Simple trigonometric substitutions with broad results

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Often, the key to solve some intricate algebraic inequality is to simplify it by employing a trigonometric substitution. When we make a clever trigonometric substitution the problem may reduce so much that we can see a direct solution immediately. Besides, trigonometric functions have well-known properties that may help in solving such inequalities. As a result, many algebraic problems can be solved by using an inspired substitution.

We start by introducing the readers to such substitutions. After that we present some well-known trigonometric identities and inequalities. Finally, we discuss some Olympiad problems and leave others for the reader to solve.

Theorem 1. Let α, β, γ be angles in $(0, \pi)$. Then α, β, γ are the angles of a triangle if and only if

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1.$$

Proof. First of all note that if $\alpha = \beta = \gamma$, then the statement clearly holds. Assume without loss of generality that $\alpha \neq \beta$. Because $0 < \alpha + \beta < 2\pi$, it follows that there exists an angle in $(-\pi, \pi)$, say γ', such that $\alpha + \beta + \gamma' = \pi$.

Using the addition formulas and the fact that $\tan x = \cot\left(\frac{\pi}{2} - x\right)$, we have

$$\tan\frac{\gamma'}{2}=\cot\frac{\alpha+\beta}{2}=\frac{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}},$$

yielding

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma'}{2}+\tan\frac{\gamma'}{2}\tan\frac{\alpha}{2}=1. \tag{1}$$

Now suppose that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1, \tag{2}$$

for some α, β, γ in $(0, \pi)$.

We will prove that $\gamma = \gamma'$, and this will imply that α, β, γ are the angles of a triangle. Subtracting (1) from (2) we get $\tan\frac{\gamma}{2} = \tan\frac{\gamma'}{2}$. Thus $\left|\frac{\gamma-\gamma'}{2}\right| = k\pi$ for some nonnegative integer k. But $\left|\frac{\gamma-\gamma'}{2}\right| \leq \left|\frac{\gamma}{2}\right| + \left|\frac{\gamma'}{2}\right| < \pi$, so it follows that $k = 0$. That is $\gamma = \gamma'$, as desired. □

Theorem 2. Let α, β, γ be angles in $(0, \pi)$. Then α, β, γ are the angles of a triangle if and only if

$$\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} + \sin^2\frac{\gamma}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = 1.$$

Proof. As $0 < \alpha + \beta < 2\pi$, there exists an angle in $(-\pi, \pi)$, say γ', such that $\alpha + \beta + \gamma' = \pi$. Using the product-to-sum and the double angle formulas we get

$$\begin{aligned}
\sin^2\frac{\gamma'}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma'}{2} &= \cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha+\beta}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) \\
&= \cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
&= \frac{\cos\alpha + \cos\beta}{2} \\
&= \frac{\left(1 - 2\sin^2\frac{\alpha}{2}\right) + \left(1 - 2\sin^2\frac{\beta}{2}\right)}{2} \\
&= 1 - \sin^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}.
\end{aligned}$$

Thus

$$\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} + \sin^2\frac{\gamma'}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma'}{2} = 1. \tag{1}$$

Now suppose that

$$\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} + \sin^2\frac{\gamma}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = 1, \tag{2}$$

for some α, β, γ in $(0, \pi)$. Subtracting (1) from (2) we obtain

$$\sin^2\frac{\gamma}{2} - \sin^2\frac{\gamma'}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\left(\sin\frac{\gamma}{2} - \sin\frac{\gamma'}{2}\right) = 0,$$

that is

$$\left(\sin\frac{\gamma}{2} - \sin\frac{\gamma'}{2}\right)\left(\sin\frac{\gamma}{2} + \sin\frac{\gamma'}{2} + 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) = 0.$$

The second factor can be written as

$$\sin\frac{\gamma}{2} + \sin\frac{\gamma'}{2} + \cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2} = \sin\frac{\gamma}{2} + \cos\frac{\alpha-\beta}{2},$$

which is clearly greater than 0. It follows that $\sin\frac{\gamma}{2} = \sin\frac{\gamma'}{2}$, and so $\gamma = \gamma'$, showing that α, β, γ are the angles of a triangle. $\square$

Substitutions and Transformations

T1. Let α, β, γ be angles of a triangle. Let

$$A = \frac{\pi - \alpha}{2}, \;\; B = \frac{\pi - \beta}{2}, \;\; C = \frac{\pi - \gamma}{2}.$$

Then $A+B+C = \pi$, and $0 \le A, B, C < \frac{\pi}{2}$. This transformation allows us to switch from angles of an arbitrary triangle to angles of an acute triangle. Note that

$$cyc(\sin\frac{\alpha}{2} = \cos A), \;\; cyc(\cos\frac{\alpha}{2} = \sin A), \;\; cyc(\tan\frac{\alpha}{2} = \cot A), \;\; cyc(\cot\frac{\alpha}{2} = \tan A),$$

where by cyc we denote a cyclic permutation of angles.

T2. Let x, y, z be positive real numbers. Then there is a triangle with sidelengths $a = x + y, \;\; b = y + z, \;\; c = z + x$. This transformation is sometimes called Dual Principle. Clearly, $s = x+y+z$ and $(x, y, z) = (s-a, s-b, s-c)$. This transformation already triangle inequality.

S1. Let a, b, c be positive real numbers such that $ab + bc + ca = 1$. Using the function $f : (0, \frac{\pi}{2}) \to (0, +\infty)$, for $f(x) = \tan x$, we can do the following substitution

$$a = \tan\frac{\alpha}{2}, \quad b = \tan\frac{\beta}{2}, \quad c = \tan\frac{\gamma}{2},$$

where α, β, γ are the angles of a triangle ABC.

S2. Let a, b, c be positive real numbers such that $ab+bc+ca = 1$. Applying **T1** to **S1**, we have

$$a = \cot A, \quad b = \cot B, \quad c = \cot C,$$

where A, B, C are the angles of an acute triangle.

S3. Let a, b, c be positive real numbers such that $a + b + c = abc$. Dividing by abc it follows that $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = 1$. Due to **S1**, we can substitute

$$\frac{1}{a} = \tan\frac{\alpha}{2}, \quad \frac{1}{b} = \tan\frac{\beta}{2}, \quad \frac{1}{c} = \tan\frac{\gamma}{2},$$

that is

$$a = \cot\frac{\alpha}{2}, \quad b = \cot\frac{\beta}{2}, \quad c = \cot\frac{\gamma}{2},$$

where α, β, γ are the angles of a triangle.

S4. Let a, b, c be positive real numbers such that $a + b + c = abc$. Applying **T1** to **S3**, we have

$$a = \tan A, \quad b = \tan B, \quad c = \tan C,$$

where A, B, C are the angles of an acute triangle.

S5. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + 2abc = 1$. Note that since all the numbers are positive it follows that $a, b, c < 1$. Usign the function $f : (0, \pi) \to (0, 1)$, for $f(x) = \sin \frac{x}{2}$, and recalling Theorem 2, we can substitute

$$a = \sin \frac{\alpha}{2}, \quad b = \sin \frac{\beta}{2}, \quad c = \sin \frac{\gamma}{2},$$

where α, β, γ are the angles of a triangle.

S6. Let a, b, c be positive real numbers such that $a^2+b^2+c^2+2abc = 1$. Applying **T1** to **S5**, we have

$$a = \cos A, \quad b = \cos B, \quad c = \cos C,$$

where A, B, C are the angles of an acute triangle.

S7. Let x, y, z be positive real numbers. Applying **T2** to expressions

$$\sqrt{\frac{yz}{(x+y)(x+z)}}, \ \sqrt{\frac{zx}{(y+z)(y+x)}}, \ \sqrt{\frac{xy}{(z+x)(z+y)}},$$

they can be substituted by

$$\sqrt{\frac{(s-b)(s-c)}{bc}}, \ \sqrt{\frac{(s-c)(s-a)}{ca}}, \ \sqrt{\frac{(s-a)(s-b)}{ab}},$$

where a, b, c are the sidelengths of a triangle. Recall the following identities

$$\sin \frac{\alpha}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos \frac{\alpha}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

Thus our expressions can be substituted by

$$\sin \frac{\alpha}{2}, \quad \sin \frac{\beta}{2}, \quad \sin \frac{\gamma}{2},$$

where α, β, γ are the angles of a triangle.

S8. Analogously to **S7**, the expressions

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}, \ \sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}}, \ \sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}},$$

can be substituted by

$$\cos \frac{\alpha}{2}, \quad \cos \frac{\beta}{2}, \quad \cos \frac{\gamma}{2},$$

where α, β, γ are the angles of a triangle.

Further we present a list of inequalities and equalities that can be helpful in solving many problems or simplify them.

Well-known inequalities

Let α, β, γ be angles of a triangle ABC. Then

1. $\cos\alpha+\cos\beta+\cos\gamma \le \sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2} \le \frac{3}{2}$

2. $\sin\alpha+\sin\beta+\sin\gamma \le \cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2} \le \frac{3\sqrt{3}}{2}$

3. $\cos\alpha\cos\beta\cos\gamma \le \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \le \frac{1}{8}$

4. $\sin\alpha\sin\beta\sin\gamma \le \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \le \frac{3\sqrt{3}}{8}$

5. $\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{C}{2} \ge 3\sqrt{3}$

6. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma \ge \sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{C}{2} \ge \frac{3}{4}$

7. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma \le \cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}+\cos^2\frac{\gamma}{2} \le \frac{9}{4}$

8. $\cot\alpha+\cot\beta+\cot\gamma \ge \tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2} \ge \sqrt{3}$

Well-known identities

Let α, β, γ be angles of a triangle ABC. Then

1. $\cos\alpha+\cos\beta+\cos\gamma = 1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

2. $\sin\alpha+\sin\beta+\sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

3. $\sin 2\alpha+\sin 2\beta+\sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$

4. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma = 2+2\cos\alpha\cos\beta\cos\gamma$

For arbitrary angles α, β, γ we have

$$\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma) = 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}.$$

$$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma) = 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}.$$

Applications

1. Let x, y, z be positive real numbers. Prove that

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{x+\sqrt{(z+x)(z+y)}} \leq 1.$$

(Walther Janous, Crux Mathematicorum)

Solution. The inequality is equivalent to

$$\sum \frac{1}{1+\sqrt{\frac{(x+y)(x+z)}{x^2}}} \leq 1.$$

Beceause the inequality is homogeneous, we can assume that $xy+yz+zx=1$. Let us apply substitution **S1**: $cyc(x=\tan\frac{\alpha}{2})$, where α, β, γ are angles of a triangle. We get

$$\frac{(x+y)(x+z)}{x^2}=\frac{\left(\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}\right)\left(\tan\frac{\alpha}{2}+\tan\frac{\gamma}{2}\right)}{\tan^2\frac{\alpha}{2}}=\frac{1}{\sin^2\frac{\alpha}{2}},$$

and similar expressions for the other terms. The inequality becomes

$$\frac{\sin\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}}+\frac{\sin\frac{\beta}{2}}{1+\sin\frac{\beta}{2}}+\frac{\sin\frac{\gamma}{2}}{1+\sin\frac{\gamma}{2}} \leq 1,$$

that is

$$2 \leq \frac{1}{1+\sin\frac{\alpha}{2}}+\frac{1}{1+\sin\frac{\beta}{2}}+\frac{1}{1+\sin\frac{\gamma}{2}}.$$

On the other hand, using the well-known inequality $\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2} \leq \frac{3}{2}$ and the Cauchy-Schwarz inequality, we have

$$2 \leq \frac{9}{(1+\sin\frac{\alpha}{2})+(1+\sin\frac{\beta}{2})+(1+\sin\frac{\gamma}{2})} \leq \sum \frac{1}{1+\sin\frac{\alpha}{2}},$$

and we are done. ■

2. Let x, y, z be real numbers greater than 1 such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$. Prove that

$$\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1} \leq \sqrt{x+y+z}.$$

(Iran, 1997)

Solution. Let $(x, y, z) = (a+1, b+1, c+1)$, with a, b, c positive real numbers. Note that the hypothesis is equivalent to $ab + bc + ca + 2abc = 1$. Then it suffices to prove that

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \le \sqrt{a + b + c + 3}.$$

Squaring both sides of the inequality and canceling some terms yields

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \le \frac{3}{2}.$$

Using substitution **S5** we get $(ab, bc, ca) = (\sin^2 \frac{\alpha}{2}, \sin^2 \frac{\beta}{2}, \sin^2 \frac{\gamma}{2})$, where ABC is an arbitrary triangle. The problem reduces to proving that

$$\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \le \frac{3}{2},$$

which is well-known and can be done using Jensen inequality. ■

3. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\sqrt{\frac{ab}{c+ab}} + \sqrt{\frac{bc}{a+bc}} + \sqrt{\frac{ca}{b+ca}} \le \frac{3}{2}.$$

(Open Olympiad of FML No-239, Russia)

Solution. The inequality is equivalent to

$$\sqrt{\frac{ab}{((c+a)(c+b)}} + \sqrt{\frac{bc}{(a+b)(a+c)}} + \sqrt{\frac{ca}{(b+c)(b+a)}} \le \frac{3}{2}.$$

Substitution **S7** replaces the three terms in the inequality by $\sin\frac{\alpha}{2}, \sin\frac{\beta}{2}, \sin\frac{\gamma}{2}$. Thus it suffices to prove $\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \le \frac{3}{2}$, which clearly holds. ■

4. Let a, b, c be positive real numbers such that $(a+b)(b+c)(c+a) = 1$. Prove that

$$ab + bc + ca \le \frac{3}{4}.$$

(Cezar Lupu, Romania, 2005)

Solution. Observe that the inequality is equivalent to

$$\left(\sum ab\right)^3 \le \left(\frac{3}{4}\right)^3 (a+b)^2(b+c)^2(c+a)^2.$$

Because the inequality is homogeneous, we can assume that $ab + bc + ca = 1$. We use substitution **S1**: $cyc(a = \tan\frac{\alpha}{2})$, where α, β, γ are the angles of a triangle. Note that

$$(a+b)(b+c)(c+a) = \prod\left(\frac{\cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}\right) = \frac{1}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}.$$

Thus it suffices to prove that

$$\left(\frac{4}{3}\right)^3 \le \frac{1}{\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}},$$

or

$$4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \le \frac{3\sqrt{3}}{2}.$$

From the identity $4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} = \sin\alpha + \sin\beta + \sin\gamma$, the inequality is equivalent to

$$\sin\alpha + \sin\beta + \sin\gamma \le \frac{3\sqrt{3}}{2}.$$

But $f(x) = \sin x$ is a concave function on $(0, \pi)$ and the conclusion follows from Jensen's inequality. ■

5. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$a^2 + b^2 + c^2 + 2\sqrt{3abc} \le 1.$$

(Poland, 1999)

Solution. Let $cyc\left(x = \sqrt{\frac{bc}{a}}\right)$. It follows that $cyc(a = yz)$. The inequality becomes

$$x^2y^2 + y^2z^2 + x^2z^2 + 2\sqrt{3}xyz \le 1,$$

where x, y, z are positive real numbers such that $xy + yz + zx = 1$. Note that the inequality is equivalent to

$$(xy + yz + zx)^2 + 2\sqrt{3}xyz \le 1 + 2xyz(x + y + z),$$

or

$$\sqrt{3} \le x + y + z.$$

Applying substitution **S1** $cyc(x = \tan\frac{\alpha}{2})$, it suffices to prove

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \ge \sqrt{3}.$$

The last inequality clearly holds, as $f(x) = \tan\frac{x}{2}$ is convex function on $(0, \pi)$, and the conclusion follows from Jensen's inequality. ■

6. Let x, y, z be positive real numbers. Prove that

$$\sqrt{x(y+z)} + \sqrt{y(z+x)} + \sqrt{z(x+y)} \geq 2\sqrt{\frac{(x+y)(y+z)(z+x)}{x+y+z}}$$

(Darij Grinberg)

Solution. Rewrite the inequality as

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}} + \sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}} + \sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}} \geq 2.$$

Applying substitution **S8**, it suffices to prove that

$$\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \geq 2,$$

where α, β, γ are the angles of a triangle. Using transformation **T1** $cyc(A = \frac{\pi-\alpha}{2})$,
where A, B, C are angles of an acute triangle, the inequality is equivalent to

$$\sin A + \sin B + \sin C \geq 2.$$

There are many ways to prove this fact. We prefer to use Jordan's inequality, that is

$$\frac{2\alpha}{\pi} \leq \sin\alpha \leq \alpha \quad \text{for all } \alpha \in (0, \frac{\pi}{2}).$$

The conclusion immediately follows. ■

7. Let a, b, c be positive real numbers such that $a+b+c+1 = 4abc$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3 \geq \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}.$$

(Daniel Campos Salas, Mathematical Reflections, 2007)

Solution. Rewrite the condition as

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} + \frac{1}{abc} = 4.$$

Observe that we can use substitution **S5** in the following way

$$\left(\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}\right) = \left(2\sin^2\frac{\alpha}{2}, 2\sin^2\frac{\beta}{2}, 2\sin^2\frac{\gamma}{2}\right),$$

where α, β, γ are angles of a triangle. It follows that

$$\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right) = \left(\frac{2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}, \frac{2\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}}, \frac{2\sin\frac{\beta}{2}\sin\frac{\alpha}{2}}{\sin\frac{\gamma}{2}}\right).$$

Then it suffices to prove that

$$\frac{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\frac{\alpha}{2}} + \frac{\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}} + \frac{\sin\frac{\beta}{2}\sin\frac{\alpha}{2}}{\sin\frac{\gamma}{2}} \geq \frac{3}{2} \geq \sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}.$$

The right-hand side of the inequality is well known. For the left-hand side we use trasnformation **T2** backwards. Denote by $x = s - a$, $y = s - b$, $z = s - c$, where s is the semiperimeter of the triangle. The left-hand side is equivalent to

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} \geq \frac{3}{2},$$

which a famous Nesbitt's inequality, and we are done. ■

8. Let $a, b, c \in (0, 1)$ be real numbers such that $ab + bc + ca = 1$. Prove that

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3}{4}\left(\frac{1-a^2}{a} + \frac{1-b^2}{b} + \frac{1-c^2}{c}\right).$$

(Calin Popa)

Solution. We apply substitution **S1** $cyc(a \equiv \tan\frac{A}{2})$, where A, B, C are angles of a triangle. Because $a, b, c \in (0, 1)$, it follows that $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2} \in (0, 1)$, that is A, B, C are angles of an acute triangle. Note that

$$cyc\left(\frac{a}{1-a^2} = \frac{\sin\frac{A}{2}\cos\frac{A}{2}}{\cos A} = \frac{\tan A}{2}\right).$$

Thus the inequality is equivalent to

$$\tan A + \tan B + \tan C \geq 3\left(\frac{1}{\tan A} + \frac{1}{\tan B} + \frac{1}{\tan C}\right).$$

Now observe that if we apply transformation **T1** and the result in Theorem 1, we get

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Hence our inequality is equivalent to

$$(\tan A + \tan B + \tan C)^2 \geq 3(\tan A \tan B + \tan B \tan C + \tan A \tan C).$$

This can be written as

$$\frac{1}{2}(\tan A - \tan B)^2 + (\tan B - \tan C)^2 + (\tan C - \tan A)^2 \geq 0,$$

and we are done. ■

9. Let x, y, z be positive real numbers. Prove that

$$\sqrt{\frac{y+z}{x}} + \sqrt{\frac{z+x}{y}} + \sqrt{\frac{x+y}{z}} \geq \sqrt{\frac{16(x+y+z)^3}{3(x+y)(y+z)(z+x)}}.$$

(Vo Quoc Ba Can, Mathematical Reflections, 2007)

Solution. Note that the inequality is equivalent to

$$\sum_{cyc}(y+z)\sqrt{\frac{(x+y)(z+x)}{x(x+y+z)}} \geq \frac{4(x+y+z)}{\sqrt{3}}.$$

Let use transfromation **T2** and substitution **S8**. We get

$$cyc\left((y+z)\sqrt{\frac{(x+y)(z+x)}{x(x+y+z)}} = \frac{a}{\cos\frac{\alpha}{2}} = 4R\sin\frac{\alpha}{2}\right),$$

and

$$\frac{4(x+y+z)}{\sqrt{3}} = \frac{4R(\sin\alpha + \sin\beta + \sin\gamma)}{\sqrt{3}},$$

where α, β, γ are angles of a triangle with circumradius R. Therefore it suffices to prove that

$$\frac{\sqrt{3}}{2}\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right) \geq \sin\frac{\alpha}{2}\cos\frac{\alpha}{2} + \sin\frac{\beta}{2}\cos\frac{\beta}{2} + \sin\frac{\gamma}{2}\cos\frac{\gamma}{2}.$$

Because $f(x) = \cos\frac{x}{2}$ is a concave function on $[0, \pi]$, from Jensen's inequality we obtain

$$\frac{\sqrt{3}}{2} \geq \frac{1}{3}\left(\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2}\right).$$

Finally, we observe that $f(x) = \sin\frac{x}{2}$ is an increasing function on $[0, \pi]$, while $g(x) = \cos\frac{x}{2}$ is a decreasing function on $[0, \pi]$. Using Chebyschev's inequality, we have

$$\frac{1}{3}\left(\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}\right)\left(\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2}\right) \geq \sum \sin\frac{\alpha}{2}\cos\frac{\alpha}{2},$$

and the conclusion follows. ■

Problems for independent study

1. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}.$$

(Romanian Mathematical Olympiad, 2005)

2. Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1}+\sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1}+\sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1} \geq 6$$

(A. Teplinsky, Ukraine, 2005)

3. Let a, b, c be positive real numbers such that $ab+bc+ca=1$. Prove that

$$\frac{1}{\sqrt{a+b}}+\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}} \geq 2+\frac{1}{\sqrt{2}}.$$

(Le Trung Kien)

4. Prove that for all positive real numbers a, b, c,

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

(APMO, 2004)

5. Let x, y, z be positive real numbers such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$. Prove that

$$\sqrt{x+yz}+\sqrt{x+yz}+\sqrt{x+yz} \geq \sqrt{xyz}+\sqrt{x}+\sqrt{y}+\sqrt{z}.$$

(APMO, 2002)

6. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} \geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

(Mircea Lascu)

7. Let a, b, c be positive real numbers, such that $a+b+c=\sqrt{abc}$. Prove that

$$ab+bc+ca \geq 9(a+b+c).$$

(Belarus, 1996)

8. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} + 2\sqrt{\frac{abc}{(a+b)(b+c)(c+a)}} \geq 2$$

(Bui Viet Anh)

9. Let a, b, c be positive real numbers such that $a+b+c = abc$. Prove that

$$(a-1)(b-1)(c-1) \leq 6\sqrt{3} - 10.$$

(Gabriel Dospinescu, Marian Tetiva)

10. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2+abc = 4$. Prove that

$$0 \leq ab+bc+ca-abc \leq 2.$$

(Titu Andreescu, USAMO, 2001)

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