Algebra Qualifying Exam, Fall 1998: Part I

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming. If you have any questions about notation, terminology the meaning of a problem or the level of detail appropriate, please do not hesitate to ask the proctor.

Notation:
$\mathbb{Z}$: Integers
$\mathbb{Q}$: Rational Field
$\mathbb{R}$: Real Field
$\mathbb{C}$: Complex Field
$GL(\ )$: Full linear group
$\mathbb{F}_q$: Finite field with q elements

1. Classify groups of order $171 = 9 \cdot 19$.

2. Find all groups which can occur as the Galois group of the splitting field over $\mathbb{F}_5$ of a polynomial of degree 9. (The polynomial is not assumed irreducible.)

3. (a) Let p be an odd prime. Explain why $-1 \in \mathbb{Z}/(p)$ is a square if and only if $p \equiv 1$ mod 4.

(b) You may assume the fact that the ring $\mathbb{Z}[i]$ of Gaussian integers is a principal ideal domain. Show that an odd prime $p \in \mathbb{Z}$ is irreducible in $\mathbb{Z}[i]$ if and only if $p \equiv 3$ mod 4. [**Hint**: Use (a).]

4. Suppose that V is a finite dimensional complex vector space and suppose that $S_1, \cdots, S_n$ are endomorphisms of V such that each S_i is diagonalizable and $S_i S_j = S_j S_i$ for all i, j. Show that there is a basis of V consisting of vectors each of which is an eigenvector for all S_i.

5. Let $F \subset K$ be subfields of the complex numbers such that K is a finite algebraic extension of F. Let $\zeta \in \mathbb{C}$.

(a) If ζ is *transcendental* over K, prove that $[K(\zeta) : F(\zeta)] = [K : F]$.

(b) Give an example of $F \subset K$ and ζ *algebraic* over K such that $[K(\zeta) : F(\zeta)]$ does not divide $[K : F]$.

Algebra Qualifying Exam, Fall 1998: Part II

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Notation:
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1. Let A be an abelian group with generators x, y and z subject to the relations

$$\begin{aligned} 2x + 2y - 16z &= 0, \\ 8x + 4y + 2z &= 0, \\ 2x + y - 22z &= 0. \end{aligned}$$

What is the structure of A as a direct sum of cyclic groups?

2. Use linear algebra to prove that if $F \subset E$ is a cyclic Galois field extension then there is an F-vector space basis of E of the form $\{\sigma(x) | \sigma \in \mathrm{Gal}(E/F)\}$, for some $x \in E$.

3. (a) Assume that A is a commutative Noetherian integral domain. Show that every nonzero noninvertible element of A can be written as a finite product of irreducible elements. [**Definition**: a noninvertible element $p \neq 0$ of A is *irreducible* if whenever $p = bc$ with $b, c \in A$ either b or c is invertible in A.]

(b) Give an example of a Noetherian integral domain which is not a unique factorization domain.

4. Let G be the group of order 20 with generators σ and τ and relations $\sigma^4 = \tau^5 = 1$, $\sigma\tau\sigma^{-1} = \tau^2$. Determine the conjugacy classes of G and compute the character table of the irreducible complex representations of G.

5. (a) Find the Galois group of $x^5 + 3x^2 + 1$ over the prime fields $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_5$.

Hint: The only irreducible quadratic over $\mathbb{F}_2$ is $x^2 + x + 1$.

(b) Find the Galois group of $x^5 + 3x^2 + 1$ over $\mathbb{Q}$.

Hint: Use part (a).