$$

$$84x - 113y = 0 \qquad 168x + 604y = 3528$$

These last two equations give:

$$(x, y) = \left(\frac{2373}{415}, \frac{1764}{415}\right)$$

Note that P is the Lemoine point of $\triangle ABC$, that is, the intersection of the symmedians. (*Editor:* A symmedian is the reflection of a median about its corresponding angle bisector.)

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky, Central Islip, NY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; John Nord, Spokane, WA; Raúl A. Simón, Santiago, Chile; Danielle Urbanowicz, Jennie Clinton, and Bill Solyst (jointly; students at Taylor University), Upland, IN; David Stone and John Hawkins (jointly), Satetesboro, GA, and the proposer.

- **5111:** *Proposed by Michael Brozinsky, Central Islip, NY.*

In Cartesianland where immortal ants live, it is mandated that any anthill must be surrounded by a triangular fence circumscribed in a circle of unit radius. Furthermore, if the vertices of any such triangle are denoted by A, B, and C, in counterclockwise order, the anthill's center must be located at the interior point P such that $\angle PAB = \angle PBC = \angle PCA$.

Show $\overline{PA} \cdot \overline{PB} \cdot \overline{PC} \leq 1$.

Solution by Kee-Wai Lau, Hong Kong, China

It is easy to check that $\angle APB = 180^\circ - B$, $\angle BPC = 180^\circ - C$, and $\angle CPA = 180^\circ - A$.

It is well known that the area of $\triangle ABC = 2R^2 \sin A \sin B \sin C$, where R is the circumradius of the triangle. Here we have $R = 1$. Since the area of $\triangle ABC$ equals the sum of the areas of triangles APB, BPC and CPA, we have

$$\text{Area } \triangle\text{ABC} = \text{Area } \triangle\text{APB} + \text{Area } \triangle\text{BPC} + \text{Area } \triangle\text{CPA}$$

$$2\sin A \sin B \sin C = \frac{1}{2}\left(\overline{PA}\cdot\overline{PB}\sin B + \overline{PB}\cdot\overline{PC}\sin C + \overline{PC}\cdot\overline{PA}\sin A\right).$$

By the arithmetic mean-geometric mean inequality, we have

$$\overline{PA}\cdot\overline{PB}\sin B + \overline{PB}\cdot\overline{PC}\sin C + \overline{PC}\cdot\overline{PA}\sin A \geq 3\left(\overline{PA}\cdot\overline{PB}\cdot\overline{PC}\right)^{2/3}(\sin A \sin B \sin C)^{1/3}.$$

It follows that

$$\left(\overline{PA}\cdot\overline{PB}\cdot\overline{PC}\right)^{2/3} \leq \frac{4}{3}(\sin A \sin B \sin C)^{2/3}. \qquad (1)$$

By the concavity of the function $\ln(\sin x)$ for $0 < x < \pi$, we obtain

$$\ln(\sin A) + \ln(\sin B) + \ln(\sin C) \leq 3\left(\sin\left(\frac{A+B+C}{3}\right)\right) = 3\ln\left(\frac{\sqrt{3}}{2}\right).$$

Therefore,

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}. \qquad (2)$$

The result $\overline{PA}\cdot\overline{PB}\cdot\overline{PC} \leq 1$ now follows easily from (1) and (2) immediately above.

Comments: The proposer, **Michael Brozinsky,** mentioned in his solution that point P is precisely the Brocard point of the triangle, and **David Stone and John Hawkins** noted in their solution that given an inscribed triangle and letting $\theta = \angle PAB = \angle PBC = \angle PCA$, then the identity

$$\sin\theta = \frac{abc}{2\sqrt{a^2b^2 + a^2c^2 + b^2c^2}}$$

allows one to find the unique angle θ and thus sides $\overline{PA}, \overline{PB}$, and $\overline{PC}$.

Also solved by David Stone and John Hawkins (jointly), Satetesboro, GA, and the proposer.

- **5112:** *Proposed by Juan-Bosco Romero Márquez, Madrid, Spain*

Let $0 < a < b$ be real numbers with a fixed and b variable. Prove that

$$\lim_{b\to a}\int_a^b \frac{dx}{\ln\dfrac{b+x}{a+x}} = \lim_{b\to a}\int_a^b \frac{dx}{\ln\dfrac{b(a+x)}{a(b+x)}}.$$

Solution by Shai Covo, Kiryat-Ono, Israel

We begin with the left-hand side limit. Writing $\ln\dfrac{b+x}{a+x}$ as $\ln(b+x) - \ln(a+x)$, we have by the mean value theorem that this expression is equal to $\dfrac{1}{\xi}(b-a)$ where $\xi = \xi(a,b,x)$ is some point between $(a+x)$ and $(b+x)$. Since x varies from a to b, it thus follows that

$$\frac{b-a}{2b} \leq \ln\frac{b+x}{a+x} \leq \frac{b-a}{2a}.$$

Hence,

$$2a = \int_a^b \frac{2a}{b-a}dx \le \int_a^b \frac{dx}{\ln\dfrac{b+x}{a+x}} \le \int_a^b \frac{2b}{b-a}dx = 2b,$$

and so

$$\lim_{b\to a}\int_a^b \frac{dx}{\ln\dfrac{b+x}{a+x}} = 2a.$$

Applying this technique to the computation of the right-hand side limit gives

$$\frac{a(b-a)}{ab+b^2} \le \ln\frac{b(a+x)}{a(b+x)} \le \frac{b(b-a)}{ab+a^2},$$

from which it follows immediately that also

$$\lim_{b\to a}\int_a^b \frac{dx}{\ln\dfrac{b(a+x)}{a(b+x)}} = 2a.$$

Also solved by Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5113:** *Proposed by Paolo Perfetti, Mathematics Department, Tor Vergata University, Rome, Italy*

  Let x, y be positive real numbers. Prove that

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2+y^2}{2}} \le \sqrt{xy} + \frac{x+y}{2} + \frac{\left(\dfrac{x+y}{6} - \dfrac{\sqrt{xy}}{3}\right)^2}{\dfrac{2xy}{x+y}}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China

By homogeneity, we may assume without loss of generality that $xy = 1$. Let $t = x + y \ge 2\sqrt{xy} = 2$. Then the inequality of the problem is equivalent to

$$\begin{aligned}
& \frac{2}{t} + \sqrt{\frac{t^2-2}{2}} \le 1 + \frac{t}{2} + \frac{t(t-2)^2}{72} \\
\Leftrightarrow\quad & 36t\sqrt{2\left(t^2-2\right)} \le t^4 - 4t^3 + 40t^2 + 72t - 144 \\
\Leftrightarrow\quad & \left(t^4 - 4t^3 + 40t^2 + 72t - 144\right) - 2592t^2\left(t^2-2\right) \ge 0 \\
\Leftrightarrow\quad & t^8 - 8t^7 + 96t^6 - 176t^5 - 1856t^4 + 6912t^3 - 1152t^2 - 20376t + 20376 \ge 0 \\
\Leftrightarrow\quad & (t-2)^2\left(t^6 - 4t^5 + 76t^4 + 144t^3 - 1584t^2 + 5184\right) \ge 0.
\end{aligned}$$

Since

$$\begin{aligned} & t^6 - 4t^5 + 76t^4 + 144t^3 - 1584t^2 + 5184 \\ = \quad & t^4(t-2)^2 + 72(t-2)^4 + \frac{16(3t-8)^2(15t+11)+832}{3} > 0, \end{aligned}$$

the inequality of the problem holds.

Solution 2 by Paul M. Harms, North Newton, KS

Le $w = \dfrac{x+y}{2\sqrt{xy}}$ and $z = \sqrt{xy}$. For x and y positive

$$\left(\sqrt{x}-\sqrt{y}\right)^2 = x+y-2\sqrt{xy} \geq 0 \Longrightarrow w = \frac{x+y}{2\sqrt{xy}} \geq 1. \text{ Also } z > 0.$$

From the substitutions we have the following expressions :

$$\begin{aligned} 2xy &= 2z^2 \\ x+y &= 2zw \\ x^2+y^2 = (x+y)^2 - 2xy &= 4z^2w^2 - 2z^2 = 2z^2(2w^2-1) \end{aligned}$$

The inequality becomes

$$\frac{2z^2}{2zw} + \sqrt{\frac{2z^2\left(2w^2-1\right)}{2}} \leq z + \frac{2zw}{2} + \frac{\left(\dfrac{2zw-2z}{6}\right)^2}{\dfrac{2z^2}{2zw}}$$

Simplifying and dividing both sides of the inequality by z yields the inequality

$$\frac{1}{w} + \sqrt{2w^2-1} \leq 1 + w + \frac{1}{9}(w-1)^2 w.$$

After multiplying both sides by $9w$ and isolating the square root term we get

$$9w\sqrt{2w^2-1} \leq -9 + 9w + 9w^2 + (w-1)^2w^2 = w^4 - 2w^3 + 10w^2 + 9w - 9.$$

Now let $w = L+1$. Since $w \geq 1$, we check the resulting inequality for $L \geq 0$. Replacing w by $L+1$ and squaring both sides of the inequality we obtain

$$\begin{aligned} 81(L+1)^2\left[2L^2+4L+1\right] &= 81\left(2L^4+8L^3+11L^2+6L+1\right) \\ &\leq \left(L^4+2L^3+10L^2+27L+9\right)^2 \\ &= L^8+4L^7+24L^6+94L^5+226L^4+576L^3+909L^2+486L+81 \end{aligned}$$

Moving all terms to the right side, we need to show for $L \geq 0$, that

$$0 \leq L^2\left[L^6+4L^5+24L^4+94L^3+64L^2-72L+18\right].$$

Let

$$g(L) = 94L^3 + 64L^2 - 72L + 18.$$

If $g(L) \geq 0$ for $L \geq 0$, then the inequality holds since all other terms and factors of the inequality not involved with $g(L)$ are non-negative.

The derivative $g'(L) = 2\left[141L^2 + 64L - 36\right]$. The zeroes of $g'(L)$ are $L = -0.7810$ and $L = 0.3297$ with a negative derivative between these two L values. It is easy to check that $g(0.3297) > 0$ is the only relative minimum and that $g(L) > 0$ for all $L \geq 0$. Thus the inequality holds.

A comment by the editor: **David Stone and John Hawkins of Statesboro, GA** sent in a solution path that was dependent on a computer, and this bothered them. They let $y = ax$ in the statement of the problem and then showed that the original inequality was equivalent to showing that

$$\frac{2a}{1+a} + \sqrt{\frac{1+a^2}{2}} \leq \frac{(\sqrt{a}+1)^2}{2} + \frac{(a+1)(\sqrt{a}-1)^4}{72a}.$$

They then had Maple graph the left and right hand sides of the inequality respectively; they analyzed the graphs and concluded that the inequality held (with equality holding for $a = 1$.) But this approach bothered them and so they let $a = z^2$ in the above inequality and they eventually obtained the following:

$$(z-1)^4\Big(z^{12} - 4z^{11} + 82z^{10} + 124z^9 - 1265z^8$$
$$+392z^7 + 2492z^6 + 392z^5 - 1265z^4 + 124z^3 + 82z^2 - 4z + 1\Big) \leq 0.$$

Again they called on Maple to factor the above polynomial, and it did into linear and irreducible quadratic factors. They then showed that there were no positive real zeros and so the inequality must be true. They also noted that equality holds if and only if $z = 1$; that is, equality holds for the original statement if and only if $x = y$. They ended their submission with the statement:

"The bottom line: with the use of a machine's assistance, we believe the original inequality to be true."

In their letter submitting the above to me David wrote:

"Last week I mentioned that our solution to Problem 5113 was dependent upon machine help. We are still in that position, so I send this to you as a comment, not as a solution. There is a nice reduction to an inequality in a single variable, but we never found an analytic verification for the inequality."

All of this reminded me of the comments in 1976 surrounding Appel and Haken's proof of the four color problem which was done with the aid of a computer. The concerns raised then, still exist today.

Also solved by Shai Covo, Kiryat-Ono, Israel; Boris Rays, Brooklyn, NY, and the proposer.

- **5114:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let M be a point in the plane of triangle ABC. Prove that

$$\frac{\overline{MA}^2 + \overline{MB}^2 + \overline{MC}^2}{\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2} \geq \frac{1}{3}.$$

When does equality hold?

Solution by Michael Brozinsky, Central Islip, NY

Without loss of generality let the vertices of the triangle be $A(0,0), B(a,0)$, and $C(b,c)$ and let M be (x,y). Now completing the square shows

$$\overline{AM}^2+\overline{BM}^2+\overline{CM}^2-\frac{1}{3}\left(\overline{AB}^2+\overline{BC}^2+\overline{AC}^2\right)$$

$$= \left(x^2+y^2+(x-a)^2+y^2+(x-b)^2+(y-c)^2-\frac{1}{3}\left(a^2+\left(b-a\right)^2+c^2+b^2+c^2\right)\right)$$

$$= 3\cdot\left(\left(x-\frac{a+b}{3}\right)+\left(y-\frac{c}{3}\right)^2\right)$$

and thus the given inequality follows at once and equality holds iff M is $\frac{2}{3}$ of the way from vertex C to side $\overline{AB}$. Relabeling thus implies that M is the centroid of the triangle.

Comments in the solutions of others: 1) From Kee-Wai Lau, Hong Kong, China. The inequality of the problem can be found at the top of p. 283, Chapter XI in *Recent Advances in Geometric Inequalities* by Mitrinovic, Pecaric, and Volenec, (Kluwer Academic Press), 1989.

The inequality was obtained using the Leibniz identity

$$\overline{MA}^2+\overline{MB}^2+\overline{MC}^2=3\overline{MG}^2+\frac{1}{3}\left(\overline{AB}^2+\overline{BC}^2+\overline{CA}^2\right)$$

where G is the centroid of triangle ABC. Equality holds if and only if $M=G$.

2) From Bruno Salgueiro Fanego, Viveiro Spain. This problem was solved for any point M in space using vectors. (See page 303 in *Problem Solving Strategies* by Arthur Engel, (Springer-Verlag), 1998.) Equality holds if, and only if, M is the centroid of ABC.

Another solution and a discussion of where the problem mostly likely originated can be found on pages 41 and 42 of

$$http://www.cpohoata.com/wp-conent/uploads/2008/10/inf081019.pdf.$$

Also, a local version of the Spanish Mathematical Olympiad of 1999 includes a version of this problem and it can be seen at
$http://platea.pntic.mec.es/\sim csanchez/local99.htm.$

3) From David Stone and John Hawkins (jointly), Statesboro, GA. Because the given problem has the sum of the squares of the triangle's sides as the denominator, one might conjecture the natural generalization

$$\frac{\sum_{i=1}^{n}\overline{MA_i}^2}{\sum_{i=1}^{n}\overline{A_iA_{i+1}}^2}\geq\frac{1}{n},$$

but this is not true. Instead, we must also allow all squares of diagonals to appear in the sum in the denominator. Of course, a triangle has no diagonals.

Also solved by Shai Covo, Kiryat-Ono, Israel; Michael N. Fried, Kibbutz Revivim, Israel; Paul M. Harms, North Newton, KS; Michael N. Fried, Kibbutz Revivim, Israel; Raúl A. Simón, Santiago, Chile, and the proposer.

- **5115:** *Proposed by Mohsen Soltanifar (student, University of Saskatchewan), Saskatoon, Canada*

Let G be a finite cyclic group. Compute the number of distinct composition series of G.

Solution 1 by Kee-Wai Lau,Hong Kong, China

Denote the order of a group S by $|S|$. Let $E = G_0, < G_1 < G_2 < \ldots < G_m = G$ be a composition series for G, where E is the subgroup of G consisting of the identity element only. A composition series is possible if and only if the factor groups $G_1/G_0, G_2/G_1, \ldots, G_m/G_{m-1}$ are simple. For cyclic group G, where all these factor groups are also cyclic, this is equivalent to saying that

$$|G_1/G_0| = p_1,\ |G_2/G_1| = p_2,\ \ldots,\ |G_m/G_{m-1}| = p_m,$$

where $p_1, p_2, \ldots, p_m$ are primes, not necessarily distinct. By the Jordan-Hölder theorem, m is uniquely determined and the prime divisors, $p_1, p_2, \cdots, p_m$ themselves are unique. Any other composition series therefore correspond with a permutation of the primes $p_1, p_2, \ldots, p_m$. Note that

$$|G| = |G_m| = \frac{|G_m|}{|G_{m-1}|}\frac{|G_{m-1}|}{|G_{m-2}|}\cdots\frac{|G_2|}{|G_1|}\frac{|G_1|}{1} = p_m p_{m-1}\cdots p_2 p_1.$$

We rewrite $|G|$ in standard form $|G| = q_1^{a_1} q_2^{a_2} \ldots q_k^{a_k}$, where $a_1, a_2, \ldots, a_k$ are positive integers and $q_1 < q_2 < \ldots q_k$ are primes. The number of distinct composition series of G then equals

$$\frac{(a_1 + a_2 + \cdots + a_k)!}{a_1! a_2! \ldots a_k!}$$

Solution 2 by David Stone and John Hawkins (jointly), Statesboro, GA

Let G have order n, where n has prime factorization $n = \prod_{i=1}^{m} p_i^{e_i}$. Then the number of distinct composition series of G is the multinomial coefficient $\binom{e_1 + e_2 + e_3 + \ldots + e_m}{e_1, e_2, e_3, \ldots, e_m}$. Letting e=$e_1 + e_2 + e_3 + \ldots + e_m$, this can be computed as

$$\binom{e}{e_1}\binom{e-e_1}{e_2}\binom{e-e_1-e_2}{e_3}\cdots\binom{e_{m-1}+e_m}{e_{m-1}}\binom{e_m}{e_m} = \frac{e!}{(e_1!)(e_2!)(e_3!)\cdots(e_m!)}.$$

Our rationale follows.

We'll simply let G be Z_n, written additively and denote the cyclic subgroup generated by a as $< a >= \{ka| \in Z\}$.

Note that $< a >$ is a subgroup of $< b >$ if and only if $a = bc$ for some c in G. We'll denote this by $< a > \leq < b >$. That is, to enlarge the subgroup $< a >$ to $< b >$, we divide a by some group element c to obtain b. In particular, if we divide a by a prime p to obtain b, then the factor group $< b > / < a >$ is isomorphic to the simple group Z_p.

In the lattice of subgroups of G, any two subgroups have a greatest lower bound, given by intersection , and a least upper bound, given by summation.The maximal length (ascending) chains are the distinct composition series. All such chain have the same length (by the Jordan-Hölder Theorem).

For a specific example, let $n = 12 = 2^2 \cdot 3^1$. In Z_{12}, the distinct subgroups are:

$$\begin{aligned}
0 &= \{0\},\\
< 2 > &= \{0, 2, 4, 6, 8, 10\},\\
< 4 > &= \{0, 4, 8\},\\
< 3 > &= \{0, 3, 6, 9\},\\
< 6 > &= \{0, 6\},\\
< 1 > &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} = Z_{12},
\end{aligned}$$

and the maximal length ascending chains (composition series) are

$$0 \leq < 4 > \ \leq \ < 2 > \ \leq \ < 1 >,$$

$$0 \leq < 6 > \ \leq \ < 2 > \ \leq \ < 1 >,$$

$$0 \leq < 6 > \ \leq \ < 3 > \ \leq \ < 1 > .$$

Note that the composition factors (the simple factor groups) of the first chain are

$$\begin{aligned}
< 4 > /0 &\cong Z_3\\
< 2 > / < 4 > &\cong Z_2, \text{ and}
\end{aligned}$$

$$< 1 > / < 2 > \quad \cong \quad Z_2.$$

Thus, the sequence of composition factors is Z_3, Z_2, Z_2.

Similarly for the second chain, the sequence of composition factors is Z_2, Z_3, Z_2, and for the third chain the sequence of composition factors is Z_2, Z_2, Z_3. The three elements of each chain are Z_2, Z_2, and Z_3, forced by the factorization of 12. The number of possible chains is simply the number of ways to arrange these three simple groups: 3. Note that

$$\binom{2+1}{2,1} = \binom{3}{2,1} = \binom{3}{2} \cdot \binom{1}{1} = 3.$$

Method: For arbitrary $n = \prod_{i=1}^{m} p_i^{e_i}$, this example demonstrates a constructive method for generating (and counting) all such maximal chains:

(i) Start with $0 =< n >$.

(ii) Divide (in the usual sense, not mod n) by one of $n's$ prime divisors, p, to obtain $k = \frac{n}{p}$, so that $0 =< n > \leq < k >$ and the factor group $< k > / < n > \cong Z_p$.

(iii) Next, divide k by any unused prime divisor, say q of n to obtain $h = \frac{k}{q}$, so that $< k > \leq < h >$ and the factor group $< h > / < k > \cong Z_q$.

(In this process, each prime factor p will be used e_i times, so there will be
$e = e_1 + e_2 + e_3 = \ldots + e_m$ steps.)

We now have the beginning of a composition series: $0 \ \leq < k > \leq < h >$. Continue with the division steps until the supply of prime divisors of n is exhausted, so the final division will produce the final element of the chain: $< 1 > = \ Z_n$. We will have thus constructed a composition series. In the procedure there will be e_1 divisions by p_1, e_2 divisions by p_2, etc.

Therefore, the number of ways to carry out this procedure is the number of ways to carry out these dvisions: choose e_1 places from e possible spots to divide by p, choose e_2 places from the remaining $e - e_1$ possible spots to divide by p_2 etc.
So we can count the total number of ways to carry out the process:

$$\binom{e}{e_1}\binom{e-e_1}{e_2}\binom{e-e_1-e_2}{e_3}\cdots\binom{e_{m-1}+e_m}{e_{m-1}}\binom{e_m}{e_m}.$$

Moreover, if we let S be the sequence of simple groups consisting of e_1 copies of Z_{p_1}, e_2 copies of Z_{p_2}, etc., then S will have $e = e_1 + e_2 + e_3 + \cdots + e_m$ elements and each of our composition series will have some rearrangement of S as its sequence of compositions factors.

Example: Let $n = 360 = 2^3 \cdot 3^2 \cdot 5^1$.
Then the sequence of divisors $3, 5, 2, 2, 3, 2$ will produce he composition series

$$0 =< 360 > \leq < 120 > \leq < 24 > \leq < 12 > \leq < 6 > \leq < 2 > \leq < 1 > = Z_{360},$$

with composition factors $Z_3, \ Z_5, \ Z_2, \ Z_2, \ Z_3, \ Z_2$.

There are $\binom{3+2+1}{3,2,1} = \binom{6}{3} \cdot \binom{3}{2} \cdot \binom{1}{1} = 60$ different ways to construct a divisors sequence from $2, 2, 2, 3, 3, 5$, so Z_{360} has 60 distinct composition series.

Also solved by the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before February 15, 2011

- **5134:** *Proposed by Kenneth Korbin, New York, NY*

  Given isosceles $\triangle ABC$ with cevian CD such that $\triangle CDA$ and $\triangle CDB$ are also isosceles, find the value of
  $$\frac{AB}{CD} - \frac{CD}{AB}.$$

- **5135:** *Proposed by Kenneth Korbin, New York, NY*

  Find a, b, and c such that
  $$\begin{cases} ab + bc + ca = -3 \\ a^2b^2 + b^2c^2 + c^2a^2 = 9 \\ a^3b^3 + b^3c^3 + c^3a^3 = -24 \end{cases}$$
  with $a < b < c$.

- **5136:** *Proposed by Daniel Lopez Aguayo (student, Institute of Mathematics, UNAM), Morelia, Mexico*

  Prove that for every positive integer n, the real number
  $$\left(\sqrt{19} - 3\sqrt{2}\right)^{1/n} + \left(\sqrt{19} + 3\sqrt{2}\right)^{1/n}$$
  is irrational.

- **5137:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let a, b, c be positive numbers such that $abc \geq 1$. Prove that
  $$\prod_{cyclic} \frac{1}{a^5 + b^5 + c^2} \leq \frac{1}{27}.$$

- **5138:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Let $n \geq 2$ be a positive integer. Prove that
  $$\frac{n}{F_nF_{n+1}} \leq \frac{1}{(n-1)F_1^2 + F_2^2} + \cdots + \frac{1}{(n-1)F_n^2 + F_1^2} \leq \frac{1}{n}\sum_{k=1}^{n} \frac{1}{F_k^2},$$
  where F_n is the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$ and for all $n \geq 2,\ F_n = F_{n-1} + F_{n-2}$.

- **5139:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Calculate
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\zeta(n+m)-1}{n+m},$$
  where ζ denotes the Riemann Zeta function.

Solutions

- **5116:** *Proposed by Kenneth Korbin, New York, NY*

  Given square $ABCD$ with point P on side AB, and with point Q on side BC such that
$$\frac{AP}{PB}=\frac{BQ}{QC}>5.$$
  The cevians DP and DQ divide diagonal AC into three segments with each having integer length. Find those three lengths, if $AC=84$.

  Solution by David E. Manes, Oneonta, NY

  Let E and F be the points of intersection of AC with DP and DQ respectively. Then $AE=40,\ EF=37$ and $FC=7$.

  Since $ABCD$ is a square with diagonal of length 84, it follows that the sides of the square have length $42\sqrt{2}$. Let $\dfrac{AP}{PB}=\dfrac{BQ}{QC}=t>5$. Then $AP=t\cdot PB$ and $AP+PB=AB=42\sqrt{2}$. Therefore,
$$\begin{aligned}PB(t+1) &= 42\sqrt{2}\\ PB &= \frac{42\sqrt{2}}{1+t},\quad\text{and}\\ AP &= \frac{42\sqrt{2}\cdot t}{1+t}.\end{aligned}$$
  Similarly, $QC=\dfrac{42\sqrt{2}}{1+t}$ and $BQ=\dfrac{42\sqrt{2}\cdot t}{1+t}$.

  Coordinatize the problem so that
$$\begin{aligned}A &= (0,0),\quad B=(42\sqrt{2},0),\quad C=(42\sqrt{2},42\sqrt{2}),\quad D=(0,42\sqrt{2}),\\ P &= \left(\frac{42\sqrt{2}\cdot t}{1+t},0\right),\ \text{and}\quad Q=\left(42\sqrt{2},\frac{42\sqrt{2}\cdot t}{1+t}\right).\end{aligned}$$
  Let L_1 be the line through the points D and P. Then the equation of L_1 is $y-42\sqrt{2}=-\left(\dfrac{1+t}{t}\right)x$. The point of intersection of L_1 and the line $y=x$ is the point E. Therefore,
$$x-42\sqrt{2} = -\left(\frac{1+t}{t}\right)x,\ \text{and so}$$

$$
\begin{aligned}
x &= \frac{42\sqrt{2}\cdot t}{2t+1}. \text{ Thus,}\\
E &= \left(\frac{42\sqrt{2}\cdot t}{2t+1}, \frac{42\sqrt{2}\cdot t}{2t+1}\right) \text{ so that}\\
AE &= \sqrt{2\left(\frac{42\sqrt{2}\cdot t}{2t+1}\right)^2} = \frac{84\cdot t}{2t+1}.
\end{aligned}
$$

Let L_2 be the line through D and Q. Then the equation of L_2 is $y - 42\sqrt{2} = -\left(\dfrac{1}{1+t}\right) x$. Since F is the point of intersection of L_2 and $y = x$, we obtain $x = \dfrac{42\sqrt{2}(t+1)}{t+2}$. Thus,

$$
\begin{aligned}
F &= \left(\frac{42\sqrt{2}(t+1)}{t+2}, \frac{42\sqrt{2}(t+1)}{t+2}\right) \text{so that}\\
AF &= \frac{84(t+1)}{t+2}.
\end{aligned}
$$

Using the distance formula, one obtains

$$
CF = \sqrt{2\left(42\sqrt{2} - \frac{42\sqrt{2}(t+1)}{t+2}\right)^2} = \frac{84}{t+2}.
$$

As a result,

$$
AE = \frac{84\cdot t}{2t+1},\ AF = \frac{84(t+1)}{t+2},\ \text{and } CF = \frac{84}{t+2}
$$

If $t = 10$, then $AE = 40, AF = 77$, and $CF = 7$. Therefore $EF = AF - AE = 37$, yielding the claimed values. Finally, one checks that for these values all triangles in the figure are defined.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paul M. Harms, North Newton, KS; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5117:** *Proposed by Kenneth Korbin, New York, NY*

Find positive acute angles A and B such that

$$
\sin A + \sin B = 2\sin A \cdot \cos B.
$$

Solution by David Stone and John Hawkins (jointly), Statesboro, GA

There are infinitely many solutions, given by

$$A = \sin^{-1}\left(\frac{\sqrt{1-t^2}}{2t-1}\right),\ \mathrm{B} = \cos^{-1} t, \quad \text{where } \frac{4}{5} < t < 1.$$

Here's why.

The given condition is equivalent to

$$2\sin A(2\cos B - 1) = \sin B$$

so we see that $2\cos B - 1 > 0$, that is, $0 < B < \dfrac{\pi}{3}$.

Solving for $\sin A$, we must have $\sin A = \dfrac{\sin B}{2\cos B - 1}$, which requires $0 \le \dfrac{\sin B}{2\cos B - 1} \le 1$.

Upon squaring, this is equivalent to

$$\begin{aligned} \sin^2 B &\le 4\cos^2 B - 4\cos B + 1 \\ 1 - \cos^2 B &\le 4\cos^2 B - 4\cos B + 1 \\ \cos B &\ge \frac{4}{5}. \end{aligned}$$

So if we choose angle B to make $\cos B \ge \dfrac{4}{5}$, then we can choose angle A to make $\sin A = \dfrac{\sin B}{2\cos B - 1}$.

Since cosine is decreasing in the first quadrant, the size condition on $\cos B$ forces $B \le \cos^{-1}\left(\dfrac{4}{5}\right) \approx 36.87°$.

In fact, for any t, with $\dfrac{4}{5} \le t \le 1$, we can let $B = \cos^{-1} t$, in which case $\sin B = \sqrt{1-t^2}$, and let $A = \sin^{-1}\left(\dfrac{\sqrt{1-t^2}}{2t-1}\right)$.

Note that the endpoint "solution" given by $t = 1$ is $A = 0, B = 0$, which we disregard.

Also, the endpoint solution given by $t = \dfrac{4}{5}$ is $A = \dfrac{\pi}{2}, B = \cos^{-1}\dfrac{4}{5}$.

It is worth noting that we thus have a right triangle solution, but it doesn't quite meet the problem's criteria, so we'll disregard this one. Thus, there are infinitely many solutions, given in terms of the parameter t for $\dfrac{4}{5} < t < 1$.

We also note that one could also say that all solutions are given by $\sin A = \dfrac{\sin B}{2\cos B - 1}$, where angle B is chosen so that $\cos B > \dfrac{4}{5}$.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Michael Brozinsky, Central Islip, NY; Shai Covo, Kiryat-Ono, Israel; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Raúl A. Simón, Santiago, Chile; Taylor University Problem Solving Group; Upland, IN, and the proposer.

- **5118:** *Proposed by David E. Manes, Oneonta, NY*

  Find the value of

$$\sqrt{2011 + 2007\sqrt{2012 + 2008\sqrt{2013 + 2009\sqrt{2014 + \cdots}}}}$$

Solution 1 by Shai Covo, Kiryat-Ono, Israel

The value is 2009. More generally, for any integer $n \geq 3$ we have

$$n = \sqrt{(n+2) + (n-2)\sqrt{(n+3) + (n-1)\sqrt{(n+4) + n\sqrt{(n+5) + \cdots}}}}$$

($n = 2009$ corresponds to the original problem.) The claim follows from an iterative application of the identity $n = \sqrt{(n+2) + (n-2)(n+1)}$, as follows:

$$\begin{aligned} n &= \sqrt{(n+2) + (n-2)(n+1)} \\ &= \sqrt{(n+2) + (n-2)\sqrt{(n+3) + (n-1)(n+2)}} \\ &= \sqrt{(n+2) + (n-2)\sqrt{(n+3) + (n-1)\sqrt{(n+4) + n(n+3)}}} \\ &= \cdots. \end{aligned}$$

Solution 2 by Taylor University Problem Solving Group, Upland, IN

We use Ramanujan's nested radical approach. Beginning with

$$(x + n + a)^2 = x^2 + n^2 + a^2 + 2ax + 2nx + 2an,$$

we see that

$$\begin{aligned} x + n + a &= \sqrt{x^2 + n^2 + a^2 + 2ax + 2nx + 2an} \\ &= \sqrt{ax + n^2 + a^2 + 2an + x\,(x + 2n + a)} \\ &= \sqrt{ax + (n+a)^2 + x\,(x + 2n + a)}. \end{aligned}$$

However, the $(x + 2n + a)$ term on the right is basically of the same form as the left (with n replaced by $2n$). We can make the corresponding substitution, and continue this process indefinitely, until we are left with $x + n + a =$

$$\sqrt{ax + (n+a)^2 + x\sqrt{a(x+n) + (n+a)^2 + (x+n)\sqrt{a\,(x+2n) + (n+a)^2 + (x+2n)\sqrt{\cdots}}}}$$

Substituting in $x = 2007$, $n = a = 1$ produces

$$\begin{aligned} 2009 &= \sqrt{2007 + 4 + 2007\sqrt{2008 + 4 + 2008\sqrt{2009 + 4 + 2009\sqrt{\cdots}}}} \\ &= \sqrt{2011 + 2007\sqrt{2012 + 2008\sqrt{2013 + 2009\sqrt{\cdots}}}}. \end{aligned}$$

Hence, the value is 2009.

Also solved by Scott H. Brown, Auburn University, Montgomery, AL; G. C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS: Kenneth Korbin, NY, NY; Charles McCracken, Dayton, OH; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; Boris Rays, Brooklyn, NY; David Stone and John Hawkins (jointly), Stateboro GA, and the proposer.

- **5119:** *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain*

Let n be a non-negative integer. Prove that

$$2 + \frac{1}{2^{n+1}} \prod_{k=0}^{n} \csc\left(\frac{1}{F_k}\right) < F_{n+1}$$

where F_n is the n^{th} Fermat number defined by $F_n = 2^{2^n} + 1$ for all $n \geq 0$.

Solution by Charles R. Diminnie, San Angelo, TX

To begin, we note that for $x \in \left(0, \frac{\pi}{3}\right)$, $\cos x$ is decreasing and the Mean Value Theorem for Derivatives implies that there is a point $c_x \in (0, x)$ such that

$$\begin{aligned} \sin x &= \sin x - \sin 0 \\ &= \cos c_x \,(x - 0) \\ &> \cos\frac{\pi}{3} \cdot x \\ &= \frac{x}{2}. \end{aligned}$$

As a result, when $x \in \left(0, \frac{\pi}{3}\right)$,

$$x \csc x < 2.$$

Since $F_n \geq F_0 = 3$ for all $n \geq 0$, it follows that $0 < \frac{1}{F_n} \leq \frac{1}{3} < \frac{\pi}{3}$ and hence,

$$\begin{aligned} \frac{1}{F_n} \csc\left(\frac{1}{F_n}\right) &< 2, \text{ or} \\ \csc\left(\frac{1}{F_n}\right) &< 2F_n \qquad (1) \end{aligned}$$

Let $P(n)$ be the statement

$$\prod_{k=0}^{n} \csc\left(\frac{1}{F_k}\right) < 2^{n+1}\left(F_{n+1} - 2\right) \qquad (2)$$

By (1),

$$\csc\left(\frac{1}{F_0}\right) < 2F_0 = 2 \cdot 3 = 2\left(F_1 - 2\right)$$

and $P(0)$ is true. If $P(n)$ is true for some $n \geq 0$, then by (1),

$$\begin{aligned}
\prod_{k=0}^{n+1} \csc\left(\frac{1}{F_k}\right) &= \csc\left(\frac{1}{F_{n+1}}\right) \prod_{k=0}^{n} \csc\left(\frac{1}{F_k}\right) \\
&< \csc\left(\frac{1}{F_{n+1}}\right) \cdot 2^{n+1}\left(F_{n+1} - 2\right) \\
&< 2F_{n+1} \cdot 2^{n+1}\left(F_{n+1} - 2\right) \\
&= 2^{n+2}\left(2^{2^{n+1}} + 1\right)\left(2^{2^{n+1}} - 1\right) \\
&= 2^{n+2}\left(2^{2^{n+2}} - 1\right) \\
&= 2^{n+2}\left(F_{n+2} - 2\right)
\end{aligned}$$

and $P(n+1)$ follows. By Mathematical Induction, $P(n)$ is true for all $n \geq 0$.

Since (2) is equivalent to the given inequality, the proof is complete.

Also solved by Shai Covo, Kiryat-Ono, Israel; Bruno Salgueiro Fanego, Viveiro, Spain; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **5120:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Calculate

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \log\left(\frac{2n-k}{2n+k}\right).$$

Solution 1 by Ovidiu Furdui, Cluj, Romania

The limit equals 0. More generally, we prove that if $f : [0,1] \to \Re$ is a continuous function then

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(\frac{k}{n}\right) = 0.$$

Before we give the solution of the problem we collect the following equality from **[1]** (Formula 0.154(3), p.4): If $p \geq 0$ is a nonnegative integer, then the following equality holds

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^p = 0. \qquad (1)$$

Now we are ready to solve the problem. First we note that for a polynomial $P(x) = \sum_{j=0}^{m} a_j x^j$ we have, based on (1), that

$$\frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} P\left(\frac{k}{n}\right) = \sum_{j=0}^{m} \frac{a_j}{n^j} \cdot \frac{1}{2^n} \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^j \right) = 0. \qquad (2)$$

Let $\epsilon > 0$ and let P_ϵ be the polynomial that uniformly approximates f, i.e. $|f(x) - P_\epsilon(x)| < \epsilon$ for all $x \in [0,1]$. We have, based on (2), that

$\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_\epsilon\left(\frac{k}{n}\right) = 0$. Thus,

$$\begin{aligned}\left|\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f\left(\frac{k}{n}\right)\right| &= \left|\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(f\left(\frac{k}{n}\right)-P_\epsilon\left(\frac{k}{n}\right)\right)\right| \\ &\leq \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left|f\left(\frac{k}{n}\right)-P_\epsilon\left(\frac{k}{n}\right)\right| \\ &\leq \frac{\epsilon}{2^n}\sum_{k=0}^{n}\binom{n}{k} \\ &= \epsilon.\end{aligned}$$

Thus, the limit is 0 and the problem is solved.

[1] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products, Sixth Edition, Alan Jeffrey, Editor, Daniel Zwillinger, Associate Editor, 2000.

Solution 2 by Shai Covo, Kiryat-Ono, Israel

We will show that

$$\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\log\left(\frac{2n-k}{2n+k}\right) = 0. \qquad (1)$$

(The log function in (1) has no significant role in the analysis below, we could replace it by any other continuous function.)

The lemma below follows straightforwadly from the Central Limit Theorem (CLT). We recall that, according to the CLT, if $X_1, X_2, \ldots$ is a sequence of independent and identically distributed (i.i.d) random variables with expectation μ and variance σ^2, then

$$P\left(a < \frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}} \leq b\right) \to \Phi(b)-\Phi(a) \qquad (2)$$

as $n\to\infty$, for any $a, b \in \Re$ with $a < b$ where Φ is the distribution function of the Normal $(0,1)$ distribution (i.e., $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-\mu^2/2}du$).

Lemma: For any $\epsilon > 0$, there exists an $r > 0$ such that

$$\frac{1}{2^n}\sum_{\substack{0\leq k\leq n/2-r\sqrt{n} \\ n/2+r\sqrt{n}<k\leq n}}\binom{n}{k} < \epsilon \qquad (3)$$

for all n sufficiently large.

Proof: Fix $\epsilon > 0$. Choose $r > 0$ sufficiently large so that $\Phi(2r)-\Phi(-2r) > 1-\epsilon$. Let $X_1, X_2\ldots$ be a sequence of i.i.d. variables with $P(X_i = 0) = P(X_i = 1) = 1/2$. Put $Y_n = \sum_{i=1}^{n} X_i$. Thus Y_n has a binomial $(n, 1/2)$ distribution. The X_i's have expectation $\mu = 1/2$ and variance $\sigma^2 = 1/4$. Hence by (2) (with $a = -2r$ and $b = 2r$),

$$P(n/2 - r\sqrt{n} < Y_n \leq n/2 + r\sqrt{n}) > 1-\epsilon$$

for all n sufficiently large. In turn, by taking complements, we conclude (3), since the distribution of Y_n is given by $P(Y_n = k) = \dfrac{1}{2^n}\dbinom{n}{k}, k = 0, \ldots, n.$

It follows from the lemma and the fact that $\left|(-1)^k \log\left(\dfrac{2n-k}{2n+k}\right)\right|$ is bounded uniformly in k (say, by 2) that (1) will be proved if we show that

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{n/2 - r\sqrt{n} < k < n/2 + r\sqrt{n}} (-1)^k \binom{n}{k} \log\left(\frac{2n-k}{2n+k}\right) = 0 \qquad (4)$$

for any fixed $r > 0$. This is shown as follows. We first write

$$\begin{aligned} &\left|(-1)^k \binom{n}{k} \log\left(\frac{2n-k}{2n+k}\right) + (-1)^k \binom{n}{k+1} \log\left(\frac{2n-(k+1)}{2n+(k+1)}\right)\right| \\ = \;& \binom{n}{k}\left|\log\left(\frac{2n-k}{2n+k}\right) - \frac{n-k}{k+1}\log\left(\frac{2n-(k+1)}{2n+(k+1)}\right)\right|. \qquad (5) \end{aligned}$$

Clearly, the expression multipying $\dbinom{n}{k}$ on the right of the equality in (5) can be made arbitrarily small uniformly in $k \in [n/2 - r\sqrt{n},\ n/2 + r\sqrt{n}]$, where $r > 0$ is fixed, by choosing n sufficiently large. Then, in view of the triangle inequality, (4) follows from $\dfrac{1}{2^n}\displaystyle\sum_{k=0}^{n}\binom{n}{k}\epsilon = \epsilon$ (where $\epsilon > 0$ is arbitrarily small) and $\dbinom{n}{k}\Big/ 2^n \overset{\text{unif.}}{\longrightarrow} 0$ (to be used if the sum in (4) consists of an odd number of terms). The desired result (1) is thus proved.

Also proved by Boris Rays, Brooklyn, NY and the proposer.

5121: *Proposed by Tom Leong, Scotrun, PA*

Let n, k and r be positive integers. It is easy to show that

$$\sum_{n_1+n_2+\cdots+n_r=n} \binom{n_1}{k}\binom{n_2}{k}\cdots\binom{n_r}{k} = \binom{n+r-1}{kr+r-1}, \quad n_1, n_2, \cdots n_r \in N$$

using generating functions. Give a combinatorial argument that proves this identity.

Solution 1 by Shai Covo, Kiryat-Ono, Israel

Suppose we have n identical boxes and kr ($\le n$) identical balls. The stated equality is trivial if $r = 1$, hence we can assume $r > 1$.

We begin with the left-hand side of the stated equality. Assuming $n_1, \ldots, n_r \ge k$, it gives the number of ways to divide the n boxes into r groups–the ith group having $n_i \ge k$ elements–and put exactly k balls in each group.

As for the right-hand side, suppose that in addition to the n boxes and the kr balls we have $r-1$ separators. This gives rise to an $(n+r-1)$-tuple of boxes and separators. We denote this tuple by M. We identify a sequence $(i_1, i_2, \ldots, i_{kr+r-1})$ such that $1 \le i_1 < i_2 < \cdots < i_{kr+r-1} \le n+r-1$ with the following arrangement: the i_jth $(j = 1, \ldots, kr+r-1)$ element of M is a separator if j is a multiple of $k+1$ and a box containing a ball otherwise. (The remaining $n-kr$ elements are empty boxes.) We thus conclude that $\binom{n+r-1}{kr+r-1}$ gives the number of ways to place $r-1$ separators between the n boxes and kr balls into the boxes, such that each of the resulting r groups contains exactly k balls. This establishes the equality of the left-and right-hand sides.

Solution 2 by the proposer

Both sides count the number of possible ways to arrange $kr+r-1$ green balls and $n-kr$ red balls in a row. This is clearly true for the right side. In the left side, note that any term in the sum with $n_i < k$ for some i is equal to zero; so we may assume $n_i \ge k$ for all i. For each *composition* $n_1 + \cdots + n_r = n$ of n, consider the row of n red and $r-1$ green balls arranged as

$$\underbrace{RR\cdots R}_{n_1 \text{ balls}} G \underbrace{RR\cdots R}_{n_2 \text{ balls}} G \underbrace{RR\cdots R}_{n_3 \text{ balls}} G \cdots G \underbrace{RR\cdots R}_{n_{r-1} \text{ balls}} G \underbrace{RR\cdots R}_{n_r \text{ balls}}$$

From each block of red balls, choose k of them and paint them green. The number of ways to do this is $\binom{n_1}{k}\binom{n_2}{k}\cdots\binom{n_r}{k}$. This results in a row consisting of $kr+r-1$ green balls and $n-kr$ red balls. Conversely, in any row consisting of $kr+r-1$ green balls and $n-kr$ red balls, we can determine a unique composition $n_1 + n_2 + \cdots + n_r = n$ of n by reversing the process.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before
March 15, 2011

- **5140:** *Proposed by Kenneth Korbin, New York, NY*

  Given equilateral triangle ABC with an interior point P such that

$$\begin{aligned}\overline{AP} &= 22+16\sqrt{2}\\ \overline{BP} &= 13+9\sqrt{2}\\ \overline{CP} &= 23+16\sqrt{2}.\end{aligned}$$

  Find $\overline{AB}$.

- **5141:** *Proposed by Kenneth Korbin, New York, NY*

  A quadrilateral with sides $259, 765, 285, 925$ is constructed so that its area is maximum. Find the size of the angles formed by the intersection of the diagonals.

- **5142:** *Proposed by Michael Brozinsky, Central Islip, NY*

  Let CD be an arbitrary diameter of a circle with center O. Show that for each point A distinct from O, C, and D on the line containing CD, there is a point B such that the line from D to any point P on the circle distinct from C and D bisects angle APB.

- **5143:** *Proposed by Valmir Krasniqi (student), Republic of Kosova*

  Show that

$$\sum_{n=1}^{\infty} Cos^{-1}\,\frac{1+\sqrt{n^2+2n}\cdot\sqrt{n^2-1}}{n\,(n+1)} \quad = \quad \frac{\pi}{2}.$$

$$(Cos^{-1} = Arccos)$$

- **5144:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  Compute

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\ln\left(\frac{k+\sqrt{n^2+k^2}}{n}\right)^{1/n}\right].$$

- **5145:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k \geq 1$ be a natural number. Find the sum of

$$\sum_{n=1}^{\infty} \left(\frac{1}{1-x} - 1 - x - x^2 - \cdots - x^n \right)^k, \quad \text{for } |x| < 1.$$

Solutions

- **5122:** *Proposed by Kenneth Korbin, New York, NY*

Partition the first 32 non-negative integers from 0 to 31 into two sets A and B so that the sum of any two distinct integers from set A is equal to the sum of two distinct integers from set B and vice versa.

Solution 1 by Michael N. Fried, Kibbutz Revivim, Israel

Suppose A contains 0. This means that any other number in A must be the sum of two numbers in B. The next number in A, therefore, must be at least 3 since 3 is the smallest number that is the sum of two positive integers. On the other hand, the next number in A cannot be greater than 3, for 1 and 2 must still be in B. This group of four numbers forms a kind of unit, which we can represent graphically as follows:

$$\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array} \quad \text{or} \quad \begin{array}{ccc} 1 & & 2 \\ & \sqcap & \\ 0 & & 3 \end{array}$$

The symmetry of the unit reflects the fact that $a + b = c + d$ if and only if $b - d = a - c$, that is if and only if there is some number k such that $c = a + k$ and $d = b - k$. Thus any four consecutive integers forming such a figure will have the property that the sum of the top pair of numbers equals the sum of the bottom pair.

(This makes the problem almost a geometrical one, for arranging the numbers in set A and B in parallel lines as in the figure above, the condition of the problem becomes that every pair of numbers in the first line corresponds to a pair of numbers in the second line.)

So our strategy for the problem will be to assemble units such as those above to produce larger units satisfying in each case the condition of the problem.

Let us then start with two. The first, as before is:

$$\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array}$$

And as we have already argued, the first two numbers of A and B *must* be arranged in this way. The second unit, then, will be either

$$\begin{array}{ccc} 4 & & 7 \\ & \sqcup & \\ 5 & & 6 \end{array} \quad \text{or} \quad \begin{array}{ccc} 5 & & 6 \\ & \sqcap & \\ 4 & & 7 \end{array}$$

The symmetrical combination,

$$\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array} \quad \begin{array}{ccc} 4 & & 7 \\ & \sqcup & \\ 5 & & 6 \end{array}$$

fails, because the pair $(0, 4)$ in the upper row has no matching pair in the second row.

However, the non-symmetrical combination works:

$$\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array} \qquad \begin{array}{ccc} 5 & & 6 \\ & \sqcap & \\ 4 & & 7 \end{array}$$

Again, these two form a new kind of unit, and, as before, any eight consecutive integers forming a unit such as the above, will have the property that any pair of numbers in the top row will have the same sum as some pair in the bottom row.

So, let us try and fit together two units of this type, and let us call them R and S. As before, there are two possibilities, one symmetric and one anti-symmetric.
Since the anti-symmetric option worked before, let us try it again and call the top row **A** and the bottom row **B**.

$$\overbrace{\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array} \quad \begin{array}{ccc} 5 & & 6 \\ & \sqcap & \\ 4 & & 7 \end{array}}^{R} \qquad\qquad \overbrace{\begin{array}{ccc} 8 & & 11 \\ & \sqcap & \\ 9 & & 10 \end{array} \quad \begin{array}{ccc} 13 & & 14 \\ & \sqcup & \\ 12 & & 15 \end{array}}^{S}$$

$$\begin{aligned} \mathbf{A} &= \{0,3,5,6,8,11,13,14\} \\ \mathbf{B} &= \{1,2,4,7,9,10,12,15\} \end{aligned}$$

Now, to check whether this combination works we do not have to check $\binom{8}{2} = 28$ pairs of numbers.

All of the subunits will satisfy the condition of the problem. Indeed, we do not have to check pairs contained in the first and second, second and third and third and fourth terms, because they represent eight consecutive integers as discussed above. And we do not have to check pairs from the first and fourth terms because these also behave like a single unit R (where for example the pair (0,13) corresponds to (1,12) just as (0,5) corresponded to (1,4). So we only have to check pairs of numbers coming from the first and third elements and the second and fourth. But here we find a problem, for (2,10) in **B** cannot have a corresponding pair in **A.**

Let us then check the symmetrical arrangement:

$$\overbrace{\begin{array}{ccc} 0 & & 3 \\ & \sqcup & \\ 1 & & 2 \end{array} \quad \begin{array}{ccc} 5 & & 6 \\ & \sqcap & \\ 4 & & 7 \end{array}}^{R} \qquad\qquad \overbrace{\begin{array}{ccc} 9 & & 10 \\ & \sqcup & \\ 8 & & 11 \end{array} \quad \begin{array}{ccc} 12 & & 15 \\ & \sqcap & \\ 13 & & 14 \end{array}}^{S}$$

$$\begin{aligned} \mathbf{A} &= \{0,3,5,6,9,10,12,15\} \\ \mathbf{B} &= \{1,2,4,7,8,11,13,14\} \end{aligned}$$

As in the anti-symmetrical arrangement, we need not check pairs of numbers in R or S, or, in this case, pairs if the first and third elements or second and fourth, which behave exactly as R and S individually. We need only check non-symmetrical pairs in the first and fourth elements and in the second and third. For the former this means (3,15) and (0,12) in **A** and (1,13) and (2,14) in **B.** For these we have corresponding pairs (3,15) to (7,8), (0,12) to (4,8), (1,13) to (5,9) and (2,14) to (6,10). Similarly, corresponding pairs exist for each non-smmetric pair in **A** and **B** in the second and third elements.

The above arrangement is then a new unit of 16 consecutive numbers satisfying the condition that every pair in the upper row **A,** has a correspnding pair of numbers in the second row **B,** with the same sum.

Finally, then, we want to join together two units, each of 16 consecutive integers as above, to partition the set of 32 consecutive integers $\{0, 1, 2, \ldots, 31\}$.
Reasoning as above, and checking only the critical elements in the unit for corresponding sums, we see that the symmetric case works.

$$\text{The symmetric case :}$$
$$\begin{array}{cccccccccccc} 0 & & 3 & 5 & & 6 & 9 & & 10 & 12 & & 15 \\ & \sqcup & & & \sqcap & & & \sqcap & & & \sqcup & \\ 1 & & 2 & 4 & & 7 & 8 & & 11 & 13 & & 14 \end{array} \quad \text{and} \quad \begin{array}{cccccccccccc} 16 & & 19 & 21 & & 22 & 25 & & 26 & 28 & & 31 \\ & \sqcup & & & \sqcap & & & \sqcap & & & \sqcup & \\ 17 & & 18 & 20 & & 23 & 24 & & 27 & 29 & & 30 \end{array}$$

Thus,

$$\begin{array}{rcl} \mathbf{A} & = & \{0, 3, 5, 6, 9, 10, 12, 15, 16, 19, 21, 22, 25, 26, 28, 31\} \\ \mathbf{B} & = & \{1, 2, 4, 7, 8, 11, 13, 14, 17, 18, 20, 23, 24, 27, 29, 30\} \end{array}$$

Editor's comment: In Michael's solution each element in the set of four consecutive integers was written as being the vetex of an isosceles trapezoid. (The trapezoids were oriented with the bases being parallel to the top and bottom edges of page; Michael then manitpulated the trapezoids by flipping their bases.)

Adoración Martínez Ruiz of the Mathematics Club of the Institute of Secondary Education (No. 1) in Requena-Valencia, Spain also approached the problem geometrically in an almost identical manner as Michael. I adopted Adoración Martínez' notation of "cups" $\sqcup$ and "caps" $\sqcap$ instead of Michael's isosceles trapezoids in writing-up Michael's solution. (If the shorter base of the trapezoid was closer to the bottom edge of the page than the longer base, then that trapezoid became a cup, $\sqcup$; whereas if the shorter base of the trapezoid was closer to the top edge of the page than the longer base, then that trapezoid became a cap, $\sqcap$.

Michael's solution and Adoración Martínez' solution were identical to one another up until the last step. At that point Michael took the symmetric extension in moving from the first 16 non-negative integers to the first 32 non-negative integers, whereas Adoración Martínez took the anti-symmetric extension, and surprisingly (at least to me), each solution worked.

$$\text{Adoración Martínez}' \text{ anti} - \text{symmetric case :}$$
$$\begin{array}{cccccccccccc} 0 & & 3 & 5 & & 6 & 9 & & 10 & 12 & & 15 \\ & \sqcup & & & \sqcap & & & \sqcap & & & \sqcup & \\ 1 & & 2 & 4 & & 7 & 8 & & 11 & 13 & & 14 \end{array} \quad \text{and} \quad \begin{array}{cccccccccccc} 17 & & 18 & 20 & & 23 & 24 & & 27 & 29 & & 30 \\ & \sqcap & & & \sqcup & & & \sqcup & & & \sqcap & \\ 16 & & 19 & 21 & & 22 & 25 & & 26 & 28 & & 31 \end{array}$$

$$\begin{array}{rcl} \mathbf{A} & = & \{0, 3, 5, 6, 9, 10, 12, 15, 17, 18, 20, 23, 24, 27, 29, 30\} \\ \mathbf{B} & = & \{1, 2, 4, 7, 8, 11, 13, 14, 16, 19, 21, 22, 25, 26, 28, 31\} \end{array}$$

So now we have two solutions to the problem, each motivated by geometry, and it was assumed (at least by me) that their were no other solutions. Michael challenged **Mayer Goldberg,** a colleague in CS here at BGU, to find other solutions, and he did; many of them! Following is his approach.

Solution 2 by Mayer Goldberg, Beer-Sheva, Israel

Notation: For any set S of integers, the set $aS + b$ is the set $\{ak + b : k \in S\}$.

Construction: We start with the set $A_0 = \{0, 4\}, b_0 = \{1, 2\}$. We define A_n, B_n inductively as follows:

$$\begin{aligned} A_{n+1} &= (2A_n + 1) \cup (2B_n) \\ B_{n+1} &= (2A_n) \cup (2B_n + 1) \end{aligned}$$

Claim: The sets A_n, B_n partition the set $\{0, \ldots, 2^{n+2}\}$ according to the requirements of the problem.

Proof: By Induction. The sets $A_0 B_0$ satisfy the requirement trivially, since they each contain one pair, and by inspection, we see that the sums are the same. Assume that A_n, B_n satisfy the requirement. Pick $x_1, x_2 \in A_{n+1}$.

- **Case I:** $x_1 = 2x_3 + 1, x_2 = 2x_4 + 1$, for $x_3, x_4 \in A_n$. Then by the induction hypothesis (IH), there exists $y_3, y_4 \in B_n$, such that $x_3 + x_4 = y_3 + y_4$. Consequently,

  $$x_1 + x_2 = 2(x_3 + x_4) + 2 = 2(y_3 + y_4) + 2 = (2y_3 + 1) + (2y_4 + 1).$$

  So let $y_1 = 2y_3 + 1, y_2 = 2y_4 + 1 \in B_{n+1}$.
- **Cases II & III:** $x_1 = 2x_3 + 1, x_2 = 2y_4$, for $x_3 \in A_n, y_4 \in B_n$.

  $$x_1 + x_2 = 2(x_3 + 1) + 2y_4 = 2x_3 + (2y_4 + 1).$$

  So let $y_1 = 2x_3, y_2 = 2y_4 + 1 \in B_{n+1}$.
- **Case IV:** $x_1 = 2y_3, x_2 = 2y_4$, for $y_3, y_4 \in B_n$.Then by the IH, there exists $x_3, x_4 \in A_n$, suc that $y_3 + y_4 = x_3 + x_4$. Consequently,

  $$x_1 + x_2 = 2y_3 + 2y_4 = 2(y_3 + y_4) = 2(x_3 + x_4) = 2x_3 + 2x_4.$$

  So let $y_1 = 2x_3, y_2 = 2y_3 \in B_{n+1}$

Editor: This leads to potentially thousands of such pairs of sets that satisfy the criteria of the problem. Mayer listed about one hundred such examples, a few of which are reproduced below:

$$\begin{aligned} A &= \{0, 3, 5, 6, 9, 10, 13, 15, 17, 18, 19, 23, 24, 27, 29, 30\}, \\ B &= \{1, 2, 4, 7, 8, 11, 12, 14, 16, 20, 21, 22, 25, 26, 28, 31\} \end{aligned}$$

$$\begin{aligned} A &= \{0, 3, 5, 6, 9, 10, 11, 15, 17, 18, 19, 23, 24, 27, 29, 30\}, \\ B &= \{1, 2, 4, 7, 8, 12, 13, 14, 16, 20, 21, 22, 25, 26, 28, 31\} \end{aligned}$$

$$\begin{aligned} A &= \{0, 3, 5, 6, 9, 10, 14, 15, 17, 18, 19, 23, 24, 27, 29, 30\}, \\ B &= \{1, 2, 4, 7, 8, 11, 12, 13, 16, 20, 21, 22, 25, 26, 28, 31\} \end{aligned}$$

$$\begin{aligned} A &= \{0, 3, 5, 6, 9, 10, 13, 15, 17, 18, 22, 23, 24, 27, 29, 30\}, \\ B &= \{1, 2, 4, 7, 8, 11, 12, 14, 16, 19, 20, 21, 25, 26, 28, 31\} \end{aligned}$$

$$\begin{aligned} A &= \{0, 3, 5, 6, 9, 10, 14, 15, 17, 18, 22, 23, 24, 27, 29, 30\}, \\ B &= \{1, 2, 4, 7, 8, 11, 12, 13, 16, 19, 20, 21, 25, 26, 28, 31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,15,16,17,18,19,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,11,12,13,14,20,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,15,16,17,18,22,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,11,12,13,14,19,20,21,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,8,9,13,15,17,18,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,10,11,12,14,16,19,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,8,9,11,15,17,18,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,10,12,13,14,16,19,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,8,9,15,16,17,18,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,10,11,12,13,14,19,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,13,15,17,19,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,11,12,14,16,18,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,11,15,17,19,20,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,12,13,14,16,18,21,22,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,13,15,17,20,22,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,11,12,14,16,18,19,21,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,11,15,17,20,22,23,24,27,29,30\}, \\ B &= \{1,2,4,7,8,12,13,14,16,18,19,21,25,26,28,31\} \end{aligned}$$

$$\begin{aligned} A &= \{0,3,5,6,9,10,14,15,17,20,22,23,24,27,29,30\}, \\ B &= \ 1,2,4,7,8,11,12,13,16,18,19,21,25,26,28,31\} \end{aligned}$$

Editor (again): **Edwin Gray of Highland Beach, FL** working together with **John Kiltinen of Marquette, MI** claimed and proved by induction the following more general theorem:

Let $S = \{0, 1, 2, 3, \ldots, 2^n - 1\},\ n > 1$. Then there is a partition of S, say A, B such that

1) $A \cup B = S,\ A \cap B = \emptyset,$ and

2) For all $x, y \in A$, there is an $r, s \in B,$ such that $x + y = r + s,$ and vice versa.

That is, the sum of any two elements in B has two elements in A equal to their sum.

David Stone and John Hawkins both of Statesboro, GA also claimed and proved a more general statement: They showed that: for $n \geq 2$, the set $S_n = \{0, 1, 2, \ldots, 2^n - 1\}$ consists of the non-negaitve integers which can be written with n or fewer binary digits. E.g.,

$$S_2 = \{0, 1, 2, 3\} = \{00, 01, 10, 11\} \text{ and}$$

$$S_3 = \{0,1,2,3,4,5,6,7\} = \{000,001,010,011,100,101,110,111\}$$

Their proof consisted of partitioning S_n into two subsets: E_n: those elements of S_n whose binary representation uses an even number of ones, and O_n: those numbers in S_n whose binary reprsentation uses an odd number of ones. Hence, for any $x \neq y$ in $E_n,\ x+y$ can be written as $x+y=w+z$ for some $w \neq z$ in O_n, and vice versa. This lead them to Adoración Martínez' solution, and they speculated on its uniqueness.

All of this seemed to be getting out-of-hand for me; at first I thought the solution is unique; then I thought that there are only two solutions, and then I thought that there are many solutions to the problem. **Shai Covo's** solutio/Users/admin/Desktop/SSM/For Jan 11/For Jan 11; Jerry.texn however, shows that the answer can be unique if one uses a notion of *sum multiplicity.*

Solution 3 by Shai Covo, Kiryat-Ono, Israel

We give two solutions, the first simple and original, the second sophisticated and more interesting, thanks to the Online Encyclopedia of Integer sequences(OEIS).

Assuming that $0 \in A$, one checks that we must have either

$$\begin{array}{ccc} \{0,3,5,6\} \cup \{25,26,28,31\} \subset A & \text{and} & \{1,2,4,7\} \cup \{24,27,29,30\} \subset B \\ & \text{or} & \\ \{0,3,5,6\} \cup \{24,27,29,30\} \subset A & \text{and} & \{1,2,4,7\} \cup \{25,26,28,31\} \subset B. \end{array}$$

In view of the first possibility, it is natural to examine the following sets:

$$A = \{0,3,5,6,9,10,13,14,17,18,21,22,25,26,28,31\}$$

$$B = \{1,2,4,7,8,11,12,15,16,19,20,23,24,27,29,30\}.$$

To see why this is natural, connect the numbers with arrows, in increasing order, starting with a vertical arrow pointing down to 1. Now, define

$$C = \{a_1 + a_2 \mid a_1, a_2 \in A, a_1 \neq a_2\} \subset \{3,4,5,\ldots,59\} \text{ and}$$

$$D = \{b_1 + b_2 \mid b_1, b_2 \in B, b_1 \neq b_2\} \subset \{3,4,5,\ldots,59\}.$$

We want to show that $C = D$, or equivalently, for every $x \in \{3,4,5,\ldots,59\}$ either $x \in C \cap D$ or $x \notin C \cup D$. Checking each x value, we find that

$$C \cap D = \{3,4,5,\ldots,59\} \setminus \{4,7,55,58\} \text{ and } \{4,7,55,58\} \cap (\mathrm{C} \cup \mathrm{D}) = \emptyset.$$

Thus, $C = D$, and so the problem is solved with A and B as above.

We now turn to the second solution. OEIS sequences $A001969$ (numbers with an even number of 1's in their binary expansion) and $A000069$ (numbers with an odd number of 1's in their binary expansion) "give the unique solution to the problem of splitting the nonnegative integers into two classes in such a way that sums of pairs of distinct elements from either class occur *with the same multiplicities.* [Lambek and Moser]." We have verified (by computer) that, in the case at hand, the sets

$$A = \{A001969(n) : A001969(n) \leq 30\}$$

$$= \{0,3,5,6,9,10,12,15,17,18,20,23,24,27,29,30\} \text{ and}$$

$$B = \{A000069(n): A000069(n) \leq 31\}$$
$$= \{1,2,4,7,8,11,13,14,16,19,21,22,25,26,28,31\}$$

split the first 32 nonnegative integers from 0 to 31 in the manner stated for splitting the nonnegative integers. (The number 32 plays an important role here.) However, this is not the case for the sets A and B from the previous solution (consider, for example,12=3+9 versus 12=1+11, 12=4+8; there are seven more such examples.)

Editor (still again): I did not understand the notion about sums having the *same multiplicity,* but this is the key for having *a unique solution* to the problem, as it states in the OEIS. So I asked Shai to elaborate on this notion. Here is what he wrote:

The point is that "given the unique solution to the problem of splitting the nonnegative integers..." refers to the infinite set $\{0, 1, 2, 3, ...\}$ and not the finite set $\{0, 1, 2, ..., 31\}$. I should have stressed this point in my solution. As far as I can recall, I considered doing so, but decided not to, based on the following: "... the manner stated for splitting the nonnegative integers" only refers to "splitting the nonnegative integers into two classes in such a way that sums of pairs of distinct elements from either class occur with the same multiplicities," and not to "give the unique solution to the problem of splitting the nonnegative integers...".

In explaining the notion of itself, Shai wrote:

Consider Michael Fried's sets:

$$A = \{0,3,5,6,9,10,12,15,16,19,21,22,25,26,28,31\}$$

$$B = \{1,2,4,7,8,11,13,14,17,18,20,23,24,27,29,30\}.$$

For set A, the number 16 can be decomposed as 0+16 and 6+10; hence the multiplicity is 2. For set B, on the other hand, 16 can only be decomposed as 2+14 (8+8 does not count, since we consider distinct elements only); hence the multiplicity is 1.

Also solved by Brian D. Beasley, Clinton, SC; Edwin Gray, Highland Beach, FL; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; John Kiltinen, Marquette, MI; Charles McCracken, Dayton, OH; Adoración Martínez Ruiz, Requena-Valencia, Spain; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5123:** *Proposed by Kenneth Korbin, New York, NY*

Given isosceles triangle ABC with $\overline{AB} = \overline{BC} = 2011$ and with cevian $\overline{BD}$. Each of the line segments $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ have positive integer length with $\overline{AD} < \overline{CD}$.

Find the lengths of those three segments when the area of the triangle is minimum.

Solution by Shai Covo, Kiryat-Ono, Israel

We begin by observing that $\overline{AC} \in \{3, 4, \ldots, 4021\}$. This follows from $\overline{AC} < \overline{AB} + \overline{BC} = 4022$ and the assumption that $\overline{AC} = \overline{AD} + \overline{CD}$ is the sum of the distinct positive integers. The area S of triangle ABC can be expressed in terms of $\overline{AC}$ as

$$S = S\left(\overline{AC}\right) = \frac{\overline{AC}}{2}\sqrt{2011^2 - \left(\frac{\overline{AC}}{2}\right)^2}.$$

Define $f(x) = x^2(2011^2 - x^2), x \in [0, 2011]$. Then $S\left(\overline{AC}\right) = \sqrt{f\left(\overline{AC}/2\right)}$. It is readily verified that the function f (and hence $\sqrt{f}$) is unimodal with mode $m = 2011/\sqrt{2}$; that is, it is increasing for $x \leq m$ and decreasing for $x \geq m$. If thus follows from $f(4021/2) < f(127/2)$ that $S(4021) < S(k)$ for any integer $127 \leq k \leq 4020$. Next by the law of cosines, we find that

$$\overline{BD}^2 = 2011^2 + \overline{AD}^2 - 2 \cdot 2011 \cdot \overline{AD} \cdot \frac{\overline{AC}/2}{2011}.$$

Hence,

$$\overline{AD}^2 - \overline{AC} \cdot \overline{AD} + \left(2011^2 - \overline{BD}^2\right) = 0.$$

The roots of this quadratic equation are given by the standard formula as

$$\overline{AD}_{1,2} = \frac{\overline{AC} \pm \sqrt{\overline{AC}^2 - 4\left(2011^2 - \overline{BD}^2\right)}}{2}.$$

However, we are given that $\overline{AD} < \overline{CD}$; hence $\overline{AD} = \overline{AD_2}$ and $\overline{CD} = \overline{AD_1}$, and we must have $\overline{AC}^2 > 4\left(2011^2 - \overline{BD}^2\right)$. Since, obviously, $\overline{BD} \leq 2010$, we must have $\overline{AC^2} > 4\left(2011^2 - 2010^2\right) = 4 \cdot 4021$; hence, $127 \leq \overline{AC} \leq 4021$.

Thus, under the condition that S is minimum, we wish to find an integer value of $\overline{BD}(\leq 2010)$ that makes $\overline{AD_{1,2}}$ (that is, $\overline{CD}$ and $\overline{AD}$) distinct integers when $\overline{AC}$ is set to 4021.

We thus look for $\overline{BD} \in \{1, 2, \ldots, 2010\}$ for which the discriminant $\Delta = 4021^2 - 4(2011^2 - BD^2)$ is a positive perfect square, say $\Delta = j^2$ with $j \in N$ (actually, $j = \overline{CD} - \overline{AD}$). This leads straightforwardly to the following equation:

$$\left(2\overline{BD} + j\right)\left(2\overline{BD} - j\right) = 3 \cdot 7 \cdot 383.$$

Since 3,7, and 383 are primes, we have to consider the following four cases:

- $\left(2\overline{BD} - j\right) = 1$ and $\left(2\overline{BD} + j\right) = 3 \cdot 7 \cdot 383$. This leads to $\overline{BD} = 2011$; however, $\overline{BD}$ must be less than 2011.
- $\left(2\overline{BD} - j\right) = 3$ and $\left(2\overline{BD} + j\right) = 7 \cdot 383$. This leads to $\overline{BD} = 671$ and $j = 1339$, and hence to our first solution:

$$\overline{AD} = 1341,\ \overline{BD} = 671,\ \overline{CD} = 2680.$$

- $\left(2\overline{BD}-j\right)=7$ and $\left(2\overline{BD}+j\right)=3\cdot 383$. This leads to $\overline{BD}=289$ and $j=571$, and hence to our second solution:

$$\overline{AD}=1725,\ \overline{BD}=289,\ \overline{CD}=2296.$$

- $\left(2\overline{BD}-j\right)=3\cdot 7$ and $\left(2\overline{BD}+j\right)=383$. This leads to $\overline{BD}=101$ and $j=181$, and hence to our third solution:

$$\overline{AD}=1920,\ \overline{BD}=101,\ \overline{CD}=2101.$$

Editor: **David Stone and John Hawkins** made two comments in their solution. They started off their solution by letting $r=\overline{AC}$, the length of the triangle's base. By Heron's formula, they obtained the triangle's area: $K=\dfrac{r}{4}\sqrt{4022^2-r^2}$ and then they made the following observations.

- **a)** $\overline{BD}=1$ and $\overline{CD}=2011$ gives us a triangle ABC with area $\left(\dfrac{1}{2}-\dfrac{1}{4\,(2011)^2}\right)\sqrt{4\,(2011^2)-1}\approx 2010.999689$ which is the smallest value that can be obtained **not** requiring $\overline{AD}$ to be an integer.
- **b)** Letting $m=\overline{AD}, n=\overline{CD}, k=\overline{BD}$, (where $1\le m<n$ and $\overline{AC}=m+n\le 4021$), and letting α be the base angle at vertex A (and at C), and dropping an altitude from B to side AC, we obtain a right triangle and see that

$$\cos\alpha=\frac{\overline{AC}/2}{2011}=\frac{m+n}{2\cdot 2011}.$$

Using the Law of Cosines in triangle BDC, we have

$$k^2=n^2+2011^2-2\cdot 2011\cdot n\cos\alpha=2011^2+n^2-n(m+n),$$

so we have a condition which the integers m,n and k must satisfy

$$k^2=2011^2-mn \qquad (\mathbf{1})$$

There are many triangles satisfying condition (**1**), some with interesting characteristics. There are no permissible triangles with base 4020, five with base 4019 and six with base 4018. All have larger areas than the champions listed above.

The altitude of each triangle in our winners group is 44.8 so the "shape ratio", altitude/base, is very small: 0.011. A wide flat triangle indeed!

One triangle with base 187 has a relatively small area: 187,825.16. This is as close as we can come to a tall, skinny triangle with small area. Its altitude/base ratio is 10.7.

In general, the largest isosceles triangle is an isosceles right triangle. With side lengths 2011, this would require a hypotenuse (our base) of $2011\sqrt{2}\approx 2843.98$. There are no permissible triangles with $r=2844$. Letting $r=2843$, we find the two largest permissible triangles:

$$m = 291,\ n=2552, \text{cevian}=1817 \text{ and area } 2,022,060.02$$

$$m = 883,\ n=1960, \text{cevian}=1521 \text{ and area } 2,022,060.02$$

The triangle with $m = 3, n = 2680$ (hence base =2683) has a large area: 2,009,788,52. The cevian has length 2009; it is very close to the side AB.

The triangle with $m = 1524, n = 1560$ and cevian=1291, comes closer than any other we found to having the cevian bisect the base. Its area is 1,990,528.49

Also solved by Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5124:** *Proposed by Michael Brozinsky, Central Islip, NY*

If $n > 2$ show that $\displaystyle\sum_{i=1}^{n} \sin^2\left(\frac{2\pi i}{n}\right) = \frac{n}{2}$.

Solution 1 by Piriyathumwong P. (student, Patumwan Demonstration School), Bangkok, Thailand

Since $\cos 2\theta = 1 - 2\sin^2\theta$, we have

$$\begin{aligned}\sum_{i=1}^{n} \sin^2\left(\frac{2\pi i}{n}\right) &= \frac{1}{2}\sum_{i=1}^{n}\left(1 - \cos\left(\frac{4\pi i}{n}\right)\right)\\ &= \frac{n}{2} - \frac{1}{2}\sum_{i=1}^{n}\cos\left(\frac{4\pi i}{n}\right)\end{aligned}$$

We now have to show that $S = \displaystyle\sum_{i=1}^{n}\cos\left(\frac{4\pi i}{n}\right) = 0$.

Multiplying both sides of S by $2\sin\left(\dfrac{2\pi}{n}\right)$, gives

$$\begin{aligned}2\sin\left(\frac{2\pi}{n}\right)\cdot S &= 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{4\pi}{n}\right) + 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{8\pi}{n}\right) + \ldots + 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{4n\pi}{n}\right)\\ &= \left(\sin\left(\frac{6\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\right) + \left(\sin\left(\frac{10\pi}{n}\right) - \sin\left(\frac{6\pi}{n}\right)\right) + \ldots\\ &\quad + \left(\sin\left(\frac{(4n+2)\pi}{n}\right) - \sin\left(\frac{(4n-2)\pi}{n}\right)\right)\\ &= \sin\left(\frac{(4n+2)\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\\ &= 0\end{aligned}$$

Hence, $S = 0$, and we are done.

Solution 2 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

To avoid confusion with the complex number $i = \sqrt{-1}$, we will consider

$$\sum_{k=1}^{n}\sin^2\left(\frac{2\pi k}{n}\right).$$

If $R = e^{\left(4\pi i/n\right)}$, with $n > 2$, then $R \neq 1$ and $R^n = e^{4\pi i} = 1$. Then, using the formula for a geometric sum, we get

$$\sum_{k=1}^{n} R^k = R\frac{R^n - 1}{R - 1} = 0,$$

and hence,

$$\sum_{k=1}^{n} \cos\left(\frac{4\pi k}{n}\right) = \sum_{k=1}^{n} Re\left(R^k\right) = Re\left(\sum_{k=1}^{n} R^k\right) = 0.$$

Therefore, by the half-angle formula,

$$\sum_{k=1}^{n} \sin^2\left(\frac{2\pi k}{n}\right) = \frac{1}{2}\sum_{k=1}^{n}\left[1 - \cos\left(\frac{4\pi k}{n}\right)\right] = \frac{n}{2}.$$

Also solved by Daniel Lopez Aguayo (student, Institute of Mathematics, UNAM), Morelia, Mexico; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Shai Covo, Kiryat-Ono, Israel; Bruno Salgueiro Fanego, Viveiro, Spain; Michael N. Fried, Kibbutz Revivim, Israel; G.C. Greubel, Newport News, VA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Pedro H. O. Pantoja, Natal-RN, Brazil; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; Boris Rays, Brooklyn, NY; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5125:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{ab}{2(c+a)+5b} + \frac{bc}{2(a+b)+5c} + \frac{ca}{2(b+c)+5a} < \frac{11}{32}.$$

Solution by Kee-Wai Lau, Hong Kong, China

We prove the sharp inequality

$$\frac{ab}{2(c+a)+5b} + \frac{bc}{2(a+b)+5c} + \frac{ca}{2(b+c)+5a} < \frac{1}{3}. \qquad (1)$$

Let $x = \dfrac{a}{a+b+c}$, $y = \dfrac{b}{a+b+c}$, $z = \dfrac{c}{a+b+c}$ so that (1) can be written as

$$(a+b+c)\left(\frac{xy}{3y+2} + \frac{yz}{3z+2} + \frac{zx}{3x+2}\right) \leq \frac{1}{3}. \qquad (2)$$

Since

$$a+b+c = \sqrt{3\left(a^2+b^2+c^2\right) - (a-b)^2 - (b-c)^2 - (c-a)^2} \leq \sqrt{3\left(a^2+b^2+c^2\right)} = 3$$

so to prove (2), we need only prove that

$$\frac{xy}{3y+2}+\frac{yz}{3z+2}+\frac{zx}{3x+2}\leq\frac{1}{9}. \qquad (3)$$

whenever x, y, z are positive and $x+y+z=1$. It is easy to check that (3) is equivalent to

$$\frac{x}{3y+2}+\frac{y}{3z+2}+\frac{z}{3x+2}\geq\frac{1}{3}. \qquad (4)$$

By the convexity of the function $\frac{1}{t}$, for $t>0$ and Jensen's inequality, we have

$$\frac{x}{3y+2}+\frac{y}{3z+2}+\frac{z}{3x+2}\geq\frac{1}{x(3y+2)+y(3z+2)+z(3x+2)}=\frac{1}{3\left(xy+yz+zx\right)+2}.$$

Now

$$xy+yz+zx=\frac{2\left(x+y+z\right)^2-\left(x-y\right)^2-\left(y-z\right)^2-\left(z-x\right)^2}{6}\leq\frac{1}{3}$$

and so (4) holds. This proves (1) and equality holds when $a=b=c=1$.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paolo Perfetti, Department of Mathematics, University of Rome, Italy, and the proposer.

- **5126:** *Proposed by Pantelimon George Popescu, Bucharest, Romania and José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c, d be positive real numbers and $f : [a, b] \to [c, d]$ be a function such that $|f(x) - f(y)| \geq |g(x) - g(y)|$, for all $x, y \in [a, b]$, where $g : R \to R$ is a given injective function, with $g(a)$, $g(b) \in \{c, d\}$.

Prove

(**i**) $f(a)=c$ and $f(b)=d$, or $f(a)=d$ and $f(b)=c$.

(**ii**) If $f(a)=g(a)$ and $f(b)=g(b)$, then $f(x)=g(x)$ for $a\leq x\leq b$.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

To avoid trivial situations, we will assume that $a < b$. Then, since $g(x)$ is injective and $g(a), g(b) \in \{c, d\}$, it follows that $c < d$ also.

First of all, the fact that $f(x) \in [c, d]$ for all $x \in [a, b]$ implies that

$$|f(x)-f(y)|\leq d-c$$

for all $x, y \in [a, b]$.

(i) In particular, since $g(a), g(b) \in \{c, d\}$, we have

$$d-c\geq|f(a)-f(b)|\geq|g(a)-g(b)|=d-c.$$

Hence, $|f(a)-f(b)|=d-c$ with $c\leq f(a), f(b)\leq d$, and we get $f(a)=c$ and $f(b)=d$, or $f(a)=d$ and $f(b)=c$.

(ii) Suppose $f(a) = g(a) = c$ and $f(b) = g(b) = d$. The proof in the other case is similar. Then, since $c \leq f(x) \leq d$ for all $x \in [a, b]$, we obtain

$$\begin{aligned} d - c &= (d - f(x)) + (f(x) - c) \\ &= |d - f(x)| + |f(x) - c| \\ &= |f(b) - f(x)| + |f(x) - f(a)| \\ &\geq |g(b) - g(x)| + |g(x) - g(a)| \\ &= |d - g(x)| + |g(x) - c| \\ &\geq |d - c| \\ &= d - c. \end{aligned}$$

Thus, for all $x \in [a, b]$,

$$|d - f(x)| = |d - g(x)| \text{ and } |f(x) - c| = |g(x) - c|.$$

If there is an $x_0 \in [a, b]$ such that $f(x_0) \neq g(x_0)$, then

$$d - f(x_0) = g(x_0) - d \text{ and } f(x_0) - c = c - g(x_0)$$

and hence,

$$2d = f(x_0) + g(x_0) = 2c.$$

This is impossible since $c \neq d$. Therefore, $f(x) = g(x)$ for all $x \in [a, b]$.

Remark. The condition that $a, b, c, d > 0$ seems unnecessary for the solution of this problem.

Editor: **Shai Covo** suggested that the problem can be made more interesting by adding a third condition. Namely:

iii) If $f(a) \neq g(a)$ (or equivalently, $f(b) \neq g(b)$), then $f(x) + g(x) = c + d$ for all $x \in [a, b]$ and, hence, $f(x) - f(y) = g(y) - g(x)$ for all $x, y \in [a, b]$.

Also solved by Shai Covo, Kiryat-Ono, Israel; Paolo Perfetti, Department of Mathematics, University of Rome, Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- **5127:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $n \geq 1$ be an integer and let $T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$, denote the $(2n-1)$th Taylor polynomial of the sine function at 0. Calculate

$$\int_0^\infty \frac{T_n(x) - \sin x}{x^{2n+1}} dx.$$

Solution by Paolo Perfetti, Department of Mathematics, University of Rome, Italy

Answer: $\dfrac{\pi(-1)^{n-1}}{2(2n)!}$

Proof: Integrating by parts:

$$\begin{aligned}
\int_0^\infty \frac{T_n(x)-\sin x}{x^{2n+1}}dx &= -\frac{1}{2n}\int_0^\infty (T_n(x)-\sin x)(x^{-2n})'dx \\
&= \frac{T_n(x)-\sin x}{-2nx^{2n}}\Big|_0^\infty + \frac{1}{2n}\int_0^\infty \frac{T_n'(x)-\cos x}{x^{2n}}dx \\
&= \frac{1}{2n}\int_0^\infty \frac{T_n'(x)-\cos x}{x^{2n}}dx
\end{aligned}$$

using $T_n(x) - \sin x = -\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}$ in the last equality.

After writing $T_n'(x) - \cos x = -\sum_{k=n+1}^{\infty}(-1)^{k-1}\frac{x^{2k-2}}{(2k-2)!}$, we do the second step.

$$\begin{aligned}
\int_0^\infty \frac{T_n'(x)-\cos x}{(2n)x^{2n}}dx &= \frac{-1}{2n(2n-1)}\int_0^\infty (T_n'(x)-\cos x)(x^{-2n+1})'dx \\
&= \frac{T_n'(x)-\cos x}{-2n(2n-1)x^{2n-1}}\Big|_0^\infty + \frac{1}{2n(2n-1)}\int_0^\infty \frac{T_n''(x)+\sin x}{x^{2n-1}}dx \\
&= \frac{1}{2n(2n-1)}\int_0^\infty \frac{T_n''(x)+\sin x}{x^{2n-1}}dx.
\end{aligned}$$

After $2n$ steps we obtain

$$\frac{(-1)^{n-1}}{(2n)!}\int_0^\infty \frac{\sin}{x}dx = \frac{\pi(-1)^{n-1}}{2(2n)!}$$

Also solved by Shai Covo, Kiryat-Ono, Israel; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://ssmj.tamu.edu>.

Solutions to the problems stated in this issue should be posted before April 15, 2011

- **5146:** *Proposed by Kenneth Korbin, New York, NY*

  Find the maximum possible value of the perimeter of an integer sided triangle with in-radius $r = \sqrt{13}$.

- **5147:** *Proposed by Kenneth Korbin, New York, NY*

  Let
  $$\begin{cases} x = 5N^2 + 14N + 23 \text{ and} \\ y = 5(N+1)^2 + 14(N+1) + 23 \end{cases}$$
  where N is a positive integer. Find integers a_i such that
  $$a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 = 0.$$

- **5148:** *Proposed by Pedro Pantoja (student, UFRN), Natal, Brazil*

  Let a, b, c be positive real numbers such that $ab + bc + ac = 1$. Prove that
  $$\frac{a^2}{\sqrt[3]{b(b+2c)}} + \frac{b^2}{\sqrt[3]{c(c+2a)}} + \frac{c^2}{\sqrt[3]{a(a+2b)}} \geq 1.$$

- **5149:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

  A regular n-gon $A_1, A_2 \cdots, A_n$ $(n \geq 3)$ has center F, the focus of the parabola $y^2 = 2px$, and no one of its vertices lies on the x axis. The rays $FA_1, FA_2, \cdots, FA_n$ cut the parabola at points $B_1, B_2, \cdots, B_n$.

  Prove that
  $$\frac{1}{n}\sum_{k=1}^{n} FB_k^2 > p^2.$$

- **5150:** *Proposed by Mohsen Soltanifar(student, University of Saskatchewan), Saskatoon, Canada*

  Let $\{A_n\}_{n=1}^{\infty}, (A_n \in M_{n\times n}(C))$ be a sequence of matrices such that $\det(A_n) \neq 0, 1$ for all $n \in N$. Calculate:
  $$\lim_{n\to\infty} \frac{n^n \ln(|\det(A_n)|)}{\ln\left(|\det\left(adj^{\circ n}\left(A_n\right)\right)|\right)},$$
  where $adj^{\circ n}$ refers to $adj \circ adj \circ \cdots \circ adj, n$ times, the n^{th} iterate of the classical adjoint.

- **5151:** *Proposed by Ovidiu Furdui, Cluj, Romania*

  Find the value of

$$\prod_{n=1}^{\infty}\left(\sqrt{\frac{\pi}{2}}\cdot\frac{(2n-1)!!\sqrt{2n+1}}{2^n n!}\right)^{(-1)^n}.$$

  More generally, if $x \neq n\pi$ is a real number, find the value of

$$\prod_{n=1}^{\infty}\left(\frac{x}{\sin x}\left(1-\frac{x^2}{\pi^2}\right)\cdots\left(1-\frac{x^2}{(n\pi)^2}\right)\right)^{(-1)^n}.$$

Solutions

- **5128:** *Proposed by Kenneth Korbin, New York, NY*

  Find all positive integers less than 1000 such that the sum of the divisors of each integer is a power of two.

  For example, the sum of the divisors of 3 is 2^2, and the sum of the divisors of 7 is 2^3.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, San Angelo, TX

For $n \geq 1$, let $\sigma(n)$ denote the sum of the positive divisors of n. The problem is to find all positive integers $n < 1000$ such that $\sigma(n) = 2^k$ for some integer $k \geq 0$. We note first that $n = 1$ is a solution since $\sigma(1) = 1 = 2^0$. For the remainder, we will assume that $n \geq 2$. Our key result is the following:

Lemma. If p is prime and k and e are positive integers such that $\sigma(p^e) = 2^k$, then $e = 1$ and $p = 2^k - 1$ (i.e., p is a Mersenne prime).

Proof. First of all, $p \neq 2$ since $\sigma(2^e) = 1 + 2 + \ldots + 2^e$, which is odd. Further, since p must be odd,

$$2^k = \sigma(p^e) = 1 + p + \ldots + p^e$$

implies that e is also odd. It follows that

$$\begin{aligned} 2^k &= (1+p) + \left(p^2+p^3\right) + \left(p^4+p^5\right) + \ldots + \left(p^{e-1}+p^e\right) \\ &= (1+p)\left(1+p^2+p^4+\ldots+p^{e-1}\right). \quad (*) \end{aligned}$$

Then, $1 + p$ divides 2^k and $1 + p > 1$, which leads us to conclude that $1 + p = 2^m$, with $1 \leq m \leq k$. Statement (*) reduces to

$$2^{k-m} = 1 + p^2 + p^4 + \ldots + p^{e-1}.$$

If $e \geq 3$, then $m < k$ and using the same reasoning as above, we get

$$\begin{aligned} 2^{k-m} &= \left(1+p^2\right) + \left(p^4+p^6\right) + \ldots + \left(p^{e-3}+p^{e-1}\right) \\ &= \left(1+p^2\right)\left(1+p^4+\ldots+p^{e-3}\right), \end{aligned}$$

which implies that $1 + p^2 = 2^i$, for some positive integer $i \leq k - m$. Thus,

$$2^i = 1 + p^2 = 1 + (2^m - 1)^2 = 2^{2m} - 2^{m+1} + 2,$$

or

$$2^{i-1} = 2^{2m-1} - 2^m + 1 = 2^m \left(2^{m-1} - 1\right) + 1.$$

This requires $i = m = 1$, which is impossible since this would entail $p = 2^m - 1 = 2 - 1 = 1$. Therefore, $e = 1$ and $2^k = \sigma(p) = p + 1$, i.e., $p = 2^k - 1$.

To return to our problem, we may write

$$n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$$

for distinct primes $p_1, \ldots, p_m$ and positive integers $e_1, \ldots, e_m$. Since σ is multiplicative and $p_1^{e_1}, \ldots, p_m^{e_m}$ are pairwise relatively prime,

$$2^k = \sigma(n) = \sigma\left(p_1^{e_1}\right) \sigma\left(p_2^{e_2}\right) \cdots \sigma\left(p_m^{e_m}\right).$$

Further, for $i = 1, \ldots, m$, $\sigma\left(p_i^{e_i}\right) \geq p_i + 1 > 1$. Hence, there are positive integers $k_1, \ldots, k_m$ such that

$$\sigma\left(p_i^{e_i}\right) = 2^{k_i}$$

for $i = 1, \ldots, m$. By the Lemma, $e_1 = e_2 = \ldots = e_m = 1$ and

$$p_i = 2^{k_i} - 1$$

for $i = 1, \ldots, m$. Therefore, $n = p_1 p_2 \cdots p_m$, where each p_i is a distinct Mersenne prime.

To solve our problem, we need to find all Mersenne primes < 1000 and all products of distinct Mersenne primes for which the product < 1000. The Mersenne primes < 1000 are $3, 7, 31$, and 127. All solutions of $\sigma(n) = 2^k$, with $n < 1000$, are listed below.

n	$\sigma(n)$
1	2^0
3	2^2
7	2^3
$21 = 3 \cdot 7$	2^5
31	2^5
$93 = 3 \cdot 31$	2^7
127	2^7
$217 = 7 \cdot 31$	2^8
$381 = 3 \cdot 127$	2^9
$651 = 3 \cdot 7 \cdot 31$	2^{10}
$889 = 7 \cdot 127$	2^{10}

Also solved by Brian D. Beasley, Clinton, SC; Pat Costello, Richmond, KY; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Charles McCracken, Dayton, OH; Boris Rays, Brooklyn, NY; Harry Sedinger, St. Bonaventure, NY; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA; Tran Trong Hoang Tuan (student, Bac Lieu High School for the Gifted), Bac Lieu City, Vietnam, and the proposer.

- **5129:** *Proposed by Kenneth Korbin, New York, NY*

Given prime number c and positive integers a and b such that $a^2+b^2=c^2$, express in terms of a and b the lengths of the legs of the primitive Pythagorean Triangles with hypotenuses c^3 and c^5, respectively.

Solution 1 by Howard Sporn, Great Neck, NY

A Pythagorean Triple (a,b,c) can be represented by the complex number $a+bi$, with modulus c. By multiplying two Pythagorean Triples in this form, one can generate another Pythagorean Triple. For instance, the complex representation of the 3-4-5 triangle is $3+4i$. By multiplying the complex number by itself, (and taking the absolute value of the real and imaginary parts), one obtains the 7-24-25 triangle:

$$\begin{aligned}(3+4i)(3+4i) &= -7+24i\\ 7^2+24^2 &= 25^2\end{aligned}$$

By cubing $a+bi$, one can obtain a Pythagorean Triple whose hypotenuse is c^3.

$$\begin{aligned}(a+bi)^3 &= (a+bi)^2(a+bi)\\ &= (a^2-b^2+2abi)(a+bi)\\ &= a^3-3ab^2+i\left(3a^2-b^3\right)\end{aligned}$$

One can verify that the modulus of this complex number is $\left(a^2+b^2\right)^3=c^3$. Thus we obtain the Pythagorean Triple $\left(\left|a^3-3ab^2\right|, \left|3a^2b-b^3\right|, c^3\right)$.

That this Pythagorean Triangle is primitive can be seen by factoring the lengths of the legs:

$$\begin{aligned}a^3-3ab^2 &= a\left(a^2-3b^2\right), \text{ and}\\ 3a^2b-b^3 &= b\left(3a^2-b^2\right),\end{aligned}$$

generally have no factors in common.

Example: If we let $(a,b,c)=(3,4,5)$, we obtain the Pythagorean Triple $(117,44,125)$.

By a similar procedure , one can obtain a Pythagorean Triple whose hypotenuse is c^5.

$$\begin{aligned}(a+bi)^5 &= (a+bi)^3(a+bi)(a+bi)\\ &= \left[a^3-3ab^2+i\left(3a^2b-b^3\right)\right](a+bi)(a+bi)\\ &= \left[a^4-6a^2b^2+b^4+i\left(4a^3b-4ab^3\right)\right](a+bi)\\ &= a^5-10a^3b^2+5ab^4+i\left(5a^4b-10a^2b^3+b^5\right).\end{aligned}$$

Thus we obtain the Pythagorean Triple

$$\left(\left|a^5-10a^3b^2+5ab^4\right|,\ \left|5a^4b-10a^2b^3+b^5\right|,\ c^5\right).$$

Example: If we let $(a,b,c)=(3,4,5)$, we obtain the Pythagorean Triple $(237,3116,3125)$.

Solution 2 by Brian D. Beasley, Clinton, SC

Given positive integers a, b, and c with c prime and $c^2 = a^2 + b^2$, we may assume without loss of generality that $a < b < c$. Also, we note that c must be odd and that c divides neither a nor b. Using the classic identity

$$(w^2 + x^2)(y^2 + z^2) = (wy + xz)^2 + (wz - xy)^2,$$

we proceed from $c^2 = a^2 + b^2$ to obtain $c^4 = (-a^2 + b^2)^2 + (2ab)^2$. Similarly, we have

$$c^6 = (-a^3 + 3ab^2)^2 + (3a^2b - b^3)^2$$

and

$$c^{10} = (a^5 - 10a^3b^2 + 5ab^4)^2 + (-5a^4b + 10a^2b^3 - b^5)^2.$$

Thus the leg lengths for the Primitive Pythagorean Triangle (PPT) with hypotenuse c^3 are

$$m = |-a^3 + 3ab^2| \quad \text{and} \quad n = |3a^2b - b^3|,$$

while the leg lengths for the PPT with hypotenuse c^5 are

$$q = |a^5 - 10a^3b^2 + 5ab^4| \quad \text{and} \quad r = |-5a^4b + 10a^2b^3 - b^5|.$$

To show that these triangles are primitive, we first note that $(-a^2 + b^2, 2ab, c^2)$ is a PPT, since c cannot divide $2ab$. Next, we prove that (m, n, c^3) is also a PPT: If not, then c divides both $a(-a^2 + 3b^2)$ and $b(3a^2 - b^2)$, so c divides $-a^2 + 3b^2$ and $3a^2 - b^2$; thus c divides the linear combination $(-a^2 + 3b^2) + 3(3a^2 - b^2) = 8a^2$, a contradiction. Similarly, we prove that (q, r, c^5) is a PPT: If not, then c divides both $a(a^4 - 10a^2b^2 + 5b^4)$ and $b(-5a^4 + 10a^2b^2 - b^4)$, so c divides $a^4 - 10a^2b^2 + 5b^4$ and $-5a^4 + 10a^2b^2 - b^4$; thus c divides the linear combinations

$$(a^4 - 10a^2b^2 + 5b^4) + 5(-5a^4 + 10a^2b^2 - b^4) = 8a^2(-3a^2 + 5b^2)$$

and

$$5(a^4 - 10a^2b^2 + 5b^4) + (-5a^4 + 10a^2b^2 - b^4) = 8b^2(-5a^2 + 3b^2).$$

But this means that c divides the linear combination $3(-3a^2 + 5b^2) - 5(-5a^2 + 3b^2) = 16a^2$, a contradiction.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; David E. Manes, Oneonta, NY, and the proposer.

- **5130:** *Proposed by Michael Brozinsky, Central Islip, NY*

In Cartesianland, where immortal ants live, calculus has not been discovered. A bride and groom start out from $A(-a, 0)$ and $B(b, 0)$ respectively where $a \neq b$ and $a > 0$ and $b > 0$ and walk at the rate of one unit per second to an altar located at the point P on line $L : y = mx$ such that the time that the first to arrive at P has to wait for the other to arrive is a maximum. Find, without calculus, the locus of P as m varies through all nonzero real numbers.

Solution 1 by Michael N. Fried, Kibbtuz Revivim, Israel

Let OQ be the line $y = mx$. Since it is the total time which must be a minimum, we might as well consider the minimum time from A to a point P on OQ and then from P to B. But since the speed is equal and constant for both the bride and groom the minimum time will be achieved for the path having the minimum distance. This, as is well-known, occurs when $\angle APO = \angle BPQ$. Accordingly, OP is the *external* angle bisector of angle APB, and, thus, $\dfrac{BP}{AP} = \dfrac{BO}{OA}$ = a constant ratio. So, P lies on a circle (an Apollonius circle) whose diameter is OAC, where OC is the harmonic mean between OA and OB.

Solution 2 by the proposer

Since the bride and groom go at the same rate, then for a given m, P is the point such that the maximum of $||AQ| - |BQ||$ for points Q on L occurs when Q is P. Let A' denote the reflection of A about this line.

Now since $||AQ| - |BQ|| = ||A'Q| - |BQ|| \geq |A'B|$ (from the triangle inequality) we have this maximum must be $|A'B|$ since it is attained when P is the point of intersection of the line through B and A', with L. (Note that the line through A' and B is not parallel to L because that would imply that the origin is the midpoint of AB because the line through the midpoint of AA' and the midpoint of AB is parallel to the line through A' and B.)

Let M be the midpoint of segment AA'. Now, since triangles $A'PM$ and APM are congruent, L is the angle bisector at P in triangle ABP, and since an angle bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides we have $\dfrac{AP}{BP} = \dfrac{a}{b}$ (1).

Denoting P by $P(X, Y)$ we thus have $Y \neq 0$ and thus $X \neq 0$ and so from (1)

$$\frac{\sqrt{(X+a)^2 + (mX)^2}}{\sqrt{(X-b)^2 + (mX)^2}} = \frac{a}{b},$$

and since $X \neq 0$, we have by squaring both sides and solving for X, that

$$\begin{aligned} X &= \frac{2ab}{(a-b)(m^2+1)}, \text{ and thus} \\ Y &= \frac{2mab}{(a-b)(m^2+1)} \end{aligned}$$

are parametric equations of the locus. Now replacing m by $\dfrac{Y}{X}$ and simplifying, we obtain

$$X = \frac{2abX^2}{(X^2+Y^2)(a-b)}$$

which is just the circle

$$(X^2+Y^2)(a-b) = 2abX$$

with the endpoints of the diameter deleted. The endpoints of the diameter occur when $Y = 0$; that is, at $(0, 0)$, and at $\left(\dfrac{2ab}{a-b}, 0\right)$.

Note that if the line $x = 0$ were a permissible altar line, then we would add $(0, 0)$ to the locus, while if the $x-$axis were a permissible altar line, then the union of the rays $(-\infty, -a] \cup [b, \infty)$ would be part of the locus, and in particular, this includes $\left(\dfrac{2ab}{a-b}, 0\right)$.

- **5131:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Let a, b, c be positive real numbers. Prove that

$$\frac{a+b+3c}{3a+3b+2c}+\frac{a+3b+c}{3a+2b+3c}+\frac{3a+b+c}{2a+3b+3c}\geq\frac{15}{8}.$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

The inequality is homogeneous, so we can assume without loss of generality that $a+b+c=1$, being equivalent to

$$\frac{1+2c}{3-c}+\frac{1+2b}{3-b}+\frac{1+2a}{3-a}\geq\frac{15}{8},$$

which is Jensen's inequality $f(c)+f(b)+f(a)\geq 3f\left(\frac{c+b+a}{3}\right)$ applied to the convex function $f(x)=\frac{1+2x}{3-x}$ and the numbers $c,\ b,\ a$ on the interval $(0,1)$; equality occurs if and only if $a=b=c$.

Solution 2 by Javier García Cavero (student, Mathematics Club of the Instituto de Educación Secundaria- N^o 1), Requena-Valencia, Spain

Changing the variables, that is to say, calling

$$\begin{aligned} x &= 2a+3b+3c, \\ y &= 3a+2b+3c, \text{ and} \\ z &= 3a+3b+2c \end{aligned}$$

it is easy to see, solving the corresponding system of equations, that

$$\begin{aligned} a+b+c &= \frac{x+y+z}{8} \text{ and that} \\ a &= \frac{-5x+3y+3z}{8} \\ b &= \frac{3x-5y+3z}{8}, \text{ and} \\ c &= \frac{3x+3y-5z}{8}. \end{aligned}$$

The numerators of the fractions will thus be:

$$a+b+3c=\frac{7x+7y-9z}{8},\quad a+3b+c=\frac{7x-9y+7z}{8},\quad 3a+b+c=\frac{-9x+7y+7z}{8}$$

Replacing everything in the initial expression:

$$\begin{aligned} &\frac{a+b+3c}{3a+3b+2c}+\frac{a+3b+c}{3a+2b+3c}+\frac{3a+b+c}{2a+3b+3c} \\ = \quad &\frac{7x+7y-9z}{8z}+\frac{7x-9y+7z}{8y}+\frac{-9x+7y+7z}{8x} \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{7x}{8z}+\frac{7y}{8z}+\frac{-9}{8}\right)+\left(\frac{7x}{8y}+\frac{-9}{8}+\frac{7z}{8y}\right)+\left(\frac{-9}{8}+\frac{7y}{8x}+\frac{7z}{8x}\right)\\
&= 3\cdot\left(\frac{-9}{8}\right)+\frac{7}{8}\left(\frac{x}{z}+\frac{y}{z}+\frac{x}{y}+\frac{z}{y}+\frac{y}{x}+\frac{z}{x}\right)\\
&\quad \frac{-27}{8}+\frac{7}{8}\left(\left(\frac{x}{z}+\frac{z}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)+\left(\frac{x}{y}+\frac{y}{x}\right)\right)\\
&\geq \frac{-27}{8}+\frac{42}{8}\\
&= \frac{15}{8},
\end{aligned}
$$

since $r+\dfrac{1}{r}\geq 2$. Equality occurs for $x=y=z$ and, therefore, for $a=b=c$.

Solution 3 by Kee-Wai Lau, Hong Kong, China

Since

$$
\begin{aligned}
&\frac{a+b+3c}{3a+3b+2c}+\frac{b+c+3a}{3b+3c+2a}+\frac{c+a+3b}{3c+3a+2b}-\frac{15}{8}\\
=\;&\frac{7\left(6a^3+6b^3+6c^3-a^2b-ab^2-b^2c-bc^2-c^2a-ca^2-12abc\right)}{8\left(3a+3b+2c\right)\left(3b+3c+2a\right)\left(3c+3a+2b\right)}\\
=\;&\frac{7\Big((3a+3b+2c)(a-b)^2+(3b+3c+2a)(b-c)^2+(3c+3a+2b)(c-a)^2\Big)}{8(3a+3b+2c)(3b+3c+2a)(3c+3a+2b)}\\
\geq\;& 0,
\end{aligned}
$$

the inequality of the problem follows.

Solution 4 by P. Piriyathumwong (student, Patumwan Demonstration School), Bangkok, Thailand

The given inequality is equivalent to the following:

$$
\begin{aligned}
\sum_{cyc}\left(\frac{a+b+3c}{3a+3b+2c}-\frac{5}{8}\right)\geq 0 \quad &\Leftrightarrow\quad \sum_{cyc}\left(\frac{-a-b+2c}{3a+3b+2c}\right)\geq 0\\
&\Leftrightarrow\quad \sum_{cyc}\left(\frac{(c-a)+(c-b)}{3a+3b+2c}\right)\geq 0\\
&\Leftrightarrow\quad \sum_{cyc}(a-b)\left(\frac{1}{2a+3b+3c}-\frac{1}{3a+2b+3c}\right)\geq 0\\
&\Leftrightarrow\quad \sum_{cyc}\frac{(a-b)^2}{(2a+3b+3c)(3a+2b+3c)}\geq 0,
\end{aligned}
$$

which is obviously true.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Valmir Bucaj (student, Texas Lutheran University), Seguin, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata", Rome, Italy; Boris Rays, Brooklyn, NY; Tran Trong Hoang Tuan (student, Bac Lieu High School for the Gifted), Bac Lieu City, Vietnam, and the proposer.

- **5132:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Find all all functions $f : C \to C$ such that $f(f(z)) = z^2$ for all $z \in C$.

Solution by Kee-Wai Lau, Hong Kong, China

We show that no such functions $f(z)$ exist by considering the values of $f(1), f(-1), f(i), f(-i)$, where $i = \sqrt{-1}$.

From the given relation

$$f(f(z)) = z^2 \qquad (1)$$

we obtain $f(f(f(z))) = f(z^2)$ so that

$$(f(z))^2 = f\left(z^2\right). \qquad (2)$$

Replacing z by z^2 in (2), we get

$$f(z^4) = (f(z))^4. \qquad (3)$$

By putting $z = 1$ into (2), we obtain $f(1) = 0$ or 1. If $f(1) = 0$, then by putting $z = i$ into (3), we get $0 = f(i^4) = (f(i))^4$, so that $f(i) = 0$. Putting $z = i$ into (1) we get $f(0) = -1$ and putting $z = 0$ into (2) we obtain $(-1)^2 = -1$ which is false. It follows that

$$f(1) = 1. \qquad (4)$$

By putting $z = -1$ into (2) we get $(f(-1))^2$ so that $f(-1) = -1$ or 1.

If $f(-1) = -1$ then by (1), $-1 = f(f(-1)) = (-1)^2 = 1$, which is false.

Hence,

$$f(-1) = 1. \qquad (5)$$

By putting $z = i$ into (3), we are $(f(i))^4 = 1$, so that $f(i) = -1, 1, i, -i$.

If $f(i) = \pm 1$, then by (1), (4) and (5), $1 = f(f(i)) = i^2 = -1$, which is false.

If $f(i) = i$, then by (1), $i = f\left(f((i)\right) = -1$, which is also false. Hence,

$$f(i) = -i \qquad (6)$$

By putting $z = -i$ into (3), we have $(f(-i))^4 = 1$, so that $f(-i) = -1, 1, i, -i$.

If $f(-i) = \pm 1$, then by (1), (4), and (5) $1 = f(f(-i)) = (-i)^2 = -1$, which is false.

If $f(-i) = \pm i$, then by (1) and (6) $-i = f(f(-i)) = (-i)^2 = -1$, which is also false.

Thus $f(-i)$ can take no value, showing that no such $f(z)$ exists.

Also solved by Howard Sporn and Michael Brozinsky (jointly), of Great Neck and Central Islip, NY (respectively), and the proposer.

- **5133:** *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $n \geq 1$ be a natural number. Calculate

$$I_n = \int_0^1 \int_0^1 (x-y)^n dxdy.$$

Solutions 1 and 2 by Valmir Bucaj (student, Texas Lutheran University), Seguin, TX

Solution 1) We first calculate $\displaystyle\int_0^1 (x-y)^n dx$.

Letting $u = x - y$ we get

$$\begin{aligned}\int_0^1 (x-y)^n &= \int_{-y}^{1-y} u^n du \\ &= \frac{1}{n+1}\Big[(1-y)^{n+1} + (-1)^n y^{n+1}\Big].\end{aligned}$$

Now,

$$\begin{aligned}I_n &= \int_0^1 \int_0^1 (x-y)^n dxdy \\ &= \frac{1}{n+1}\int_0^1 \Big[(1-y)^{n+1} + (-1)^n y^{n+1}\Big] dy \\ &= \begin{cases} \dfrac{2}{(n+1)(n+2)} & : n \text{ even} \\ 0 & : n \text{ odd}\end{cases}\end{aligned}$$

Solution 2) Using the fact that

$$(x-y)^n = \sum_{k=0}^{n} C_n^k \Big(-1\Big)^k x^{n-k} y^k,$$

we get

$$\begin{aligned}I_n &= \int_0^1 \int_0^1 (x-y)^n dxdy \\ &= \int_0^1 \int_0^1 \sum_{k=0}^{n} C_n^k (-1)^k x^{n-k} y^k dxdy\end{aligned}$$

$$= \sum_{k=0}^{n} C_n^k \, (-1)^k \, \frac{1}{(n-k+1)(k+1)}.$$

Comment: Comparing Solution 1 with Solution 2, we obtain an interesting *side-result:* namely the identity

$$\sum_{k=0}^{n} C_n^k \, (-1)^k \, \frac{1}{(n-k+1)(k+1)} = \begin{cases} \dfrac{2}{(n+1)(n+2)} & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases},$$

which one can verify directly, as well.

Solution 3 by Paul M. Harms, North Newton, KS

Let $f(x,y) = (x-y)^n$. The integration region is the square in the x, y plane with vertices at $(0,0), (1,0), (1,1)$, and $(0,1)$. The line $y = x$ divides this region into two congruent triangles. I will use the terms *lower triangle* and *upper triangle,* for these two congruent triangles.

The points (x,y) and (y,x) are symmetric with respect to the line $y = x$. Let n be an odd integer. For each point (x,y) in the lower (upper) triangle we have a point (y,x) in the upper (lower) triangle such that $f(y,x) = -f(x,y)$. Thus the value of $I_n = 0$ when n is an odd integer.

When n is an even integer, $f(y,x) = f(x,y)$ and the value of the original double integral should equal $2\int_0^1 \int_y^1 (x-y)^n dxdy$ where the region of the integration is the lower triangle. The first integration of the last double integral yields

$$\left. \frac{(x-y)^{n+1}}{n+1} \right|_y^1 = \frac{(1-y)^{n+1}}{n+1}.$$

The second integration of the double integral then yields the expression

$$\left. \frac{-2(1-y)^{n+2}}{(n+1)(n+2)} \right|_0^1 = \frac{2}{(n+1)(n+2)} = I_n$$

when n is an even integer.

Also solved by Brian D. Beasley, Clinton, SC; Michael C. Faleski, University Center, MI; G. C. Greubel, Newport News, VA; David E. Manes, Oneonta, NY; Paolo Perfetti, Department of Mathematics, University "Tor Vergata," Rome, Italy; James Reid (student, Angelo State University), San Angelo, TX; Raúl A. Simón, Santiago, Chile; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.