Chuyªn ®Ò: **BÊt ®¼ng thøc**

T¸c gi¶ : NguyÔn –V¨n –Thñy
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B¸c tÆng ch¸u - chóc ch¸u thµnh c«ng

A- Më ®Çu:

BÊt ®¼ng thøc lµ mét trong nh÷ng m¶ng kiÕn thøc khã nhÊt cña to¸n häc phæ th«ng .
Nhng th«ng qua c¸c bµi tËp vÒ chøng minh bÊt ®¼ng thøc häc sinh hiÓu kü vµ s©u s¾c h¬n vÒ gi¶i vµ biÖn luËn ph¬ng tr×nh , bÊt ph¬ng tr×nh ,vÒ mèi liªn hÖ gi÷a c¸c yÕu tè
cña tam gi¸c vÒ t×m gi¸ trÞ lín nhÊt vµ nhá nhÊt cña mét biÓu thøc. Trong qu¸ tr×nh gi¶i bµi tËp , n¨ng lùc suy nghÜ , s¸ng t¹o cña häc sinh ®îc phat triÓn ®a dang vµ phong phó
v× c¸c bµi tËp vÒ bÊt ®¼ng thøc cã c¸ch gi¶i kh«ng theo quy t¾c hoÆc khu«n mÉu nµo c¶.
Nã ®ßi hái ngêi ®äc ph¶i cã c¸ch suy nghÜ l«gic s¸ng t¹o biÕt kÕt hîp kiÕn thøc cò víi kiÕn thøc míi mét c¸ch l«gÝc cã hÖ thèng.

Còng v× to¸n vÒ bÊt ®¼ng thøc kh«ng cã c¸ch gi¶i mÉu , kh«ng theo mét ph¬ng ph¸p nhÊt ®Þnh nªn häc sinh r©t lóng tóng khi gi¶i to¸n vÒ bÊt ®¼ng thøc v× vËy häc sinh sÏ kh«ng biÕt b¾t ®Çu tõ ®©u vµ ®i theo h¬ng nµo .Do ®ã hÇu hÕt häc sinh kh«ng biÕt lµm to¸n vÒ bÊt ®¼ng thøcvµ kh«ng biÕt vËn dông bÊt ®¼ng thøc ®Ó gi¶i quyÕt c¸c lo¹i bµi tËp kh¸c.

Trong thùc tÕ gi¶ng d¹y to¸n ë trêng THCS viÖc lµm cho häc sinh biÕt chøng minh bÊt ®¼ng thøc vµ vËn dông c¸c bÊt ®¼ng thøc vµo gi¶i c¸c bµi tËp cã liªn quan lµ c«ng viÖc rÊt quan trängvµ kh«ng thÓ thiÕu ®îc cña ngêi d¹y to¸n ,th«ng qua ®ã rÌn luyÖn T duy l«gic vµ kh¶ n¨ng s¸ng t¹o cho häc sinh .§Ó lµm ®îc ®iÒu ®ã ngêi thÇy gi¸o ph¶i cung cÊp cho häc sinh mét sè kiÕn thøc c¬ b¶n vµ mét sè ph¬ng ph¸p suy nghÜ ban ®Çu vÒ bÊt ®¼ng thøc .

ChÝnh v× lÝ do trªn nªn t«i tù tham kh¶o biªn so¹n chuyªn ®Ò bÊt ®¼ng thøc nh»m môc ®Ých gióp häc sinh häc tèt h¬n.

Danh môc cña chuyªn ®Ò

B- néi dung

PhÇn 1 : c¸c kiÕn thøc cÇn lu ý

1- §Þnh nghÜa
2- TÝnh chÊt
3-Mét sè h»ng bÊt ®¼ng thøc hay dïng

PhÇn 2:mét sè ph¬ng ph¸p chøng minh bÊt ®¼ng thøc

1-Ph¬ng ph¸p dïng ®Þnh nghÜa
2- Ph¬ng ph¸p dïng biÕn ®æi t¬ng ®¬ng
3- Ph¬ng ph¸p dïng bÊt ®¼ng thøc quen thuéc
4- Ph¬ng ph¸p sö dông tÝnh chÊt b¾c cÇu
5- Ph¬ng ph¸p dïng tÝnh chÊt tØ sè
6- Ph¬ng ph¸p lµm tréi
7- Ph¬ng ph¸p dïng bÊt ®¼ng thøc trong tam gi¸c
8- Ph¬ng ph¸p ®æi biÕn sè
9- Ph¬ng ph¸p dïng tam thøc bËc hai
10- Ph¬ng ph¸p quy n¹p
11- Ph¬ng ph¸p ph¶n chøng

PhÇn 3 :c¸c bµi tËp n©ng cao

PHÇN 4 : øng dông cña bÊt ®¼ng thøc

1- Dïng bÊt ®¼ng thøc ®Ó t×m cùc trÞ
2-Dïng bÊt ®¼ng thøc ®Ó gi¶i ph¬ng tr×nh vµ bÊt ph¬ng tr×nh
3-Dïng bÊt ®¼ng thøc gi¶i ph¬ng tr×nh nghiÖm nguyªn

PhÇn I : c¸c kiÕn thøc cÇn lu ý

1-§inhnghÜa

$$\begin{cases} A \geq B \Leftrightarrow A - B \geq 0 \\ A \leq B \Leftrightarrow A - B \leq 0 \end{cases}$$

2-<u>tÝnh chÊt</u>

+ A>B $\Leftrightarrow B < A$
+ A>B vµ B >C $\Leftrightarrow A > C$
+ A>B $\Rightarrow$ A+C >B + C
+ A>B vµ C > D $\Rightarrow$ A+C > B + D
+ A>B vµ C > 0 $\Rightarrow$ A.C > B.C
+ A>B vµ C < 0 $\Rightarrow$ A.C < B.C
+ 0 < A < B vµ 0 < C <D $\Rightarrow$ 0 < A.C < B.D
+ A > B > 0 $\Rightarrow$ $A^n > B^n$ $\forall n$
+ A > B $\Rightarrow$ $A^n > B^n$ víi n lÎ
+ $|A| > |B|$ $\Rightarrow$ $A^n > B^n$ víi n ch½n
+ m > n > 0 vµ A > 1 $\Rightarrow$ $A^m > A^n$
+ m > n > 0 vµ 0 <A < 1 $\Rightarrow$ $A^m < A^n$
+A < B vµ A.B > 0 $\Rightarrow$ $\frac{1}{A} > \frac{1}{B}$

3-<u>mét sè h»ng bÊt ®¼ng thøc</u>

-

+ $A^2 \geq 0$ víi $\forall$A (dÊu = x¶y ra khi A = 0)
+ $A^n \geq 0$ víi $\forall$A (dÊu = x¶y ra khi A = 0)
+ $|A| \geq 0$ víi $\forall A$ (dÊu = x¶y ra khi A = 0)
+ $-|A| < A = |A|$
+ $|A + B| \geq |A| + |B|$ (dÊu = x¶y ra khi A.B > 0)
+ $|A - B| \leq |A| - |B|$ (dÊu = x¶y ra khi A.B < 0)

<u>PhÇn II</u> : mét sè ph¬ng ph¸p chøng minh bÊt ®¼ng thøc

<u>Ph¬ng ph¸p</u> 1 : dïng ®Þnh nghÜa

<u>KiÕn thøc</u> : §Ó chøng minh A > B
Ta chøng minh A –B > 0
Lu ý dïng h»ng bÊt ®¼ng thøc $M^2 \geq 0$ víi $\forall$ M

<u>VÝ dô 1</u> $\forall$ x, y, z chøng minh r»ng :
a) $x^2 + y^2 + z^2 \geq xy + yz + zx$
b) $x^2 + y^2 + z^2 \geq 2xy - 2xz + 2yz$
c) $x^2 + y^2 + z^2 + 3 \geq 2(x + y + z)$

<u>Gi¶i:</u>

a) Ta xÐt hiÖu
$x^2 + y^2 + z^2 - xy - yz - zx$

$=\frac{1}{2}.2.(x^2+y^2+z^2-xy-yz-zx)$

$=\frac{1}{2}\left[(x-y)^2+(x-z)^2+(y-z)^2\right]\geq 0$ ®óng víi mäi x;y;z$\in R$

V× $(x-y)^2\geq 0$ víi $\forall x$; y DÊu b»ng x¶y ra khi x=y

$(x-z)^2\geq 0$ víi $\forall x$; z DÊu b»ng x¶y ra khi x=z

$(y-z)^2\geq 0$ víi $\forall$ z; y DÊu b»ng x¶y ra khi z=y

VËy $x^2+y^2+z^2\geq xy+yz+zx$

DÊu b»ng x¶y ra khi x = y =z

b)Ta xÐt hiÖu

$x^2+y^2+z^2-(2xy-2xz+2yz)$

$=x^2+y^2+z^2-2xy+2xz-2yz$

$=(x-y+z)^2\geq 0$ ®óng víi mäi x;y;z$\in R$

VËy $x^2+y^2+z^2\geq 2xy-2xz+2yz$ ®óng víi mäi x;y;z$\in R$

DÊu b»ng x¶y ra khi x+y=z

c) Ta xÐt hiÖu

$x^2+y^2+z^2+3-2(x+y+z)$

$=x^2-2x+1+y^2-2y+1+z^2-2z+1$

$=(x-1)^2+(y-1)^2+(z-1)^2\geq 0$

DÊu(=)x¶y ra khi x=y=z=1

VÝ dô 2: chøng minh r»ng :

a) $\frac{a^2+b^2}{2}\geq\left(\frac{a+b}{2}\right)^2$;b) $\frac{a^2+b^2+c^2}{3}\geq\left(\frac{a+b+c}{3}\right)^2$

c) H·y tæng qu¸t bµi to¸n

gi¶i

a) Ta xÐt hiÖu $\frac{a^2+b^2}{2}-\left(\frac{a+b}{2}\right)^2$

$=\frac{2(a^2+b^2)}{4}-\frac{a^2+2ab+b^2}{4}$

$=\frac{1}{4}(2a^2+2b^2-a^2-b^2-2ab)$

$=\frac{1}{4}(a-b)^2\geq 0$

VËy $\frac{a^2+b^2}{2}\geq\left(\frac{a+b}{2}\right)^2$

DÊu b»ng x¶y ra khi a=b

b)Ta xÐt hiÖu

$\frac{a^2+b^2+c^2}{3}-\left(\frac{a+b+c}{3}\right)^2$

$=\frac{1}{9}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\geq 0$

VËy $\frac{a^2+b^2+c^2}{3}\geq\left(\frac{a+b+c}{3}\right)^2$

DÊu b»ng x¶y ra khi a = b =c

c)Tæng qu¸t

$$\frac{a_1^2+a_2^2+....+a_n^2}{n}\ge\left(\frac{a_1+a_2+....+a_n}{n}\right)^2$$

Tãm l¹i c¸c bíc ®Ó chøng minh A≥B tho ®Þnh nghÜa

Bíc 1: Ta xÐt hiÖu H = A - B

Bíc 2:BiÕn ®æi H=(C+D)^2 hoÆc H=(C+D)^2+….+(E+F)^2

Bíc 3:KÕt luËn A ≥ B

VÝ dô:(chuyªn Nga- Ph¸p 98-99)

Chøng minh ∀m,n,p,q ta ®Òu cã

$$m^2+n^2+p^2+q^2+1\ge m(n+p+q+1)$$

Gi¶i:

$$\Leftrightarrow\left(\frac{m^2}{4}-mn+n^2\right)+\left(\frac{m^2}{4}-mp+p^2\right)+\left(\frac{m^2}{4}-mq+q^2\right)+\left(\frac{m^2}{4}-m+1\right)\ge 0$$

$\Leftrightarrow\left(\frac{m}{2}-n\right)^2+\left(\frac{m}{2}-p\right)^2+\left(\frac{m}{2}-q\right)^2+\left(\frac{m}{2}-1\right)^2\ge 0$ (lu«n ®óng)

DÊu b»ng x¶y ra khi $\begin{cases}\frac{m}{2}-n=0\\ \frac{m}{2}-p=0\\ \frac{m}{2}-q=0\\ \frac{m}{2}-1=0\end{cases}\Leftrightarrow\begin{cases}n=\frac{m}{2}\\ p=\frac{m}{2}\\ q=\frac{m}{2}\\ m=2\end{cases}\Leftrightarrow\begin{cases}m=2\\ n=p=q=1\end{cases}$

Bµi tËp bæ xung

ph¬ng ph¸p 2 : Dïng phÐp biÕn ®æi t¬ng ®¬ng

Lu ý:

Ta biÕn ®æi bÊt ®¼ng thøc cÇn chøng minh t¬ng ®¬ng víi bÊt ®¼ng thøc ®óng hoÆc bÊt ®¼ng thøc ®· ®îc chøng minh lµ ®óng.

Chó ý c¸c h»ng ®¼ng thøc sau:

$$(A+B)^2=A^2+2AB+B^2$$

$$(A+B+C)^2=A^2+B^2+C^2+2AB+2AC+2BC$$

$$(A+B)^3=A^3+3A^2B+3AB^2+B^3$$

VÝ dô 1:

Cho a, b, c, d,e lµ c¸c sè thùc chøng minh r»ng

a) $a^2+\frac{b^2}{4}\ge ab$

b) $a^2+b^2+1\ge ab+a+b$

c) $a^2+b^2+c^2+d^2+e^2\ge a(b+c+d+e)$

Gi¶i:

a) $a^2 + \frac{b^2}{4} \ge ab$

$\Leftrightarrow 4a^2 + b^2 \ge 4ab \Leftrightarrow 4a^2 - 4a + b^2 \ge 0$

$\Leftrightarrow (2a - b)^2 \ge 0$ (bÊt ®¼ng thøc nµy lu«n ®óng)

VËy $a^2 + \frac{b^2}{4} \ge ab$ (dÊu b»ng x¶y ra khi 2a=b)

b) $a^2 + b^2 + 1 \ge ab + a + b$

$\Leftrightarrow 2(a^2 + b^2 + 1) > 2(ab + a + b)$

$\Leftrightarrow a^2 - 2ab + b^2 + a^2 - 2a + 1 + b^2 - 2b + 1 \ge 0$

$\Leftrightarrow (a - b)^2 + (a - 1)^2 + (b - 1)^2 \ge 0$ BÊt ®¼ng thøc cuèi ®óng.

VËy $a^2 + b^2 + 1 \ge ab + a + b$

DÊu b»ng x¶y ra khi a=b=1

c) $a^2 + b^2 + c^2 + d^2 + e^2 \ge a(b + c + d + e)$

$\Leftrightarrow 4(a^2 + b^2 + c^2 + d^2 + e^2) \ge 4a(b + c + d + e)$

$\Leftrightarrow (a^2 - 4ab + 4b^2) + (a^2 - 4ac + 4c^2) + (a^2 - 4ad + 4d^2) + (a^2 - 4ac + 4c^2) \ge 0$

$\Leftrightarrow (a - 2b)^2 + (a - 2c)^2 + (a - 2d)^2 + (a - 2c)^2 \ge 0$

BÊt ®¼ng thøc ®óng vËy ta cã ®iÒu ph¶i chøng minh

VÝ dô 2:

Chøng minh r»ng: $(a^{10} + b^{10})(a^2 + b^2) \ge (a^8 + b^8)(a^4 + b^4)$

Gi¶i:

$(a^{10} + b^{10})(a^2 + b^2) \ge (a^8 + b^8)(a^4 + b^4) \Leftrightarrow a^{12} + a^{10}b^2 + a^2b^{10} + b^{12} \ge a^{12} + a^8b^4 + a^4b^8 + b^{12}$

$\Leftrightarrow a^8b^2(a^2 - b^2) + a^2b^8(b^2 - a^2) \ge 0$

$\Leftrightarrow a^2b^2(a^2-b^2)(a^6-b^6) \ge 0 \Leftrightarrow a^2b^2(a^2-b^2)^2(a^4+a^2b^2+b^4) \ge 0$

BÊt ®¼ng thøccuèi ®óng vËy ta cã ®iÒu ph¶i chøng minh

VÝ dô 3: cho x.y =1 vµ x.y

Chøng minh $\frac{x^2 + y^2}{x - y} \ge 2\sqrt{2}$

Gi¶i:

$\frac{x^2 + y^2}{x - y} \ge 2\sqrt{2}$ v× :x〉y nªn x- y 〉 0 $\Rightarrow x^2+y^2 \ge 2\sqrt{2}(x-y)$

$\Rightarrow x^2+y^2 - 2\sqrt{2}x + 2\sqrt{2}y \ge 0 \Leftrightarrow x^2+y^2+2 - 2\sqrt{2}x + 2\sqrt{2}y - 2 \ge 0$

$\Leftrightarrow x^2+y^2+(\sqrt{2})^2 - 2\sqrt{2}x + 2\sqrt{2}y - 2xy \ge 0$ v× x.y=1 nªn 2.x.y=2

$\Rightarrow (x-y-\sqrt{2})^2 \ge 0$ §iÒu nµy lu«n lu«n ®óng . VËy ta cã ®iÒu ph¶i chøng minh

VÝ dô 4:

1)CM: P(x,y)= $9x^2y^2 + y^2 - 6xy - 2y + 1 \ge 0 \quad \forall x, y \in R$

2)CM: $\sqrt{a^2 + b^2 + c^2} \le |a| + |b| + |c|$ (gîi ý :b×nh ph¬ng 2 vÕ)

3)choba sè thùc kh¸c kh«ng x, y, z tháa m·n:

$$\begin{cases} x.y.z = 1 \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < x + y + z \end{cases}$$

Chøng minh r»ng :cã ®óng mét trong ba sè x,y,z lín h¬n 1

(®Ò thi Lam S¬n 96-97)

Gi¶i:

XÐt (x-1)(y-1)(z-1)=xyz+(xy+yz+zx)+x+y+z-1

=(xyz-1)+(x+y+z)-xyz($\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$)=x+y+z - ($\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$) > 0 (v× $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} <$ x+y+z theo gt)

$\rightarrow$ 2 trong 3 sè x-1 , y-1 , z-1 ©m hoÆc c¶ ba sç-1 , y-1, z-1 lµ d¬ng.

NÕñ trêng hîp sau x¶y ra th× x, y, z >1 $\Rightarrow$ x.y.z>1 M©u thuÉn gt x.y.z=1 b¾t buéc ph¶i x¶y ra trêng hîp trªn tøc lµ cã ®óng 1 trong ba sè x ,y ,z lµ sè lín h¬n 1

Ph¬ng ph¸p 3: dïng bÊt ®¼ng thøc quen thuéc

A/ mét sè bÊt ®¼ng thøc hay dïng

1) C¸c bÊt ®¼ng thøc phô:

a) $x^2+y^2 \ge 2xy$

b) $x^2+y^2 \ge |xy|$ dÊu(=) khi x = y = 0

c) $(x+y)^2 \ge 4xy$

d) $\frac{a}{b}+\frac{b}{a} \ge 2$

2)BÊt ®¼ng thøc C« sy: $\frac{a_1+a_2+a_3+....+a_n}{n} \ge \sqrt[n]{a_1a_2a_3....a_n}$ Víi $a_i>0$

3)BÊt ®¼ng thøc Bunhiacopski

$$\left(a_1^2+a_2^2+....+a_n^2\right)\left(x_1^2+x_2^2+....+x_n^2\right) \ge \left(a_1x_1+a_2x_2+....+a_nx_n\right)^2$$

4) BÊt ®¼ng thøc Trª- b-sÐp:

NÕu $\begin{cases} a \le b \le c \\ A \le B \le C \end{cases} \Rightarrow \frac{aA+bB+cC}{3} \ge \frac{a+b+c}{3}.\frac{A+B+C}{3}$

NÕu $\begin{cases} a \le b \le c \\ A \ge B \ge C \end{cases} \Rightarrow \frac{aA+bB+cC}{3} \le \frac{a+b+c}{3}.\frac{A+B+C}{3}$

DÊu b»ng x¶y ra khi $\begin{cases} a=b=c \\ A=B=C \end{cases}$

b/ c¸c vÝ dô

vÝ dô 1 Cho a, b ,c lµ c¸c sè kh«ng ©m chøng minh r»ng

(a+b)(b+c)(c+a)$\ge$8abc

Gi¶i:

C¸ch 1:Dïng bÊt ®¼ng thøc phô: $(x+y)^2 \ge 4xy$

Tacã $(a+b)^2 \ge 4ab$; $(b+c)^2 \ge 4bc$; $(c+a)^2 \ge 4ac$

$\Rightarrow (a+b)^2(b+c)^2(c+a)^2 \ge 64a^2b^2c^2 = (8abc)^2$

$\Rightarrow$ (a+b)(b+c)(c+a)$\ge$8abc

DÊu "=" x¶y ra khi a = b = c

vÝ dô 2(tù gi¶i): 1)Cho a,b,c>0 vµ a+b+c=1 CMR: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 9$ (403-1001)

2)Cho x,y,z>0 vµ x+y+z=1 CMR:x+2y+z $\ge 4(1-x)(1-y)(1-z)$

3)Cho a>0 , b>0, c>0

CMR: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}$

4)Cho x≥ 0,y≥ 0 tháa m·n $2\sqrt{x}-\sqrt{y}=1$;CMR: x+y$\ge \frac{1}{5}$

vÝ dô 3: Cho a>b>c>0 vµ $a^2+b^2+c^2=1$ chøng minh r»ng

$$\frac{a^3}{b+c}+\frac{b^3}{a+c}+\frac{c^3}{a+b} \ge \frac{1}{2}$$

Gi¶i:

Do a,b,c ®èi xøng ,gi¶ sö a$\ge$b$\ge$c $\Rightarrow$ $\begin{cases} a^2 \ge b^2 \ge c^2 \\ \dfrac{a}{b+c} \ge \dfrac{b}{a+c} \ge \dfrac{c}{a+b} \end{cases}$

¸p dông B§T Trª- b-sÐp ta cã

$$a^2.\frac{a}{b+c}+b^2.\frac{b}{a+c}+c^2.\frac{c}{a+b} \ge \frac{a^2+b^2+c^2}{3}.\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right) = \frac{1}{3}.\frac{3}{2} = \frac{1}{2}$$

VËy $\dfrac{a^3}{b+c}+\dfrac{b^3}{a+c}+\dfrac{c^3}{a+b} \ge \dfrac{1}{2}$ DÊu b»ng x¶y ra khi a=b=c=$\dfrac{1}{\sqrt{3}}$

vÝ dô 4:

Cho a,b,c,d>0 vµ abcd =1 .Chøng minh r»ng :

$$a^2+b^2+c^2+d^2+a(b+c)+b(c+d)+d(c+a) \ge 10$$

Gi¶i:

Ta cã $a^2+b^2 \ge 2ab$

$c^2+d^2 \ge 2cd$

Do abcd =1 nªn cd =$\dfrac{1}{ab}$ (dïng $x+\dfrac{1}{x} \ge \dfrac{1}{2}$)

Ta cã $a^2+b^2+c^2 \ge 2(ab+cd) = 2(ab+\dfrac{1}{ab}) \ge 4$ (1)

MÆt kh¸c: $a(b+c)+b(c+d)+d(c+a)$

=(ab+cd)+(ac+bd)+(bc+ad)

$= \left(ab+\dfrac{1}{ab}\right)+\left(ac+\dfrac{1}{ac}\right)+\left(bc+\dfrac{1}{bc}\right) \ge 2+2+2$

VËy $a^2+b^2+c^2+d^2+a(b+c)+b(c+d)+d(c+a) \ge 10$

vÝ dô 5: Cho 4 sè a,b,c,d bÊt kú chøng minh r»ng:

$$\sqrt{(a+c)^2+(b+d)^2} \le \sqrt{a^2+b^2}+\sqrt{c^2+d^2}$$

Gi¶i: Dïng bÊt ®¼ng thøc Bunhiacopski

tacã ac+bd$\le \sqrt{a^2+b^2}.\sqrt{c^2+d^2}$

mµ $(a+c)^2+(b+d)^2 = a^2+b^2+2(ac+bd)+c^2+d^2$

$\le (a^2+b^2)+2\sqrt{a^2+b^2}.\sqrt{c^2+d^2}+c^2+d^2$

$\Rightarrow \sqrt{(a+c)^2+(b+d)^2} \le \sqrt{a^2+b^2}+\sqrt{c^2+d^2}$

vÝ dô 6: Chøng minh r»ng

$$a^2+b^2+c^2 \ge ab+bc+ac$$

Gi¶i: Dïng bÊt ®¼ng thøc Bunhiacopski

C¸ch 1: XÐt cÆp sè (1,1,1) vµ (a,b,c) ta cã

$$(1^2+1^2+1^2)(a^2+b^2+c^2) \ge (1.a+1.b+1.c)^2$$

$\Rightarrow$ $3(a^2+b^2+c^2) \ge a^2+b^2+c^2+2(ab+bc+ac)$

$\Rightarrow$ $a^2+b^2+c^2 \ge ab+bc+ac$ §iÒu ph¶i chøng minh DÊu b»ng x¶y ra khi a=b=c

Ph¬ng ph¸p 4: Sö dông tÝnh chÊt b¾c cÇu

Lu ý: A>B vµ b>c th× A>c

0< x <1 th× x^2<x

vÝ dô 1:

Cho a, b, c ,d >0 tháa m·n a> c+d , b>c+d

Chøng minh r»ng ab >ad+bc

Gi¶i:

Tacã $\begin{cases} a > c + d \\ b > c + d \end{cases}$ $\Rightarrow$ $\begin{cases} a - c > d > 0 \\ b - d > c > 0 \end{cases}$

$\Rightarrow$ (a-c)(b-d) > cd

$\Leftrightarrow$ ab-ad-bc+cd >cd

$\Leftrightarrow$ ab> ad+bc (®iÒu ph¶i chøng minh)

vÝ dô 2:

Cho a,b,c>0 tháa m·n $a^2 + b^2 + c^2 = \frac{5}{3}$

Chøng minh $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{abc}$

Gi¶i:

Ta cã :(a+b- c)^2= a^2+b^2+c^2+2(ab –ac – bc) 〉 0

$\Rightarrow$ ac+bc-ab $<$ $\frac{1}{2}$(a^2+b^2+c^2)

$\Rightarrow$ ac+bc-ab $\le \frac{5}{6} <$ 1 Chia hai vÕ cho abc > 0 ta cã $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} < \frac{1}{abc}$

vÝ dô 3

Cho 0 < a,b,c,d <1 Chøng minh r»ng (1-a).(1-b) (1-c).(1-d) > 1-a-b-c-d

Gi¶i:

Ta cã (1-a).(1-b) = 1-a-b+ab

Do a>0 , b>0 nªn ab>0

$\Rightarrow$ (1-a).(1-b) > 1-a-b (1)

Do c <1 nªn 1- c >0 ta cã

$\Rightarrow$(1-a).(1-b) (1-c) > 1-a-b-c

$\Rightarrow$ (1-a).(1-b) (1-c).(1-d) > (1-a-b-c) (1-d)

=1-a-b-c-d+ad+bd+cd

$\Rightarrow$ (1-a).(1-b) (1-c).(1-d) > 1-a-b-c-d

(§iÒu ph¶i chøng minh)

vÝ dô 4

1- Cho 0 <a,b,c <1 . Chøng minh r»ng

$$2a^3 + 2b^3 + 2c^3 < 3 + a^2b + b^2c + c^2a$$

Gi¶i :

Do a < 1 $\Rightarrow$ $a^2 < 1$ vµ

Ta cã $(1 - a^2)(1 - b) < 0$ $\Rightarrow$ 1-b-a^2+a^2b > 0

$\Rightarrow$ 1+a^2b^2 > a^2 + b

mµ 0< a,b <1 $\Rightarrow$ a^2 > a^3, b^2 > b^3

Tõ (1) vµ (2) $\Rightarrow$ 1+a^2b^2 > $a^3 + b^3$

VËy $a^3 + b^3$ < 1+a^2b^2

T¬ng tù $b^3 + c^3 \le 1 + b^2c$

$c^3 + a^3 \le 1 + c^2a$

Céng c¸c bÊt ®¼ng thøc ta cã :

$$2a^3 + 2b^3 + 2c^3 \le 3 + a^2b + b^2c + c^2a$$

b)Chøng minh r»ng : NÕu $a^2 + b^2 = c^2 + d^2 = 1998$ th× $|$ac+bd$|$=1998

(Chuyªn Anh –98 – 99)

Gi¶i:

Ta cã $(ac + bd)^2 + (ad – bc)^2 = a^2c^2 + b^2d^2 + 2abcd + a^2d^2 + b^2c^2 - 2abcd =$
$= a^2(c^2+d^2)+b^2(c^2+d^2) =(c^2+d^2).(a^2+ b^2) = 1998^2$

rá rµng $(ac+bd)^2 \leq (ac+bd)^2 + (ad-bc)^2 = 1998^2$

$\Rightarrow |ac+bd| \leq 1998$

2-<u>Bµi tËp</u> : 1, Cho c¸c sè thùc : $a_1; a_2; a_3 \ldots; a_{2003}$ tháa m·n : $a_1 + a_2 + a_3 + \ldots + a_{2003} = 1$

c høng minh r»ng : $a_1^2 + a_2^2 + a_3^2 + \ldots + a_{2003}^2 \geq \frac{1}{2003}$ (®Ò thi vµo chuyªn nga ph¸p 2003- 2004Thanh hãa)

2,Cho a;b;c ≥ 0 tháa m·n :a+b+c=1(?)

Chøng minh r»ng: $\left(\frac{1}{a} - 1\right).\left(\frac{1}{b} - 1\right).\left(\frac{1}{c} - 1\right) \geq 8$

<u>Ph¬ng ph¸p 5:</u> dïng tÝnh chÊtcña tû sè

<u>KiÕn thøc</u>

1) Cho a, b ,c lµ c¸c sè d¬ng th×

a – NÕu $\frac{a}{b} > 1$ th× $\frac{a}{b} > \frac{a+c}{b+c}$

b – NÕu $\frac{a}{b} < 1$ th× $\frac{a}{b} < \frac{a+c}{b+c}$

2)NÕu b,d >0 th× tõ

$$\frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

`

<u>vÝ dô 1</u> :

Cho a,b,c,d > 0 .Chøng minh r»ng

$$1 < \frac{a}{a+b+c} + \frac{b}{b+c+d} + \frac{c}{c+d+a} + \frac{d}{d+a+b} < 2$$

<u>Gi¶i</u> :

Theo tÝnh chÊt cña tØ lÖ thøc ta cã

$$\frac{a}{a+b+c}<1 \Rightarrow \frac{a}{a+b+c}<\frac{a+d}{a+b+c+d} \quad (1)$$

MÆt kh¸c : $\frac{a}{a+b+c}>\frac{a}{a+b+c+d}$ (2)

Tõ (1) vµ (2) ta cã

$$\frac{a}{a+b+c+d}<\frac{a}{a+b+c}<\frac{a+d}{a+b+c+d} \quad (3)$$

T¬ng tù ta cã

$$\frac{b}{a+b+c+d}<\frac{b}{b+c+d}<\frac{b+a}{a+b+c+d} \quad (4)$$

$$\frac{c}{a+b+c+d}<\frac{c}{c+d+a}<\frac{b+c}{a+b+c+d} \quad (5)$$

$$\frac{d}{a+b+c+d}<\frac{d}{d+a+b}<\frac{d+c}{a+b+c+d} \quad (6)$$

céng vÕ víi vÕ cña (3); (4); (5); (6) ta cã

$1<\frac{a}{a+b+c}+\frac{b}{b+c+d}+\frac{c}{c+d+a}+\frac{d}{d+a+b}<2$ ®iÒu ph¶i chøng minh

<u>vÝ dô 2</u> :

Cho: $\frac{a}{b}<\frac{c}{d}$ vµ b,d > 0 .Chøng minh r»ng $\frac{a}{b}<\frac{ab+cd}{b^2+d^2}<\frac{c}{d}$

<u>Gi¶i:</u> Tõ $\frac{a}{b}<\frac{c}{d} \Rightarrow \frac{ab}{b^2}<\frac{cd}{d^2} \Rightarrow \frac{ab}{b^2}<\frac{ab+cd}{b^2+d^2}<\frac{cd}{d^2}=\frac{c}{d}$

VËy $\frac{a}{b}<\frac{ab+cd}{b^2+d^2}<\frac{c}{d}$ ®iÒu ph¶i chøng minh

<u>vÝ dô 3</u> : Cho a;b;c;dlµ c¸c sè nguyªn d¬ng tháa m·n : a+b = c+d =1000

t×m gi¸ trÞ lín nhÊt cña $\frac{a}{c}+\frac{b}{d}$

<u>gi¶i :</u> Kh«ng mÊt tÝnh tæng qu¸t ta gi¶ sö : $\frac{a}{c}\le\frac{b}{d}$ Tõ : $\frac{a}{c}\le\frac{b}{d} \Rightarrow \frac{a}{c}\le\frac{a+b}{c+d}\le\frac{b}{d}$

$\frac{a}{c}\le 1$ v× a+b = c+d

a, NÕu :b ≤ 998 th× $\frac{b}{d}\le 998 \Rightarrow \frac{a}{c}+\frac{b}{d}\le 999$

b, NÕu: b=998 th× a=1 $\Rightarrow \frac{a}{c}+\frac{b}{d}=\frac{1}{c}+\frac{999}{d}$ §¹t gi¸ trÞ lín nhÊt khi d= 1; c=999

VËy gi¸ trÞ lín nhÊt cña $\frac{a}{c}+\frac{b}{d}=999+\frac{1}{999}$ khi a=d=1; c=b=999

Ph¬ng ph¸p 6: Ph¬ng ph¸plµm tréi

Lu ý:

Dïng c¸c tÝnh bÊt ®¼ng thøc ®Ó ®a mét vÕ cña bÊt ®¼ng thøc vÒ d¹ng tÝnh ®îc tæng h÷u h¹n hoÆc tÝch h÷u h¹n.

(*) Ph¬ng ph¸p chung ®Ó tÝnh tæng h÷u h¹n :

S = $u_1 + u_2 + ... + u_n$

Ta cè g¾ng biÕn ®æi sè h¹ng tæng qu¸t u_k vÒ hiÖu cña hai sè h¹ng liªn tiÕp nhau:

$u_k = a_k - a_{k+1}$

Khi ®ã :

S = $(a_1 - a_2) + (a_2 - a_3) + ... + (a_n - a_{n+1}) = a_1 - a_{n+1}$

(*) Ph¬ng ph¸p chung vÒ tÝnh tÝch h÷u h¹n

P = $u_1 u_2 .... u_n$

BiÕn ®æi c¸c sè h¹ng u_k vÒ th¬ng cña hai sè h¹ng liªn tiÕp nhau:

$$u_k = \frac{a_k}{a_{k+1}}$$

Khi ®ã P = $\frac{a_1}{a_2} \cdot \frac{a_2}{a_3} ..... \frac{a_n}{a_{n+1}} = \frac{a_1}{a_{n+1}}$

VÝ dô 1 :

Víi mäi sè tù nhiªn n >1 chøng minh r»ng

$$\frac{1}{2} < \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+n} < \frac{3}{4}$$

Gi¶i:

Ta cã $\frac{1}{n+k} > \frac{1}{n+n} = \frac{1}{2n}$ víi k = 1,2,3,…,n-1

Do ®ã:

$$\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}>\frac{1}{2n}+...+\frac{1}{2n}=\frac{n}{2n}=\frac{1}{2}$$

VÝ dô 2 :

Chøng minh r»ng:

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+....+\frac{1}{\sqrt{n}}>2\left(\sqrt{n+1}-1\right)$$ Víi n lµ sè nguyªn

Gi¶i :

Ta cã $\frac{1}{\sqrt{k}}=\frac{2}{2\sqrt{k}}>\frac{2}{\sqrt{k}+\sqrt{k+1}}=2\left(\sqrt{k+1}-\sqrt{k}\right)$

Khi cho k ch¹y tõ 1 ®Õn n ta cã

$1>2\left(\sqrt{2}-1\right)$

$\frac{1}{\sqrt{2}}>2\left(\sqrt{3}-\sqrt{2}\right)$

....

$\frac{1}{\sqrt{n}}>2\left(\sqrt{n+1}-\sqrt{n}\right)$

Céng tõng vÕ c¸c bÊt ®¼ng thøc trªn ta cã

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+....+\frac{1}{\sqrt{n}}>2\left(\sqrt{n+1}-1\right)$$

VÝ dô 3 :

Chøng minh r»ng $\sum_{k=1}^{n}\frac{1}{k^2}<2$ $\forall n\in Z$

Gi¶i:

Ta cã $\frac{1}{k^2}<\frac{1}{k(k-1)}=\frac{1}{k-1}-\frac{1}{k}$

Cho k ch¹y tõ 2 ®Õn n ta cã

$\frac{1}{2^2}<1-\frac{1}{2}$

$\frac{1}{3^2}<\frac{1}{2}-\frac{1}{3}$

....

$\frac{1}{n^2}<\frac{1}{n-1}-\frac{1}{n}$

$\Rightarrow\frac{1}{2^2}+\frac{1}{3^2}+....+\frac{1}{n^2}<1$

VËy $\sum_{k=1}^{n}\frac{1}{k^2}<2$

Ph¬ng ph¸p 7:

Dïng bÊt ®¼ng thøc trong tam gi¸c

Lu ý: NÕu a;b;clµ sè ®o ba c¹nh cña tam gi¸c th× : a;b;c> 0

Vµ |b-c| < a < b+c ; |a-c| < b < a+c ; |a-b| < c < b+a

VÝ dô1: Cho a;b;clµ sè ®o ba c¹nh cña tam gi¸c chøng minh r»ng

a, $a^2+b^2+c^2< 2(ab+bc+ac)$

b, abc>(a+b-c).(b+c-a).(c+a-b)

Gi¶i

a)V× a,b,c lµ sè ®o 3 c¹nh cña mét tam gi¸c nªn ta cã

$$\begin{cases} 0 < a < b+c \\ 0 < b < a+c \\ 0 < c < a+b \end{cases} \Rightarrow \begin{cases} a^2 < a(b+c) \\ b^2 < b(a+c) \\ c^2 < c(a+b) \end{cases}$$

Céng tõng vÕ c¸c bÊt ®¼ng thøc trªn ta cã

$$a^2+b^2+c^2< 2(ab+bc+ac)$$

b) Ta cã a > |b-c | $\Rightarrow a^2 > a^2 - (b-c)^2 > 0$

b > |a-c | $\Rightarrow b^2 > b^2 - (c-a)^2 > 0$

c > |a-b | $\Rightarrow c^2 > c^2 - (a-b)^2 > 0$

Nh©n vÕ c¸c bÊt ®¼ng thøc ta ®îc

$$\Rightarrow a^2b^2c^2 > \left[a^2 - (b-c)^2\right]\left[b^2 - (c-a)^2\right]\left[c^2 - (a-b)^2\right]$$

$$\Rightarrow a^2b^2c^2 > (a+b-c)^2(b+c-a)^2(c+a-b)^2$$

$$\Rightarrow abc > (a+b-c).(b+c-a).(c+a-b)$$

VÝ dô2: (404 – 1001)

1) Cho a,b,c lµ chiÒu dµi ba c¹nh cña tam gi¸c

Chøng minh r»ng $ab+bc+ca < a^2+b^2+c^2 < 2(ab+bc+ca)$

2) Cho a,b,c lµ chiÒu dµi ba c¹nh cña tam gi¸c cã chu vi b»ng 2

Chøng minh r»ng $a^2+b^2+c^2+2abc < 2$

Ph¬ng ph¸p 8: ®æi biÕn sè

VÝ dô1:

Cho a,b,c > 0 Chøng minh r»ng $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$ (1)

Gi¶i :

§Æt x=b+c ; y=c+a ;z= a+b ta cã a= $\frac{y+z-x}{2}$; b = $\frac{z+x-y}{2}$; c = $\frac{x+y-z}{2}$

ta cã (1) $\Leftrightarrow$ $\frac{y+z-x}{2x}+\frac{z+x-y}{2y}+\frac{x+y-z}{2z}\geq\frac{3}{2}$

$\Leftrightarrow$ $\frac{y}{x}+\frac{z}{x}-1+\frac{x}{y}+\frac{z}{y}-1+\frac{x}{z}+\frac{y}{z}-1\geq 3$

$\Leftrightarrow$ ($\frac{y}{x}+\frac{x}{y})+(\frac{z}{x}+\frac{x}{z})+(\frac{z}{y}+\frac{y}{z})\geq 6$

BÊt ®¼ng thøc cuèi cïng ®óng v× ($\frac{y}{x}+\frac{x}{y}\geq 2;$ $\frac{z}{x}+\frac{x}{z}\geq 2$; $\frac{z}{y}+\frac{y}{z}\geq 2$ nªn ta cã ®iÒu ph¶i chøng minh

VÝ dô2:

Cho a,b,c > 0 vµ a+b+c <1

Chøng minh r»ng

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ac}+\frac{1}{c^2+2ab}\geq 9 \qquad (1)$$

Gi¶i:

§Æt x = a^2+2bc ; y = b^2+2ac ; z = c^2+2ab

Ta cã $x+y+z=(a+b+c)^2<1$

(1) $\Leftrightarrow$ $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 9$ Víi x+y+z < 1 vµ x ,y,z > 0

Theo bÊt ®¼ng thøc C«si ta cã

$x+y+z\geq 3.\sqrt[3]{xyz}$

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 3.\,.\sqrt[3]{\frac{1}{xyz}}$

$\Rightarrow$ $(x+y+z).\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9$

Mµ x+y+z < 1

VËy $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 9$ (®pcm)

VÝ dô3:

Cho x ≥ 0 , y ≥ 0 tháa m·n $2\sqrt{x} - \sqrt{y} = 1$ CMR $x + y \geq \frac{1}{5}$

Gîi ý:
§Æt $\sqrt{x} = u$, $\sqrt{y} = v$ $\Rightarrow$ 2u-v =1 vµ S = x+y = $u^2 + v^2 \Rightarrow$ v = 2u-1 thay vµo tÝnh S min

Bµi tËp

1) Cho a > 0 , b > 0 , c > 0 CMR: $\frac{25a}{b+c} + \frac{16b}{c+a} + \frac{c}{a+b} > 8$

2)Tæng qu¸t m, n, p, q, a, b >0
CMR
$$\frac{ma}{b+c} + \frac{nb}{c+a} + \frac{pc}{a+b} \geq \frac{1}{2}\left(\sqrt{m} + \sqrt{n} + \sqrt{p}\right)^2 - (m + n + p)$$

Ph¬ng ph¸p 9: dïng tam thøc bËc hai

Lu ý :

Cho tam thøc bËc hai $f(x) = ax^2 + bx + c$

NÕu $\Delta < 0$ th× $a.f(x) > 0$ $\forall x \in R$

NÕu $\Delta = 0$ th× $a.f(x) > 0$ $\forall x \neq - \dfrac{b}{a}$

NÕu $\Delta > 0$ th× $a.f(x) > 0$ víi $x < x_1$ hoÆc $x > x_2$ ($x_2 > x_1$)

$a.f(x) < 0$ víi $x_1 < x < x_2$

VÝ dô1:

Chøng minh r»ng

$$f(x, y) = x^2 + 5y^2 - 4xy + 2x - 6y + 3 > 0 \qquad (1)$$

Gi¶i:

Ta cã (1) $\Leftrightarrow$ $x^2 - 2x(2y - 1) + 5y^2 - 6y + 3 > 0$

$$\Delta' = (2y - 1)^2 - 5y^2 + 6y - 3$$
$$= 4y^2 - 4y + 1 - 5y^2 + 6y - 3$$
$$= -(y - 1)^2 - 1 < 0$$

VËy $f(x, y) > 0$ víi mäi x, y

VÝ dô2:

Chøng minh r»ng

$$f(x, y) = x^2y^4 + 2(x^2 + 2).y^2 + 4xy + x^2 > 4xy^3$$

Gi¶i:

BÊt ®¼ng thøc cÇn chøng minh t¬ng ®¬ng víi

$$x^2y^4 + 2(x^2 + 2).y^2 + 4xy + x^2 - 4xy^3 > 0$$

$$\Leftrightarrow (y^2 + 1)^2.x^2 + 4y(1 - y)^2x + 4y^2 > 0$$

Ta cã $\Delta' = 4y^2(1 - y^2)^2 - 4y^2(y^2 + 1)^2 = -16y^2 < 0$

V× a = $(y^2 + 1)^2 > 0$ vËy $f(x, y) > 0$ (®pcm)

Ph¬ng ph¸p 10: dïng quy n¹p to¸n häc

KiÕn thøc:

§Ó chøng minh bÊt ®¼ng thøc ®óng víi $n > n_0$ ta thùc hiÖn c¸c bíc sau :

1 – KiÓm tra bÊt ®¼ng thøc ®óng víi $n = n_0$

2 - Gi¶ sö B§T ®óng víi n =k (thay n =k vµo B§T cÇn chøng minh ®îc gäi lµ gi¶ thiÕt quy n¹p)

3- Ta chøng minh bÊt ®¼ng thøc ®óng víi n = k +1 (thay n = k+1vµo B§T cÇn chøng minh råi biÕn ®æi ®Ó dïng gi¶ thiÕt quy n¹p)

4 – kÕt luËn B§T ®óng víi mäi $n > n_0$

VÝ dô1:

Chøng minh r»ng

$$\frac{1}{1^2}+\frac{1}{2^2}+....+\frac{1}{n^2}<2-\frac{1}{n} \qquad \forall n \in N; n>1 \qquad (1)$$

Gi¶i :

Víi n =2 ta cã $1+\frac{1}{4}<2-\frac{1}{2}$ (®óng)

VËy B§T (1) ®óng víi n =2

Gi¶ sö B§T (1) ®óng víi n =k ta ph¶i chøng minh

B§T (1) ®óng víi n = k+1

ThËt vËy khi n =k+1 th×

(1) $\Leftrightarrow \frac{1}{1^2}+\frac{1}{2^2}+....+\frac{1}{k^2}+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$

Theo gi¶ thiÕt quy n¹p

$$\Leftrightarrow \frac{1}{1^2}+\frac{1}{2^2}+....+\frac{1}{k^2}+\frac{1}{(k+1)^2}<2-\frac{1}{k}+\frac{1}{(k+1)^2}<2-\frac{1}{k+1}$$

$$\Leftrightarrow \frac{1}{1^2}+....+\frac{1}{(k+1)^2}<\frac{1}{k+1}+\frac{1}{(k+1)^2}<\frac{1}{k}$$

$\Leftrightarrow \frac{k+1+1}{(k+1)^2}<\frac{1}{k} \Leftrightarrow k(k+2)<(k+1)^2 \Leftrightarrow$ $k^2+2k<k^2+2k+1$ §iÒu nµy ®óng .VËy bÊt ®¼ng thøc (1)®îc chøng minh

VÝ dô2: Cho $n \in N$ vµ a+b> 0

Chøng minh r»ng $\left(\frac{a+b}{2}\right)^n \leq \frac{a^n+b^n}{2}$ (1)

Gi¶i

Ta thÊy B§T (1) ®óng víi n=1

Gi¶ sö B§T (1) ®óng víi n=k ta ph¶i chøng minh B§T ®óng víi n=k+1

ThËt vËy víi n = k+1 ta cã

(1) $\Leftrightarrow \left(\frac{a+b}{2}\right)^{k+1} \leq \frac{a^{k+1}+b^{k+1}}{2}$

$$\Leftrightarrow \left(\frac{a+b}{2}\right)^{k}.\frac{a+b}{2} \leq \frac{a^{k+1}+b^{k+1}}{2} \qquad (2)$$

$\Leftrightarrow$ VÕ tr¸i (2) $\leq \frac{a^k+b^k}{2}.\frac{a+b}{2}=\frac{a^{k+1}+ab^k+a^kb+b^{k+1}}{4}\leq\frac{a^{k+1}+b^{k+1}}{2}$

$\Leftrightarrow \frac{a^{k+1}+b^{k+1}}{2}-\frac{a^{k+1}+ab^k+a^kb+b^{k+1}}{4}\geq 0$

$\Leftrightarrow (a^k-b^k)(a-b)\geq 0$ (3)

Ta chøng minh (3)

(+) Gi¶ sö a $\geq$ b vµ gi¶ thiÕt cho a $\geq$ -b $\Leftrightarrow$ a $\geq |b|$

$\Leftrightarrow \quad a^k \geq |b|^k \geq b^k \quad \Rightarrow \quad (a^k - b^k)(a - b) \geq 0$

(+) Gi¶ sö a < b vµ theo gi¶ thiÕt - a<b $\Leftrightarrow$ $|a|^k < b^k \Leftrightarrow a^k < b^k \Leftrightarrow (a^k - b^k)(a - b) \geq 0$

VËy B§T (3)lu«n ®óng ta cã (®pcm)

Ph¬ng ph¸p 11: Chøng minh ph¶n chøng

Lu ý:

1) Gi¶ sö ph¶i chøng minh bÊt ®¼ng thøc nµo ®ã ®óng , ta h·y gi¶ sö bÊt ®¼ng thøc ®ã sai vµ kÕt hîp víi c¸c gi¶ thiÕt ®Ó suy ra ®iÒu v« lý , ®iÒu v« lý cã thÓ lµ ®iÒu tr¸i víi gi¶ thiÕt , cã thÓ lµ ®iÒu tr¸i ngîc nhau .Tõ ®ã suy ra bÊt ®¼ng thøc cÇn chøng minh lµ ®óng

2) Gi¶ sö ta ph¶i chøng minh luËn ®Ò "G $\Rightarrow$ K"

phÐp to¸n mÖnh ®Ò cho ta :

Nh vËy ®Ó phñ ®Þnh luËn ®Ò ta ghÐp tÊt c¶ gi¶ thiÕt cña luËn ®Ò víi phñ ®Þnh kÕt luËn cña nã .

Ta thêng dïng 5 h×nh thøc chøng minh ph¶n chøng sau :

A - Dïng mÖnh ®Ò ph¶n ®¶o : $\overline{K} \Rightarrow \overline{G}$

B – Phñ ®Þnh r«i suy tr¸i gi¶ thiÕt :

C – Phñ ®Þnh råi suy tr¸i víi ®iÒu ®óng
D – Phñ ®Þnh råi suy ra 2 ®iÒu tr¸i ngîc nhau
E – Phñ ®Þnh råi suy ra kÕt luËn :

VÝ dô 1:

Cho ba sè a,b,c tháa m·n a +b+c > 0 , ab+bc+ac > 0 , abc > 0
Chøng minh r»ng a > 0 , b > 0 , c > 0

Gi¶i :

Gi¶ sö a $\le$ 0 th× tõ abc > 0 $\Rightarrow$ a$\neq$ 0 do ®ã a < 0
Mµ abc > 0 vµ a < 0 $\Rightarrow$ cb < 0
Tõ ab+bc+ca > 0 $\Rightarrow$ a(b+c) > -bc > 0
V× a < 0 mµ a(b +c) > 0 $\Rightarrow$ b + c < 0
a < 0 vµ b +c < 0 $\Rightarrow$ a + b +c < 0 tr¸i gi¶ thiÕt a+b+c > 0
VËy a > 0 t¬ng tù ta cã b > 0 , c > 0

VÝ dô 2:

Cho 4 sè a , b , c ,d tháa m·n ®iÒu kiÖn
ac $\geq$ 2.(b+d) .Chøng minh r»ng cã Ýt nhÊt mét trong c¸c bÊt ®¼ng thøc sau lµ sai:

$a^2 < 4b$, $c^2 < 4d$

Gi¶i :

Gi¶ sö 2 bÊt ®¼ng thøc : $a^2 < 4b$, $c^2 < 4d$ ®Òu ®óng khi ®ã céng c¸c vÕ ta ®îc
$a^2 + c^2 < 4(b+d)$ (1)
Theo gi¶ thiÕt ta cã 4(b+d) $\leq$ 2ac (2)
Tõ (1) vµ (2) $\Rightarrow$ $a^2 + c^2 < 2ac$ hay $(a-c)^2 < 0$ (v« lý)
VËy trong 2 bÊt ®¼ng thøc $a^2 < 4b$ vµ $c^2 < 4d$ cã Ýt nhÊt mét c¸c bÊt ®¼ng thøc sai

VÝ dô 3:

Cho x,y,z > 0 vµ xyz = 1. Chøng minh r»ng
NÕu x+y+z > $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ th× cã mét trong ba sè nµy lín h¬n 1

Gi¶i :

Ta cã (x-1).(y-1).(z-1) =xyz – xy- yz + x + y+ z –1
=x + y + z – ($\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$) v× xyz = 1
theo gi¶ thiÕt x+y +z > $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$
nªn (x-1).(y-1).(z-1) > 0
Trong ba sè x-1 , y-1 , z-1 chØ cã mét sè d¬ng
ThËt vËy nÕu c¶ ba sè d¬ng th× x,y,z > 1 $\Rightarrow$ xyz > 1 (tr¸i gi¶ thiÕt)
Cßn nÕu 2 trong 3 sè ®ã d¬ng th× (x-1).(y-1).(z-1) < 0 (v« lý)
VËy cã mét vµ chØ mét trong ba sè x , y,z lín h¬n 1

PhÇn iii : c¸c bµi tËp n©ng cao

1/dïng ®Þnh nghÜa

1) Cho abc = 1 vµ $a^3 > 36$. . Chøng minh r»ng $\frac{a^2}{3}$+b²+c²> ab+bc+ac

Gi¶i

Ta cã hiÖu: $\frac{a^2}{3}+b^2+c^2-ab-bc-ac$

$=\frac{a^2}{4}+\frac{a^2}{12}+b^2+c^2-ab-bc-ac$

$=\left(\frac{a^2}{4}+b^2+c^2-ab-ac+2bc\right)+\frac{a^2}{12}-3bc$

$=\left(\frac{a}{2}-b-c\right)^2+\frac{a^3-36abc}{12a}$

$=\left(\frac{a}{2}-b-c\right)^2+\frac{a^3-36abc}{12a}>0$ (v× abc=1 vµ $a^3 > 36$ nªn a >0)

VËy : $\frac{a^2}{3}+b^2+c^2>ab+bc+ac$ §iÒu ph¶i chøng minh

2) Chøng minh r»ng

a) $x^4+y^4+z^2+1\geq 2x.(xy^2-x+z+1)$

b) víi mäi sè thùc a , b, c ta cã

$a^2 + 5b^2 - 4ab + 2a - 6b + 3 > 0$

c) $a^2 + 2b^2 - 2ab + 2a - 4b + 2 \geq 0$

Gi¶i :

a) XÐt hiÖu

H = $x^4 + y^4 + z^2 + 1 - 2x^2y^2 + 2x^2 - 2xz - 2x$

= $(x^2 - y^2)^2 + (x - z)^2 + (x - 1)^2$

H$\geq$0 ta cã ®iÒu ph¶i chøng minh

b) VÕ tr¸i cã thÓ viÕt

H = $(a - 2b + 1)^2 + (b - 1)^2 + 1$

$\Rightarrow$ H > 0 ta cã ®iÒu ph¶i chøng minh

c) vÕ tr¸i cã thÓ viÕt

H = $(a - b + 1)^2 + (b - 1)^2$

$\Rightarrow$ H $\geq$ 0 ta cã ®iÒu ph¶i chøng minh

Ii / Dïng biÕn ®æi t¬ng ®¬ng

1) Cho x > y vµ xy =1 .Chøng minh r»ng

$$\frac{(x^2 + y^2)^2}{(x - y)^2} \geq 8$$

Gi¶i :

Ta cã $x^2 + y^2 = (x - y)^2 + 2xy = (x - y)^2 + 2$ (v× xy = 1)

$\Rightarrow$ $(x^2 + y^2)^2 = (x - y)^4 + 4.(x - y)^2 + 4$

Do ®ã B§T cÇn chøng minh t¬ng ®¬ng víi

$(x - y)^4 + 4(x - y)^2 + 4 \geq 8.(x - y)^2$

$\Leftrightarrow$ $(x - y)^4 - 4(x - y)^2 + 4 \geq 0$

$\Leftrightarrow$ $\left[(x - y)^2 - 2\right]^2 \geq 0$

B§T cuèi ®óng nªn ta cã ®iÒu ph¶i chøng minh

2) Cho xy $\geq$ 1 .Chøng minh r»ng

$$\frac{1}{1 + x^2} + \frac{1}{1 + y^2} \geq \frac{2}{1 + xy}$$

Gi¶i :

Ta cã $\frac{1}{1 + x^2} + \frac{1}{1 + y^2} \geq \frac{2}{1 + xy}$

$\Leftrightarrow$ $\left(\frac{1}{1 + x^2} - \frac{1}{1 + y^2}\right) + \left(\frac{1}{1 + y^2} - \frac{1}{1 + xy}\right) \geq 0$

$\Leftrightarrow$ $\frac{xy - x^2}{(1 + x^2).(1 + xy)} + \frac{xy - y^2}{(1 + y^2).(1 + xy)} \geq 0$

$\Leftrightarrow$ $\frac{x(y - x)}{(1 + x^2).(1 + xy)} + \frac{y(x - y)}{(1 + y^2).(1 + xy)} \geq 0$

$$\Leftrightarrow \quad \frac{(y-x)^2(xy-1)}{(1+x^2).(1+y^2).(1+xy)} \geq 0$$

B§T cuèi nµy ®óng do xy > 1 .VËy ta cã ®iÒu ph¶i chøng minh

<u>Iii / dïng bÊt ®¼ng thøc phô</u>

1) Cho a , b, c lµ c¸c sè thùc vµ a + b +c =1

Chøng minh r»ng $a^2+b^2+c^2 \geq \frac{1}{3}$

<u>Gi¶i</u> :

¸p dông B§T BunhiaC«pski cho 3 sè (1,1,1) vµ (a,b,c)

Ta cã $(1.a+1.b+1.c)^2 \leq (1+1+1).(a^2+b^2+c^2)$

$\Leftrightarrow \quad (a+b+c)^2 \leq 3.(a^2+b^2+c^2)$

$\Leftrightarrow \quad a^2+b^2+c^2 \geq \frac{1}{3}$ (v× a+b+c =1) (®pcm)

2) Cho a,b,c lµ c¸c sè d¬ng

Chøng minh r»ng $(a+b+c).\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$ (1)

<u>Gi¶i</u> :

(1) $\Leftrightarrow \quad 1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+1+\frac{b}{c}+\frac{c}{a}+\frac{c}{a}+1 \geq 9$

$\Leftrightarrow \quad 3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right) \geq 9$

¸p dông B§T phô $\frac{x}{y}+\frac{y}{x} \geq 2$ Víi x,y > 0

Ta cã B§T cuèi cïng lu«n ®óng

VËy $(a+b+c).\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$ (®pcm)

Iv <u>/ dïng ph¬ng ph¸p b¾c cÇu</u>

1) Cho 0 < a, b,c <1 .Chøng minh r»ng :

$2a^3+2b^3+2c^3 < 3+a^2b+b^2c+c^2a$

<u>Gi¶i</u> :

Do a <1 ⇒ a^2 <1 vµ b <1

Nªn $(1-a^2)(1-b^2) > 0 \Rightarrow 1 + a^2 b - a^2 - b > 0$

Hay $1 + a^2 b > a^2 + b$ (1)

MÆt kh¸c 0 <a,b <1 ⇒ $a^2 > a^3$; $b > b^3$

⇒ $1 + a^2 > a^3 + b^3$

VËy $a^3 + b^3 < 1 + a^2 b$

T¬ng tù ta cã

$$b^3 + c^3 < 1 + b^2 c$$

$$a^3 + c^3 < 1 + c^2 a$$

⇒ $2a^3 + 2b^3 + 2c^3 < 3 + a^2 b + b^2 c + c^2 a$ (®pcm)

2) So s¸nh 31^{11} vµ 17^{14}

<u>Gi¶i</u> :

Ta thÊy $31^{11} < 32^{11} = (2^5)^{11} = 2^{55} < 2^{56}$

MÆt kh¸c $2^{56} = 2^{4.14} = (2^4)^{14} = 16^{14} < 17^{14}$

Vëy $31^{11} < 17^{14}$ (®pcm)

V/ <u>dïng tÝnh chÊt tØ sè</u>

1) Cho a ,b ,c ,d > 0 .Chøng minh r»ng :

$$2 < \frac{a+b}{a+b+c} + \frac{b+c}{b+c+d} + \frac{c+d}{c+d+a} + \frac{d+a}{d+a+b} < 3$$

<u>Gi¶i :</u>

V× a ,b ,c ,d > 0 nªn ta cã

$$\frac{a+b}{a+b+c+d} < \frac{a+b}{a+b+c} < \frac{a+b+d}{a+b+c+d} \quad (1)$$

$$\frac{b++c}{a+b+c+d} < \frac{b+c}{b+c+d} < \frac{b+c+a}{a+b+c+d} \quad (2)$$

$$\frac{d+a}{a+b+c+d} < \frac{d+a}{d+a+b} < \frac{d+a+c}{a+b+c+d} \quad (3)$$

Céng c¸c vÕ cña 4 bÊt ®¼ng thøc trªn ta cã :

$$2 < \frac{a+b}{a+b+c} + \frac{b+c}{b+c+d} + \frac{c+d}{c+d+a} + \frac{d+a}{d+a+b} < 3 \qquad (®pcm)$$

2) Cho a ,b,c lµ sè ®o ba c¹nh tam gi¸c

Chøng minh r»ng

$$1 < \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$$

<u>Gi¶i</u> :

V× a ,b ,c lµ sè ®o ba c¹nh cña tam gi¸c nªn ta cã a,b,c > 0

Vµ a < b +c ; b <a+c ; c < a+b

Tõ (1) ⇒ $\frac{a}{b+c} < \frac{a+a}{a+b+c} = \frac{2a}{a+b+c}$

MÆt kh¸c $\frac{a}{b+c} > \frac{a}{a+b+c}$

VËy ta cã $\frac{a}{a+b+c} < \frac{a}{b+c} < \frac{2a}{a+b+c}$ T¬ng tù ta cã $\frac{b}{a+b+c} < \frac{b}{a+c} < \frac{2b}{a+b+c}$

$$\frac{c}{a+b+c} < \frac{c}{b+a} < \frac{2c}{a+b+c}$$

Céng tõng vÕ ba bÊt ®¼ng thøc trªn ta cã :

$$1 < \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$$ (®pcm)

V/ ph¬ng ph¸p lµm tréi :

1) Chøng minh B§T sau :

a) $\frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2n-1).(2n+1)} < \frac{1}{2}$

b) $1 + \frac{1}{1.2} + \frac{1}{1.2.3} + ... + \frac{1}{1.2.3.....n} < 2$

Gi¶i :

a) Ta cã

$$\frac{1}{(2n-1).(2n+1)} = \frac{1}{2}.\frac{(2k+1)-(2k-1)}{(2k-1).(2k+1)} = \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$$

Cho n ch¹y tõ 1 ®Õn k .Sau ®ã céng l¹i ta cã

$$\frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2n-1).(2n+1)} = \frac{1}{2}.\left(1 - \frac{2}{2n+1}\right) < \frac{1}{2}$$ (®pcm)

b) Ta cã

$$1 + \frac{1}{1.2} + \frac{1}{1.2.3} + ... + \frac{1}{1.2.3.....n} < 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + ... + \frac{1}{(n-1).n}$$

$$< 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n-1} - \frac{1}{n}\right) < 2 - \frac{1}{n} < 2$$ (®pcm)

PhÇn iv : øng dông cña bÊt ®¼ng thøc

1/ dïng bÊt ®¼ng thøc ®Ó t×m cc trÞ

Lu ý

- NÕu f(x) $\ge$ A th× f(x) cã gi¸ trÞ nhá nhÊt lµ A
- NÕu f(x) $\le$ B th× f(x) cã gi¸ trÞ lín nhÊt lµ B

VÝ dô 1 :

T×m gi¸ trÞ nhá nhÊt cña :

T = |x-1| + |x-2| +|x-3| + |x-4|

Gi¶i :

Ta cã |x-1| + |x-4| = |x-1| + |4-x| $\ge$ |x-1+4-x| = 3 (1)

Vµ $\left|x-2\right|+\left|x-3\right|=\left|x-2\right|+\left|3-x\right|\ge\left|x-2+3-x\right|=1$ (2)

VËy T = |x-1| + |x-2| +|x-3| + |x-4| $\ge$ 1+3 = 4

Ta cã tõ (1) $\Rightarrow$ DÊu b»ng x¶y ra khi $1\le x\le 4$

(2) $\Rightarrow$ DÊu b»ng x¶y ra khi $2\le x\le 3$

VËy T cã gi¸ trÞ nhá nhÊt lµ 4 khi $2\le x\le 3$

VÝ dô 2 :

T×m gi¸ trÞ lín nhÊt cña

S = xyz.(x+y).(y+z).(z+x) víi x,y,z > 0 vµ x+y+z =1

Gi¶i :

V× x,y,z > 0 , ¸p dông B§T C«si ta cã

x+ y + z $\ge 3\sqrt[3]{xyz}$

$$\Rightarrow \sqrt[3]{xyz}\le\frac{1}{3}\Rightarrow xyz\le\frac{1}{27}$$

¸p dông bÊt ®¼ng thøc C«si cho x+y ; y+z ; x+z ta cã

$$(x+y).(y+z).(z+x)\ge 3\sqrt[3]{(x+y).(y+z).(x+z)}$$

$$\Rightarrow 2\ge 3\sqrt[3]{(x+y).(y+z).(z+x)}$$

DÊu b»ng x¶y ra khi x=y=z=$\frac{1}{3}$

VËy S $\le \frac{8}{27}.\frac{1}{27}=\frac{8}{729}$

VËy S cã gi¸ trÞ lín nhÊt lµ $\frac{8}{729}$ khi x=y=z=$\frac{1}{3}$

VÝ dô 3 : Cho xy+yz+zx = 1

T×m gi¸ trÞ nhá nhÊt cña $x^4+y^4+z^4$

Gi¶i :

¸p dông B§T Bunhiacèpski cho 6 sè (x,y,z) ;(x,y,z)

Ta cã $(xy+yz+zx)^2\le\left(x^2+y^2+z^2\right)^2$

$\Rightarrow 1\le\left(x^2+y^2+z^2\right)^2$ (1)

Ap dông B§T Bunhiacèpski cho (x^2,y^2,z^2) vµ (1,1,1)

Ta cã $(x^2+y^2+z^2)^2 \le (1^2+1^2+1^2)(x^4+y^4+z^4)$

$\rightarrow (x^2+y^2+z^2)^2 \le 3(x^4+y^4+z^4)$

Tõ (1) vµ (2) $\Rightarrow 1 \le 3(x^4+y^4+z^4)$

$\Rightarrow x^4+y^4+z^4 \le \frac{1}{3}$

VËy $x^4+y^4+z^4$ cã gi¸ trÞ nhá nhÊt lµ $\frac{1}{3}$ khi x=y=z= $\pm\frac{\sqrt{3}}{3}$

<u>VÝ dô 4</u> :

Trong tam gi¸c vu«ng cã cïng c¹nh huyÒn , tam gi¸c vu«ng nµo cã diÖn tÝch lín nhÊt

<u>Gi¶i</u> :

Gäi c¹nh huyÒn cña tam gi¸c lµ 2a

§êng cao thuéc c¹nh huyÒn lµ h

H×nh chiÕu c¸c c¹nh gãc vu«ng lªn c¹nh huyÒn lµ x,y

Ta cã S = $\frac{1}{2}.(x+y).h = a.h = a.\sqrt{h^2} = a.\sqrt{xy}$

V× a kh«ng ®æi mµ x+y = 2a

VËy S lín nhÊt khi x.y lín nhÊt $\Leftrightarrow x = y$

VËy trong c¸c tam gi¸c cã cïng c¹nh huyÒn th× tam gi¸c vu«ng c©n cã diÖn tÝch lín nhÊt

II/ <u>dïng b.®.t ®Ó gi¶i ph¬ng tr×nh vµ hÖ ph¬ng tr×nh</u>

VÝ dô 1 :

Gi¶i ph¬ng tr×nh sau

$$4\sqrt{3x^2+6x+19}+\sqrt{5x^2+10x+14}=4-2x-x^2$$

Gi¶i :

Ta cã $3x^2+6x+19=3.(x^2+2x+1)+16$

$$=3.(x+1)^2+16\geq 16$$

$$5x^2+10x+14=5.(x+1)^2+9\geq 9$$

VËy $4.\sqrt{3x^2+6x+19}+\sqrt{5x^2+10x+14}\geq 2+3=5$

DÊu (=) x¶y ra khi x+1 = 0 $\Rightarrow$ x = -1

VËy $4\sqrt{3x^2+6x+19}+\sqrt{5x^2+10x+14}=4-2x-x^2$ khi x = -1

VËy ph¬ng tr×nh cã nghiÖm duy nhÊt x = -1

VÝ dô 2 :

Gi¶i ph¬ng tr×nh

$$x+\sqrt{2-x^2}=4y^2+4y+3$$

Gi¶i :

¸p dông B§T BunhiaCèpski ta cã :

$$x+\sqrt{2-x^2}\leq\sqrt{1^2+1^2}.\sqrt{x^2+\left(2-x^2\right)}\leq\sqrt{2}.\sqrt{2}=2$$

DÊu (=) x¶y ra khi x = 1

MÆt kh¸c $4y^2+4y+3=(2y+1)^2+2\geq 2$

DÊu (=) x¶y ra khi y = $-\frac{1}{2}$

VËy $x+\sqrt{2-x^2}=4y^2+4y+3=2$ khi x =1 vµ y =$-\frac{1}{2}$

VËy nghiÖm cña ph¬ng tr×nh lµ $\begin{cases} x=1 \\ y=-\dfrac{1}{2} \end{cases}$

VÝ dô 3 :

Gi¶i hÖ ph¬ng tr×nh sau:

$$\begin{cases} x+y+z=1 \\ x^4+y^4+z^4=xyz \end{cases}$$

Gi¶i : ¸p dông B§T C«si ta cã

$$\mathrm{x}^4+y^4+z^4=\frac{x^4+y^4}{2}+\frac{y^4+z^4}{2}+\frac{z^4+x^4}{2}$$

$$\geq x^2y^2+y^2z^2+z^2x^2$$

$$\geq\frac{x^2y^2+y^2z^2}{2}+\frac{z^2y^2+z^2z^2}{2}+\frac{x^2z^2+y^2x^2}{2}$$

$$\geq y^2xz+z^2xy+x^2yz$$

$$\geq xyz.(x+y+z)$$

V× x+y+z = 1)

Nªn $x^4 + y^4 + z^4 \ge xyz$

DÊu (=) x¶y ra khi x = y = z = $\frac{1}{3}$

VËy $\begin{cases} x + y + z = 1 \\ x^4 + y^4 + z^4 = xyz \end{cases}$ cã nghiÖm x = y = z = $\frac{1}{3}$

<u>VÝ dô 4</u> : Gi¶i hÖ ph¬ng tr×nh sau

$$\begin{cases} |xy - 4| = 8 - y^2 & (1) \\ xy = 2 + x^2 & (2) \end{cases}$$

Tõ ph¬ng tr×nh (1) $\Rightarrow 8 - y^2 \ge 0$ hay $|y| \le \sqrt{8}$

Tõ ph¬ng tr×nh (2) $\Rightarrow x^2 + 2 = |x|.|y| \le 2\sqrt{2}\,|x|$

$\Rightarrow x^2 - 2\sqrt{2}\,|x| + \sqrt{2^2} \le 0$

$\Rightarrow (|x| - \sqrt{2})^2 \le 0$

$\Rightarrow |x| = \sqrt{2}$

$\Rightarrow x = \pm\sqrt{2}$

NÕu x = $\sqrt{2}$ th× y = 2$\sqrt{2}$

NÕu x = -$\sqrt{2}$ th× y = -2$\sqrt{2}$

VËy hÖ ph¬ng tr×nh cã nghiÖm $\begin{cases} x = \sqrt{2} \\ y = -\sqrt{2} \end{cases}$ vµ $\begin{cases} x = 2\sqrt{2} \\ y = -2\sqrt{2} \end{cases}$

Iii/ <u>dïng B.§.t ®Ó gi¶i ph¬ng tr×nh nghiÖm nguyªn</u>

1) T×m c¸c sè nguyªn x,y,z tho¶ m·n

$$x^2 + y^2 + z^2 \le xy + 3y + 2z - 3$$

Gi¶i :

V× x,y,z lµ c¸c sè nguyªn nªn

$$x^2 + y^2 + z^2 \le xy + 3y + 2z - 3$$

$$\Leftrightarrow x^2 + y^2 + z^2 - xy - 3y - 2z + 3 \le 0$$

$$\Leftrightarrow \left(x^2 - xy + \frac{y^2}{4}\right) + \left(\frac{3y^2}{4} - 3y + 3\right) + \left(z^2 - 2z + 1\right) \le 0$$

$$\Leftrightarrow \left(x - \frac{y}{2}\right)^2 + 3\left(\frac{y}{2} - 1\right)^2 + (z-1)^2 \le 0 \qquad (*)$$

Mµ $\left(x - \frac{y}{2}\right)^2 + 3\left(\frac{y}{2} - 1\right)^2 + (z-1)^2 \ge 0 \qquad \forall x, y \in R$

$$\Leftrightarrow \left(x - \frac{y}{2}\right)^2 + 3\left(\frac{y}{2} - 1\right)^2 + (z-1)^2 = 0$$

$$\Leftrightarrow \begin{cases} x - \frac{y}{2} = 0 \\ \frac{y}{2} - 1 = 0 \\ z - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = 2 \\ z = 1 \end{cases}$$

C¸c sè x,y,z ph¶i t×m lµ $\begin{cases} x = 1 \\ y = 2 \\ z = 1 \end{cases}$

VÝ dô 2:

T×m nghiÖm nguyªn d¬ng cña ph¬ng tr×nh

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$$

Gi¶i :

Kh«ng mÊt tÝnh tæng qu¸t ta gi¶ sö $x \ge y \ge z$

Ta cã $2 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \le \frac{3}{z} \Rightarrow 2z \le 3$

Mµ z nguyªn d¬ng vËy z = 1

Thay z = 1 vµo ph¬ng tr×nh ta ®îc $\frac{1}{x} + \frac{1}{y} = 1$

Theo gi¶ sö $x \ge y$ nªn 1 = $\frac{1}{x} + \frac{1}{y} \le \frac{1}{y} \Rightarrow y \le 2$ mµ y nguyªn d¬ng

Nªn y = 1 hoÆc y = 2

Víi y = 1 kh«ng thÝch hîp

Víi y = 2 ta cã x = 2

VËy (2 ,2,1) lµ mét nghiÖm cña ph¬ng tr×nh

Ho¸n vÞ c¸c sè trªn ta ®îc c¸c nghiÖm cña ph¬ng tr×nh

lµ (2,2,1) ; (2,1,2) ; (1,2,2)

<u>VÝ dô 3</u> :

T×m c¸c cÆp sè nguyªn tho¶ m·n ph¬ng tr×nh

$$\sqrt{x+\sqrt{x}} = y \qquad \textbf{(*)}$$

<u>Gi¶i :</u>

(*) Víi x < 0 , y < 0 th× ph¬ng tr×nh kh«ng cã nghÜa

(*) Víi x > 0 , y > 0

Ta cã $\sqrt{x+\sqrt{x}} = y \quad \Leftrightarrow x+\sqrt{x} = y^2$

$\Leftrightarrow \sqrt{x} = y^2 - x > 0$

§Æt $\sqrt{x} = k$ (k nguyªn d¬ng v× x nguyªn d¬ng)

Ta cã $k.(k+1) = y^2$

Nhng $k^2 < k(k+1) < (k+1)^2$

$\Rightarrow k < y < k+1$

Mµ gi÷a k vµ k+1 lµ hai sè nguyªn d¬ng liªn tiÕp kh«ng tån t¹i mét sè nguyªn d¬ng nµo c¶

Nªn kh«ng cã cÆp sè nguyªn d¬ng nµo tho¶ m·n ph¬ng tr×nh .

VËy ph¬ng tr×nh cã nghiÖm duy nhÊt lµ : $\begin{cases} x = 0 \\ y = 0 \end{cases}$

Tµi liÖu tham kh¶o

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