Problems
in
Matrix Calculus

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Preface

The purpose of this book is to supply a collection of problems in matrix calculus.

Prescribed books for problems.

1) Matrix Calculus and Kronecker Product:
A Pratcial Approach to Linear and Multilinear Algebra, 2nd edition

by Willi-Hans Steeb and Yorick Hardy
World Scientific Publishing, Singapore 2011
ISBN 978 981 4335 31 7
`http://www.worldscibooks.com/mathematics/8030.html`

2) Problems and Solutions in Introductory and Advanced Matrix Calculus

by Willi-Hans Steeb
World Scientific Publishing, Singapore 2006
ISBN 981 256 916 2
`http://www.worldscibooks.com/mathematics/6202.html`

3) Continous Symmetries, Lie Algebras, Differential Equations and Computer Algebra, second edition

by Willi-Hans Steeb
World Scientific Publishing, Singapore 2007
ISBN 981-256-916-2
`http://www.worldscibooks.com/physics/6515.html`

4) Problems and Solutions in Quantum Computing and Quantum Information, second edition

by Willi-Hans Steeb and Yorick Hardy
World Scientific, Singapore, 2006
ISBN 981-256-916-2
`http://www.worldscibooks.com/physics/6077.html`

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Contents

Notation

$:=$	is defined as
$\in$	belongs to (a set)
$\notin$	does not belong to (a set)
$\cap$	intersection of sets
$\cup$	union of sets
$\emptyset$	empty set
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
	space of column vectors with n real components
$\mathbb{C}^n$	n-dimensional complex linear space
	space of column vectors with n complex components
$\mathcal{H}$	Hilbert space
i	$\sqrt{-1}$
$\Re z$	real part of the complex number z
$\Im z$	imaginary part of the complex number z
$\lvert z\rvert$	modulus of complex number z
	$\lvert x+iy\rvert = (x^2+y^2)^{1/2}, \ \ x, y \in \mathbf{R}$
$T \subset S$	subset T of set S
$S \cap T$	the intersection of the sets S and T
$S \cup T$	the union of the sets S and T
$f(S)$	image of set S under mapping f
$f \circ g$	composition of two mappings $(f \circ g)(x) = f(g(x))$
$\mathbf{x}$	column vector in $\mathbf{C}^n$
$\mathbf{x}^T$	transpose of $\mathbf{x}$ (row vector)
$\mathbf{0}$	zero (column) vector
$\lVert . \rVert$	norm
$\mathbf{x} \cdot \mathbf{y} \equiv \mathbf{x}^* \mathbf{y}$	scalar product (inner product) in $\mathbf{C}^n$
$\mathbf{x} \times \mathbf{y}$	vector product in $\mathbf{R}^3$
A, B, C	$m \times n$ matrices
$\det(A)$	determinant of a square matrix A
tr(A)	trace of a square matrix A
$\mathrm{rank}(A)$	rank of matrix A
A^T	transpose of matrix A

$\overline{A}$	conjugate of matrix A
A^*	conjugate transpose of matrix A
$A^\dagger$	conjugate transpose of matrix A (notation used in physics)
A^{-1}	inverse of square matrix A (if it exists)
I_n	$n \times n$ unit matrix
I	unit operator
0_n	$n \times n$ zero matrix
AB	matrix product of $m \times n$ matrix A and $n \times p$ matrix B
$A \bullet B$	Hadamard product (entry-wise product) of $m \times n$ matrices A and B
$[A, B] := AB - BA$	commutator for square matrices A and B
$[A, B]_+ := AB + BA$	anticommutator for square matrices A and B
$A \otimes B$	Kronecker product of matrices A and B
$A \oplus B$	Direct sum of matrices A and B
δ_{jk}	Kronecker delta with $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$
λ	eigenvalue
ϵ	real parameter
t	time variable
$\hat{H}$	Hamilton operator

The Pauli spin matrices are used extensively in the book. They are given by

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In some cases we will also use σ_1, σ_2 and σ_3 to denote σ_x, σ_y and σ_z .

Chapter 1

Basic Operations

Problem 1. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \equiv \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \equiv \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

where $\cdot$ denotes the scalar product and $\times$ the vector product.

Problem 2. Consider the three linear independent normalized column vectors in $\mathbb{R}^3$

$$\mathbf{a}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{a}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

(i) Find the volume

$$V_{\mathbf{a}} := \mathbf{a}_1^T (\mathbf{a}_2 \times \mathbf{a}_3).$$

(ii) From the three vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ we form the matrix

$$\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}.$$

Find the determinant. Discuss.
(iii) Find the vectors

$$\mathbf{b}_1 = \frac{1}{V_{\mathbf{a}}} \mathbf{a}_2 \times \mathbf{a}_3, \quad \mathbf{b}_2 = \frac{1}{V_{\mathbf{a}}} \mathbf{a}_3 \times \mathbf{a}_1, \quad \mathbf{b}_3 = \frac{1}{V_{\mathbf{a}}} \mathbf{a}_1 \times \mathbf{a}_2$$

where $\times$ denotes the vector product. Are the vectors linearly independent?

Problem 3. Consider the normalized vector $\mathbf{v}_0 = (1 \quad 0 \quad 0)^T$ in $\mathbb{R}^3$. Find three normalized vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ such that

$$\sum_{j=0}^{3} \mathbf{v}_j = \mathbf{0}, \qquad \mathbf{v}_j^T \mathbf{v}_k = -\frac{1}{3} \quad (j \neq k)\,.$$

Problem 4. Let $\mathbf{x} \in \mathbb{R}^3$.
(i) Find all solutions of

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(ii) Find all solutions of

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}.$$

Problem 5. (i) Find four normalized vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ in $\mathbb{R}^3$ such that

$$\mathbf{a}_j^T \mathbf{a}_k = \frac{4}{3}\delta_{jk} - \frac{1}{3} = \begin{cases} \;\;1 & \text{for } j = k \\ -1/3 & \text{for } j \neq k \end{cases}.$$

(ii) Calculate the vector and the matrix

$$\sum_{j=1}^{4} \mathbf{a}_j, \qquad \frac{3}{4}\sum_{j=1}^{4} \mathbf{a}_j \mathbf{a}_j^T\,.$$

Discuss.

Problem 6. Find the set of all four (column) vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ in $\mathbb{R}^2$ such that the following conditions are satisfied

$$\mathbf{v}_1^T \mathbf{u}_2 = 0, \quad \mathbf{v}_2^T \mathbf{u}_1 = 0, \quad \mathbf{v}_1^T \mathbf{u}_1 = 1, \quad \mathbf{v}_2^T \mathbf{u}_2 = 1\,.$$

Problem 7. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the standard basis in $\mathbb{R}^3$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(i) Consider the normalized vectors

$$\mathbf{a} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), \qquad \mathbf{b} = \frac{1}{\sqrt{3}}(-\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3),$$

$$\mathbf{c} = \frac{1}{\sqrt{3}}(-\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3), \qquad \mathbf{d} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)\,.$$

These vectors are the unit vectors giving the direction of the four bonds of an atom in the diamond lattice. Show that the four vectors are linearly dependent.
(ii) Find the scalar products $\mathbf{a}^T\mathbf{b}$, $\mathbf{b}^T\mathbf{c}$, $\mathbf{c}^T\mathbf{d}$, $\mathbf{d}^T\mathbf{a}$.

Problem 8. Let $\mathbf{u}$, $\mathbf{v}$ be (column) vectors in $\mathbb{R}^n$. What does

$$A = \sqrt{|(\mathbf{u}^T\mathbf{u})(\mathbf{v}^T\mathbf{v}) - (\mathbf{u}^T\mathbf{v})^2|}$$

calculate?

Problem 9. Let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

be two normalized vectors in $\mathbb{R}^3$. Assume that $\mathbf{x}^T\mathbf{y} = 0$, i.e. the vectors are orthogonal. Is the vector $\mathbf{x} \times \mathbf{y}$ a unit vector again? Here $\times$ denotes the vector product.

Problem 10. Consider the 2×2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{11} \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $a_{11}, a_{12} \in \mathbb{R}$. Can the expression

$$A^3 + 3AC(A + C) + C^3$$

be simplified for computation?

Problem 11. Let A, B be 2×2 matrices. Let $AB = 0_2$ and $BA = 0_2$. Can we conclude that at least one of the two matrices is the 2×2 zero matrix? Prove or disprove.

Problem 12. Let A, C be $n \times n$ matrices over $\mathbb{R}$. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{b}$, $\mathbf{d}$ be column vectors in $\mathbb{R}^n$. Write the system of equations

$$(A + iC)(\mathbf{x} + i\mathbf{y}) = (\mathbf{b} + i\mathbf{d})$$

as a $2n \times 2n$ set of real equations.

Problem 13. (i) Consider the Hilbert space $M_2(\mathbb{R})$ of the 2×2 matrices over $\mathbb{R}$. Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

are linearly independent.
(ii) Use the Gram-Schmidt orthonormalization technqiue to find an orthonormal basis for $M_2(\mathbb{R})$.

Problem 14. Let A, B be symmetric matrices over $\mathbb{R}$. What is the condition on A, B such that AB is symmetric?

Problem 15. Let A, B be positive definite matrices. Is AB also positive definite? If not, what is the condition on A, B such that AB is positive definite.

Problem 16. Let $m \geq 1$ and $N \geq 2$. Assume that $N > m$. Let X be an $N \times m$ matrix over $\mathbb{R}$ such that $X^* X = I_m$, where I_m is the $m \times m$ unit matrix.
(i) We define

$$P := XX^* .$$

Calculate P^2, P^* and $\text{tr}P$.
(ii) Give an example for such a matrix X, where $m = 1$ and $N = 2$.

Problem 17. (i) Compute the matrix product

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 4 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

(ii) Write the quadratic polynomial

$$3x_1^2 - 8x_1x_2 + 2x_2^2 + 6x_1x_3 - 3x_3^2$$

in matrix form.

Problem 18. Consider the 4×4 matrix

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

Calculate N^2, N^3, N^4. Is the matrix nilpotent?

Problem 19. Is the product of two $n \times n$ nilpotent matrices nilpotent?

Problem 20. Given the 2×2 matrix A. Find all 2×2 matrics X such that

$$AX = XA\,.$$

Problem 21. Consider the matrix A and the vector $\mathbf{b}$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Are the vectors $\mathbf{b}$, $A\mathbf{b}$, $A^2\mathbf{b}$, $A^3\mathbf{b}$ linearly independent?

Problem 22. Let A be an $n \times n$ matrix over $\mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$. Show that

$$\Re(\mathbf{x}^* A\mathbf{x}) \equiv \frac{1}{2}\mathbf{x}^*(A + A^*)\mathbf{x}\,.$$

Problem 23. Let A be an $n \times n$ hermitian matrix and P be an $n \times n$ projection matrix. Then PAP is again a hermitian matrix. Is this still true if A is a normal matrix, i.e. $AA^* = A^*A$?

Problem 24. Let A, B be normal $n \times n$ matrices. Assume that $AB^* = B^*A$ and $BA^* = A^*B$.
(i) Show that their sum $A + B$ is normal.
(ii) Show that their product AB is normal.

Problem 25. An $n \times n$ matrix over $\mathbb{C}$ is called normal if $MM^* = M^*M$. Let $a, b \in \mathbb{C}$. What is the condition on a, b such that the 2×2 matrix

$$M = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

is normal?

Problem 26. An $n \times n$ matrix is called *nilpotent* if some power of it is th zero matrix, i.e. there is a postive integer p such that $A^p = 0_n$. Show that every nonzero nilpotent matrix is nondiagonable.

Problem 27. Let B be an $n \times n$ hermitian matrix. Is iB skew-hermitian?

Problem 28. Let A be an $n \times n$ *normal matrix*, i.e. $AA^* = A^*A$. Show that $\ker A = \ker A^*$, where ker denotes the kernel.

Problem 29. Let A be an $n \times n$ hermitian matrix. Show that A^m is a hermitian matrix for all $m \in \mathbb{N}$.

Problem 30. Let A be a hermitian $n \times n$ matrix and $A \neq 0$. Show that $A^m \neq 0$ for all $m \in \mathbb{N}$.

Problem 31. Show that if hermitian matrices S and T are positive semi-definite and commute ($ST = TS$), then their product ST is also positive semi-definite. We have to show that

$$(ST\mathbf{u})^*\mathbf{u} \geq 0$$

for all $\mathbf{u} \in \mathbb{C}^n$.

Problem 32. An $n \times n$ matrix is called *normal* if $AA^* = A^*A$. Obviously, a hermitian matrix is normal. Give a 3×3 matrix which is normal but not hermitian.

Problem 33. Let A be an $n \times n$ matrix with $A^2 = 0$. Is the matrix $I_n + A$ invertible?

Problem 34. Let A be an $n \times n$ matrix with $A^3 = 0$. Show that $I_n + A$ has an inverse.

Problem 35. Let A, B be $n \times n$ matrices and c a constant. Assume that the inverses of $(A - cI_n)$ and $(A + B - cI_n)$ exist. Show that

$$(A - cI_n)^{-1}B(A + B - cI_n)^{-1} \equiv (A - cI_n)^{-1} - (A + B - cI_n)^{-1}.$$

Problem 36. Represent the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{pmatrix} \qquad \text{(relative to the natural basis)}$$

relative to the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Problem 37. Consider the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Let n be a positive integer. Calculate $R^n(\theta)$.

Problem 38. An $n \times n$ matrix K is called a *Cartan matrix* if it satisfies the following properties
(i) $K_{jj} = 2$ for $j = 1, \ldots, N$.
(ii) K_{jk} is a nonpositive integer if $j \neq k$.
(iii) $K_{jk} = 0$ if and only if $K_{jk} = 0$.
(iv) K is postive definite, i.e. it has rank n.

Find a 2×2 Cartan matrix.

Problem 39. Let B, C be $n \times n$ matrices and 0_n the $n \times n$ zero matrix. Consider the $2n \times 2n$ matrix

$$A = \begin{pmatrix} 0_n & B \\ C & 0_n \end{pmatrix}.$$

Find A^2.

Problem 40. Find a 2×2 matrix which is normal but not hermitian.

Problem 41. Consider the 2×2 matrix

$$A = \begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{R}$. Find the condition on α, β such that the inverse matrix exists. Find the inverse in this case.

Problem 42. An $n \times n$ matrix over $\mathbb{R}$ is orthogonal if and only if the columns of A form an orthogonal basis in $\mathbb{R}^3$. Show that the matrix

$$\begin{pmatrix} \sqrt{3}/3 & 0 & -\sqrt{6}/3 \\ \sqrt{3}/3 & \sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{3}/3 & -\sqrt{2}/2 & \sqrt{6}/6 \end{pmatrix}$$

is orthogonal.

Problem 43. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Can one find a permutation matrix such that $A = PBP^T$?

Problem 44. Let A be an $n \times n$ matrix. Then A can be written as $A = HU$, where H is an $n \times n$ positive semi-definite matrix and U a unitary matrix. Show that $H^2U = UH^2$ if A is normal, i.e. $A^*A = AA^*$.

Problem 45. Can one find an orthogonal matrix over $\mathbb{R}$ such that

$$R^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ?$$

Problem 46. Let 0_n be the $n \times n$ zero matrix and I_n be the $n \times n$ identity matrix. Find an invertible $2n \times 2n$ matrix T such that

$$T^{-1} \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} T = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} .$$

Problem 47. Find all 2×2 matrices g over $\mathbb{C}$ such that

$$\det g = 1, \qquad \eta g^* \eta = g^{-1}$$

where η is the diagonal matrix $\eta = \mathrm{diag}(1, -1)$.

Problem 48. The $(n+1) \times (n+1)$ *Hadamard matrix* $H(n)$ of any dimension is generated recursively as follows

$$H(n) = \begin{pmatrix} H(n-1) & H(n-1) \\ H(n-1) & -H(n-1) \end{pmatrix}$$

where $n = 1, 2, \ldots$ and

$$H(0) = (1) .$$

Find $H(1)$, $H(2)$, and $H(3)$.

Problem 49. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and σ_1, σ_2, σ_3 be the Pauli spin matrices. We define

$$\mathbf{a} \cdot \boldsymbol{\sigma} := a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 .$$

What is the condition on $\mathbf{a}$, $\mathbf{b}$ such that

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) \equiv (\mathbf{a} \cdot \mathbf{b})I_2 + i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}?$$

Here $\times$ denotes the vector product and I_2 is the 2×2 identity matrix.

Problem 50. Let M be an $2n \times 2n$ matrix with $n \geq 1$. Then M can be written in *block form*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $n \times n$ matrices. Assume that M^{-1} exists and that the $n \times n$ matrix D is also nonsingular. Find M^{-1} using this condition.

Problem 51. Let A be an $m \times n$ matrix with $m \geq n$. Assume that A has rank n. Show that there exists an $m \times n$ matrix B such that the $n \times n$ matrix B^*A is nonsingular. The matrix B can be chosen such that $B^*A = I_n$.

Problem 52. Let A be an $m \times n$ matrix over $\mathbb{R}$. Let

$$\mathbf{x} := (1, x, x^2, \ldots, x^{m-1})^T, \qquad \mathbf{y} := (1, y, y^2, \ldots, y^{n-1})^T.$$

Find the extrema of the function

$$p(x, y) = \mathbf{x}^T A \mathbf{y}.$$

Problem 53. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ considered as column vectors. Is

$$\mathbf{v}^* A \mathbf{u} = \mathbf{u}^* A^* \mathbf{v}\,?$$

Problem 54. Two $n \times n$ matrices A, B are called *similar* if there exists an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Show that the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are similar.

Problem 55. Let $\mathbf{u}$, $\mathbf{v}$ be normalized (column) vectors in $\mathbb{C}^n$. Let A be an $n \times n$ positive semidefinite matrix over $\mathbb{C}$. Show that

$$(\mathbf{u}^* \mathbf{v})(\mathbf{u}^* A \mathbf{v}) \geq 0.$$

Problem 56. Let $\epsilon \in [0, 1]$. Show that the 2×2 matrix

$$\Pi = \begin{pmatrix} \epsilon & \sqrt{\epsilon - \epsilon^2} \\ \sqrt{\epsilon - \epsilon^2} & 1 - \epsilon \end{pmatrix}$$

is a projection matrix.

Problem 57. Let $A \in \mathbb{R}^{m\times n}$ be a nonzero matrix. Let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ be vectors such that $c := \mathbf{y}^T A\mathbf{x} \neq 0$. Show that the matrix

$$B := A - c^{-1}A\mathbf{x}\mathbf{y}^T A$$

has rank exactly one less than the rank of A.

Problem 58. Let A be an arbitrary $n\times n$ matrix over $\mathbb{R}$. Can we conclude that A^2 is positive semi-definite?

Problem 59. Let A, B be $n \times n$ idempotent matrices. Show that $A+B$ are idempotent if and only if $AB = BA = 0$.

Problem 60. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that $A+B$ is invertible. Show that

$$(A+B)^{-1}A = I_n - (A+B)^{-1}B$$
$$A(A+B)^{-1} = I_n - B(A+B)^{-1} .$$

Problem 61. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Show that

$$(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}\times\mathbf{v}) \equiv (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})^2 .$$

Problem 62. Consider the vector space of 2×2 matrices over $\mathbb{R}$ and the matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} .$$

Are these matrices linearly independent? Which of these matrices are normal matrices?

Problem 63. Let $\alpha, \beta \in \mathbb{C}$. What is the condition on α, β such that

$$A(\alpha,\beta) = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$$

is a normal matrix?

Problem 64. Find all invertible 2×2 matrices such that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} S \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} S \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Problem 65. (i) Consider the two-dimensional Euclidean space and let $\mathbf{e}_1$, $\mathbf{e}_2$ be the standard basis

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consider the vectors

$$\mathbf{v}_0 = \mathbf{0}, \quad \mathbf{v}_1 = \frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2, \quad \mathbf{v}_2 = -\frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2.$$

$$\mathbf{v}_3 = -\frac{1}{2}\mathbf{e}_1 + \frac{\sqrt{3}}{2}\mathbf{e}_2, \quad \mathbf{v}_4 = \frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2, \quad \mathbf{v}_5 = -\mathbf{e}_1, \quad \mathbf{v}_6 = \mathbf{e}_1 .$$

Find the distance between the vectors and select the vectors pairs with the shortest distance.

Problem 66. Given four points in $\mathbb{R}^2$ $\mathbf{x}_i$, $\mathbf{x}_j$, $\mathbf{x}_k$, $\mathbf{x}_\ell$ (pairwise different). One can define their *cross-ratio*

$$r_{ijk\ell} := \frac{|\mathbf{x}_i - \mathbf{x}_j||\mathbf{x}_k - \mathbf{x}_\ell|}{|\mathbf{x}_i - \mathbf{x}_\ell||\mathbf{x}_k - \mathbf{x}_j|}.$$

Show that the cross-rations are invariant under *conformal transformation.*

Problem 67. Consider the vector space $M_2(\mathbb{R})$ of 2×2 matrices over $\mathbb{R}$. Can one find a basis of $M_2(\mathbb{R})$ such that all four matrices are normal and invertible?

Problem 68. Find all 2×2 matrices A over $\mathbb{C}$ such that

$$A^2 = -I_2, \qquad A^* = -A.$$

Extend to 3×3 matrices.

Problem 69. Let $a, b \in \mathbb{R}$ and $a \neq 0$. Find the inverse of the transformation

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}.$$

Problem 70. Let A be an $n \times n$ matrix over $\mathbb{C}$ with $A^2 = I_n$. Can we conclude that A is *normal*?

Problem 71. Consider the 4×4 matrix

$$A(\alpha,\beta,\gamma) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ -\sin\beta\sinh\alpha & \cos\beta & 0 & -\sin\beta\cosh\alpha \\ \sin\gamma\cos\beta\sinh\alpha & \sin\gamma\sin\beta & \cos\gamma & \sin\gamma\cos\beta\cosh\alpha \\ \cos\gamma\cos\beta\sinh\alpha & \cos\gamma\sin\beta & -\sin\gamma & \cos\gamma\cos\beta\cosh\alpha \end{pmatrix}.$$

(i) Is each column a normalized vector in $\mathbb{R}^4$?
(ii) Calculate the scalar product between the column vectors. Discuss.

Problem 72. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be vectors in $\mathbb{R}^3$. Show that (*Lagrange identity*)

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \det\begin{pmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{pmatrix}.$$

Problem 73. Consider the normalized state $\mathbf{u}$ and the permutation matrix P, respectively

$$\mathbf{u} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Are the vectors $\mathbf{u}$, $P\mathbf{u}$, $P^2\mathbf{u}$ linearly independent?

Problem 74. (i) Consider the 3×3 permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find all 3×3 matrices A such that $PAP^T = A$.
(ii) Consider the 4×4 permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Find all 4×4 matrices A such that $PAP^T = A$.

Problem 75. Let $0 \le \theta < \pi/4$. Note that $\sec(x) := 1/\cos(x)$. Consider the matrix

$$A(\theta) = \begin{pmatrix} \sec(2\theta) & -i\tan(2\theta) \\ i\tan(2\theta) & \sec(2\theta) \end{pmatrix}.$$

Is the matrix hermitian? Is the matrix orthogonal? Is the matrix unitary? Is the inverse of $A(\theta)$ given by $A(-\theta)$?

Problem 76. Are the 4×4 matrices

$$P = \frac{1}{2} \begin{pmatrix} 1 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 1 \end{pmatrix}, \qquad \widetilde{P} = I_4 - P$$

projection matrices? If so describe the subspaces of $\mathbb{R}^4$ they project into.

Problem 77. Let A be a positive definite $n \times n$ matrix over $\mathbb{R}$. Let $\mathbf{x} \in \mathbb{R}^n$. Show that $A + \mathbf{x}\mathbf{x}^T$ is positive definite.

Problem 78. Let A be a positive definite $n \times n$ matrix over $\mathbb{R}$. Let $\mathbf{x} \in \mathbb{R}^n$. Show that $A + \mathbf{x}\mathbf{x}^T$ is positive definite.

Problem 79. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that $A^2 = 0_n$. Find the inverse of $I_n + iA$.

Problem 80. Write the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

as a linear combination of the Pauli spin matrices and the 2×2 identity matrix.

Problem 81. (i) Let $x \in \mathbb{R}$. Show that the matrix

$$A(x) = \begin{pmatrix} \cos(x) & 0 & -\sin(x) & 0 \\ 0 & \cos(x) & 0 & -\sin(x) \\ \sin(x) & 0 & \cos(x) & 0 \\ 0 & \sin(x) & 0 & \cos(x) \end{pmatrix}$$

is invertible. Find the inverse.
(ii) Let $x \in \mathbb{R}$. Show that the matrix

$$B(x) = \begin{pmatrix} \cosh(x) & 0 & \sinh(x) & 0 \\ 0 & \cosh(x) & 0 & \sinh(x) \\ \sinh(x) & 0 & \cosh(x) & 0 \\ 0 & \sinh(x) & 0 & \cosh(x) \end{pmatrix}$$

is invertible. Find the inverse.

Problem 82. Consider the 2×2 matrices

$$S = \begin{pmatrix} r & t \\ t & r \end{pmatrix}, \qquad R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Calculate RSR^T. Discuss.

Problem 83. The vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ point to the vertices of a equilateral triangle

$$\mathbf{u} = \begin{pmatrix} 1/\sqrt{3} \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -1/(2\sqrt{3}) \\ 1/2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1/(2\sqrt{3}) \\ -1/2 \end{pmatrix}.$$

Find the area of this triangle.

Problem 84. One can describe a *tetrahedron* in the vector space $\mathbb{R}^3$ by specifying vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ normal to its faces with lengths equal to the faces' area. Give an example.

Problem 85. Consider the 4×4 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & * \\ 1 & -1 & -1 & * \\ 1 & -1 & 1 & * \\ 1 & 1 & -1 & * \end{pmatrix}$$

Find the 4-th column non-zero vector in the matrix A so that this vector is orthogonal to each of three other column vectors of the matrix.

Problem 86. Assume that two planes in $\mathbb{R}^3$ given by

$$kx_1 + \ell x_2 + mx_3 + n = 0, \qquad k'x_1 + \ell' x_2 + m'x_3 + n' = 0$$

be the mirror images with respect to a third plane in $\mathbb{R}^3$ given by

$$ax_1 + bx_2 + cx_3 + d = 0.$$

Show that

$$\begin{pmatrix} k' \\ \ell' \\ m' \end{pmatrix} = \frac{1}{a^2+b^2+c^2} \begin{pmatrix} a^2-b^2-c^2 & 2ab & 2ac \\ 2ab & -a^2+b^2-c^2 & 2bc \\ 2ac & 2bc & -a^2-b^2+c^2 \end{pmatrix} \begin{pmatrix} k \\ \ell \\ m \end{pmatrix}.$$

Problem 87. (i) Consider a tetrahedron defined by the triple of linearly independent vectors $\mathbf{v}_j \in \mathbb{R}^3$, $j = 1, 2, 3$. Show that the normal vectors to the faces defined by two of these vectors, normalized to the area of the face, is given by

$$\mathbf{n}_1 = \frac{1}{2}\mathbf{v}_2 \times \mathbf{v}_3, \quad \mathbf{n}_2 = \frac{1}{2}\mathbf{v}_3 \times \mathbf{v}_1, \quad \mathbf{n}_3 = \frac{1}{2}\mathbf{v}_1 \times \mathbf{v}_2\,.$$

(ii) Show that

$$\mathbf{v}_1 = \frac{2}{3V}\mathbf{n}_2 \times \mathbf{n}_3, \quad \mathbf{v}_2 = \frac{2}{3V}\mathbf{n}_3 \times \mathbf{n}_1, \quad \mathbf{v}_3 = \frac{2}{3V}\mathbf{n}_1 \times \mathbf{n}_2$$

where V is the volume of the tetrahedron given by

$$V = \frac{1}{3!}(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 = \sqrt{\frac{2}{9}(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{n}_3}\,.$$

Problem 88. (i) Find the area of the set

$$S_2 := \{\, (x_1, x_2) \,:\, 1 \geq x_1 \geq x_2 \geq 0 \,\}\,.$$

(ii) Find the volume of the set

$$S_3 := \{\, (x_1, x_2, x_3) \,:\, 1 \geq x_1 \geq x_2 \geq x_3 \geq 0 \,\}\,.$$

Extend the n-dimensions.

Problem 89. Let A be a hermitian $n \times n$ matrix over $\mathbb{C}$ with $A^2 = I_n$. Find the matrix

$$(A^{-1} + iI_n)^{-1}\,.$$

Chapter 2

Linear Equations

Problem 1. (i) Find all 2×2 matrices A over $\mathbb{R}$ with

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} = 1 \qquad (1)$$

and

$$A\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \qquad (2)$$

(ii) Do these matrices form a group under matrix multiplication?

Problem 2. Find all solutions of the linear system

$$\begin{aligned} x_1 + 2x_2 - 4x_3 + x_4 &= 3 \\ 2x_1 - 3x_2 + x_3 + 5x_4 &= -4 \\ 7x_1 - 10x_3 + 13x_4 &= 0\,. \end{aligned}$$

Problem 3. Consider the area-preserving map of the two-dimensional torus (modulo 1)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \qquad A = \begin{pmatrix} 4 & 15 \\ 1 & 4 \end{pmatrix}$$

where $\det A = 1$ (area-preserving). Consider a rational point on the torus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} n_1/p \\ n_2/p \end{pmatrix}$$

where p is a prime number (except 2, 3, 5) and n_1, n_2 are integers between 0 and $p-1$. One finds that the orbit has the following property. It is perdioc and its period T depends on p alone. Consider $p = 7$, $n_1 = 2$, $n_2 = 3$. Find the orbit and the period T.

Problem 4. Solve the linear equation

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} .$$

Problem 5. *Gordan's theorem* tells us the following. Let A be an $m \times n$ matrix over $\mathbb{R}$ and $\mathbf{c}$ be an n-vector in $\mathbb{R}^n$. Then exactly one of the following systems has a solution:

System 1: $A\mathbf{x} < \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$.
System 2: $A^T\mathbf{p} = \mathbf{0}$ and $\mathbf{p} \geq \mathbf{0}$ for some $\mathbf{p} \in \mathbb{R}^m$.

Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

Find out whether system (1) or system (2) has a solution.

Problem 6. *Gordan's theorem* tells us the following: Let A be an $m \times n$ matrix over $\mathbb{R}$. Exactly one of the following systems has a solution:

System 1: $A\mathbf{x} < \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$
System 2: $A^T\mathbf{p} = \mathbf{0}$ and $\mathbf{p} \geq \mathbf{0}$ for some nonzero $\mathbf{p} \in \mathbb{R}^n$.

Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} .$$

Find out whether system (1) or system (2) has a solution.

Problem 7. *Farkas' theorem* tells us the following. Let A be an $m \times n$ matrix over $\mathbb{R}$ and $\mathbf{c}$ be an n-vector in $\mathbb{R}^n$. Then exactly one of the following systems has a solution:

System 1: $A\mathbf{x} \leq \mathbf{0}$ and $\mathbf{c}^T\mathbf{x} > 0$ for some $\mathbf{x} \in \mathbb{R}^n$.

System 2: $A^T\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$ for some $\mathbf{y} \in \mathbb{R}^m$.

Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find out whether system (1) or system (2) has a solution.

Problem 8. Apply the Gauss-Seidel method to solve the linear system

$$\begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Problem 9. Let A be an $n \times n$ matrix. Consider the linear equation $A\mathbf{x} = \mathbf{0}$. If the matrix A has rank r, then there are $n-r$ linearly independent solutions of $A\mathbf{x} = \mathbf{0}$. Let $n = 3$ and

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find the rank of A and the linearly independent solutions.

Problem 10. Consider the curve described by the equation

$$2x^2 + 4xy - y^2 + 4x - 2y + 5 = 0 \tag{1}$$

relative to the natural basis (standard basis $\mathbf{e}_1 = (1\ \ 0)^T$, $\mathbf{e}_2 = (0\ \ 1)^T$).
(i) Write the equation in matrix form.
(ii) Find an orthogonal change of basis so that the equation relative to the new basis has no crossterms, i.e. no $x'y'$ term. This change of coordinate system does not change the origin.

Problem 11. Consider the 2×2 matrix

$$\begin{pmatrix} b & -a \\ a & b \end{pmatrix}$$

with $a, b \in \mathbb{R}$ and positive determinant, i.e. $a^2 + b^2 > 0$.
(i) Solve the equation

$$\begin{pmatrix} b & -a \\ a & b \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} b & a \\ -a & b \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

for the vector $(x_1 \;\; y_1)^T$ with a given vector $(x_0 \;\; y_0)^T$.
(ii) Let

$$M := \begin{pmatrix} b & -a \\ a & b \end{pmatrix}^{-1} \begin{pmatrix} b & a \\ -a & b \end{pmatrix}$$

and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Calculate $M^T J M$.

Problem 12. Suppose that V is a vector space over a field $\mathbb{F}$ and $U \subset V$ is a subspace. We define an equivalence relation $\sim$ on V by $x \sim y$ iff $x - y \in U$. Let $V/U = V/\sim$. Define addition and scalar multiplication on V/U by $[x] + [y] = [x] + [y]$, $c[x] = [cx]$, where $c \in \mathbb{F}$ and

$$[x] = \{\, y \in V \; : \; y \sim x \,\}.$$

Show that these operations do not depend on which representative x we choose.

Problem 13. Consider the vector space $V = \mathbb{C}^2$ and the subspace $U = \{(x_1, x_2) \; : \; x_1 = 2x_2\}$. Find V/U.

Problem 14. Find all solutions of the system of linear equations

$$\begin{pmatrix} 5 & -2 & -4 \\ -2 & 2 & 2 \\ -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Problem 15. Let $b > a$. Consider the system of linear equations

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & x_2^n & \dots & x_n^n \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} b - a \\ (b^2 - a^2)/2 \\ (b^3 - a^3)/3 \\ \vdots \\ (b^{n+1} - a^{n+1})/(n+1) \end{pmatrix}.$$

Let $n = 2$, $a = 0$, $b = 1$, $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$. Find w_0, w_1, w_2.

Problem 16. Let Y, X, A, B, C, E $n \times n$ matrices over $\mathbb{R}$. Consider the system of matrix equations

$$Y + CE + DX = 0_n, \qquad AE + BX = 0_n.$$

Assume that A has an inverse. Eliminate the matrix E and solve the system for Y

Problem 17. Let V be a vector space over a field $\mathbb{F}$. Let W be a subspace of V. We define an *equivalence relation* $\sim$ on V by stating that $v_1 \sim v_2$ if $v_1 - v_2 \in W$. The *quotient space* V/W is the set of equivalence classes $[v]$ where $v_1 - v_2 \in W$. Thus we can say that v_1 is equivalent to v_2 modulo W if $v_1 = v_2 + w$ for some $w \in W$. Let

$$V = \mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \; : \; x_1, x_2 \in \mathbb{R} \right\}$$

and

$$W = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \; : \; x_1 \in \mathbb{R} \right\}.$$

(i) Is

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 4 \\ 1 \end{pmatrix} \sim \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 3 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 4 \\ 1 \end{pmatrix} ?$$

(ii) Give the quotient space.

Problem 18. For the *three-body problem* the following linear transformation plays a role

$$\begin{aligned} X(x_1, x_2, x_3) &= \frac{1}{3}(x_1 + x_2 + x_3) \\ x(x_1, x_2, x_3) &= \frac{1}{\sqrt{2}}(x_1 - x_2) \\ y(x_1, x_2, x_3) &= \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3). \end{aligned}$$

(i) Find the inverse transformation.
(ii) Introduce polar coordinates

$$x(r, \phi) = r \sin \phi, \quad y(r, \phi) = r \cos \phi, \quad r^2 = \frac{1}{3}((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2).$$

Express $(x_1 - x_2)$, $(x_2 - x_3)$, $(x_3 - x_1)$ using this coordinates.

Problem 19. Let $\alpha \in [0, 2\pi)$. Find all solutions of the linear equation

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Thus x_1 and x_2 depends on α.

Problem 20. Consider the partial differential equation (*Laplace equation*)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on} \quad [0,1] \times [0,1]$$

with the boundary conditions

$$u(x,0) = 1, \quad u(x,1) = 2, \quad u(0,y) = 1, \quad u(1,y) = 2\,.$$

Apply the *central difference scheme*

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{j,k} \approx \frac{u_{j-1,k} - 2u_{j,k} + u_{j+1,k}}{(\Delta x)^2}, \qquad \left(\frac{\partial^2 u}{\partial y^2}\right)_{j,k} \approx \frac{u_{j,k-1} - 2u_{j,k} + u_{j,k+1}}{(\Delta y)^2}$$

and then solve the linear equation. Consider the cases $\Delta x = \Delta y = 1/3$ and $\Delta x = \Delta y = 1/4$.

Problem 21. Let $\mathbf{n}$ and $\mathbf{p}$ be vectors in $\mathbb{R}^n$ with $\mathbf{n} \neq \mathbf{0}$. The set of all vectors $\mathbf{x}$ in $\mathbb{R}^n$ which satisfy the equation

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$$

is called a hyperplane through the point $\mathbf{p} \in \mathbb{R}$. We call $\mathbf{n}$ a normal vector for the hyperplane and call $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ a normal equation for the hyperplane. Find $\mathbf{n}$ and $\mathbf{p}$ in $\mathbb{R}^4$ such that we obtain the hyperplane given by

$$x_1 + x_2 + x_3 + x_4 = \frac{7}{2}\,.$$

Note that any hyperplane of the Euclidean space $\mathbb{R}^n$ has exactly two unit normal vectors.

Problem 22. (i) The equation of a line in the Euclidean space $\mathbb{R}^2$ passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$(y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1)\,.$$

Apply this equation to the points in $\mathbb{R}^2$ given by $(x_1, y_1) = (1, 1/2)$, $(x_2, y_2) = (1/2, 1)$. Consider the unit square with the corner points $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ and the map

$$(0,0) \to 0, \;\; (0,1) \to 0, \;\; (1,0 \to 0, \;\; (1,1) \to 1\,.$$

We can consider this as a 2 input AND-gate. Show that the line constructed above classifies this map.

(ii) The equation of a plane in $\mathbb{R}^3$ passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) in $\mathbb{R}^3$ is given by

$$\det \begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{pmatrix} = 0 \, .$$

Apply this equations to the points

$$(1, 1, 1/2), \quad (1, 1/2, 1), \quad (1/2, 1, 1) \, .$$

Consider the unit cube in $\mathbb{R}^3$ with the corner points (vertices)

$$(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1)$$

$$(1,0,0),\ (1,0,1),\ (1,1,0),\ (1,1,1)$$

and the map where all corner points are mapped to 0 except for $(1,1,1)$ which is mapped to 1. We can consider this as a 3 input AND-gate. Show that the plane constructed in (i) separates these solutions.

Chapter 3

Traces, Determinants and Hyperdeterminants

Problem 1. Find all 2×2 matrices over $\mathbb{C}$ that satify the conditions

$$\mathrm{tr}A = 0, \quad A = A^*, \quad A^2 = I_2 \,.$$

Problem 2. Find all 2×2 matrices A such that

$$A^2 = \mathrm{tr}(A)A \,.$$

Calculate $\det(A)$ and $\det(A^2)$ of such a matrix.

Problem 3. Let σ_x, σ_y, σ_z be the Pauli spin matrices. Calculate the trace of σ_x, σ_y, σ_z, $\sigma_x\sigma_y$, $\sigma_x\sigma_z$, $\sigma_y\sigma_z$, $\sigma_x\sigma_y\sigma_z$.

Problem 4. Let A be an $n \times n$ matrix with $A^2 = I_n$. Let B be a matrix with $AB = -BA$, i.e. $[A, B]_+ = 0_n$.
(i) Show that $\mathrm{tr}(B) = 0$.
(ii) Find $\mathrm{tr}(A \otimes B)$.

Problem 5. (i) Consider the two 2×2 matrices

$$A = \begin{pmatrix} a_{11} & 1 \\ a_{21} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 1 \end{pmatrix} .$$

The first column of the matrices A and B agree, but the second column of the two matrices differ. Is

$$\det(A+B) = 2(\det(A)+\det(B))?$$

Problem 6. Let A, B be 2×2 matrices. Assume that $\det(A)=0$ and $\det(B)=0$. Can we conclude that $\det(A+B)=0$?

Problem 7. The oriented volume of an n-simplex in n-dimensional Euclidean space with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, $\ldots$, $\mathbf{v}_n$ is given by

$$\frac{1}{n!}\det(S)$$

where S is the $n\times n$ matrix

$$S := (\mathbf{v}_1-\mathbf{v}_0 \;\; \mathbf{v}_2-\mathbf{v}_0 \;\; \ldots \;\; \mathbf{v}_{n-1}-\mathbf{v}_0 \;\; \mathbf{v}_n-\mathbf{v}_0\,).$$

Thus each column of the $n\times n$ matrix is the difference between the vectors representing two vertices.
(i) Let

$$\mathbf{v}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}.$$

Find the oriented volume.
(ii) Let

$$\mathbf{v}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find the oriented volume.

Problem 8. The area A of a *triangle* given by the coordinates of its vertices

$$(x_0,y_0), \qquad (x_1,y_1), \qquad (x_2,y_2)$$

is

$$A = \frac{1}{2}\det\begin{pmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}.$$

(i) Let $(x_0,y_0)=(0,0)$, $(x_1,y_1)=(1,0)$, $(x_2,y_2)=(0,1)$. Find A.
(ii) A *tetrahedron* is a polyhedron composed of four triangular faces, three of which meet at each vertex. A tetrahedron can be defined by the coordinates of the vertices

$$(x_0,y_0,z_0), \quad (x_1,y_1,z_1), \quad (x_2,y_2,z_2), \quad (x_3,y_3,z_3).$$

The volume V of the tetrahedron is given by

$$V = \frac{1}{6}\det\begin{pmatrix} x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix}.$$

Let

$$(x_0, y_0, z_0) = (0, 0, 0), \quad (x_1, y_1, z_1) = (0, 0, 1),$$

$$(x_2, y_2, z_2) = (0, 1, 0), \quad (x_3, y_3, z_3) = (1, 0, 0)\,.$$

Find the volume V.
(iii) Let

$$(x_0, y_0, z_0) = (+1, +1, +1), \quad (x_1, y_1, z_1) = (-1, -1, +1),$$

$$(x_2, y_2, z_2) = (-1, +1, -1), \quad (x_3, y_3, z_3) = (+1, -1, -1)\,.$$

Find the volume V.

Problem 9. Let A, B be $n \times n$ matrices. Assume that $[A, B] = A$. What can be said about the trace of A?

Problem 10. Find all linearly independent diagonal 3×3 matrices over $\mathbb{R}$ with trace zero.

Problem 11. Let A be a 2×2 matrix over $\mathbb{C}$. Assume that $A^2 = I_2$ and thus $\mathrm{tr}(A^2) = 2$. What can be said about the trace of A?

Problem 12. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that

$$\mathrm{tr}(AB) = 0\,.$$

(i) Can we conclude that $\mathrm{tr}(AB^*) = 0$?
(ii) Consider the case that B is skew-hermitian.

Problem 13. Let

$$A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}, \mathbf{u}), \qquad B = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}, \mathbf{v})$$

be $n \times n$ matrices, where the first $n-1$ columns $\mathbf{a}_1$, $\ldots$, $\mathbf{a}_{n-1}$ are the same and for the last column $\mathbf{u} \neq \mathbf{v}$. Show that

$$\det(A + B) = 2^{n-1}(\det(A) + \det(B))\,.$$

Problem 14. An $n \times n$ tridiagonal matrix ($n \geq 3$) has nonzero elements only in the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal. The determiant of an $n \times n$ tridiagonal matrix can be calculated by the recursive formula

$$\det(A) = a_{n,n} \det[A]_{\{1,\ldots,n-1\}} - a_{n,n-1} a_{n-1,n} \det[A]_{\{1,\ldots,n-2\}}$$

where $\det[A]_{\{1,\ldots,k\}}$ denotes the k-th principal minor, that is, $[A]_{\{1,\ldots,k\}}$ is the submatrix by the first k rows and columns of A. The cost of computing the determinant of a tridiagonal matrix using this recursion is linear in n, while the cost is cubic for a general matrix. Apply this recursion relation to calculate the determinant of the 4×4 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix}.$$

Problem 15. (i) Let A be a 2×2 matrix over $\mathbb{R}$. Assume that

$$\operatorname{tr}(A) = \operatorname{tr}(A^2) = \operatorname{tr}(A^3) = \operatorname{tr}(A^4) = 0\,.$$

Can we conclude that A is the 2×2 zero matrix?
(ii) Assume that A is a normal matrix and satisfies these conditions. Can we conclude that A is the 2×2 zero matrix?

Problem 16. Consider the symmetric $n \times n$ band matrix ($n \geq 3$)

$$M_n = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 1 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 1 \end{pmatrix}$$

with the elements in the field $\mathbb{F}_2$. Show that

$$\det(M_n) = \det(M_{n-1}) - \det(M_{n-2})$$

with the initial conditions $\det M_3 = \det M_4 = 1$. Show that the solution is

$$\det(M_n) = \frac{2\sqrt{3}}{3} \cos\left(\frac{n\pi}{3} - \frac{\pi}{6}\right) \pmod 2\,.$$

Problem 17. Let H be the 8×8 matrix

$$H = \begin{pmatrix} \Sigma & 0 \\ 0 & I_4 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Let A, D, X, Y be 4×4 matrices over $\mathbb{C}$ and

$$M = \begin{pmatrix} A & X \\ Y & D \end{pmatrix}.$$

Find the conditions on the matrix M such that

$$HM + M^* H = 0_8$$

and $\text{tr}A - \text{tr}D = 0$.

Problem 18. Consider the symmetric 3×3 matrix

$$A(\alpha) = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{pmatrix}, \qquad \alpha \in \mathbb{R}.$$

(i) Find the maxima and minima of the function

$$f(\alpha) = \det(A(\alpha)).$$

(ii) For which values of α is the matrix noninvertible?

Problem 19. Let A, B be $n \times n$ hermitian positive definite matrices. Show that

$$\text{tr}(AB) > 0.$$

Problem 20. Let A, B be $n \times n$ matrices over $\mathbb{R}$. Assume that A is invertible. Let t be a nonzero real number. Show that

$$\det(A + tB) = t^n \det(A) \det(A^{-1}B + t^{-1}I_n).$$

Problem 21. Let A be an $n \times n$ invertible matrix over $\mathbb{R}$. Show that A^T is also invertible. Is $(A^T)^{-1} = (A^{-1})^T$?

Problem 22. Let A be an 2×2 matrix over $\mathbb{R}$. Let I_2 be the 2×2 unit matrix and $\mu \in \mathbb{R}$. Find the determinant of the 4×4 matrix

$$\begin{pmatrix} -\mu I_2 & A \\ A^T & -\mu I_2 \end{pmatrix}.$$

Problem 23. Let A be an 2×2 matrix over $\mathbb{R}$. Calculate

$$r = \operatorname{tr}(A^2) - (\operatorname{tr}(A))^2 .$$

What are the conditions on a_{jk} such that $r = 0$?

Problem 24. Let A be a 2×2 symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

over $\mathbb{R}$. We define

$$\frac{\partial}{\partial a_{12}} A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Show that

$$\frac{\partial}{\partial a_{12}} \operatorname{tr} A^2 = \operatorname{tr}\left(\frac{\partial}{\partial a_{12}} A^2 \right) = \operatorname{tr}\left(2A \frac{\partial A}{\partial a_{12}} \right) .$$

Problem 25. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Is

$$\operatorname{tr}(A^* B) = \operatorname{tr}(AB^*) \,?$$

Problem 26. Let A, B be 2×2 matrices. Show that

$$[A, B]_+ \equiv AB + BA = (\operatorname{tr}(AB) - \operatorname{tr}(A)\operatorname{tr}(B)) I_2 + \operatorname{tr}(A) B + \operatorname{tr}(B) A .$$

Can this identity be extended to 3×3 matrices?

Problem 27. Find all nonzero 2×2 matrices A and B such that

$$BA^* = \operatorname{tr}(AA^*) A .$$

Problem 28. Consider the Hilbert space $M_4(\mathbb{C})$ of all 4×4 matrices over $\mathbb{C}$ with the scalar product $\langle A, B \rangle := \operatorname{tr}(AB^*)$, where $A, B \in M_4(\mathbb{C})$. The γ-matrices are given by

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We define the 4×4 matrices

$$\sigma_{jk} := \frac{i}{2}[\gamma_j, \gamma_k], \qquad j < k$$

where $j = 1, 2, 3,\ k = 2, 3, 4$ and $[\,,\,]$ denotes the commutator.
(i) Calculate σ_{12}, σ_{13}, σ_{14}, σ_{23}, σ_{24}, σ_{34}.
(ii) Do the 16 matrices

$$I_4,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_5\gamma_1,\ \gamma_5\gamma_2,\ \gamma_5\gamma_3,\ \gamma_5\gamma_4,\ \sigma_{12},\ \sigma_{13},\ \sigma_{14},\ \sigma_{23},\ \sigma_{24},\ \sigma_{34}$$

form a basis in the Hilbert space $M_4(\mathbb{C})$? If so is the basis orthogonal?

Problem 29. Let $n \geq 2$. An invertible integer matrix, $A \in GL(n, \mathbb{Z})$, generates a toral automorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ via the formula

$$f \circ \pi = \pi \circ A, \qquad \pi : \mathbb{R}^n \to \mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n\,.$$

The set of fixed points of f is given by

$$\mathrm{Fix}(f) := \{\, x^* \in \mathbb{T}^n \,:\, f(x^*) = x^* \,\}\,.$$

Let $\sharp\mathrm{Fix}(f)$ be the number of fixed points of f. Now we have: if $\det(I_n - A) \neq 0$, then

$$\sharp\mathrm{Fix}(f) = |\det(I_n - A)|\,.$$

Let $n = 2$ and

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Show that $\det(I_2 - A) \neq 0$ and find $\sharp\mathrm{Fix}(f)$.

Problem 30. Calculate the determinant of the 4×4 matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

using the *exterior product*. This means calculate

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Problem 31. (i) Let $\alpha \in \mathbb{R}$. Find the determinant of the matrices

$$A(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad B(\alpha) = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix},$$

(ii) Let $\alpha \in \mathbb{R}$. Find the determinant of the matrices

$$A(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}, \qquad B(\alpha) = \begin{pmatrix} \cosh\alpha & i\sinh\alpha \\ -i\sinh\alpha & \cosh\alpha \end{pmatrix},$$

Problem 32. The 3×3 diagonal matrices over $\mathbb{R}$ with trace equal to 0 form a vector space. Provide a basis for this vector space. Using the scalar product $\mathrm{tr}(AB^T)$ for $n \times n$ matrices A, B over $\mathbb{R}$ the elements of the basis should be orthogonal to each other.

Problem 33. Let A be a $n \times n$ matrix with $\det A = -1$. Find $\det(A^{-1})$.

Problem 34. The *Hilbert-Schmidt norm* of an $n \times n$ matrix over $\mathbb{C}$ is given by

$$\|A\|_2 = \sqrt{\mathrm{tr}(A^* A)}\,.$$

Another norm is the *trace norm* given by

$$\|A\|_1 = \mathrm{tr}\sqrt{(A^* A)}\,.$$

Calculate the two norms for the matrix

$$A = \begin{pmatrix} 0 & -2i \\ i & 0 \end{pmatrix}.$$

Problem 35. The $n \times n$ permutation matrices form a group under matrix multiplications. Show that

$$\det(I_n - P) = 0$$

for any $n \times n$ permutation matrices.

Problem 36. Let A be a 3×3 matrix over $\mathbb{R}$. Consider the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Assume that $AP = A$. Is A invertible?

Problem 37. Let $A = (a_{ij})$ be a $2n \times 2n$ skew-symmetric matrix. The *Pfaffian* is defined as

$$\mathrm{Pf}(A) := \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} a_{\sigma(2j-1),\sigma(2j)}$$

where S_{2n} is the symmetric group and $\mathrm{sgn}(\sigma)$ is the signature of permutation σ. Consider the case with $n = 2$, i.e.

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Calculate $\mathrm{Pf}(A)$.

Problem 38. Let A be a skew-symmetric $2n \times 2n$ matrix. For the Pfaffian we have the properties

$$(\mathrm{Pf(A)})^2 = \det(A), \qquad \mathrm{Pf}(BAB^T) = \det(B)\mathrm{Pf}(A)$$
$$\mathrm{Pf}(\lambda A) = \lambda^n \mathrm{Pf}(A), \qquad \mathrm{Pf}(A^T) = (-1)^n \mathrm{Pf}(A).$$

where B is an arbitrary $2n \times 2n$ matrix. Let J be a $2n \times 2n$ skew-symmetric matrix with $\mathrm{Pf}(J) \neq 0$. Let B be a $2n \times 2n$ matrix such that $B^T J B = J$. Show that $\det(B) = 1$.

Problem 39. Consider the *Legendre polynomials* P_j, where

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$p_3(x) = \frac{1}{2}(5x^3 - 3x), \quad p_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Show that

$$\det \begin{pmatrix} p_0(x) & p_1(x) & p_2(x) \\ p_1(x) & p_2(x) & p_3(x) \\ p_2(x) & p_3(x) & p_4(x) \end{pmatrix} = (1 - x^2)^3 \begin{pmatrix} p_0(0) & 0 & p_2(0) \\ 0 & p_2(0) & 0 \\ p_2(0) & 0 & p_4(0) \end{pmatrix}.$$

Problem 40. Let $n \geq 2$. Consider the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ 2 & 3 & 4 & \dots & n & 1 \\ 3 & 4 & 5 & \dots & 1 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & 1 & 2 & \dots & n-2 & n-1 \end{pmatrix}.$$

Show that

$$\det(A) = (-1)^{n(n-1)/2}\frac{1}{2}(n+1)n^{n-1}\,.$$

Problem 41. Let $n \geq 2$. Consider the $n \times n$ matrix

$$A(x) = \begin{pmatrix} c_1 & x & x & \dots & x & x \\ x & c_2 & x & \dots & x & x \\ x & x & c_3 & \dots & x & x \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x & x & x & \dots & x & c_n \end{pmatrix}.$$

Show that

$$\det(A) = (-1)^n (P(n) - xP'(x))$$

where

$$P(x) = (x - c_1)(x - c_2)\cdots(x - c_n)\,.$$

Problem 42. Let V_1 be a hermitian $n \times n$ matrix. Let V_2 be a positive semidefinite $n \times n$ matrix. Let k be a positive integer. Show that

$$\operatorname{tr}((V_2 V_1)^k)$$

can be written as $\operatorname{tr}(V^k)$, where $V := V_2^{1/2} V_1 V_2^{1/2}$.

Problem 43. Consider the 2×2 matrix

$$M = \begin{pmatrix} \cosh(r) - \sinh(r)\cos(2\theta) & -\sinh(r)\sin(2\theta) \\ -\sinh(r)\sin(2\theta) & \cosh(r) + \sinh(r)\cos(2\theta) \end{pmatrix}.$$

Find the determinant of M. Thus show that the inverse of M exists. Find the inverse of M.

Problem 44. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Is $\operatorname{tr}(AB^*) = \operatorname{tr}(A^*B)$?

Problem 45. Let A be an $n \times n$ matrix. Assume that the inverse of A exists, i.e. $\det(A) \neq 0$. Then the inverse $B = A^{-1}$ can be calculated as

$$\frac{\partial}{\partial a_{jk}} \ln(\det(A)) = b_{kj}\,.$$

Apply this formula to the 2×2 matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$.

Problem 46. Show that the determinant of the matrix

$$A = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

is nonzero. Find the inverse of the matrix.

Problem 47. Consider the 2×2 matrix over $\mathbb{C}$

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Calculate $\det(CC^*)$ and show that $\det(CC^*) \geq 0$.

Problem 48. Let $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be an analytic function, where $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^n \times \mathbb{R}^n$. The *Monge-Ampere determinant* $M(\Phi)$ is defined by

$$M(\Phi) := \det \begin{pmatrix} \Phi & \partial\Phi/\partial x_1 & \ldots & \partial\Phi/\partial x_n \\ \partial\Phi/\partial y_1 & \partial^2\Phi/\partial x_1 \partial y_1 & \ldots & \partial^2\Phi/\partial x_n \partial y_1 \\ \vdots & \vdots & \ddots & \vdots \\ \partial\Phi/\partial y_n & \partial^2\Phi/\partial x_1 \partial y_n & \ldots & \partial^2\Phi/\partial x_n \partial y_n \end{pmatrix}.$$

Let $n = 2$ and

$$\Phi(x_1, x_2, y_1, y_2) = x_1^2 + x_2^2 + (x_1 y_1)^2 + (x_2 y_2)^2 + y_1^2 + y_2^2 .$$

Find the Monge-Ampere determinant and the conditions on x_1, x_2, y_1, y_2 such that $M(\Phi) = 0$.

Problem 49. (i) Let $z \in \mathbb{C}$. Find the determinant of

$$A = \begin{pmatrix} 1 & z \\ \bar{z} & z\bar{z} \end{pmatrix}.$$

Is the matrix

$$P_2 = I_2 - \frac{1}{1 + z\bar{z}} A$$

a projection matrix?
(ii) Let $z_1, z_2 \in \mathbb{C}$. Find the determinant of

$$B = \begin{pmatrix} 1 & z_1 & z_2 \\ \bar{z}_1 & z_1\bar{z}_1 & z_2\bar{z}_1 \\ \bar{z}_2 & z_1\bar{z}_2 & z_2\bar{z}_2 \end{pmatrix}.$$

Is the matrix

$$P_3 = I_3 - \frac{1}{1 + z_1\bar{z}_1 + z_2\bar{z}_2} B$$

a projection matrix?

Problem 50. Let T be the 2×2 matrix

$$T = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Calculate $\ln(\det(I_2 - T))$ using the right-hand side of the identity

$$\ln(\det(I_2 - T)) = -\sum_{k=1}^{\infty} \frac{1}{k} \mathrm{tr}(T^k).$$

Problem 51. Let A be an $n \times n$ matrix. Assume that

$$\mathrm{tr}(A^j) = 0, \quad \text{for} \quad j = 1, 2, \dots, n.$$

Can we conclude that $\det(A) = 0$?

Problem 52. Consider the golden mean number $\tau = (\sqrt{5} - 1)/2$ and the matrix

$$F = \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}.$$

Find $\mathrm{tr}(F)$ and $\det(F)$. Since $\det(F) \neq 0$ we have an inverse. Find F^{-1}.

Problem 53. Let A be an $n \times n$ matrix and B be an invertible $n \times n$ matrix. Show that

$$\det(I_n + A) = \det(I_n + BAB^{-1}).$$

Problem 54. Let A be an 2×2 matrix. Show that

$$\det(I_2 + A) = 1 + \mathrm{tr}(A) + \det(A).$$

Can the result extended to $\det(I_3 + A)$?

Problem 55. Let A, B be $n \times n$ matrices over $\mathbb{R}$. Assume that $A = A^T$ (symmetric) and $B = -B^T$ (skew-symmetric). Show that $[A, B]$ is symmetric.

Problem 56. Let A be a 2×2 matrix over $\mathbb{C}$. Let

$$t_1 = \operatorname{tr}(A), \quad t_2 = \operatorname{tr}(A^2), \quad t_3 = \operatorname{tr}(A^3), \quad t_4 = \operatorname{tr}(A^4).$$

Can we reconstruct A from t_1, t_2, t_3, t_4?

Problem 57. The *Levi-Civita symbol* (also called completely antisymmetric constant tensor) is defined by

$$\epsilon_{j_1,j_2,\ldots,j_n} := \begin{cases} +1 & \text{if} \quad j_1, j_2, \ldots, j_n \text{ is an even permutation of } 12\cdots n \\ -1 & \text{if} \quad j_1, j_2, \ldots, j_n \text{ is an odd permutation of } 12\cdots n \\ 0 & \text{otherwise} \end{cases}$$

Let δ_{jk} be the Kronecker delta. Show that

$$\epsilon_{j_1,j_2,\ldots,j_n}\epsilon_{k_1,k_2,\ldots,k_n} = \det \begin{pmatrix} \delta_{j_1k_1} & \delta_{j_2k_1} & \ldots & \delta_{j_nk_1} \\ \delta_{j_1k_2} & \delta_{j_2k_2} & \ldots & \delta_{j_nk_2} \\ \vdots & & & \vdots \\ \delta_{j_1k_n} & \delta_{j_2k_n} & \ldots & \delta_{j_nk_n} \end{pmatrix}.$$

Problem 58. Let $x, \epsilon \in \mathbb{R}$. Find the determinant of the symmetric $n \times n$ matrix

$$A = \begin{pmatrix} x+\epsilon & x & \cdots & x \\ x & x+\epsilon & & x \\ \vdots & & \ddots & \\ x & x & & x+\epsilon \end{pmatrix}.$$

Problem 59. Let $\epsilon \in \mathbb{R}$. Let $A(\epsilon)$ be an invertible $n \times n$ matrix. Assume that the entries a_{jk} are analytic functions of ϵ. Show that

$$\operatorname{tr}\left(A^{-1}(\epsilon)\frac{d}{d\epsilon}A(\epsilon)\right) = \frac{1}{\det(A(\epsilon))}\frac{d}{d\epsilon}\det(A(\epsilon)).$$

Problem 60. Let $\{\, \mathbf{e}_j \,\}$ be the three orthonormal vectors in $\mathbb{Z}^3$

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We consider the face-centered cubic lattice as a sublattice of $\mathbb{Z}^3$ generated by the three primitive vectors

$$\mathbf{e}_1 + \mathbf{e}_2, \qquad \mathbf{e}_1 + \mathbf{e}_3, \qquad \mathbf{e}_2 + \mathbf{e}_3.$$

Form the 3×3 matrix

$$(\mathbf{e}_1 + \mathbf{e}_2 \quad \mathbf{e}_1 + \mathbf{e}_3 \quad \mathbf{e}_2 + \mathbf{e}_3) \, .$$

Show that this matrix has an inverse and find the inverse.

Problem 61. Let A, B, C be $n \times n$ matrices. Show that

$$\mathrm{tr}([A, B]C) = \mathrm{tr}(A[B, C]) \, .$$

Problem 62. (i) Let M be a 2×2 matrix over $\mathbb{R}$. Assume that $\mathrm{tr}(M) = 0$. Show that

$$M^2 = -\det(M) I_2 \, .$$

(ii) Show that

$$e^M = \cos(\sqrt{M}) I_2 + \frac{\sin(\sqrt{\det(M)})}{\sqrt{\det(M)}} M \, .$$

If $\det(M) = 0$ then $\sin(0)/0 = 1$. Both $cos\alpha)$ and $\sin(\alpha)/\alpha$ are even functions of α and thus $\exp(M)$ is independent of the choice of the square root of $\det(M)$.

Problem 63. Consider the $m \times m$ matrix $F(\mathbf{x}) = (f_{jk}(\mathbf{x}))$ $(j, k = 1, 2, \ldots, m)$, where $f_{jk} : \mathbb{R}^n \to \mathbb{R}$ are analytic functions. Assume that $F(\mathbf{x})$ is invertible for all $\mathbf{x} \in \mathbb{R}^n$. Then we have the identities $(j = 1, 2, \ldots, m)$

$$\frac{\partial(\det(F(\mathbf{x})))}{\partial x_j} \equiv \det(F(\mathbf{x})) \mathrm{tr}\left(F^{-1}(\mathbf{x}) \frac{\partial F(\mathbf{x})}{\partial x_j} \right)$$

and

$$\frac{\partial F^{-1}(\mathbf{x})}{\partial x_j} \equiv -F^{-1}(\mathbf{x}) \frac{\partial F(\mathbf{x})}{\partial x_j} F^{-1}(\mathbf{x}) \, .$$

The differentiation is understood entrywise. Apply the identities to the matrix $(m = 2, n = 1)$

$$F(x) = \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix} \, .$$

Problem 64. Let $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable functions. Find the determinant of the 3×3 matrix $A = (a_{jk})$

$$a_{jk} := \frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j} \, .$$

Problem 65. Consider the 3×3 permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Can we find a 3×3 matrix A such that $[P, A]$ is invertible?

Problem 66. Let $n \geq 2$. Consider the $n \times n$ symmetric tridiagonal matrix over $\mathbb{R}$

$$A_n = \begin{pmatrix} c & 1 & 0 & 0 & \cdots & & & & 0 \\ 1 & c & 1 & 0 & \cdots & & & & 0 \\ 0 & 1 & c & 1 & \cdots & & & & 0 \\ \cdots & & & & \cdots & & & & \cdots \\ & & & & \cdots & & & & \\ & & & & \cdots & & & & \\ \cdots & & & & \cdots & & & & \\ 0 & 0 & 0 & & \cdots & 1 & c & 1 & 0 \\ 0 & 0 & 0 & & \cdots & 0 & 1 & c & 1 \\ 0 & 0 & 0 & & \cdots & 0 & 0 & 1 & c \end{pmatrix}$$

where $c \in \mathbb{R}$. Find the determinant of A_n.

Problem 67. An $n \times n$ matrix A is called *idempotent* if $A^2 = A$. Show that

$$\text{rank}(A) = \text{tr}(A).$$

Problem 68. Let A, B, C be $n \times n$ matrices over $\mathbb{C}$. Let $\mathbf{a}_j$, $\mathbf{b}_j$, $\mathbf{c}_j$ $(j = 1, 2, \ldots, n)$ be the j-th column of A, B, C, respectively. Show that if for some $k \in \{1, 2, \ldots, n\}$

$$\mathbf{c}_k = \mathbf{a}_k + \mathbf{b}_k$$

and

$$\mathbf{c}_j = \mathbf{a}_j = \mathbf{b}_j, \quad j = 1, \ldots, k-1, k+1, \ldots, n$$

then

$$\det(C) = \det(A) + \det(B).$$

Problem 69. Let R be an nonsingular $n \times n$ matrix over $\mathbb{C}$. Let A be an $n \times n$ matrix over $\mathbb{C}$ of rank one.
(i) Show that the matrix $R + A$ is nonsingular if and only if

$$\text{tr}(R^{-1}A) \neq -1.$$

(ii) Show that in this case we have

$$(R+A)^{-1} = R^{-1} - (1+\mathrm{tr}(R^{-1}A))^{-1}R^{-1}AR^{-1}\,.$$

(iii) Simplify to the case that $R = I_n$.

Problem 70. Let A be an $n \times n$ diagonal matrix over $\mathbb{C}$. Let B be an $n \times n$ matrix over $\mathbb{C}$ with $b_{jj} = 0$ for all $j = 1, \ldots, n$. Can we conclude that all diagonal elements of the commutator $[A, B]$ are 0?

Problem 71. (i) Find a nonzero 2×2 matrix V such that

$$V^2 = \mathrm{tr}(V)V\,.$$

(ii) Can such a matrix be invertible?

Problem 72. Consider the $(n+1) \times (n+1)$ matrix over $\mathbb{C}$

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots & \ldots & 0 & z_1 \\ 0 & 1 & 0 & \ldots & \ldots & 0 & z_2 \\ 0 & 0 & 1 & \ldots & \ldots & 0 & z_3 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \ldots & 1 & z_n \\ z_1 & z_2 & z_3 & \ldots & \ldots & z_n & 1 \end{pmatrix}.$$

Find the determinant. What is the condition on the z_j's such that A is invertible?

Problem 73. Let $\phi \in \mathbb{R}$. Consider the unitary matrix

$$U(\phi) = \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix}.$$

Find the minima and maxima of the function $\mathrm{tr}(U(\phi))$.

Problem 74. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that B is invertible. Find

$$\det(I_n + BAB^{-1})\,.$$

Problem 75. Let $z_k = x_k + iy_k$, where $x_k, y_k \in \mathbb{R}$ and $k = 1, \dots, n$. Find the $2n \times 2n$ matrix A such that

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \\ \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{pmatrix} .$$

Find the determinant of the matrix A.

Problem 76. Let A, B be $n \times n$ matrices over $\mathbb{C}$. We define the product

$$A \star B := \frac{1}{2}(AB + BA) - \frac{1}{n}\mathrm{tr}(AB)I_n \,.$$

(i) Find the trace of $A \star B$.
(ii) Is the product commutative? Is the product associative?

Problem 77. Let A, B be $n \times n$ matrices over $\mathbb{R}$ with $\det(A) = 1$ and $\det(B) = 1$. This means A, B are elements of the Lie group $SL(n, \mathbb{R})$. Can we conclude that

$$\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\mathrm{tr}(B) \,.$$

Problem 78. Let A, B be two 2×2 matrices. We define the product

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix} .$$

(i) Find the determinant and trace of $A \star B$. Express the result using $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\det(A)$, $\det(B)$.
(ii) Assume that the inverse of A and B exists. Is

$$(A \star B)^{-1} = A^{-1} \star B^{-1} \,?$$

Problem 79. Let A be an $n \times n$ invertible matrix over $\mathbb{C}$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Then we have the identity

$$\det(A + \mathbf{x}\mathbf{y}^*) \equiv \det(A)(1 + \mathbf{y}^* A^{-1}\mathbf{x}) \,.$$

Can we conclude that $A+\mathbf{x}\mathbf{y}^*$ is also invertible?

Problem 80. Consider a triangle embedded in $\mathbb{R}^3$. Let $\mathbf{v}_j = (x_j, y_j, z_j)$ ($j = 1, 2, 3$) be the coordinates of the vertices. Then the area A of the triangle is given by

$$A = \frac{1}{2}\|(\mathbf{v}_2 - \mathbf{v}_1) \times (\mathbf{v}_1 - \mathbf{v}_3)\| = \frac{1}{2}\|(\mathbf{v}_3 - \mathbf{v}_1) \times (\mathbf{v}_3 - \mathbf{v}_2)\|$$

where $\times$ denotes the vector product and $\|.\|$ denotes the Euclidean norm. The area of the triangle can also be found via

$$A = \frac{1}{2}\sqrt{\left(\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}\right)^2 + \left(\det\begin{pmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{pmatrix}\right)^2 + \left(\det\begin{pmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{pmatrix}\right)^2}.$$

Consider

$$\mathbf{v}_1 = (1, 0, 0), \quad \mathbf{v}_2 = (0, 1, 0), \quad \mathbf{v}_3 = (0, 0, 1).$$

Find the area of the triangle using both expressions. Discuss. The triangle could be one of the faces of a tetrahedron.

Problem 81. A *tetrahedron* has four triangular faces. Given the coordinates of the four vertices

$$(x_0, y_0, z_0), \quad (x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad (x_3, y_3, z_3)$$

the volume of the tretrahedron is given by

$$V = \frac{1}{3!}\det\begin{pmatrix} x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix}$$

(i) Given the four vertices $(1, 0, -\sqrt{2})$, $(2, 0, 0)$, $(0, 0, 0)$, $(1, -\sqrt{2}, 0)$ find the volume.
(ii) Derive an equation for surface area of a tretrahedron given by coordinates. Apply it to the vertices given in (i).

Problem 82. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_m$ be vectors in $\mathbb{R}^n$. Show that the parallelepiped determined by those vectors has m-dimensional area

$$\sqrt{\det(V^T U)}$$

where V is the $n \times m$ matrix with $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_m$ as its columns.

Problem 83. The *hyperdeterminant* $\mathrm{Det}(A)$ of the three-dimensional array $A = (a_{ijk}) \in \mathbb{R}^{2\times 2\times 2}$ can be calculated as follows

$$\mathrm{Det}(A) = \frac{1}{4}\left(\det\left(\begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} + \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix}\right)\right.$$
$$\left. - \det\left(\begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} - \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix}\right)\right)^2$$
$$-4\det\begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix}\det\begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix}.$$

Assume that only one of the coefficients a_{ijk} is nonzero. Calculate the hyperdeterminant.

Problem 84. Let $\epsilon_{00} = \epsilon_{11} = 0$, $\epsilon_{01} = 1$, $\epsilon_{10} = -1$, i.e. we consider the 2×2 matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the determinant of a 2×2 matrix $A_2 = (a_{ij})$ with $i, j = 0, 1$ can be defined as

$$\det A_2 := \frac{1}{2}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{\ell=0}^{1}\sum_{m=0}^{1}\epsilon_{ij}\epsilon_{\ell m}a_{i\ell}a_{jm}.$$

Thus

$$\det A_2 = a_{00}a_{11} - a_{01}a_{10}.$$

In analogy the *hyperdeterminant* of the $2 \times 2 \times 2$ array $A_3 = (a_{ijk})$ with $i, j, k = 0, 1$ is defined as

$$\mathrm{Det}A_3 := -\frac{1}{2}\sum_{ii'=0}^{1}\sum_{jj'=0}^{1}\sum_{kk'=0}^{1}\sum_{mm'=0}^{1}\sum_{nn'=0}^{1}\sum_{pp'=0}^{1}\epsilon_{ii'}\epsilon_{jj'}\epsilon_{kk'}\epsilon_{mm'}\epsilon_{nn'}\epsilon_{pp'}a_{ijk}a_{i'j'm}a_{npk'}a_{n'p'm'}.$$

Calculate $\mathrm{Det}A_3$.

Problem 85. Given a $2 \times 2 \times 2$ *hypermatrix*

$$A = (a_{jk\ell}), \qquad j, k, \ell = 0, 1$$

and the 2×2 matrix

$$S = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix}.$$

The multiplication AS which is again a 2×2 hypermatrix is defined by

$$(AS)_{jk\ell} := \sum_{r=0}^{1} a_{jkr}s_{r\ell}.$$

Assume that $\det(S)=1$, i.e. $S \in SL(2,\mathbb{C})$. Show that $\text{Det}(AS)=\text{Det}(A)$. This is a typical problem to apply computer algebra. Write a SymbolicC++ program or Maxima program that solves the problem.

Problem 86. Let $a_j \in \mathbb{R}$ with $j=1,2,3$. Consider the 4×4 matrices

$$A = \frac{1}{2}\begin{pmatrix} 0 & a_1 & 0 & 0 \\ a_1 & 0 & a_2 & 0 \\ 0 & a_2 & 0 & a_3 \\ 0 & 0 & a_3 & 0 \end{pmatrix}, \qquad B = \frac{1}{2i}\begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & a_2 & 0 \\ 0 & -a_2 & 0 & a_3 \\ 0 & 0 & -a_3 & 0 \end{pmatrix}.$$

Find the spectrum of A and B. Find the spectrum of $[A,B]$.

Problem 87. (i) Let A be a 2×2 matrix over $\mathbb{C}$. Given

$$t_1 = \text{tr}(A), \quad t_2 = \text{tr}(A^2), \quad t_3 = \text{tr}(A^3), \quad t_4 = \text{tr}(A^4).$$

Can we reconstruct A from t_1, t_2, t_3, t_4. Does it depend on whether the matrix A is normal?
(ii) Let A be a 2×2 matrix over $\mathbb{C}$. Given

$$d_1 = \det(A), \quad d_2 = \det(A^2), \quad d_3 = \det(A^3), \quad d_4 = \det(A^4).$$

Can we reconstruct A from d_1, d_2, d_3, d_4. Does it depend on whether the matrix A is normal?

Problem 88. Let $A=(a_{jk})$ be an $n\times n$ skew-symmetric matrix over $\mathbb{R}$, i.e. $j,k=1,\ldots,n$. Let $B=(b_{jk})$ be an $n\times n$ symmetric matrix over $\mathbb{R}$ defined by $b_{jk}=b_jb_k$, i.e. $j,k=1,\ldots,n$. Let n be even. Show that

$$\det(A+B)=\det(A).$$

Problem 89. Consider the 3×3 matrix M with entries

$$(M)_{jk} = x_k^{j-1}, \quad j,k=1,2,3$$

Find the determinant of this matrix.

Chapter 4

Eigenvalues and Eigenvectors

Problem 1. (i) Let A, B be 2×2 matrices over $\mathbb{R}$ and vectors $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^2$ such that

$$A\mathbf{x} = \mathbf{y}, \qquad B\mathbf{y} = \mathbf{x}$$

$\mathbf{x}^T\mathbf{y} = 0$ and $\mathbf{x}^T\mathbf{x} = 1$, $\mathbf{y}^T\mathbf{y} = 1$. Show that AB and BA have an eigenvalue $+1$.
(ii) Find all 2×2 matrices A, B which satisfy the conditions given in (i). Use

$$\mathbf{x} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}.$$

Problem 2. Find all the eigenvalues of the 4×4 matrix.

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Problem 3. Let A be an arbitrary 2×2 matrix. Show that

$$A^2 - A\mathrm{tr}A + I_2 \det A = 0$$

and therefore

$$(\mathrm{tr}A)^2 = \mathrm{tr}A^2 + 2\det A\,.$$

Hint. Apply the *Cayley-Hamilton theorem*.

Problem 4. Let A be an $n \times n$ matrix. The matrix A is called *nilpotent* if there is a positive integer r such that $A^r = 0_n$.
(i) Show that the smallest integer r such that $A^r = 0_n$ is smaller or equal to n.
(ii) Find the characteristic polynomial of A.

Problem 5. Find all 2×2 matrices over $\mathbb{R}$ that admit only one eigenvector.

Problem 6. Let $\mathbf{x}$ be a nonzero column vector in $\mathbb{R}^n$ and $n \geq 2$. Consider the $n \times n$ matrix $\mathbf{x}\mathbf{x}^T$. Find one nonzero eigenvalue and the corresponding eigenvector of this matrix.

Problem 7. Consider the 2×2 matrix

$$A(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \qquad a \in \mathbb{R}\,.$$

Can one find a condition on the parameter a so that A has only one eigenvector?

Problem 8. If $\{\, A_j \,\}_{j=1}^{m}$ is a commuting family of matrices that is to say $A_j A_k = A_k A_j$ for every pair from the set, then there exists a unitary matrix V such that for all A_j in the set the matrix

$$\widetilde{A}_j = V^* A_j V$$

is upper triangular. Apply this to the matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Problem 9. Consider the 2×2 matrix

$$A = \begin{pmatrix} 1/4 & 1/2 \\ 1/2 & 1/4 \end{pmatrix}.$$

Let (spectral radius)

$$\rho(A) := \max_{1 \leq j \leq 2} |\lambda_j|$$

where λ_j are the eigenvalues of A.
(i) Check that $\rho(A) < 1$.
(ii) If $\rho(A) < 1$, then

$$(I_2 - A)^{-1} = I_2 + A + A^2 + \cdots$$

Calculate $(I_2 - A)^{-1}$.
(iii) Calculate

$$(I_2 - A)(I_2 + A + A^2 + \cdots + A^k)\,.$$

Problem 10. Consider a symmetric 2×2 matrix A over $\mathbb{R}$ with $a_{11} > 0$, $a_{22} > 0$, $a_{12} < 0$ and $a_{jj} > |a_{12}|$ for $j = 1, 2$. Is the matrix A positive definite?

Problem 11. Let A be a positive definite $n \times n$ matrix. Show that A^{-1} exists and is also positive definite.

Problem 12. Let $c_j \in \mathbb{R}$. Find the eigenvalues of the matrices

$$\begin{pmatrix} 0 & 1 \\ c_1 & c_2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_1 & c_2 & c_3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}.$$

Generalize to the $n \times n$ case.

Problem 13. Let n be a positive integer. Consider the 3×3 matrix with rows of elements summing to unity

$$M = \frac{1}{n}\begin{pmatrix} n-a-b & a & b \\ a & n-2a-c & a+c \\ c & a & n-a-c \end{pmatrix}$$

where the values of a, b, c are such that, $0 \le a$, $0 \le b$, $a+b \le n$, $2a+c \le n$. Thus the matrix is a stochastic matrix. Find the eigenvalues of M.

Problem 14. (i) Find the eigenvalues and normalized eigenvectors of the 3×3 matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(ii) Use the normalized eigenvectors to construct a 3×3 matrix R such that RMR^{-1} is a diagonal matrix.

(iii) Can M be written as

$$M = \sum_{j=1}^{3} \lambda_j \mathbf{v}_j \mathbf{v}_j^T$$

where λ_1, λ_2, λ_3 are the eigenvalues and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are the (column) normalized eigenvectors of M. Prove or disprove.

Problem 15. (i) Find the eigenvalues of the symmetric matrices

$$A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(ii) Extend the results from (i) to find the largest eigenvalue of the symmetric $n \times n$ matrix

$$A_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}.$$

Problem 16. Find the eigenvalues of the 4×4 symmetric matrix

$$\begin{pmatrix} -1 & \alpha & 0 & 0 \\ \alpha & -1/2 & \alpha & 0 \\ 0 & \alpha & 1/2 & \alpha \\ 0 & 0 & \alpha & 1 \end{pmatrix}.$$

Discuss the eigenvalues $\lambda_j(\alpha)$ as functions of α. Can the eigenvalues cross as function of α?

Problem 17. Consider the $n \times n$ cyclic matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \ldots & a_{1n-1} & a_{1n} \\ a_{1n} & a_{11} & a_{12} & a_{13} & \ldots & a_{1n-2} & a_{1n-1} \\ a_{1n-1} & a_{1n} & a_{11} & a_{12} & \ldots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{12} & a_{13} & a_{14} & a_{15} & \ldots & a_{1n} & a_{11} \end{pmatrix}$$

where $a_{jk} \in \mathbb{R}$. Show that

$$\frac{1}{\sqrt{n}} \begin{pmatrix} \epsilon^{2k} \\ \epsilon^{4k} \\ \vdots \\ \epsilon^{2(n-1)k} \\ 1 \end{pmatrix}, \qquad \epsilon \equiv e^{i\pi/n}, \quad 1 \leq k \leq n\,.$$

is a normalized eigenvector of A. Find the eigenvalues.

Problem 18. Let $a, b \in \mathbb{R}$. Find on inspection two eigenvectors and the corresponding eigenvalues of the 4×4 matrix

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & a & 0 & b \\ 0 & 0 & a & b \\ b & b & b & 0 \end{pmatrix}.$$

Problem 19. Let $a, b \in \mathbb{R}$. Find on inspection two eigenvectors and the corresponding eigenvalues of the 4×4 matrix

$$\begin{pmatrix} a & 0 & 0 & b \\ 0 & a & 0 & b \\ 0 & 0 & -a & b \\ b & b & b & 0 \end{pmatrix}.$$

Problem 20. Let $z \in \mathbb{C}$. Find the eigenvalues and eigenvectors of the 3×3 matrix

$$A = \begin{pmatrix} 0 & z & \bar{z} \\ \bar{z} & 0 & z \\ z & \bar{z} & 0 \end{pmatrix}.$$

Discuss the dependence of the eigenvalues on z.

Problem 21. Find the eigenvalues and normalized eigenvectors of the matrix ($\phi \in [0, 2\pi)$)

$$A(\phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\phi} \\ 1 & e^{-i\phi} \end{pmatrix}.$$

Is the matrix invertible? Make the decision by looking at the eigenvalues. If so find the inverse matrix.

Problem 22. Consider the two permutation matrices

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Show that the two matrices have the same (normalized) eigenvectors. Find the commutator $[S, T]$.

Problem 23. Consider the following 3×3 matrix A and vector $\mathbf{v}$ in $\mathbb{R}^3$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} \sin(\alpha) \\ \sin(2\alpha) \\ \sin(3\alpha) \end{pmatrix}$$

where $\alpha \in \mathbb{R}$ and $\alpha \neq n\pi$ with $n \in \mathbb{Z}$. Show that using this vector we can find the eigenvalues and eigenvectors of A.

Problem 24. Consider the symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Find a invertible matrix B such that $B^{-1}AB$ is a diagonal matrix.

Problem 25. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the 4×4 gamma matrices

$$\gamma_1 = \begin{pmatrix} 0_2 & \sigma_1 \\ -\sigma_1 & 0_2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0_2 & \sigma_2 \\ -\sigma_2 & 0_2 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0_2 & \sigma_3 \\ -\sigma_3 & 0_2 \end{pmatrix}$$

and

$$\gamma_0 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}.$$

Find $\gamma_1\gamma_2\gamma_3\gamma_0$ and $\operatorname{tr}(\gamma_1\gamma_2\gamma_3\gamma_0)$.

Problem 26. Let $c \in \mathbb{R}$ and consider the symmetric 3×3 matrix

$$A = \begin{pmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{pmatrix}.$$

(i) Show that c is an eigenvalue of A and find the corresponding eigenvector.

(ii) Find the two other eigenvalues and eigenvectors.

Problem 27. Let $c \in \mathbb{R}$. Consider the symmetric 4×4 matrix

$$A = \begin{pmatrix} 1 & c & 0 & 0 \\ c & 2 & 2c & 0 \\ 0 & 2c & 3 & c \\ 0 & 0 & c & 4 \end{pmatrix}.$$

(i) Find the characteristic equation.
(ii) Show that

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 &= 10 \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 &= 35 - 6c^2 \\ \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 &= 50 - 30c^2 \\ \lambda_1\lambda_2\lambda_3\lambda_4 &= 24 - 30c^2 + c^4 \end{aligned}$$

where λ_1, λ_2, λ_3, λ_4 denote the eigenvalues.

Problem 28. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(i) Find the rank of the matrix. Explain.
(ii) Find the determinant and trace of the matrix.
(iii) Find all eigenvalues of the matrix.
(iv) Find one eigenvector.
(v) Is the matrix positive semidefinite?

Problem 29. Find the eigenvalues of the 3×3 matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

using the trace and the determinant of the matrix and the information that two eigenvalues are the same.

Problem 30. Let B be a 2×2 matrix with eigenvalues λ_1 and λ_2. Find the eigenvalues of the 4×4 matrix

$$X = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b_{11} & b_{12} & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{v}$ be an eigenvector of B with eigenvalue λ. What can be said about an eigenvector of the 4×4 matrix X given by eigenvector $\mathbf{v}$ and eigenvalue of B.

Problem 31. Consider the 2×2 identity matrix I_2 and the 2×2 matrix

$$N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the eigenvalues λ_1, λ_2 and the corresponding normalized eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ of I_2. Then find the eigenvalues μ_1, μ_2 and the corresponding normalized eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of N. Using the normalized eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{v}_1$, $\mathbf{v}_2$ form the 2×2 matrix

$$H = \begin{pmatrix} \mathbf{u}_1^*\mathbf{v}_1 & \mathbf{u}_1^*\mathbf{v}_2 \\ \mathbf{u}_2^*\mathbf{v}_1 & \mathbf{u}_2^*\mathbf{v}_2 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of H. Discuss.

Problem 32. Let A, B be two $n \times n$ matrices over $\mathbb{C}$. The set of all matrices of the form $A - \lambda B$ with $\lambda \in \mathbb{C}$ is said to be a *pencil*. The eigenvalues of the pencil are elements of the set $\lambda(A, B)$ defined by

$$\lambda(A,B) := \{ z \in \mathbb{C} : \det(A - zB) = 0 \}.$$

If $\lambda \in \lambda(A, B)$ and

$$A\mathbf{x} = \lambda B\mathbf{x}, \quad \mathbf{x} \neq \mathbf{0}$$

then $\mathbf{x}$ is referred to as an eigenvector of $A - \lambda B$. Note that λ may be finite, empty or infinite.

Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Find the eigenvalue of the pencil.

Problem 33. Let $a, b \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$M = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{pmatrix}.$$

Problem 34. Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the matrix

$$A(\alpha) = \begin{pmatrix} -1 & \alpha & 0 \\ \alpha & 0 & \alpha \\ 0 & \alpha & 1 \end{pmatrix}.$$

Discuss the dependence of the eigenvalues and eigenvectors of α.

Problem 35. Let $\mathbf{u}$, $\mathbf{v}$ be nonzero column vectors in $\mathbb{R}^n$ and $\mathbf{u} \neq \mathbf{v}$. Consider the $n \times n$ matrix A over $\mathbb{R}$

$$A = \mathbf{u}\mathbf{u}^T + \mathbf{u}\mathbf{v}^T - \mathbf{v}\mathbf{u}^T - \mathbf{v}\mathbf{v}^T.$$

Find the nonzero eigenvalues of A and the corresponding eigenvector.

Problem 36. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $c \in \mathbb{C} \setminus \{0\}$. What are the eigenvalues of cA?

Problem 37. Let A be an $n \times n$ matrix over $\mathbb{R}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues. What can be said about the eigenvalues of the $2n \times 2n$ matrix

$$\begin{pmatrix} 0_n & A \\ A^T & 0_n \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix?

Problem 38. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Let $\alpha, \beta \in \mathbb{C}$. Assume that $A^2 = I_n$ and $B^2 = I_n$ and $AB + BA = 0$. What can be said about the eigenvalues of $\alpha A + \beta B$?

Problem 39. Let A be an $n \times n$ normal matrix, i.e. $AA^* = A^*A$. Let $\mathbf{u}$ be an eigenvector of A, i.e. $A\mathbf{u} = \lambda\mathbf{u}$. Show that $\mathbf{u}$ is also an eigenvector of A^* with eigenvalue $\overline{\lambda}$, i.e.

$$A^*\mathbf{u} = \overline{\lambda}\mathbf{u}.$$

Problem 40. Show that eigenvectors of a normal matrix A corresponding to distinct eigenvalues are orthogonal.

Problem 41. Let A, B be square matrices. Show that AB and BA have the same eigenvalues.

Problem 42. Show that if A is an $n \times m$ matrix and if B is an $m \times n$ matrix, then $\lambda \neq 0$ is an eigenvalue of the $n \times n$ matrix AB if and only if λ is an eigenvalue of the $m \times m$ matrix BA. Show that if $m = n$ then the conclusion is true even for $\lambda = 0$.

Problem 43. Let $A^T = (1/2\,,\, 1/2)^T$. Find the eigenvalues of AA^T and $A^T A$.

Problem 44. We know that a hermitian matrix has only real eigenvalues. Can we conclude that a matrix with only real eigenvalues is hermitian?

Problem 45. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that the eigenvalues of $A^* A$ are nonnegative.

Problem 46. Let $0 \leq x < 1$. Consider the $N \times N$ matrix C (correlation matrix) with the entries

$$C_{jk} := x^{|j-k|}, \quad j, k = 1, \ldots, N\,.$$

Find the eigenvalues of C. Show that if $N \to \infty$ the distribution of its eigenvalues becomes a continuous function of $\phi \in [0, 2\pi]$

$$\lambda(\phi) = \frac{1 - x^2}{1 - 2x\cos\phi + x^2}\,.$$

Problem 47. Let n be a positive integer. Consider the 2×2 matrix

$$T_n = \begin{pmatrix} 2n & 4n^2 - 1 \\ 1 & 2n \end{pmatrix}.$$

Show that the eigenvalues of T_n are real and not of absolute value 1.

Problem 48. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that A is normal if and only if there exists an $n \times n$ unitary matrix U and an $n \times n$ diagonal matrix D such that $D = U^{-1}AU$. Note that $U^{-1} = U^*$.

Problem 49. Let A be a normal $n \times n$ matrix over $\mathbb{C}$.
(i) Show that A has a set of n orthonormal eigenvectors.
(ii) Show that if A has a set of n orthonormal eigenvectors, then A is normal.

Problem 50. The Leverrier's method finds the characteristic polynomial of an $n \times n$ matrix. Find the characteristic polynomial for

$$A \otimes B, \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

using this method. How are the coefficients c_i of the polynomial related to the eigenvalues?

Problem 51. Consider the symmetric matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

over $\mathbb{R}$. Write down the characteristic polynomial $\det(\lambda I_3 - A)$ and express it using the trace and determinant of A.

Problem 52. Let L_n be the $n \times n$ matrix

$$L_n = \begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \ddots & -1 \\ -1 & -1 & -1 & \cdots & n-1 \end{pmatrix}.$$

Find the eigenvalues.

Problem 53. The *Pascal matrix* of order n is defined as

$$P_n := \left(\frac{(i+j-2)!}{(i-1)!(j-1)!} \right), \qquad i, j = 1, \dots, n.$$

Thus

$$P_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

(i) Find the determinant of P_2, P_3, P_4. Find the inverse of P_2, P_3, P_4.
(ii) Find the determinant for P_n. Is P_n an element of the group $SL(n, \mathbb{R})$?

Problem 54. Let A be an $m \times n$ matrix ($m < n$) over $\mathbb{R}$.
(i) Show that at least one eigenvalue of the $n \times n$ matrix $A^T A$ is equal to 0.

(ii) Show that the eigenvalues of the $m \times m$ matrix AA^T are also eigenvalues of $A^T A$.

Problem 55. Find the determinant and eigenvalues of the matrices

$$A_2 = \begin{pmatrix} 0 & a_{12} \\ 1 & a_{22} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & a_{13} \\ 1 & 0 & a_{23} \\ 0 & 1 & a_{33} \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 1 & 0 & 0 & a_{24} \\ 0 & 1 & 0 & a_{34} \\ 0 & 0 & 1 & a_{44} \end{pmatrix}.$$

Extend to the n-dimensional case.

Problem 56. Let A be a hermitian $n \times n$ matrix. Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ are pairwise different. Then the normalized eigenvectors $\mathbf{u}_j$ $(j = 1, \ldots, n)$ satisfy $\mathbf{u}_j^* \mathbf{u}_k = 0$ for $j \neq k$ and $\mathbf{u}_j^* \mathbf{u}_j = 1$. We have (spectral theorem)

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j \mathbf{u}_j^* .$$

Let $\mathbf{e}_k$ $(k = 1, \ldots, n)$ be the standard basis in $\mathbb{C}^n$. Calculate $U^* A U$, where

$$U = \sum_{k=1}^{n} \mathbf{u}_k \mathbf{e}_k^* .$$

Problem 57. Let A be a positive definite $n \times n$ matrix. Thus all the eigenvalues are real and positive. Assume that all the eigenvalues $\lambda_1, \ldots, \lambda_n$ are pairwise different. Then the normalized eigenvectors $\mathbf{u}_j$ $(j = 1, \ldots, n)$ satisfy $\mathbf{u}_j^* \mathbf{u}_k = 0$ for $j \neq k$ and $\mathbf{u}_j^* \mathbf{u}_j = 1$. We have (spectral theorem)

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j \mathbf{u}_j^* .$$

Let $\mathbf{e}_k$ $(k = 1, \ldots, n)$ be the standard basis in $\mathbb{C}^n$. Calculate

$$\ln(A) .$$

Note that the unitary matrix

$$U = \sum_{k=1}^{n} \mathbf{u}_k \mathbf{e}_k^*$$

transforms A into a diagonal matrix, i.e. $\widetilde{A} = U^* A U$ is a diagonal matrix.

Problem 58. Let $j = 1/2, 1, 3/2, 2, \ldots$ and $\phi \in \mathbb{R}$. Consider the $(2j+1) \times (2j+1)$ matrices

$$H = \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ e^{i\phi} & 0 & & & & 0 \end{pmatrix}$$

$$D = \mathrm{diag}(1,\ \omega,\ \omega^2,\ \ldots, \omega^{2j})$$

where $\omega := \exp(i2\pi/(2j+1)$. Is H unitary? Find $\omega DH - HD$.

Problem 59. (i) Find the eigenvalues of the 3×3 matrix

$$A(\alpha) = \begin{pmatrix} e^\alpha & 1 & 1 \\ 1 & e^\alpha & 1 \\ 1 & 1 & e^\alpha \end{pmatrix}.$$

For which values of α is the matrix $A(\alpha)$ not invertible.
(ii) Extend (i) to the $n \times n$ matrix

$$B(\alpha) = \begin{pmatrix} e^\alpha & 1 & \cdots & 1 \\ 1 & e^\alpha & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & e^\alpha \end{pmatrix}.$$

This matrix plays a role for the *Potts model*.

Problem 60. Let A be an 2×2 matrix over $\mathbb{R}$. Let

$$\mathrm{tr}(A) = c_1, \qquad \mathrm{tr}(A^2) = c_2\,.$$

Can $\det(A)$ be calculated from c_1, c_2?

Problem 61. Let A be an $n \times n$ matrix with entries $a_{jk} \geq 0$ and with positive spectral radius ρ. Then there is a (column) vector $\mathbf{x}$ with $x_j \geq 0$ and a (column) vector $\mathbf{y}$ such that the following conditions hold:

$$A\mathbf{x} = \rho\mathbf{x}, \quad \mathbf{y}^T A = \rho\mathbf{y}, \quad \mathbf{y}^T\mathbf{x} = 1\,.$$

Consider the 2×2 matrix

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Show that B has a positive spectral radius. Find the vectors $\mathbf{x}$ and $\mathbf{y}$.

Problem 62. Let A be an $n \times n$ matrix over $\mathbb{C}$. If λ is not an eigenvalue of A, then the matrix $(A - \lambda I_n)$ has an inverse, namely the *resolvent*

$$R_\lambda = (A - \lambda I_n)^{-1}.$$

Let λ_j be the eigenvalues of A. For $|\lambda| \geq a$, where a is any positive constant greater than all the $|\lambda_j|$ the resolvent can be expanded as

$$R_\lambda = -\frac{1}{\lambda}\left(I_n + \frac{1}{\lambda}A + \frac{1}{\lambda^2}A^2 + \cdots\right).$$

Calculate

$$-\frac{1}{2\pi i}\oint_{|\lambda|=a} \lambda^m R_\lambda d\lambda, \qquad m = 0, 1, 2, \dots .$$

Problem 63. Show that the resolvent satisfies the so-called *resolvent equation*

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu .$$

Problem 64. Let $\tau = (1 + \sqrt{5})/2$ be the *golden ratio*. Consider the modular matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the eigenvalues λ_1, λ_2 and the corresponding normalized eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$. Find the projection matrices Π_1 and Π_2 onto the associated eigendirections.

Problem 65. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that $A^2 = -I_n$. What can be said about the eigenvalues of A?

Problem 66. An $n \times n$ unitary matrix U is defined by

$$UU^* = I_n \quad \text{or} \quad U^* = U^{-1}.$$

What can be concluded about the eigenvalues of U if $U^* = U^T$?

Problem 67. Let $\alpha \in \mathbb{R}$. Consider the symmetric matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+\alpha & 0 & 1-\alpha & 0 \\ 0 & 1+\alpha & 0 & 1-\alpha \\ 1-\alpha & 0 & 1+\alpha & 0 \\ 0 & 1-\alpha & 0 & 1+\alpha \end{pmatrix}.$$

Find an inertible matrix B such that

$$A = B^{-1} D B$$

where D is a diagonal matrix and thus find the eigenvalues of A.

Problem 68. The additive inverse eigenvalue problem is as follows: Let A be an $n \times n$ symmetric matrix over $\mathbb{R}$ with $a_{jj} = 0$ for $j = 1, 2, \ldots, n$. Find a real diagonal $n \times n$ matrix D such that the matrix $A + D$ has the prescribed eigenvalues $\lambda_1, \ldots, \lambda_n$. The number of solutions for the real matrix D varies from 0 to $n!$. Consider the 2×2 matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the prescribed eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$. Can one find a D?

Problem 69. The *spectral theorem* for $n \times n$ normal matrices over $\mathbb{C}$ is as follows: A matrix A is normal if and only if there exists an $n \times n$ unitary matrix U and a diagonal matrix D such that $D = U^* A U$. Use this theorem to prove that the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is not normal.

Problem 70. Let A be an $n \times n$ matrix over $\mathbb{A}$. Assume that A is normal. Show that A has a set of n orthonormal eigenvectors.

Problem 71. Let $\phi \in \mathbb{R}$. Consider the $n \times n$ matrix

$$H = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & & \\ & & & & & 1 \\ e^{i\phi} & & & & & 0 \end{pmatrix}.$$

(i) Show that the matrix is unitary.
(ii) Find the eigenvalues of H.
(iii) Consider the $n \times n$ diagonal matrix

$$G = \mathrm{diag}(1, \omega, \omega^2, \cdots \omega^{n-1})$$

where $\omega := \exp(i2\pi/n$. Find $\omega GH - HG$.

Problem 72. Consider the symmetric 6×6 matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

This matrix plays a role in the construction of the *icosahedron* which is a regular polyhedron with 20 identical equilateral triangular faces, 30 edges and 12 vertices.
(i) Find the eigenvalues of this matrix.
(ii) Consider the matrix $A + \sqrt{5}I_6$. Find the eigenvalues.
(iii) The matrix $A + \sqrt{5}I_6$ induces an Euclidean structure on the quotient space $\mathbb{R}^6/\ker(A + \sqrt{5}I_6)$. Find the dimension of $\ker(A + \sqrt{5}I_6)$.

Problem 73. Let $\alpha \in \mathbb{R}$. Find the eigenvalues and eigenvectors of

$$A(\alpha) = \frac{1}{\sqrt{1+\alpha^2}} \begin{pmatrix} \alpha & 1 \\ 1 & \alpha \end{pmatrix}.$$

For which α is $A(\alpha)$ not invertible?

Problem 74. Let $\epsilon \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \epsilon & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \epsilon & 0 & 0 & 0 \end{pmatrix}.$$

Extend to the $n \times n$ case.

Problem 75. Let A, B be two $n \times n$ matrices over $\mathbb{C}$.
(i) Show that every eigenvalue of AB is also an eigenvalue of BA.
(ii) Can we conclude that every eigenvector of AB is also an eigenvector of BA?

Problem 76. (i) Find the eigenvalues and eigenvectors of the orthogonal matrices

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}, \qquad S = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of RS.

Problem 77. Find the eigenvalues and normalized eigenvectors of the 2×2 matrix

$$\begin{pmatrix} \cos\theta & -e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & \cos\theta \end{pmatrix}.$$

Problem 78. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Extend to the general case n odd.

Problem 79. (i) Consider the matrices

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \qquad B = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & 0 & 0 \\ a_3 & 0 & a_4 \end{pmatrix}.$$

How are the eigenvalues of A and B related?
(ii) Let

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

be a density matrix. Is

$$\rho = \begin{pmatrix} \rho_{11} & 0 & \rho_{12} \\ 0 & 0 & 0 \\ \rho_{21} & 0 & \rho_{22} \end{pmatrix}$$

a density matrix?

Problem 80. Let $a, b, c \in \mathbb{R}$. Find the eigenvalues of the 4×4 matrix

$$A = \begin{pmatrix} 0 & -a & -b & -c \\ a & 0 & -ic & ib \\ b & ic & 0 & -ia \\ c & -ib & ia & 0 \end{pmatrix}.$$

Problem 81. Find all 2×2 matrices over the real numbers with only one 1-dimensional eigenspace, i.e. all eigenvectors are linearly dependent.

Problem 82. Let A be an $n \times n$ matrix over $\mathbb{C}^n$. Let λ be an eigenvalue of A. A generalized eigenvector $\mathbf{x} \in \mathbb{C}^n$ of A corresponding to the eigenvalue λ is a nontrivial solution of

$$(A - \lambda I_n)^j \mathbf{x} = \mathbf{0}_n$$

for some $j \in \{1, 2, \ldots\}$, where I_n is the $n \times n$ identity matrix and $\mathbf{0}_n$ is the n-dimensional zero vector. For $j = 1$ we find the eigenvectors. It follows that $\mathbf{x}$ is a generalized eigenvector of A corresponding to λ if and only if

$$(A - \lambda I_n)^n \mathbf{x} = \mathbf{0}_n.$$

Find the eigenvectors and generalized eigenvectors of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Problem 83. Find all 2×2 matrices over $\mathbb{R}$ which commute with

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

What is the relation between the eigenvectors of these matrices?

Problem 84. Let n be odd and $n \geq 3$. Consider the matrices

$$A_3 = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}, \qquad A_5 = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 & 0 & -1/\sqrt{2} \end{pmatrix}$$

and generally

$$A_n = \begin{pmatrix} 1/\sqrt{2} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & \ldots & 0 & 0 & 0 & \ldots & 1/\sqrt{2} & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1/\sqrt{2} & 0 & 1/\sqrt{2} & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1/\sqrt{2} & 0 & -1/\sqrt{2} & \ldots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 1/\sqrt{2} & \ldots & 0 & 0 & 0 & \ldots & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & -1/\sqrt{2} \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A_3, A_5. The solve the general case.

Problem 85. Assume we know the eigenvalues λ_1, λ_2 of the 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

over $\mathbb{C}$. What can be said about the eigenvalues of the 3×3 matrix

$$B = \begin{pmatrix} a_{11} & 0 & a_{12} \\ 0 & c & 0 \\ a_{21} & 0 & a_{22} \end{pmatrix}$$

where $c \in \mathbb{C}$.

Problem 86. Let $\epsilon \in \mathbb{R}$. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 1 & \epsilon \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & \epsilon \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & \epsilon \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Extend to $n \times n$ matrices.

Problem 87. Let A, B, C, D, E, F, G, H be 2×2 matrices over $\mathbb{C}$. We define the product

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \star \begin{pmatrix} E & F \\ G & H \end{pmatrix} := \begin{pmatrix} A & O_2 & O_2 & B \\ 0_2 & E & F & 0_2 \\ 0_2 & G & H & 0_2 \\ C & 0_2 & 0_2 & D \end{pmatrix}.$$

Thus the right-hand side is an 8×8 matrix. Assume we know the eigenvalues and eigenvectors of the two 4×4 matrices on the left-hand side. What can be said about the eigenvalues and eigenvectors of the 8×8 matrix of the right-hand side.

Problem 88. The symmetric 3×3 matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

plays a role for the chemical compounds ZnS and $NaCl$. Find the eigenvalues and eigenvectors of A. Then find the inverse of A. Find all $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{x}$.

Problem 89. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be a normal matrix over $\mathbb{C}$ with eigenvalues λ_1, λ_2 and corresponding eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, respectively. What can be said about the eigenvalues and eigenvectors of the 3×3 matrices

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{11} & a_{12} \\ 0 & a_{21} & a_{22} \end{pmatrix}, \quad B_2 = \begin{pmatrix} a_{11} & 0 & a_{12} \\ 0 & 1 & 0 \\ a_{21} & 0 & a_{22} \end{pmatrix}, \quad B_3 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} ?$$

Problem 90. Let I_n be the $n \times n$ identity matrix and

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \in \mathbb{R}^{2n \times 2n} .$$

A matrix $S \in \mathbb{R}^{2n \times 2n}$ is called a *symplectic matrix* if

$$S^T J S = J .$$

(i) Show that symplectic matrices are nonsingular.
(ii) Show that the product of two symplectic matrices S_1 and S_2 is also symplectic.
(iii) Show that if S is symplectic S^{-1} and S^T are also symplectic.
(iv) Let S be a symplectic matrix. Show that if $\lambda \in \sigma(S)$, then $\lambda^{-1} \in \sigma(S)$, where $\sigma(S)$ denotes the spectrum of S.

Problem 91. Let A, B be $n \times n$ matrices over $\mathbb{C}$ and $\mathbf{u}$ a nonzero vector in $\mathbb{C}^n$. Assume that $[A, B] = A$ and $A\mathbf{u} = \lambda \mathbf{u}$. Find $(AB)\mathbf{u}$.

Problem 92. Consider the Hilbert space $\mathbb{C}^n$. Let A, B, C be $n \times n$ matrices acting in $\mathbb{C}^n$. We consider the nonlinear eigenvalue problem

$$A\mathbf{u} = \lambda B\mathbf{u} + \lambda^2 C\mathbf{u}$$

where $\mathbf{u} \in \mathbb{C}^n$ and $\mathbf{u} \neq \mathbf{0}$.
(i) Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Find the solutions of the nonlinear eigenvalue problem

$$\sigma_1 \mathbf{u} = \lambda \sigma_2 \mathbf{u} + \lambda^2 \sigma_3 \mathbf{u}$$

where $\mathbf{u} \in \mathbb{C}^2$ and $\mathbf{u} \neq \mathbf{0}$.
(ii) Consider the basis of the simple Lie algebra $s\ell(2, \mathbb{R})$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Solve the nonlinear eigenvalue problem

$$H\mathbf{u} = \lambda E\mathbf{u} + \lambda^2 F\mathbf{u}$$

where $\mathbf{u} \in \mathbb{C}^2$ and $\mathbf{u} \neq \mathbf{0}$.
(iii) Consider the basis of the simple Lie algebra $so(3,\mathbb{R})$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solve the nonlinear eigenvalue problem.

Problem 93. (i) Let A, B be $n \times n$ matrices over $c \in \mathbb{C}$ with $[A,B] = 0$, where $[A,B]$ denotes the commutator of A and B. Calculate $[A + cI_n, B + cI_n]$, where $c \in \mathbb{C}$ and I_n is the $n \times n$ identity matrix.
(ii) Let $\mathbf{x}$ be an eigenvector of the $n \times n$ matrix A with eigenvalue λ. Show that $\mathbf{x}$ is also an eigenvector of $A + cI_n$, where $c \in \mathbb{C}$.

Problem 94. Consider the $n \times n$ tridiagonal matrix

$$\hat{H} = \begin{pmatrix} \epsilon_1 & 1 & . & . & & \\ 1 & \epsilon_2 & 1 & . & & \\ & 1 & \epsilon_3 & 1 & . & \\ & & & & \ddots & \\ & & & & . & 1 \\ & & & & 1 & \epsilon_n \end{pmatrix}.$$

It is used to describe an electron on a linear chain of length n. Find the eigenvalues. Find the eigenvectors. Make the ansatz

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

for the eigenvectors and find a recursion relation for c_j/c_{j+1}.

Problem 95. Find the eigenvalues and eigenvectors of the Hamilton operator

$$\hat{H} = E_0 I_2 - B_1\sigma_1 - B_2\sigma_2 - B_3\sigma_3\,.$$

Problem 96. Let $\epsilon \in \mathbb{R}$. Find the eigenvalues of

$$A(\epsilon) = \begin{pmatrix} 0 & \epsilon & 0 \\ \epsilon & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix}.$$

Do the eigenvalues cross as a function of ϵ?

Problem 97. Let $\epsilon \in [0,1]$. Consider the 2×2 matrix

$$A(\epsilon) = \frac{1}{\sqrt{1+\epsilon^2}} \begin{pmatrix} 1 & \epsilon \\ \epsilon & -1 \end{pmatrix} .$$

For $\epsilon = 0$ we have the Pauli spin matrix σ_z and for $\epsilon = 1$ we have the Hadamad matrix. Find the eigenvalues and eigenvectors of $A(\epsilon)$.

Problem 98. (i) Consider the matrix

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

Find the function (characteristic polynomial)

$$p(\lambda) = \det(A - \lambda I_2) .$$

Find the eigenvalues of A by solving $p(\lambda) = 0$. Find the minima of the function

$$f(\lambda) = |p(\lambda)| .$$

Discuss.
(ii) Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

Find the function (characteristic polynomial)

$$p(\lambda) = \det(A - \lambda I_3) .$$

Find the eigenvalues of A by solving $p(\lambda) = 0$. Find the minima of the function

$$f(\lambda) = |p(\lambda)| .$$

Discuss.

Problem 99. Let A be an $n \times n$ normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and pairwise orthogonal eigenvectors $\mathbf{u}_j$ $(j = 1, 2, \ldots, n)$. Then

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j \mathbf{u}_j^* .$$

Find $\exp(A)$ and $\sin(A)$.

Problem 100. Consider the normalized vector in $\mathbb{C}^3$

$$\mathbf{n} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}.$$

(i) Calculate the 2×2 matrix

$$U(\theta,\phi) = \mathbf{n} \cdot \boldsymbol{\sigma} \equiv n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$$

where σ_1, σ_2, σ_3 are the Pauli spin matrices.
(ii) Is the matrix $U(\theta,\phi)$ unitary? Find the trace and the determinant. Is the matrix $U(\theta,\phi)$ hermitian?
(iii) Find the eigenvalues and normalized eigenvectors of $U(\theta,\phi)$.

Problem 101. Consider the normal matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of A and thus the spectral decomposition. Thus this result to calculate $\exp(zA)$, where $z \in \mathbb{C}$.

Problem 102. Let A be an $n \times n$ normal matrix. Assume that λ_j ($j = 1, \ldots, n$) are the eigenvalues of A. Calculate

$$\prod_{k=1}^{n}(1+\lambda_k)$$

without using the eigenvalues.

Problem 103. Let A be an $n \times n$ matrix over $\mathbb{C}$. Then any eigenvalue of A satisfies the inequality

$$|\lambda| \le \max_{1 \le j \le n} \sum_{k=1}^{n} |a_{jk}|.$$

Write a C++ program that calculates the right-hand side of the inequality for a given matrix. Apply the complex class of STL. Apply it to the matrix

$$A = \begin{pmatrix} i & 0 & 0 & i \\ 0 & 2i & 2i & 0 \\ 0 & 3i & 3i & 0 \\ 4i & 0 & 0 & 4i \end{pmatrix}.$$

Problem 104. (i) Find the eigenvalues of the matrices

$$A_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 4 & 0 & 0 & 4 \end{pmatrix}.$$

Extend to the $n \times n$ case.
(ii) Find the eigenvalues of the matrices

$$B_2 = \begin{pmatrix} 1 & 1 \\ 1/2 & 1/2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 1/4 & 0 & 0 & 1/4 \end{pmatrix}.$$

Extend to the $n \times n$ case.

Problem 105. The $2n \times 2n$ symplectic matrix is defined by

$$S = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. The matrix S is unitary and skew-hermitian. Find the eigenvalues of S from this information.

Problem 106. Find the condition on a_{11}, a_{12}, b_{11}, b_{12} such that

$$\begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{12} & b_{11} & 0 \\ a_{12} & 0 & 0 & a_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

i.e. we have an eigenvalue equation.

Problem 107. Find the eigenvalues of the 4×4 matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 1 & 0 & 0 & a_{24} \\ 0 & 1 & 0 & a_{34} \\ 0 & 0 & 1 & a_{44} \end{pmatrix}.$$

Problem 108. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ a_{13} & a_{23} & 0 & a_{34} \\ a_{14} & a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Problem 109. Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}.$$

Problem 110. (i) Find the spectral decomposition of the normal matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(ii) Find the spectral decomposition of $\exp(A)$.

Problem 111. Find the eigenvalues and eigenvectors of $\sigma_x \sigma_y \sigma_z$.

Problem 112. Find the eigenvalues of the 7×7 matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} & 0 & 0 & a_{17} \\ 0 & a_{22} & a_{23} & 0 & a_{25} & a_{26} & 0 \\ 0 & a_{32} & a_{33} & 0 & a_{35} & a_{36} & 0 \\ a_{41} & 0 & 0 & a_{44} & 0 & 0 & a_{47} \\ 0 & a_{52} & a_{53} & 0 & a_{55} & a_{56} & 0 \\ 0 & a_{62} & a_{63} & 0 & a_{65} & a_{66} & 0 \\ a_{71} & 0 & 0 & a_{74} & 0 & 0 & a_{77} \end{pmatrix}.$$

Problem 113. (i) Find the eigenvalues and eigenvectors of

$$\sigma_z + i\sigma_x = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

(ii) Is this matrix normal?

Problem 114. Let A be an $n \times n$ matrix over $\mathbb{C}$, which satisfies

$$A^2 \equiv AA = cA$$

where $c \in \mathbb{C}$ is a constant. Obviously the equation is satisfied by the zero matrix with $c = 0$. Assume that $A \neq 0_n$. Then we have a "type of eigenvalue equation".
(i) Is c an eigenvalue of A.

(ii) Take the determinant of both sides of the equation. Discuss. Study the cases that A is invertible and non-invertible.
(iii) Study the case

$$A(z) = \begin{pmatrix} e^{-z} & 1 \\ 1 & e^{z} \end{pmatrix}, \quad z \in \mathbb{C}.$$

(iv) Study

$$(A \otimes A)^2 = c(A \otimes A).$$

(v) Let A be a 2×2 matrix and

$$A \star A := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Study the case $(A \star A)^2 = c(A \star A)$.
(vi) Study the case that $A^3 = cA$.

Problem 115. (i) Consider the Pauli spin matrices for describing a spin-$\frac{1}{2}$ system

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider the matrix

$$\sigma_3 + i\sigma_1 .$$

Is the matrix normal? Find the eigenvalues and eigenvectors of the matrix. Discuss. Find the eigenvalues and eigenvectors of $\sigma_3 \otimes \sigma_3 + i\sigma_1 \otimes \sigma_1$.
(ii) Consider the Pauli spin matrices for describing a spin-1 system

$$s_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider the matrix

$$s_3 + is_1 .$$

Is the matrix normal? Find the eigenvalues and eigenvectors of the matrix. Discuss. Find the eigenvalues and eigenvectors of $s_3 \otimes s_3 + is_1 \otimes s_1$.

Problem 116. Let s_x, s_y, s_z be the $(2s+1) \times (2s+1)$ spin matrices for spin $s = 1/2, s = 1, s = 3/2, s = 2, \ldots$.

(i) For $s = 1/2$ we have the 2×2 matrices

$$s_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$. Calculate the eigenvalues and eigenvectors of

$$n_1 s_x + n_2 s_y + n_3 s_z \,.$$

(ii) For $s = 1$ we have the 3×3 matrices

$$s_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad s_y = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \quad s_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $\mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$. Calculate the eigenvalues and eigenvectors of

$$n_1 s_x + n_2 s_y + n_3 s_z \,.$$

Problem 117. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 0 & 0 & a_{13} \\ 1 & 0 & a_{23} \\ 0 & 1 & a_{33} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 1 & 0 & 0 & a_{24} \\ 0 & 1 & 0 & a_{34} \\ 0 & 0 & 1 & a_{44} \end{pmatrix}.$$

Problem 118. Let K be an $n \times n$ skew-hermitian matrix with eigenvalues $\mu_1, \ldots, \mu_n$ (counted according to multiplicty) and the corresponding normalized eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$, where $\mathbf{u}_j^* \mathbf{u}_k = 0$ for $k \neq j$. Then K can be written as

$$K = \sum_{j=1}^{n} \mu_j \mathbf{u}_j \mathbf{u}_j^*$$

and $\mathbf{u}_j \mathbf{u}_j^* \mathbf{u}_k \mathbf{u}_k^* = 0$ for $k \neq j$ and $j, k = 1, 2, \ldots, n$. Note that the matrices $\mathbf{u}_j \mathbf{u}_j^*$ are projection matrices and

$$\sum_{j=1}^{n} \mathbf{u}_j \mathbf{u}_j^* = I_n \,.$$

(i) Calculate $\exp(K)$.
(ii) Every $n \times n$ unitary matrix can be written as $U = \exp(K)$, where K is a skew-hermitian matrix. Find U from a given K.
(iii) Use the result from (ii) to find for a given U a possible K.

(iv) Apply the result from (ii) and (iii) to the unitary 2×2 matrix

$$U(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

(v) Apply the result from (ii) and (iii) to the 2×2 unitary matrix

$$V(\theta,\phi) = \begin{pmatrix} \cos\theta & -e^{i\phi}\sin\theta \\ e^{-i\phi}\sin\theta & \cos\theta \end{pmatrix}.$$

(vi) Every hermitian matrix H can be written as $H = iK$, where K is a skew-hermitian matrix. Find H for the examples given above.

Problem 119. Consider a symmetric matrix over $\mathbb{R}$. We impose the following conditions. The diagonal elements are all zero. The non-diagonal elements can only be $+1$ or -1. Show that such a matrix can only have integer values as eigenvalues. An example would be

$$\begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

with eigenvalues 3 and -1 (three times).

Problem 120. Let A be an $n \times n$ normal matrix over $\mathbb{C}$. How would one apply genetic algorithms to find the eigenvalues of A. This means we have to construct a fitness function f with the minima as the eigenvalues. The eigenvalue equation is given by $A\mathbf{x} = z\mathbf{x}$ ($z \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$ with $\mathbf{x} \neq \mathbf{0}$). The characteristic equation is

$$p(z) \equiv \det(A - zI_n) = 0.$$

What would be a fitness function? Apply it to the matrices

$$B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Problem 121. Let A, B be hermitian matrices over $\mathbb{C}$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$, respectively. Assume that $\mathrm{tr}(AB) = 0$ (scalar product). What can be said about the eigenvalues of $A + B$?

Problem 122. Consider the skew-symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. Find the eigenvalues. Let 0_3 be the 3×3 zero matrix. Let A_1, A_2, A_3 be skew-symmetric 3×3 matrices over $\mathbb{R}$. Find the eigenvalues of the 9×9 matrix

$$B = \begin{pmatrix} 0_3 & -A_3 & A_2 \\ A_3 & 0_3 & -A_1 \\ -A_2 & A_1 & 0_3 \end{pmatrix}.$$

Problem 123. Consider the 4×4 Haar matrix

$$K = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}.$$

Find all 4×4 hermitian matrices H such that $KHK^T = H$.

Problem 124. Consider the reverse-diagonal $n \times n$ matrix

$$A(\phi_1, \dots, \phi_n) = \begin{pmatrix} 0 & 0 & \dots & 0 & e^{i\phi_1} \\ 0 & 0 & \dots & e^{i\phi_2} & 0 \\ \vdots & \vdots & & & \\ 0 & e^{i\phi_{n-1}} & \dots & 0 & 0 \\ e^{i\phi_n} & 0 & \dots & 0 & 0 \end{pmatrix}$$

where $\phi_j \in \mathbb{R}$ $(j = 1, \dots, n)$. Find the eigenvalues and eigenvectors. Is the matrix unitary?

Problem 125. Let $a_{11}, a_{22} \in \mathbb{R}$ and $a_{12} \in \mathbb{C}$. Consider the hermitian matrix

$$H = \begin{pmatrix} a_{11} & a_{12} \\ \bar{a}_{12} & a_{22} \end{pmatrix}$$

with the real eigenvalues λ_1 and λ_2. What conditions are impost on the matrix elements of H if $\lambda_1 = \lambda_2$?

Problem 126. (i) Consider the spin matrices for describing a spin-$\frac{1}{2}$ system

$$s_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and the spin matrices for describing a spin-1 system

$$p_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad p_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find the spectrum (eigenvalues and eigenvector) of the hermitian matrix

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = s_1 \otimes p_1 \otimes s_1 + s_2 \otimes p_2 \otimes s_2 + s_3 \otimes p_3 \otimes s_3 \,.$$

Thus $\hat{K}$ is a 12×12 matrix with $\mathrm{tr}(\hat{K}) = 0$.

Problem 127. $s\ell(3, \mathbb{R})$ is the rank 2 Lie algebra with Cartan matrix

$$C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Find the eigenvalues and normalized eigenvectors of C.

Problem 128. Find the eigenvalues and normalized eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} .$$

Problem 129. Let $\phi_k \in \mathbb{R}$. Consider the matrices

$$A(\phi_1, \phi_2, \phi_3, \phi_4) = \begin{pmatrix} 0 & e^{i\phi_1} & 0 & 0 \\ e^{i\phi_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\phi_3} \\ 0 & 0 & e^{i\phi_4} & 0 \end{pmatrix},$$

$$B(\phi_5, \phi_6, \phi_7, \phi_8) = \begin{pmatrix} 0 & 0 & e^{i\phi_5} & 0 \\ 0 & 0 & 0 & e^{i\phi_6} \\ e^{i\phi_7} & 0 & 0 & 0 \\ 0 & e^{i\phi_8} & 0 & 0 \end{pmatrix}$$

and $A(\phi_1, \phi_2, \phi_3, \phi_4)B(\phi_5, \phi_6, \phi_7, \phi_8)$. Find the eigenvalues of these matrices.

Problem 130. Let U be an $n \times n$ unitary matrix, i.e. $UU^* = I_n$. Assume that $U = U^T$. What can be said about the eigenvalues of such a matrix?

Problem 131. Let I_n be the $n \times n$ identity matrix. Find the eigenvalues of the $2n \times 2n$ matrix

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix} .$$

Problem 132. Consider the unitary matrix

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the skew-hermitian matrix K such that $U = \exp(K)$.

Problem 133. Given the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Prove or disprove that exactly two eigenvalues are 0.

Problem 134. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that the eigenvectors corresponding to distinct eigenvalues are linearly independent.

Problem 135. We know that any $n \times n$ unitary matrix has only eigenvalues λ with $|\lambda| = 1$. Assume that a given $n \times n$ matrix has only eigenvalues with $|\lambda| = 1$. Can we conclude that the matrix is unitary?

Problem 136. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & 0 & 0 \\ a_{13} & 0 & 0 & 0 \\ a_{14} & 0 & 0 & 0 \end{pmatrix}.$$

Problem 137. (i) Let $t_j \in \mathbb{R}$ for $j = 1, 2, 3, 4$. Find the eigenvalues and eigenvectors of

$$\hat{H} = \begin{pmatrix} 0 & t_1 & 0 & t_4 e^{i\phi} \\ t_1 & 0 & t_2 & 0 \\ 0 & t_2 & 0 & t_3 \\ t_4 e^{-i\phi} & 0 & t_3 & 0 \end{pmatrix}.$$

(ii) Let $t_j \in \mathbb{R}$ for $j = 1, \ldots, 5$. Find the eigenvalues and eigenvectors of

$$\hat{H} = \begin{pmatrix} 0 & t_1 & 0 & 0 & t_5 e^{i\phi} \\ t_1 & 0 & t_2 & 0 & 0 \\ 0 & t_2 & 0 & t_3 & 0 \\ 0 & 0 & t_3 & 0 & t_4 \\ t_5 e^{-i\phi} & 0 & 0 & t_4 & 0 \end{pmatrix}.$$

Problem 138. Let $\mathbf{v}$ be a nonzero column vector in $\mathbb{R}^n$. Matrix multiplication is associative. Then we have

$$(\mathbf{v}\mathbf{v}^T)\mathbf{v} = \mathbf{v}(\mathbf{v}^T\mathbf{v}).$$

Discuss.

Problem 139. (i) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} b_{11} & 0 & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 & 0 & 0 \\ 0 & b_{32} & b_{33} & 0 & 0 & 0 \\ 0 & 0 & b_{43} & b_{44} & 0 & 0 \\ 0 & 0 & 0 & b_{54} & b_{55} & 0 \\ 0 & 0 & 0 & 0 & b_{65} & b_{66} \end{pmatrix}.$$

These matrices are the so-called staircase matrices. Extend the results to the $n \times n$ case.

Problem 140. (i) Let A be an invertible $n \times n$ matrix over $\mathbb{C}$. Assume we know the eigenvalues and eigenvectors of A. What can be said about the eigenvalues and eigenvectors of $A + A^{-1}$?
(ii) Apply the result from (i) to the permutation matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Problem 141. (i) Find the eigenvalues and eigenvectors of the 4×4 matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}.$$

(ii) Find the eigenvalues and eigenvectors of the 6×6 matrix

$$B = \begin{pmatrix} b_{11} & 0 & 0 & 0 & 0 & a_{16} \\ 0 & b_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{55} & 0 \\ b_{61} & 0 & 0 & 0 & 0 & b_{66} \end{pmatrix}.$$

Problem 142. Find the eigenvalues and eigenvectors of the matrices

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}.$$

Extend to the $n \times n$ case.

Problem 143. Find the eigenvalues of the 6×6 matrix

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 & 1/10 \\ 1/6 & 1/7 & 1/8 & 1/9 & 1/10 & 1/11 \end{pmatrix}.$$

Problem 144. (i) Let $\alpha \in \mathbb{R}$. Consider the matrices

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \qquad B(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

Find the trace and determinant of these matrices. Show that for the matrix $A(\alpha)$ the eigenvalues depend on α but the eigenvectors do not. Show that for the matrix $B(\alpha)$ the eigenvalues do not depend on α but the eigenvectors do.
(ii) Let $\alpha \in \mathbb{R}$. Consider the matrices

$$C(\alpha) = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}, \qquad D(\alpha) = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ \sinh\alpha & -\cosh\alpha \end{pmatrix}.$$

Find the trace and determinant of these matrices. Show that for the matrix $C(\alpha)$ the eigenvalues depend on α but the eigenvectors do not. Show that for the matrix $D(\alpha)$ the eigenvalues do not depend on α but the eigenvectors do.

Problem 145. Find the lowest eigenvalue of the 4×4 symmetric matrix ($x \in \mathbb{R}$)

$$\begin{pmatrix} 0 & -x\sqrt{5} & 0 & 0 \\ -x\sqrt{5} & 4 & -2x & -2x \\ 0 & -2x & 4-2x & -x \\ 0 & -2x & -x & 8-2x \end{pmatrix} .$$

Problem 146. Let $m > 0$ and $\theta \in \mathbb{R}$. Consider the three 3×3 matrices

$$M_1 = m \begin{pmatrix} 0 & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \\ 0 & \cos\theta & 0 \end{pmatrix}, \quad M_2 = m \begin{pmatrix} 0 & 0 & \sin\theta \\ 0 & 0 & \cos\theta \\ \sin\theta & \cos\theta & 0 \end{pmatrix},$$

$$M_3 = m \begin{pmatrix} 0 & \sin\theta & \cos\theta \\ \sin\theta & 0 & 0 \\ \cos\theta & 0 & 0 \end{pmatrix} .$$

Find the eigenvalues and eigenvectors of the matrices. These matrices play a role for the Majorana neutrino.

Problem 147. Let A, B be real symmetric and block tridiagonal 4×4 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 \\ 0 & a_{23} & a_{33} & a_{34} \\ 0 & 0 & a_{34} & a_{44} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 \\ b_{12} & b_{22} & b_{23} & 0 \\ 0 & b_{23} & b_{33} & b_{34} \\ 0 & 0 & b_{34} & b_{44} \end{pmatrix} .$$

Assume that B is positive definite. Solve the eigenvalue problem

$$A\mathbf{v} = \lambda B\mathbf{v} .$$

Problem 148. (i) Find the eigenvalues, normalized eigenvectors and spectral decomposition of the permutation matrices

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

(ii) Use the spectral decomposition to find the matrices A_1 and A_2 such that $P_1 = \exp(A_1)$, $P_2 = \exp(A_2)$.

Problem 149. Let A be an $n \times n$ matrix over $\mathbb{C}$. What is the condition on A such that all eigenvalues are 0 and A admits only one eigenvector.

Problem 150. Let A be a 2×2 matrix. Assume that $\det(A) = 0$ and $\text{tr}(A) = 0$. What can be said about the eigenvalues of A. Is such a matrix normal?

Chapter 5

Commutators and Anticommutators

Problem 1. Let A, B be 2×2 symmetric matrices over $\mathbb{R}$. Assume that $AA^T = I_2$ and $BB^T = I_2$. Is

$$[A, B] = 0_2\,?$$

Prove or disprove.

Problem 2. Let A, B, X, Y be $n \times n$ matrices over $\mathbb{C}$. Assume that

$$AX - XB = Y\,.$$

(i) Let $z \in \mathbb{C}$. Show that

$$(A - zI_n)X - X(B - zI_n) = Y\,.$$

(ii) Assume that $A - zI_n$ and $B - zI_n$ are invertible. Show that

$$X(B - zI_n)^{-1} - (A - zI_n)^{-1}X = (A - zI_n)^{-1}Y(B - zI_n)^{-1}\,.$$

Problem 3. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that $[A, B] = 0_n$. Let U be a unitary matrix. Calculate $[U^*AU, U^*BU]$.

Problem 4. Can we find nonzero symmetric 2×2 matrices H and A over $\mathbb{R}$ such that

$$[H, A] = \mu A$$

where $\mu \in \mathbb{R}$ and $\mu \neq 0$?

Problem 5. Let A, B be $n \times n$ hermitian matrices. Is $i[A, B]$ hermitian?

Problem 6. A *truncated Bose annihilation operator* is defined as the $n \times n$ ($n \geq 2$) matrix

$$B_n = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & \sqrt{n-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

(i) Calculate $B_n^* B_n$.
(ii) Calculate the commutator $[B_n, B_n^*]$.

Problem 7. Find nonzero 2×2 matrices A, B such that $[A, B] \neq 0_2$, but

$$[A, [A, B]] = 0_2, \qquad [B, [A, B]] = 0_2\,.$$

Problem 8. Let A, B be symmetric $n \times n$ matrices over $\mathbb{R}$. Show that $[A, B]$ is skew-symmetric over $\mathbb{R}$.

Problem 9. Let A, B be $n \times n$ matrices. Show that

$$AB \equiv \frac{1}{2}([A, B] + [A, B]_+)\,.$$

Problem 10. Let A, B be $n \times n$ matrices. Suppose that

$$[A, B] = 0_n, \qquad [A, B]_+ = 0_n$$

and that A is invertible. Show that B must be the zero matrix.

Problem 11. Let

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find all 2×2 matrices A such that $[A, C] = 0_2$, where 0_2 is the 2×2 zero matrix.

Problem 12. Let A, B, C be $n \times n$ matrices. Show that

$$\operatorname{tr}([A, B]C) \equiv \operatorname{tr}(A[B, C])\,.$$

Problem 13. Find all nonzero 2×2 matrices A, B such that

$$[A, B] = A + B\,.$$

Problem 14. Find all nonzero 2×2 matrices J_+, J_-, J_z such that

$$[J_z, J_+] = J_+, \quad [J_z, J_-] = -J_-, \quad [J_+, J_-] = 2J_z$$

where $(J_+)^* = J_-$.

Problem 15. Find all nonzero 2×2 matrices K_+, K_-, K_z such that

$$[K_z, K_+] = K_+, \quad [K_z, K_-] = -K_-, \quad [K_+, K_-] = -2K_z$$

where $(K_+)^* = K_-$.

Problem 16. Find all nonzero 2×2 matrices A_1, A_2, A_3 such that

$$[A_1, A_2] = 0, \quad [A_1, A_3] = A_1, \quad [A_2, A_3] = A_2\,.$$

Problem 17. Let H be a nonzero $n \times n$ hermitian matrix. Let E be a nonzero $n \times n$ matrix. Assume that

$$[H, E] = aE$$

where $a \in \mathbb{R}$ and $a \neq 0$. Show that E cannot be hermitian.

Problem 18. Let A and B be positive semi-definite matrices. Can we conclude that $[A, B]_+ \equiv AB + BA$ is positive semi-definite.

Problem 19. Let A, B be $n \times n$ matrices. Given the expression

$$A^2B + AB^2 + B^2A + BA^2 - 2ABA - 2BAB\,.$$

Write the expression in a more compact form using commutators.

Problem 20. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that $A^2 = I_n$ and $B^2 = I_n$.
(i) Find the commutators $[AB + BA, A]$, $[AB + BA, B]$.
(ii) Give an example of such matrices for $n = 2$ and $A \neq B$.

Problem 21. Consider the 2×2 matrices

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad B(\beta) = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{R}$. Calculate the commutator $[A(\alpha), B(\beta)]$. What is the condition on α, β such that $[A(\alpha), B(\beta)] = 0_2$?

Problem 22. Let A_1, A_2, A_3 be $n \times n$ matrices over $\mathbb{C}$. The *ternary commutator* $[A_1, A_2, A_3]$ (also called the *ternutator*) is defined as

$$\begin{aligned}[A_1, A_2, A_3] &:= \sum_{\pi \in S_3} \text{sgn} A_{\pi(1)} A_{\pi(2)} A_{\pi(3)} \\ &\equiv A_1A_2A_3 + A_2A_3A_1 + A_3A_1A_2 - A_1A_3A_2 - A_2A_1A_3 - A_3A_2A_1 \,.\end{aligned}$$

(i) Let $n = 2$ and consider the Pauli spin matrices σ_x, σ_y, σ_z. Calculate the ternutator

$$[\sigma_x, \sigma_y, \sigma_z] \,.$$

(ii) Calculate

$$A_1 \otimes A_2 \otimes A_3 + A_2 \otimes A_3 \otimes A_1 + A_3 \otimes A_1 \otimes A_2 - A_1 \otimes A_3 \otimes A_2 - A_2 \otimes A_1 \otimes A_3 - A_3 \otimes A_2 \otimes A_1 \,.$$

Problem 23. Let A, B, C be 2×2 matrices. Find the conditions such that $[A, B, C] = 0$.

Problem 24. Let A, B, H be $n \times n$ matrices such that

$$[H, A] = 0, \qquad [H, B] = 0 \,.$$

Show that

$$[H \oplus I_n + I_n \oplus H, A \oplus B] = 0$$

where $\oplus$ denotes the direct sum.

Problem 25. Show that any two 2×2 matrices which commute with the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

commute with each other.

Problem 26. Let A_1, A_2 be $m \times m$ matrices over $\mathbb{C}$. Let B_1, B_2 be $n \times n$ matrices over $\mathbb{C}$. Show that

$$[A_1 \oplus B_1, A_2 \oplus B_2] = ([A_1, A_2]) \oplus ([B_1, B_2])$$

where $\oplus$ denotes the direct sum.

Problem 27. Let A, B be an $n \times n$ matrix over $\mathbb{C}$. Let $\mathbf{u} \in \mathbb{C}^n$ considered as column vector. Is

$$[\mathbf{u}^* A \boldsymbol{\partial}, \mathbf{u}^* B \boldsymbol{\partial}] = \mathbf{u}^* [A, B] \boldsymbol{\partial}\,?$$

Here $[\,,\,]$ denotes the commutator and

$$\boldsymbol{\partial} = \begin{pmatrix} \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix}.$$

Problem 28. Let $c \in \mathbb{R}$ and A be an 2×2 matrix over $\mathbb{R}$. Find the commutator of the 3×3 matrices

$$c \oplus A, \qquad A \oplus c$$

where $\oplus$ denotes the direct sum.

Problem 29. Consider the 3×3 matrices

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & b_1 \\ 0 & b_2 & 0 \\ b_3 & 0 & 0 \end{pmatrix}.$$

Can we find a_j, b_j $(j = 1, 2, 3)$ such that the commutator $[A, B]$ is invertible?

Problem 30. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Can we conclude that

$$\|[A, B]\| \le \|A\| \, \|B\|\,?$$

Problem 31. Consider $(m + n) \times (m + n)$ matrices of the form

$$\begin{pmatrix} m \times m & m \times n \\ n \times m & n \times n \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} \widetilde{B}_1 & 0 \\ 0 & \widetilde{B}_2 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0 & F_1 \\ F_2 & 0 \end{pmatrix}, \quad \widetilde{F} = \begin{pmatrix} 0 & \widetilde{F}_1 \\ \widetilde{F}_2 & 0 \end{pmatrix}.$$

Find the commutators $[B.\widetilde{B}]$, $[B, F]$ and the anticommutator $[F, \widetilde{F}]_+$.

Problem 32. Can one find non-invertible 2×2 matrices A and B such the commutator $[A, B]$ is invertible?

Problem 33. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that B is invertible. Show that

$$[A, B^{-1}] \equiv -B^{-1}[A, B]B^{-1} .$$

Problem 34. Let A, B be $n \times n$ matrices. Show that

$$AB \equiv \frac{1}{2}[A, B] + \frac{1}{2}[A, B]_+ .$$

Problem 35. (i) Let A, B be 2×2 skew-symetric matrices over $\mathbb{R}$. Find the commtutor $[A, B]$.
(ii) Let A, B be 3×3 skew-symetric matrices over $\mathbb{R}$. Find the commtutor $[A, B]$.

Problem 36. Find two linearly independent 2×2 matrices A, B such that

$$-A = [B, [B, A]], \qquad -B = [A, [A, B]] .$$

Problem 37. Let A be an $n \times n$ matrix and 0_n be the $n \times n$ zero matrix. Find the commutator

$$\left[\begin{pmatrix} 0_n & A \\ A & 0_n \end{pmatrix}, \begin{pmatrix} 0_n & A \\ -A & 0_n \end{pmatrix} \right]$$

and the anticommutator

Problem 38. Find all 2×2 matrices over $\mathbb{C}$ such that the commutator is an invertible diagonal matrix D, i.e. $d_{11} \neq 0$ and $d_{22} \neq 0$.

Problem 39. Let A, B be invertible $n \times n$ matrices over $\mathbb{C}$. Assume that $[A, B] = 0_n$. Can we conclude that $[A^{-1}, B^{-1}] = 0_n$?

Problem 40. Consider the 3×3 matrices over $\mathbb{C}$

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix} .$$

(i) Calculate the commutator $[A, B]$ and $\det([A, B])$.
(ii) Set $a_{11} = e^{i\phi_1}$, $a_{22} = e^{i\phi_2}$, $a_{33} = e^{i\phi_3}$. Find the condition on ϕ_1, ϕ_2, ϕ_3, b_{12}, b_{13}, b_{21}, b_{23}, b_{31}, b_{32} such that $[A, B]$ is unitary.

Problem 41. Let A, B be $n \times n$ matrices and T a (fixed) invertible $n \times n$ matrix. We define the bracket

$$[A, B]_T := ATB - BTA\,.$$

Let

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Find $[X, Y]_T$, $[X, H]_T$, $[Y, H]_T$.

Problem 42. Can one find a 2×2 matrix A over $\mathbb{R}$ such that

$$[A^T, A] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Problem 43. Consider the set of 3×3 matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Calculate the anticommutator and thus show that we have a basis of a *Jordan algebra*.

Problem 44. Consider the invertible matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2i\pi/3} & 0 \\ 0 & 0 & e^{-2i\pi/3} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Is the matrix $[A, B]$ invertible?

Problem 45. A classical 3×3 matrix representation of the algebra $iso(1,1)$ is given by

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}, \quad P_+ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the commutators and anticommuators.

Problem 46. Find the conditions on the two 2×2 hermitian matrices A, B such that

$$[A \otimes B, P] = 0$$

where P is the permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem 47. Let A be an arbitrary $n \times n$ matrix over $\mathbb{C}$ with $\mathrm{tr}(A) = 0$. Show that A can be written as commutator, i.e. there are $n \times n$ matrices X and Y such that

$$A = XY - YX.$$

Problem 48. Let A be an arbitrary $n \times n$ matrix over $\mathbb{C}$ with $\mathrm{tr}A = 0$. Show that A can be written as commutator, i.e., there are $n \times n$ matrices X and Y such that $A = [X, Y]$.

Problem 49. Let A be an arbitrary $n \times n$ matrix over $\mathbb{C}$ with $\mathrm{tr}A = 0$. Show that A can be written as commutator, i.e., there are $n \times n$ matrices X and Y such that $A = [X, Y]$.

Problem 50. Let A, B be hermitian matrices, i.e. $A^* = A$ and $B^* = B$. Then in general $A + iB$ is non-normal. What are the conditions on A and B such that $A + iB$ is normal?

Problem 51. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that if A and B commute and if A is normal, then A^* and B commute.

Chapter 6

Decomposition of Matrices

Problem 1. We consider 3×3 matrices over $\mathbb{R}$. An orthogonal matrix Q such that $\det Q = 1$ is called a *rotation matrix*. Let $1 \leq p < r \leq 3$ and ϕ be a real number. An orthogonal 3×3 matrix $Q_{pr}(\phi) = (q_{ij})_{1 \leq i,j \leq 3}$ given by

$$\begin{aligned} q_{pp} &= q_{rr} = \cos\phi \\ q_{ii} &= 1 \quad \text{if } i \neq p, r \\ q_{pr} &= -q_{rp} = -\sin\phi \\ q_{ip} &= q_{pi} = q_{ir} = q_{ri} = 0 \quad i \neq p, r \\ q_{ij} &= 0 \quad \text{if } i \neq p, r \text{ and } j \neq p, r \end{aligned}$$

will be called a plane rotation through ϕ in the plane span (e_p, e_r). Let $Q = (q_{ij})_{1 \leq i,j \leq 3}$ be a rotation matrix. Show that there exist angles $\phi \in [0, \pi)$, $\theta, \psi \in (-\pi, \pi]$ called the *Euler angles* of Q such that

$$Q = Q_{12}(\phi) Q_{23}(\theta) Q_{12}(\psi) \,. \tag{1}$$

Problem 2. For any $n \times n$ matrix A over $\mathbb{C}$, there exists a positive semi-definite matrix H and a unitary matrix such that $A = HU$ (*polar decomposition*). If A is nonsingular, then H is positive definite and U and

H are unique. Find the polar decomposition for

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{pmatrix}.$$

Problem 3. If $A \in \mathbb{R}^{n\times n}$, then there exists an orthogonal $Q \in \mathbb{R}^{n\times n}$ such that

$$Q^T AQ = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ 0 & R_{22} & \dots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{mn} \end{pmatrix}$$

where each R_{ii} is either a 1×1 matrix or a 2×2 matrix having complex conjugate eigenvalues. Find Q for the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 4 & 0 \end{pmatrix}.$$

Then calculate $Q^T AQ$.

Problem 4. Let $n \geq 2$ and $n = 2k$. Let A be an $n \times k$ matrix and

$$A^* A = I_k$$

where I_k is the $k\times k$ unit matrix. Find the $n\times n$ matrix AA^* using the singular value decomposition. Calculate $\mathrm{tr}(AA^*)$.

Problem 5. Let $n \geq 2$ and $n = 2k$. Let A be an $n \times k$ matrix and

$$A^* A = I_k$$

where I_k is the $k\times k$ unit matrix. Let S be a positive definite $n\times n$ matrix. Show that

$$1 \leq \frac{\mathrm{tr}(A^* S^2 A)}{\mathrm{tr}((A^* SA)^2)}.$$

Problem 6. Consider the symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and the orthogonal matrix

$$O = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Calculate $\widetilde{A} = O^{-1}AO$. Can we find an angle ϕ such that $\widetilde{a}_{12} = \widetilde{a}_{21} = 0$?

Problem 7. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that A^{-1} exits. Given the singular value decomposition of A, i.e. $A = UWV^T$. Find the singular value decomposition for A^{-1}.

Problem 8. Find the cosine-sine decomposition of the 4×4 unitary matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} .$$

Problem 9. Find a cosine-sine decomposition of the Hadamard matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

Problem 10. (i) Consider the 4×4 matrices

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} .$$

Can one find 4×4 permutation matrices P, Q such that

$$\Omega = P\widetilde{\Omega}Q\,?$$

(ii) Consider the $2n \times 2n$ matrices

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 0 & 1 & \dots & 0 & 0 & \\ 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \dots & -1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix}$$

and

$$\widetilde{\Omega} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. Can one find $2n \times 2n$ permutation matrices P, Q such that $\Omega = P\widetilde{\Omega}Q$?

Problem 11. Let A be an $m \times m$ matrix. Let B be an $n \times n$ matrix. Let X be an $m \times n$ matrix such that

$$AX = XB. \qquad (1)$$

We can find non-singular matrices V and W such that

$$V^{-1}AV = J_A, \qquad W^{-1}BW = J_B$$

where J_A, J_B are the Jordan canonical form of A and B, respectively. Show that from (1) it follows that

$$J_A Y = Y J_B$$

where $Y := V^{-1}XW$.

Chapter 7

Functions of Matrices

Problem 1. Consider the matrix ($\zeta \in \mathbb{R}$)

$$S(\zeta) = \begin{pmatrix} \cosh\zeta & 0 & 0 & \sinh\zeta \\ 0 & \cosh\zeta & \sinh\zeta & 0 \\ 0 & \sinh\zeta & \cosh\zeta & 0 \\ \sinh\zeta & 0 & 0 & \cosh\zeta \end{pmatrix}.$$

(i) Show that the matrix is invertible, i.e. find the determinant.
(ii) Calculate the inverse of $S(\zeta)$.
(iii) Calculate

$$A := \frac{d}{d\zeta} S(\zeta)\Big|_{\zeta=0}$$

and then calculate $\exp(\zeta A)$.

Problem 2. Let A be an $n \times n$ positive definite matrix over $\mathbb{R}$. Let $\mathbf{q}$ and $\mathbf{J}$ be column vectors in $\mathbb{R}^n$. Calculate

$$Z(\mathbf{J}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dq_1 \cdots dq_n \exp\left(-\frac{1}{2}\mathbf{q}^T A \mathbf{q} + \mathbf{J}^T \mathbf{q}\right).$$

Note that

$$\int_{-\infty}^{\infty} dq e^{-(aq^2+bq+c)} = \sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/(4a)}. \qquad (1)$$

Problem 3. Let A be an $n \times n$ positive definite matrix over $\mathbb{R}$, i.e. $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Calculate

$$\int_{\mathbb{R}^n} \exp(-\mathbf{x}^T A\mathbf{x}) d\mathbf{x}\,.$$

Problem 4. Consider the 2×2 matrix

$$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}.$$

Calculate A^n, where $n \in \mathbb{N}$.

Problem 5. Let A be an $n \times n$ matrix over $\mathbb{C}$. The $n \times n$ matrix B over $\mathbb{C}$ is a square root of A iff $B^2 = A$. The number of square roots of a given matrix A may be zero, finite or infinite. Does the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

admit a square root?

Problem 6. Let A be an $n \times n$ matrix. Let

$$c(z) := \det(zI_n - A) = z^n - \sum_{k=0}^{n-1} c_k z^k$$

be the characteristic polynomial of A. Apply the *Cayley-Hamilton theorem* $c(A) = 0$ to calculate $\exp(A)$.

Problem 7. (i) Let A be an $n \times n$ matrix with $A^3 = I_n$. Calculate $\exp(A)$ using

$$\exp(A) = \sum_{j=0}^{\infty} A^j/(j!)\,.$$

(ii) Let

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

with $B^3 = I_3$. Calculate $\exp(B)$ using the result from (i). Calculate $\exp(B)$ applying the *Cayley-Hamilton theorem.*

Problem 8. Show that for any *Pauli spin matrix* σ_1, σ_2, σ_3 we have

$$\sin(\theta\sigma_j) = \sin(\theta)\sigma_j\,.$$

Problem 9. Let M be an $n \times n$ matrix with $m_{jk} = 1$ for all $j, k = 1, 2, \ldots, n$. Let $s \in \mathbb{C}$. Find $\exp(sM)$. Then consider the special case $sn = i\pi$

Problem 10. Let X, Y be $n \times n$ matrices. Show that

$$[e^X, Y] = \sum_{k=1}^{\infty} \frac{[X^k, Y]}{k!} \,.$$

Problem 11. Let A, B be $n \times n$ matrices. Show that

$$e^{A+B} - e^A \equiv \int_0^1 e^{(1-t)A} B e^{t(A+B)} dt \,.$$

Problem 12. Let A be an $n \times n$ matrix. Then $\exp(A)$ can also be calculated as

$$e^A = \lim_{m \to \infty} \left(I_n + \frac{A}{m} \right)^m \,.$$

Use this definition to show that

$$\det(e^A) \equiv e^{\mathrm{tr(A)}} \,.$$

Problem 13. Let $A_1, A_2, \ldots, A_p$ be $n \times n$ matrices over $\mathbb{C}$. The generalized *Trotter formula* is given by

$$\exp\left(\sum_{j=1}^{n} A_j \right) = \lim_{n \to \infty} f_n(\{ A_j \}) \qquad (1)$$

where the n-th approximant $f_n(\{ A_j \})$ is defined by

$$f_n(\{ A_j \}) := \left(\exp\left(\frac{1}{n} A_1 \right) \exp\left(\frac{1}{n} A_2 \right) \cdots \exp\left(\frac{1}{n} A_p \right) \right)^n \,.$$

Let $p = 2$ and

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,.$$

Calculate the left and right-hand side of (1).

Problem 14. Let $\alpha, \beta \in \mathbb{R}$. Calculate

$$\exp \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \,.$$

Problem 15. Consider the 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $t \in \mathbb{R}$. Find $\exp(tA)$.

Problem 16. Let A, B be $n \times n$ matrices over $\mathbb{C}$ and $\alpha \in \mathbb{C}$. The *Baker-Campbell-Hausdorff formula* states that

$$e^{\alpha A} B e^{-\alpha A} = B + \alpha[A,B] + \frac{\alpha^2}{2!}[A,[A,B]] + \cdots = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!}\{A^j, B\} = \widetilde{B}(\alpha)$$

where $[A,B] := AB - BA$ and

$$\{A^j, B\} := [A, \{A^{j-1}, B\}]$$

is the repeated commutator.
(i) Extend the formula to

$$e^{\alpha A} B^k e^{-\alpha A}$$

where $k \geq 1$.
(ii) Extend the formula to

$$e^{\alpha A} e^B e^{-\alpha A}.$$

Problem 17. Consider the $n \times n$ matrix ($n \geq 2$)

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Calculate

$$f(0)I_n + \frac{f'(0)}{1!}A + \frac{f''(0)}{2!}A^2 + \cdots + \frac{f^{n-1}}{(n-1)!}A^{n-1}$$

where $'$ denotes differentiation. Discuss.

Problem 18. Let A, B be $n \times n$ matrices. Show that

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

if $AB = BA$.

Problem 19. Consider the 3×3 matrix

$$A(\alpha) = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

Find $\exp(A)$.

Problem 20. Let A, B be $n \times n$ matrices with $A^2 = I_n$ and $B^2 = I_n$. Assume that the anticommutator of A and B vanishes, i.e.

$$[A, B]_+ = AB + BA = 0_n .$$

Let $a, b \in \mathbb{C}$. Calculate

$$e^{aA+bB} .$$

Problem 21. Let A, B be $n \times n$ matrices with $A^2 = I_n$ and $B^2 = I_n$. Assume that the commutator of A and B vanishes, i.e.

$$[A, B] = AB - BA = 0_n .$$

Let $a, b \in \mathbb{C}$. Calculate

$$e^{aA+bB} .$$

Problem 22. Consider the 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} .$$

Find $\exp(tA)$.

Problem 23. Can one find $n \times n$ matrices A such that ($\epsilon \in \mathbb{R}$)

$$\exp(i\epsilon A) = I_n + (\cos(\epsilon) - 1)A^2 + i\sin(\epsilon)A\,?$$

Problem 24. Let $\alpha, \beta \in \mathbb{C}$. Let

$$M(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix} .$$

(i) Calculate $\exp(M(\alpha, \beta))$.
(ii) For which values of $\alpha, \beta \in \mathbb{C}$ is the matrix nonnormal? Simplify the result for $\alpha = i\pi$ and β arbitrary. Is the matrix $M(\alpha = i\pi, \beta)$ nonnormal? Is the matrix $\exp(M(\alpha = i\pi, \beta)$ nonnormal?

Problem 25. Consider the two-dimensional rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

with $0 \le \theta \le \pi$. Find $R^{1/2}$.

Problem 26. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. Let $\theta \in \mathbb{R}$, $\mathbf{n}$ a normalized vector in $\mathbb{R}^3$ and σ_1, σ_2, σ_3 the Pauli spin matrices. We define

$$\mathbf{n} \cdot \boldsymbol{\sigma} := n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 \,.$$

Then

$$f(\theta\mathbf{n} \cdot \boldsymbol{\sigma}) \equiv \frac{1}{2}(f(\theta) + f(-\theta))I_2 + \frac{1}{2}(f(\theta) - f(-\theta))(\mathbf{n} \cdot \boldsymbol{\sigma})\,.$$

Apply this identity to $f(x) = \sin(x)$.

Problem 27. Let $a, b \in \mathbb{R}$ and

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Calculate $\exp(M)$.

Problem 28. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that all eigenvalues λ_1, λ_2, $\ldots$, λ_n are pairwise distinct. Then e^{tA} can be calculated as follows (*Lagrange interpolation*)

$$e^{tA} = \sum_{j=1}^{n} e^{\lambda_j t} \prod_{\substack{k=1 \\ k \ne j}}^{n} \frac{(A - \lambda_k I_n)}{(\lambda_j - \lambda_k)}\,.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate e^{tA} using this method.

Problem 29. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that all eigenvalues λ_1, λ_2, $\ldots$, λ_n are pairwise distinct. Then e^{tA} can be calculated as follows (*Newton interpolation*)

$$e^{tA} = e^{\lambda_1 t} I_n + \sum_{j=2}^{n} [\lambda_1, \ldots, \lambda_j] \prod_{k=1}^{j-1} (A - \lambda_k I_n)\,.$$

The divided differences $[\lambda_1, \dots, \lambda_j]$ depend on t and are defined recursively by

$$[\lambda_1, \lambda_2] := \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}$$
$$[\lambda_1, \dots, \lambda_{k+1}] := \frac{[\lambda_1, \dots, \lambda_k] - [\lambda_2, \dots, \lambda_{k+1}]}{\lambda_1 - \lambda_{k+1}}, \quad k \geq 2\,.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate e^{tA} using this method.

Problem 30. Apply the *Cayley-Hamilton theorem* to the 3×3 matrix A and express the result using the trace and determinant of A.

Problem 31. Let A, B, C be $n \times n$ matrices over $\mathbb{C}$ such that $A^2 = I_n$, $B^2 = I_n$ and $C^2 = I_n$. Furthermore assume that

$$[A, B]_+ \equiv AB+BA = 0_n, \quad [B, C]_+ \equiv BC+CB = 0_n, \quad [C, A]_+ \equiv CA+AC = 0_n$$

i.e. the anticommutators vanish. Let $\alpha, \beta, \gamma \in \mathbb{C}$. Calculate $e^{\alpha A + \beta B + \gamma C}$ using

$$e^{\alpha A + \beta B + \gamma C} = \sum_{j=0}^{\infty} \frac{(\alpha A + \beta B + \gamma C)^j}{j!}\,.$$

Problem 32. Let A, B be $n \times n$ matrices. Then we have the identity

$$\det(e^A e^B e^{-A} e^{-B}) \equiv \exp(\operatorname{tr}([A, B]))$$

where $[A, B] := AB - BA$ defines the commutator. Show that

$$\det(e^A e^B e^{-A} e^{-B}) = 1\,.$$

Problem 33. Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Show that

$$\exp(e) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \exp(-f) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

(ii) Show that

$$\exp(e)\exp(-f)\exp(e) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Problem 34. Let A be an $n \times n$ matrix. The *characteristic polynomial*

$$\det(\lambda I_n - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = p(\lambda)$$

is closely related the the *resolvent* $(\lambda I_n - A)^{-1}$ trough the formula

$$(\lambda I_n - A)^{-1} = \frac{N_1\lambda^{n-1} + N_2\lambda^{n-2} + \cdots + N_n}{\lambda^n + a_1\lambda^{n-1} + \cdots + a_n} = \frac{N(\lambda)}{p(\lambda)}$$

where the adjugate matrix $N(\lambda)$ is a polynomial in λ of degree $n-1$ with constant $n \times n$ coefficient matrices $N_1, \ldots, N_n$. The *Laplace transform* of the matrix exponential is the resolvent

$$\mathcal{L}(e^{tA}) = (\lambda I_n - A)^{-1}.$$

The N_k matrices and a_k coefficients may be computed recursively as follows

$$\begin{aligned}
N_1 &= I_n, \qquad a_1 = -\frac{1}{1}\mathrm{tr}(AN_1) \\
N_2 &= AN_1 + a_1 I_n, \qquad a_2 = -\frac{1}{2}\mathrm{tr}(AN_2) \\
&\vdots \\
N_n &= AN_{n-1} + a_{n-1}I_n, \qquad a_n = -\frac{1}{n}\mathrm{tr}(AN_n) \\
0 &= AN_n + a_n I_n .
\end{aligned}$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Find the N_k matrices and the coefficients a_k and thus calculate the resolvent.

Problem 35. Let A be an $n \times n$ matrix. The *characteristic polynomial*

$$\det(\lambda I_n - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = p(\lambda)$$

is closely related the the *resolvent* $(\lambda I_n - A)^{-1}$ trough the formula

$$(\lambda I_n - A)^{-1} = \frac{N_1\lambda^{n-1} + N_2\lambda^{n-2} + \cdots + N_n}{\lambda^n + a_1\lambda^{n-1} + \cdots + a_n} = \frac{N(\lambda)}{p(\lambda)}$$

where the adjugate matrix $N(\lambda)$ is a polynomial in λ of degree $n-1$ with constant $n \times n$ coefficient matrices $N_1, \ldots, N_n$. The Laplace transform of the matrix exponential is the resolvent

$$\mathcal{L}(e^{tA}) = (\lambda I_n - A)^{-1}\,.$$

The N_k matrices and a_k coefficients may be computed recursively as follows

$$\begin{aligned}
N_1 &= I_n, \qquad a_1 = -\frac{1}{1}\mathrm{tr}(AN_1) \\
N_2 &= AN_1 + a_1 I_n, \qquad a_2 = -\frac{1}{2}\mathrm{tr}(AN_2) \\
&\vdots \\
N_n &= AN_{n-1} + a_{n-1} I_n, \qquad a_n = -\frac{1}{n}\mathrm{tr}(AN_n) \\
0 &= AN_n + a_n I_n\,.
\end{aligned}$$

Show that

$$\mathrm{tr}(\mathcal{L}(e^t A)) = \frac{p'(\lambda)}{p(\lambda)}$$

where $p'(\lambda) = dp(\lambda)/d\lambda$.

Problem 36. Let A be an $n \times n$ positive definite matrix over $\mathbb{R}$, i.e. all the eigenvalues, which are real, are positive. We also have $A^T = A$. Consider the analytic function $f : \mathbb{R}^n \to \mathbb{R}$

$$f(\mathbf{x}) = \exp\left(-\frac{1}{2}\mathbf{x}^T A^{-1}\mathbf{x}\right)\,.$$

Calculate the *Fourier transform* of f. The Fourier transform is defined by

$$\hat{f}(\mathbf{k}) := \int_{\mathbb{R}^n} f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}$$

where $\mathbf{k}\cdot\mathbf{x} \equiv \mathbf{k}^T\mathbf{x} \equiv k_1x_1 + \cdots + k_nx_n$ and $d\mathbf{x} = dx_1 \ldots dx_n$. The inverse Fourier transform is given by

$$f(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}$$

where $d\mathbf{k} = dk_1 \ldots dk_n$. Note that we have with $a > 0$

$$\int_{\mathbb{R}} e^{-(ax^2+bx+c)} dx = \sqrt{\frac{\pi}{a}} e^{(b^2-4ac)/(4a)}\,.$$

Problem 37. Let A be an $n \times n$ matrix. Suppose f is an analytic function inside on a closed contour Γ which encircles $\lambda(A)$, where $\lambda(A)$ denotes the eigenvalues of A. We define $f(A)$ to be the $n \times n$ matrix

$$f(A) = \frac{1}{2\pi i} \oint_{\Gamma} f(z)(zI_n - A)^{-1} dz\,.$$

This is a matrix version of the *Cauchy integral theorem*. The integral is defined on an element-by-element basis $f(A) = (f_{jk})$, where

$$f_{jk} = \frac{1}{2\pi i} \oint_{\Gamma} f(z)\mathbf{e}_j^T (zI_n - A)^{-1}\mathbf{e}_k dz$$

where $\mathbf{e}_j$ $(j = 1, 2, \ldots, n)$ is the standard basis in $\mathbb{C}^n$. Let $f(z) = z^2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate $f(A)$.

Problem 38. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Calculate

$$e^A B e^A\,.$$

Set $f(\epsilon) = e^{\epsilon A} B e^{\epsilon A}$, where ϵ is a real parameter. Then differentiate with respect to ϵ. For $\epsilon = 1$ we have $e^A B e^A$.

Problem 39. Let A, B be positive definite matrices. Then we have the integral representation $(x \geq 0)$

$$\ln(A + xB) - \ln A \equiv \int_0^\infty (A + uI_n)^{-1} xB(A + xB + uI_n)^{-1} du\,.$$

Let

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Calculate the left and right-hand side of the integral representation.

Problem 40. Let $\epsilon \in \mathbb{R}$. Calculate

$$f(\epsilon) = e^{-\epsilon\sigma_y} \sigma_z e^{\epsilon\sigma_y}\,.$$

Hint. Differentiate the matrix-valued function f with respect to ϵ and solve the initial value problem of the resulting ordinary differential equation.

Problem 41. Find the square root of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

i.e. find the matrices X such that $X^2 = A$.

Problem 42. Let A be an arbitrary $n \times n$ matrix. Can we conclude that

$$\exp(A^*) = (\exp(A))^* \,?$$

Problem 43. Let A be an invertible $n \times n$ matrix over $\mathbb{R}$. Consider the functions

$$E_j = \frac{1}{2}(A\mathbf{c}_j - \mathbf{e}_j)^T(A\mathbf{c}_j - \mathbf{e}_j)$$

where $j = 1, \ldots, n$, $\mathbf{c}_j$ is the j-th column of the inverse matrix of A, $\mathbf{e}_j$ is the j-th column of the $n \times n$ identity matrix. This means $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis (as column vectors) in $\mathbb{R}^n$. The $\mathbf{c}_j$ are determined by minimizing the E_j with respect to the $\mathbf{c}_j$. Apply this method to find the inverse of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Problem 44. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that A is hermitian, i.e. $A^* = A$. Thus A has only real eigenvalues. Assume that

$$A^5 + A^3 + A = 3I_n \,.$$

Show that $A = I_n$.

Problem 45. Let $\mathbf{f}$ be a function from U, an open subset of $\mathbb{R}^m$, to $\mathbb{R}^n$. Assume that the component function f_j $(j = 1, \ldots, n)$ possess first order partial derivatives. Then we can associate the $n \times m$ matrix

$$\left(\left. \frac{\partial f_j}{\partial x_k} \right|_{\mathbf{p}} \right), \qquad j = 1, \ldots, n \quad k = 1, \ldots, m$$

where $\mathbf{p} \in U$. The matrix is is called the *Jacobian matrix* of $\mathbf{f}$ at the point $\mathbf{p}$. When $m = n$ the determinant of the square matrix $\mathbf{f}$ is called the *Jacobian* of $\mathbf{f}$. Let

$$A = \{\, r \in \mathbb{R} \,:\, r > 0 \,\}, \qquad B = \{\, \theta \in \mathbb{R} \,:\, 0 \le \theta < 2\pi \,\}$$

and $\mathbf{f} : A \times B \to \mathbb{R}^2$ with $f_1(r,\theta) = r\cos\theta$, $f_2(r,\theta) = r\sin\theta$. Find the Jacobian matrix and the Jacobian.

Problem 46. Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Find all nonzero 2×2 matrices A such that

$$AJ = JA.$$

Problem 47. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let T be a nilpotent matrix over $\mathbb{C}$ satisfying

$$T^* A + AT = 0.$$

Show that

$$(e^T)^* A e^T = A.$$

Problem 48. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Let $\beta, \epsilon \in \mathbb{R}$. Show that

$$\exp(\beta(A+B)) \equiv \exp(\beta A)\left(I_n + \int_0^\beta d\epsilon e^{-\epsilon A} B e^{\epsilon(A+B)}\right).$$

Problem 49. Consider the matrix

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $\alpha \in \mathbb{R}$. Find $\exp(\alpha A)$.

Problem 50. Let $\epsilon \in \mathbb{R}$. Let

$$I_n - \epsilon A$$

be a positive definite matrix. Calculate

$$\exp(\mathrm{tr}(\ln(I_n - \epsilon A))$$

using the identity $\det e^M \equiv \exp(\mathrm{tr}(M))$.

Problem 51. Consider the Pauli spin matrices σ_x, σ_y and σ_z. Can one find an $\alpha \in \mathbb{R}$ such that

$$\exp(i\alpha\sigma_z)\sigma_x \exp(-i\alpha\sigma_z) = \sigma_y\,?$$

Problem 52. (i) Let $a, b \in \mathbb{R}$. Let

$$K = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Find $\exp(iK)$.
(ii) Use the result to find a, b such that

$$\exp(iK) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 53. Let P be an $n \times n$ projection matrix. Let $\epsilon \in \mathbb{R}$. Calculate

$$\exp(\epsilon P).$$

Problem 54. (i) Let $P_1, P_2, \ldots, P_n$ be an $n \times n$ projection matrices. Assume that $P_j P_k = 0$ $(j \neq k)$ for all $j, k = 1, 2, \ldots, n$. Let $\epsilon_j \in \mathbb{R}$ with $j = 1, 2, \ldots, n$. Calculate

$$\exp(\epsilon_1 P_1 + \epsilon_2 P_2 + \cdots + \epsilon_n P_n).$$

(ii) Assume additionally that

$$P_1 + P_2 + \cdots + P_n = I_n.$$

Simplify the result from (i) using this condition.

Problem 55. Let A, B be $n \times n$ hermitian matrices. There exists $n \times n$ unitary matrices U and V (depending on A and B) such that

$$\exp(iA)\exp(iB) = \exp(iUAU^{-1} + iVBV^{-1}).$$

Consider $n = 2$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Find U and V. Note that A and B are also unitary and represend the NOT-gate and Hadamard gate, respectively. Furthermore

$$[A, B] = \sqrt{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Problem 56. Let $a, b \in \mathbb{C}$ and

$$M(a, b) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Calculate $\exp(M(a,b))$.

Problem 57. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that A and B commute with the commutator $[A,B]$. Then

$$\exp(A+B) = \exp(A)\exp(B)\exp\left(-\frac{1}{2}[A,B]\right).$$

Can this formula be applied to the matrices

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 58. Let $\epsilon \in \mathbb{R}$. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Expand

$$e^{\epsilon A} e^{\epsilon B} e^{-\epsilon A} e^{-\epsilon B}$$

up to second order in ϵ.

Problem 59. Let $\alpha, \beta \in \mathbb{R}$. Consider the 2×2 matrix

$$B = \begin{pmatrix} -i\alpha & -\beta \\ -\beta & i\alpha \end{pmatrix}.$$

Find $\exp(tB)$, where $t \in \mathbb{R}$ and thus solve the initial value problem of the matrix differential equation

$$\frac{dA}{dt} = BA(t).$$

Problem 60. Let A be an $n \times n$ matrix over $\mathbb{C}$. Assume that for all eigenvalues λ we have $\Re(\lambda) < 0$. Let B be an arbitrary $n \times n$ matrix over $\mathbb{C}$. Let

$$R := \int_0^\infty e^{tA^*} B e^{tA}.$$

Show that the matrix R satisfies the matrix equation

$$RA + A^* R = -B.$$

Problem 61. Let A, B be $n \times n$ matrices over $\mathbb{C}$ such that $[A,B] = A$. What can be said about the commutator

$$[e^A, e^B]$$

Problem 62. Consider the positive semidefinite matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Use the right-hand side of the identity

$$\det(A) \equiv \exp(\mathrm{tr}(\ln(A)))$$

to calculate $\det(A)$.

Problem 63. Let A, B be $n \times n$ matrices over $\mathbb{C}$ and $A^2 = I_n$, $B^2 = I_n$. Calculate

$$\exp(z_1 A + z_2 B)$$

where $z_1, z_2 \in \mathbb{C}$.

Problem 64. The Cayley-Hamilton theorem can also be used to calculate $\exp(A)$ and other entire functions for an $n \times n$ matrix. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let f be an *entire function*, i.e., an analytic function on the whole complex plane, for example $\exp(z)$, $\sin(z)$, $\cos(z)$. An infinite series expansion for $f(A)$ is not generally useful for computing $f(A)$. Using the *Cayley-Hamilton theorem* we can write

$$f(A) = a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_2 A^2 + a_1 A + a_0 I_n \qquad (1)$$

where the complex numbers a_0, a_1, $\ldots$, a_{n-1} are determined as follows: Let

$$r(\lambda) := a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

which is the right-hand side of (1) with A^j replaced by λ^j, where $j = 0, 1, \ldots, n-1$. or each distinct eigenvalue λ_j of the matrix A, we consider the equation

$$f(\lambda_j) = r(\lambda_j). \qquad (2)$$

If λ_j is an eigenvalue of multiplicity k, for $k > 1$, then we consider also the following equations

$$\begin{aligned} f'(\lambda)|_{\lambda=\lambda_j} &= r'(\lambda)|_{\lambda=\lambda_j} \\ f''(\lambda)|_{\lambda=\lambda_j} &= r''(\lambda)|_{\lambda=\lambda_j} \\ \cdots &= \cdots \\ f^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_j} &= r^{(k-1)}(\lambda)\Big|_{\lambda=\lambda_j}. \end{aligned}$$

(i) Apply this technique to find $\exp(A)$ with

$$A = \begin{pmatrix} c & c \\ c & c \end{pmatrix}, \qquad c \in \mathbb{R}, \quad c \neq 0.$$

(ii) Use the method given above to calculate $\exp(iK)$, where the hermitian 2×2 matrix K is given by

$$K = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}, \qquad a, c \in \mathbb{R}, \quad b \in \mathbb{C}.$$

Problem 65. Let $z \in \mathbb{C}$. Let A, B be $n \times n$ matrices over $\mathbb{C}$. We say that B is invariant with respect to A if

$$e^{zA} B e^{-zA} = B.$$

Obviously e^{-zA} is the inverse of e^{zA}. Show that, if this condition is satisfied, one has $[A, B] = 0_n$, where 0_n is the $n \times n$ zero matrix. If e^{zA} would be unitary we have $UBU^* = B$.

Problem 66. Let $z \in \mathbb{C}$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{11} \end{pmatrix}.$$

(i) Calculate $\exp(zA)$, $\exp(-zA)$ and $\exp(zA) B \exp(-zA)$.
(ii) Calculate the commutator $[A, B]$.

Problem 67. Let $z \in \mathbb{C}$ and

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ -b_{12} & b_{11} \end{pmatrix}.$$

(i) Calculate $\exp(zA)$, $\exp(-zA)$ and $\exp(zA) B \exp(-zA)$.
(ii) Calculate the commutator $[A, B]$.

Problem 68. Let $z \in \mathbb{C}$ and

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{11} \end{pmatrix}.$$

(i) Calculate $\exp(zA)$, $\exp(-zA)$ and $\exp(zA) B \exp(-zA)$.
(ii) Calculate the commutator $[A, B]$.

Problem 69. Consider the Pauli spin matrices σ_x, σ_y, σ_z. Find the skew-hermitian matrices Σ_x, Σ_y, Σ_z such that

$$\sigma_x = \exp(\Sigma_x), \quad \sigma_y = \exp(\Sigma_y), \quad \sigma_z = \exp(\Sigma_z).$$

Find the commutators $[\Sigma_x, \Sigma_y]$, $[\Sigma_y, \Sigma_z]$, $[\Sigma_z, \Sigma_x]$ and compare with the commutators $[\sigma_x, \sigma_y]$, $[\sigma_y, \sigma_z]$, $[\sigma_z, \sigma_x]$.

Problem 70. Let $\alpha \in \mathbb{R}$. Consider the matrix

$$A(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} .$$

(i) Show that the matrix is orthogonal.
(ii) Find the determinant of $A(\alpha)$. Is the matrix an element of $SO(2,\mathbb{R})$?
(iii) Do these matrices form a group under matrix multiplication?
(iv) Calculate

$$X = \frac{d}{d\alpha} A(\alpha)\bigg|_{\alpha=0} .$$

Calculate $\exp(\alpha X)$ and compare this matrix with $A(\alpha)$. Discuss.
(v) Let $\beta \in \mathbb{R}$ and

$$B(\beta) = \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix} .$$

Is the matrix $A(\alpha) \otimes B(\beta)$ orthogonal? Find the determinant of $A(\alpha) \otimes B(\alpha)$. Is this matrix an element of $SO(4,\mathbb{R})$?

Problem 71. We know that for any $n \times n$ matrix A over $\mathbb{C}$ the matrix $\exp(A)$ is invertible with the inverse $\exp(-A)$. What about $\cos(A)$ and $\cosh(A)$?

Problem 72. (i) Let $\epsilon \in \mathbb{R}$. Let A be an $n \times n$ matrix over $\mathbb{C}$. Find

$$\lim_{\epsilon \to 0} \frac{\sinh(2\epsilon A)}{\sinh(\epsilon)} .$$

(ii) Assume that $A^2 = I_n$. Calculate

$$\frac{\sinh(2\epsilon A)}{\sinh(\epsilon)} .$$

(iii) Assume that $A^2 = 0_n$. Calculate

$$\frac{\sinh(2\epsilon A)}{\sinh(\epsilon)} .$$

Problem 73. Let A be an $n \times n$ normal matrix over $\mathbb{C}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding pairwise orthonormal eigenvectors $\mathbf{u}_j$ ($j = 1, \ldots, n$). Then the matrix A can be written as (spectral decomposition)

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{u}_j \mathbf{u}_j^* \equiv \sum_{j=1}^{n} \lambda_j |u_j\rangle\langle u_j| .$$

(i) Let $z \in \mathbb{C}$. Use this spectral decomposition to calculate $\exp(zA)$.
(ii) Apply it to $A = \sigma_x$.

Problem 74. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Is $\sin(A)$ invertible? Is $\cos(A)$ invertible? Is $\sin(B)$ invertible? Is $\cos(B)$ invertible?

Problem 75. Is $\cos(A)$ invertible for all $n \times n$ matrices A over $\mathbb{C}$?

Problem 76. Let A be a nilpotent matrix. Is the matrix $\cos(A)$ invertible?

Problem 77. Consider the 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find $\cosh(A)$, $\sinh(A)$, $\cosh(B)$, $\sinh(B)$. Which of these matrices are invertible?

Problem 78. (i) Consider the permutation matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors and thus the spectral decomposition of P.
(ii) Find the matrix X such that $\exp(X) = P$.

Problem 79. (i) Consider the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the eigenvalues and normalized eigenvectors of P and thus the spectral decomposition of P.
(ii) Find the matrix X such that $\exp(X) = P$.

Problem 80. Let A be an $n \times n$ matrix over $\mathbb{C}$. Find

$$\lim_{\epsilon \to 0} \frac{\sinh(\epsilon A)}{\sinh(\epsilon)}.$$

Problem 81. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Write the matrix A in the form

$$A = I_3 + B, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and calculate e^A using $e^A = e^{I_3} e^B$.

Problem 82. Consider the 3×3 matrix

$$A = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

Calculate $\exp(i\phi A)$, where $\phi \in \mathbb{R}$.

Problem 83. Let A be an $n \times n$ matrix. The *characteristic polynomial*

$$\det(\lambda I_n - A) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = p(\lambda)$$

is closely related the the *resolvent* $(\lambda I_n - A)^{-1}$ trough the formula

$$(\lambda I_n - A)^{-1} = \frac{N_1 \lambda^{n-1} + N_2 \lambda^{n-2} + \cdots + N_n}{\lambda^n + a_1 \lambda^{n-1} + \cdots + a_n} = \frac{N(\lambda)}{p(\lambda)}$$

where the adjugate matrix $N(\lambda)$ is a polynomial in λ of degree $n-1$ with constant $n \times n$ matrices $N_1, \ldots, N_n$. The *Laplace transform* of the matrix exponential is the resolvent

$$\mathcal{L}(e^{tA}) = (\lambda I_n - A)^{-1}.$$

The N_k matrices and a_k coefficients may be computed recursively as follows

$$\begin{aligned} N_1 &= I_n, \qquad a_1 = -\frac{1}{1}\mathrm{tr}(AN_1) \\ N_2 &= AN_1 + a_1 I_n, \qquad a_2 = -\frac{1}{2}\mathrm{tr}(AN_2) \\ &\vdots \\ N_n &= AN_{n-1} + a_{n-1} I_n, \qquad a_n = -\frac{1}{n}\mathrm{tr}(AN_n) \\ 0 &= AN_n + a_n I_n. \end{aligned}$$

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Find the N_k matrices and the coefficients a_k and thus calculate the resolvent.

Problem 84. Let A be a normal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding normalized pairwise orthogonal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Let $\mathbf{w}, \mathbf{v} \in \mathbb{C}^n$ (column vectors). Find

$$\mathbf{w}^* e^A \mathbf{v}$$

by expanding $\mathbf{w}$ and $\mathbf{v}$ with respect to the basis $\mathbf{u}_j$ $(j = 1, \ldots, n)$.

Problem 85. Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find $\cos(A)$ and the inverse of this matrix. Find $\cos(B)$ and the inverse of this matrix. Find the commutators $[A, B]$ and $[\cos(A), \cos(B)]$. Discuss.

Problem 86. Let V be the 2×2 matrix

$$V = v_0 I_2 + v_1 \sigma_1 + v_2 \sigma_2 + v_3 \sigma_3$$

where $v_0, v_1, v_2, v_3 \in \mathbb{R}$. Consider the equation

$$\exp(i\epsilon V) = (I_2 - iW)(I_2 + iW)^{-1}$$

where ϵ is real. Find W as a function of V.

Problem 87. Consider the rotation matrix

$$R(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

where ω is the fixed frequency. Find the matrix

$$H(t) = i\hbar \frac{dR(t)}{dt} R^T(t)$$

and show it is hermitian.

Problem 88. (i) Let σ_1 be the Pauli matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Calculate

$$\exp\left(-\frac{1}{2} i\pi(\sigma_1 - I_2)\right) .$$

(ii) Find all 2×2 matrices A and $c \in \mathbb{C}$ such that

$$\exp(c(A - I_2)) = A .$$

Problem 89. Let B be an $n \times n$ matrix with $B^2 = I_n$. Show that

$$\exp\left(-\frac{1}{2} i\pi(B - I_n)\right) \equiv B .$$

Problem 90. Let A, B be $n \times n$ matrices over $\mathbb{C}$ and

$$\exp(A)\exp(B) = \exp(C) .$$

Then the matrix C can be given as an infinite series of commutators of A and B. Let $z \in \mathbb{C}$. We write

$$\exp(zA)\exp(zB) = \exp(C(zA, zB))$$

where

$$C(zA, zB) = \sum_{j=1}^{\infty} c_j(A, B) z^j .$$

Show that the expansion up to fourth order is given by

$$\begin{aligned} c_1(A, B) &= A + B \\ c_2(A, B) &= \frac{1}{2}[A, B] \\ c_3(A, B) &= \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B] \\ c_4(A, B) &= -\frac{1}{24}[A, [B, [A, B] . \end{aligned}$$

Problem 91. The 2×2 matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are nonnormal, i.e. $AA^* \neq A^*A$ and $BB^* \neq B^*B$. Note that $A^* = B$. Are the matrices

$$\exp(A), \qquad \exp(B)$$

normal? Are the matrices $\sin(A)$, $\sin(B)$, $\cos(A)$, $\cos(B)$ normal?

Problem 92. Let $a, b \in \mathbb{C}$. Find

$$\exp \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} .$$

Problem 93. Find the unitary matrix

$$U(t) = e^{i\phi \sin(\omega t)\sigma_x}$$

with the Pauli spin matrix σ_x.

Chapter 8

Linear Differential Equations

Problem 1. Let A be an $n \times n$ matrix over $\mathbb{R}$. Consider the initial value problem of the system of linear differential equations

$$\frac{d\mathbf{u}(t)}{dt} + A\mathbf{u}(t) = \mathbf{g}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0 \tag{1}$$

where $\mathbf{g}(t) = (g_1(t), g_2(t), \ldots, g_n(t))^T$. The solution of the initial value problem is

$$\mathbf{u}(t) = e^{-tA}\mathbf{u}_0 + \int_0^t e^{-(t-\tau)A}\mathbf{g}(\tau)d\tau\,. \tag{2}$$

(i) Discretize the system with the implicit Euler method with step size h.
(ii) Compare the two solutions of the two systems for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

initial values $\mathbf{u}_0 = (1\,,\,1\,,\,1)^T$ with $\mathbf{g}(t) = (1,\,0,\,1)^T$ and the step size $h = 0.1$.

Problem 2. Let L and K be two $n \times n$ matrices. Assume that the entries depend on a parameter t and are differentiable with repect to t. Assume that $K^{-1}(t)$ exists for all t. Assume that the time-evolution of L is given by

$$L(t) = K(t)L(0)K^{-1}(t)\,.$$

(i) Show that $L(t)$ satisfies the matrix differential equation

$$\frac{dL}{dt} = [L, B](t)$$

where $[\,,\,]$ denotes the commutator and

$$B = -\frac{dK}{dt}K^{-1}(t)\,.$$

(ii) Show that if $L(t)$ is hermitian and $K(t)$ is unitary, then the matrix $B(t)$ is skew-hermitian.

Problem 3. Consider a system of linear ordinary differential equations with periodic coefficients

$$\frac{d\mathbf{u}}{dt} = A(t)\mathbf{u}, \qquad \mathbf{u} = \begin{pmatrix} u_2 \\ u_2 \end{pmatrix}$$

where $A(t)$ is a 2×2 matrix of periodic functions with period T. By the classical *Floquet theory*, any fundamental matrix $\Phi(t)$, which is defined as a nonsingular matrix satisfying the matrix differential equation

$$\frac{d\Phi}{dt} = A(t)\Phi(t)$$

can be expressed as

$$\Phi(t) = P(t)\exp(TR)\,.$$

Here $P(t)$ is nonsingular matrix of periodic functions with the same period T, and R, a constant matrix, whose eigenvalues λ_1 and λ_2 are called the characteristic exponents of the periodic sytem (1). For a choice of fundamental matrix $\Phi(t)$, we have

$$\exp(TR) = \Phi(t_0)\Phi(t_0 + T)$$

which does not depend on the initial time t_0. The matrix $\exp(TR)$ is called the *monodromy matrix* of the periodic system (1). Calculate

$$\operatorname{tr}\exp(TR)\,.$$

Problem 4. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(i) Calculate $\exp(tA)$, where $t \in \mathbb{R}$.
(ii) Find the solution of the initial value problem of the differential equation

$$\begin{pmatrix} du_1/dt \\ du_2/dt \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with the initial conditions $u_1(t=0) = u_{10}$, $u_2(t=0) = u_{20}$. Use the result from (i).

Problem 5. Solve the initial value problem for the matrix differential equation

$$[B, A(\epsilon)] = \frac{dA}{d\epsilon}$$

where $A(\epsilon)$ and B are 2×2 matrices with

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 6. Consider the initial problem of the matrix differential equation

$$\frac{dX}{dt} = A(t)X, \qquad X(0) = I_n$$

where $A(t)$ is an $n \times n$ matrix which depends smoothly on t and I_n is the $n \times n$ identity matrix. It is known that the solution of this matrix differential equation can locally be written as

$$X(t) = \exp(\Omega(t))$$

where $\Omega(t)$ is obtained as an infinite series

$$\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t).$$

This is the so-called Magnus expansion.
Implement this recursion in SymbolicC++ and apply it to

$$A(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Problem 7. Let $a, b \in \mathbb{R}$. Consider the linear matrix differential equation

$$\frac{d^2X}{dt^2} + a\frac{dX}{dt} + bX = 0.$$

Find the solution of the initial value problem.

Problem 8. Let A be an $n \times n$ matrix over $\mathbb{R}$. The autonomous system of first order differential equations $d\mathbf{u}/dt = A\mathbf{u}$ admits the solution of the initial value problem $\mathbf{u}(t) = \exp(A)\mathbf{u}(0)$. Differentiation of the differential equations yields the second order system

$$\frac{d^2\mathbf{u}}{dt^2} = A\frac{d\mathbf{u}}{dt} = A^2\mathbf{u}\,.$$

Thus we can write

$$\frac{d\mathbf{u}}{dt} = \mathbf{v} = A\mathbf{u}, \quad \frac{d\mathbf{v}}{dt} = A^2\mathbf{u} = A\mathbf{v}$$

or in matrix form

$$\begin{pmatrix} d\mathbf{u}/dt \\ d\mathbf{v}/dt \end{pmatrix} = \begin{pmatrix} 0_n & I_n \\ A^2 & 0_n \end{pmatrix} \begin{pmatrix} \mathbf{u}(0) \\ \mathbf{v}(0) \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix and I_n is the $n \times n$ identity matrix. Find the solution of the initial value problem. Assume that A is invertible.

Chapter 9

Norms and Scalar Products

Problem 1. Let U_1, U_2 be unitary $n \times n$ matrices. Let $\mathbf{v}$ be a normalized vector in $\mathbb{C}^n$. Consider the norm of a $k \times k$ matrix M

$$\|M\| = \max_{\|\mathbf{x}\|=1} \|M\mathbf{x}\|$$

where $\|\mathbf{x}\|$ denotes the Euclidean norm. Show that if $\|U_1 - U_2\| \le \epsilon$ then

$$\|U_1\mathbf{v} - U_2\mathbf{v}\| \le \epsilon\,.$$

Problem 2. Given the 2×2 matrix

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Calculate

$$\|A(\alpha)\| = \sup_{\|\mathbf{x}\|=1} \|A(\alpha)\mathbf{x}\|\,.$$

Problem 3. Let A be an $n \times n$ matrix. Let $\rho(A)$ be the spectral radius of A. Then we have

$$\rho(A) \le \min\{\max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|,\ \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|\}\,.$$

Let

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}.$$

Calculate $\rho(A)$ and the right-hand side of the inequality.

Problem 4. Consider the Hilbert space $\mathbb{C}^n$. We define a norm of an $n \times n$ matrix A over $\mathbb{C}$

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

where the right-hand side denotes the Euclidean norm. Let U be an $n \times n$ unitary matrix. Show that $\|U\| = 1$.

Problem 5. Let A be an $n \times n$ positive semidefinite (and thus hermitian) matrix. Is

$$\|A^{1/2}\| = \|A\|^{1/2}\,?$$

Problem 6. Let A be an $n \times n$ positive semidefinite matrix. Show that

$$|\mathbf{x}^* A \mathbf{y}| \le \sqrt{\mathbf{x}^* A \mathbf{x}}\sqrt{\mathbf{y}^* A \mathbf{y}}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}$.

Problem 7. Let $t \in \mathbb{R}$. Consider the symmetric matrix over $\mathbb{R}$

$$A(t) = \begin{pmatrix} t & 1 & 0 \\ 1 & t & 1 \\ 0 & 1 & t \end{pmatrix}.$$

Find the condition on t such that $\rho(A(t)) < 1$, where $\rho(A(t))$ denotes the spectral radius of $A(t)$.

Problem 8. (i) Let A be an $n \times n$ positive semidefinite matrix. Show that $(I_n + A)^{-1}$ exists.
(ii) Let B be an arbitrary $n \times n$ matrix. Show that the inverse of $I_n + B^* B$ exists.

Problem 9. Let A be an $n \times n$ matrix. One approach to calculate $\exp(A)$ is to compute an eigenvalue decomposition $A = XBX^{-1}$ and then apply the formula $e^A = Xe^B X^{-1}$. We have using the *Schur decomposition*

$$U^* A U = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) + N$$

where U is unitary, the matrix $N = (n_{jk})$ is a strictly upper triangular ($n_{jk} = 0, j \geq k$) and $\lambda(A) = \{ \lambda_1, \ldots, \lambda_n \}$ is the spectrum of A. Using the *Padé approximation* to calculate e^A we have

$$R_{pq} = (D_{pq}(A))^{-1} N_{pq}(A)$$

where

$$N_{pq}(A) := \sum_{j=0}^{p} \frac{(p+q-j)!p!}{(p+q)!j!(p-j)!} A^j$$

$$D_{pq}(A) := \sum_{j=0}^{q} \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!} (-A)^j .$$

Let

$$A = \begin{pmatrix} 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Calculate $\|R_{11} - e^A\|$, where $\| \cdot \|$ denotes the 2-norm.

Problem 10. Let A be an $n \times n$ matrix with $\|A\| < 1$. Then $\ln(I_n + A)$ exists. Show that

$$\| \ln(I_n + A) \| \leq \frac{\|A\|}{1 - \|A\|} .$$

Problem 11. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Show that

$$\|[A, B]\| \leq 2\|A\| \, \|B\|$$

where $[\, , \,]$ denotes the commutator.

Problem 12. Let A be an $n \times n$ matrix. Let

$$B = \begin{pmatrix} A & I_n & 0_n \\ I_n & A & I_n \\ 0_n & I_n & A \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix. Calculate B^2 and B^3.

Problem 13. Denote by $\| \cdot \|_{HS}$ the Hilbert-Schmidt norm and by $\| \cdot \|_O$ the operator norm, i.e.

$$\|A\|_{HS} := \sqrt{\operatorname{tr}(AA^*)}, \qquad \|A\|_O := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = \sup_{\substack{\lambda \in \mathbb{C}, \, \|\mathbf{x}\|=1 \\ (A^*A)\mathbf{x} = \lambda \mathbf{x}}} \sqrt{\lambda}$$

where A is an $m \times n$ matrix over $\mathbb{C}$, $\mathbf{x} \in \mathbb{C}^n$ and $\|\mathbf{x}\|$ is the Euclidean norm.
(i) Calculate

$$\left\| \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \right\|_{HS} \quad \text{and} \quad \left\| \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \right\|_{O} .$$

(ii) Let A be an $m \times n$ matrix over $\mathbb{C}$. Find the conditions on A such that

$$\|A\|_{HS} = \|A\|_O .$$

Problem 14. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Then

$$\|[A, B]\| \leq \|AB\| + \|BA\| \leq 2\|A\| \, \|B\|$$

where $\| \; \|$ denotes the norm. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

Calculate $\|[A, B]\|$, $\|AB\|$, $\|BA\|$, $\|A\|$, $\|B\|$ and thus verify the inequality for these matrices. The norm is given by $\|C\| = \sqrt{\operatorname{tr}(CC^*)}$.

Problem 15. Let A be an $n \times n$ matrix. The *logarithmic norm* is defined by

$$\mu[A] := \lim_{h \to 0^+} \frac{\|I_n + hA\| - 1}{h} .$$

Let

$$\|A\| := \sup_{\mathbf{x}=1} |A\mathbf{x}\| .$$

Let A be the $n \times n$ identity matrix I_n. Find $\mu[I_n]$.

Problem 16. Find a 2×2 unitary matrix U such that

$$U^* \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

Problem 17. Consider the Hilbert space $\mathbb{R}^n$. The scalar product $\langle \mathbf{x}, \mathbf{y} \rangle$ $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^T \mathbf{y} = \sum_{j=1}^{n} x_j y_j .$$

Thus the norm is given by $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. Show that

$$|\mathbf{x}^T \mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\| .$$

Problem 18. Let $\{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}$ be a linearly independent set of vectors in the normed space $\mathbb{R}^n$ with $m \leq n$.
(i) Show that there is a number $c > 0$ such that for every choice of real numbers $c_1, \ldots, c_m$ we have

$$\|c_1 \mathbf{v}_1 + \cdots + c_m \mathbf{v}_m\| \geq c \left(|c_1| + \cdots + |c_m| \right) . \tag{1}$$

(ii) Consider $\mathbb{R}^2$ and

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Find a c for this case.

Problem 19. Let A be an $n \times n$ hermitian matrix. Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Consider the equation

$$A\mathbf{u} - \lambda \mathbf{u} = \mathbf{v} .$$

(i) Show that for λ nonreal (i.e. it has an imaginary part) the vector $\mathbf{v}$ cannot vanish unless $\mathbf{u}$ vanishes.
(ii) Show that for λ nonreal we have

$$\|(A - \lambda I_n)^{-1} \mathbf{v}\| \leq \frac{1}{|\Im \lambda|} \|\mathbf{v}\| .$$

Problem 20. (i) Let A and C be invertible $n \times n$ matrices over $\mathbb{R}$. Let B be an $n \times n$ matrix over $\mathbb{R}$. Assume that

$$\|A\| \leq \|B\| \leq \|C\| .$$

Is B invertible?
(ii) Let A, B, C be invertible $n \times n$ matrices over $\mathbb{R}$ with

$$\|A\| \leq \|B\| \leq \|C\| .$$

Is

$$\|A^{-1}\| \geq \|B^{-1}\| \geq \|C^{-1}\| ?$$

Problem 21. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that $\|A\| < 1$, where

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| .$$

Show that the matrix $B = I_n + A$ is invertible, i.e. $B \in GL(n.\mathbb{R})$. To show that the expansion

$$I_n - A + A^2 - A^3 + \cdots$$

converges apply

$$\|A^m - A^{m+1} + A^{m+2} - \cdots \pm A^{m+k-1}\| \leq \|A^m\| \cdot \|1 + \|A\| + \cdots + \|A\|^{k-1}\|$$
$$= \|A\|^m \frac{1 - \|A\|^k}{1 - \|A\|} .$$

Then calculate $(I_n + A)(I_n - A + A^2 - A^3 + \cdots)$.

Problem 22. Let A be an $n \times n$ matrix over $\mathbb{C}$. Let $\|.\|$ be a subordinate matrix norm for which $\|I_n\| = 1$. Assume that $\|A\| < 1$.
(i) Show that $(I_n - A)$ is nonsingular.
(ii) Show that

$$\|(I_n - A)^{-1}\| \leq (1 - \|A\|)^{-1} .$$

Problem 23. Let A be an $n \times n$ matrix. Assume that $\|A\| < 1$. Show that

$$\|(I_n - A)^{-1} - I_n\| \leq \frac{\|A\|}{1 - \|A\|} .$$

Problem 24. Let A be an $n \times n$ nonsingular matrix and B an $n \times n$ matrix. Assume that $\|A^{-1}B\| < 1$.
(i) Show that $A - B$ is nonsingular.
(ii) Show that

$$\frac{\|A^{-1} - (A - B)^{-1}\|}{\|A^{-1}\|} \leq \frac{\|A^{-1}B\|}{1 - \|A^{-1}B\|} .$$

Problem 25. Let M be an $m \times n$ matrix over $\mathbb{C}$. The Frobenius norm of M is given by

$$\|M\|_F := \sqrt{\operatorname{tr}(M^*M)} = \sqrt{\operatorname{tr}(MM^*)}.$$

Let U_m be $m \times m$ unitary matrix and U_n be an $n \times n$ unitary matrix. Show that

$$\|U_m M\|_F = \|MU_n\|_F = \|M\|.$$

Show that $\|M\|_F$ is the square root of the sum of the squares of the singular values of M.

Problem 26. Let M be an $m \times n$ matrix over $\mathbb{C}$. Find the rank-1 $m \times n$ matrix A over $\mathbb{C}$ which minimizes

$$\|M - A\|_F.$$

Hint: Find the singular value decomposition of $M = U\Sigma V^*$ and find A' with rank 1 which minimizes

$$\|\Sigma - A'\|_F.$$

Apply the method to

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 27. Let A be an $n \times n$ nonsingular matrix and B an $n \times n$ matrix. Assume that $\|A^{-1}B\| < 1$.
(i) Show that $A - B$ is nonsingular.
(ii) Show that

$$\frac{\|A^{-1} - (A-B)^{-1}\|}{\|A^{-1}\|} \leq \frac{\|A^{-1}B\|}{1 - \|A^{-1}B\|}.$$

Problem 28. Let A be an $n \times n$ matrix over $\mathbb{C}$. The spectral radius of the matrix A is the non-negative number $\rho(A)$ defined by

$$\rho(A) := \max\{\, |\lambda_j(A)| \,:\, 1 \leq j \leq n \,\}$$

where $\lambda_j(A)$ ($j = 1, 2, \ldots, n$) are the eigenvalues of A. We define the norm of A as

$$\|A\| := \sup_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

where $\|A\mathbf{x}\|$ denotes the Euclidean norm. Is $\rho(A) \leq \|A\|$? Prove or disprove.

Chapter 10

Graphs and Matrices

Problem 1. A graph $G(V, E)$ is a set of nodes V (points, vertices) connected by a set of links E (edges, lines). We assume that there are n nodes. The adjancy $(n \times n)$ matrix $A = A(G)$ takes the form with 1 in row i, column j if i is connected to j, and 0 otherwise. Thus A is a symmetric matrix. Associated with A is the degree distribution, a diagonal matrix with row-sums of A along the diagonal, and 0's elsewhere. We assume that $d_{ii} > 0$ for all $i = 1, 2, \ldots, n$. We define the Laplacian as $L := D - A$. Let

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

(i) Give an interpretation of A, A^2, A^3.
(ii) Find D and L.
(iii) Show that L admits the eigenvalue $\lambda_0 = 0$ (lowest eigenvalue) with eigenvector $\mathbf{x} = (1, 1, 1, 1, 1, 1, 1)^T$.

Problem 2. A graph $G(V, E)$ is a set of nodes V (points, vertices) connected by a set of links E (edges, lines). We assume that there are n nodes. The adjacency $(n \times n)$ matrix $A = A(G)$ takes the form with 1 in row i, column j if i is connected to j, and 0 otherwise. Thus A is a symmetric matrix. Associated with A is the degree distribution D, a diagonal matrix with row-sums of A along the diagonal, and 0's elsewhere. D describes how many connections each node has. We define the *Laplacian* as $L := D - A$. Let $A = (a_{ij})$, i.e. a_{ij} are the entries of adjacency matrix. Find the mimimum of the weighted sum

$$S = \frac{1}{2} \sum_{i,j=1}^{n} (x_i - x_j)^2 a_{ij}$$

with the constraint $\mathbf{x}^T \mathbf{x} = 1$, where $\mathbf{x}^T = (x_1, x_2, \ldots, x_n)$. Use the Lagrange multiplier method. The sum is over all pairs of squared distances between nodes which are connected, and so the solution should result in nodes with large numbers of inter-connections being clustered together.

Chapter 11

Hadamard Product

Problem 1. Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times m$ matrices with entries in some fields. Then the *Hadamard product* is the entrywise product of A and B, that is, the $m \times n$ matrix $A \bullet B$ whose (i, j) entry is $a_{ij} b_{ij}$. We have the properties. Suppose A, B, C are matrices of the same size and λ is a scalar. Then

$$\begin{aligned} A \bullet B &= B \bullet A \\ A \bullet (B + C) &= A \bullet B + A \bullet C \\ A \bullet (\lambda B) &= \lambda (A \bullet B)\,. \end{aligned}$$

If A, B be $n \times n$ diagonal matrices, then $A \bullet B = AB$. If A, B are $n \times n$ positive definite matrices and (a_{jj}) are the diagonal entries of A, then

$$\det(A \bullet B) \geq \det B \prod_{j=1}^{n} a_{jj} \qquad (1)$$

with equality if and only if A is a diagonal matrix. Let

$$A = \begin{pmatrix} 5 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 13 & 4 \\ 4 & 4 \end{pmatrix}.$$

First show that A and B are positive definite and then calculate the left and right-hand side of (1).

Problem 2. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Calculate the Hadamard product $A \bullet B$. Show that

$$\|A \bullet B\| \leq \|A^*A\| \, \|B^*B\|$$

where the norm is given by the Hilbert-Schmidt norm.

Problem 3. Let A, B, C and D^T be $n \times n$ matrices over $\mathbb{R}$. The Hadamard product is defined by $(A \bullet B)_{ij} := a_{ij}b_{ij}$. Show that

$$\text{tr}((A \bullet B)(C^T \bullet D) = \text{tr}((A \bullet B \bullet C)D)\,.$$

Problem 4. If V and W are matrices of the same order, then their Schur product $V \bullet W$ is defined by (entrywise multiplication)

$$(V \bullet W)_{j,k} := V_{j,k}W_{j,k}\,.$$

If all entries of V are nonzero, then we say that X is Schur invertible and define its Schur inverse, $V^{(-)}$, by $V^{(-)} \bullet V = J$, where J is the matrix with all 1's.

The vector space $M_n(\mathbb{F})$ of $n \times n$ matrices acts on itself in three distinct ways: if $C \in M_n(\mathbb{F})$ we can define endomorphisms X_C, Δ_C and Y_C by

$$X_C M := CM, \qquad \Delta_C M := C \bullet M, \qquad Y_C := MC^T\,.$$

Let A, B be $n \times n$ matrices. Assume that X_A is invertible and Δ_B is invertible in the sense of Schur. Note that X_A is invertible if and only if A is, and Δ_B is invertible if and only if the Schur inverse $B^{(-)}$ is defined. We say that (A, B) is a *one-sided Jones pair* if

$$X_A \Delta_B X_A = \Delta_B X_A \Delta_B\,.$$

We call this the *braid relation*. Give an example for a one-sided Jones pair.

Problem 5. Let A, B be $n \times n$ matrices. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors in $\mathbb{C}^n$. We form the n^2 column vectors

$$(A\mathbf{e}_j) \bullet (B\mathbf{e}_k), \qquad j, k = 1, \ldots, n\,.$$

If A is invertible and B is Schur invertible, then for any j

$$\{ (A\mathbf{e}_1) \bullet (B\mathbf{e}_j), \ (A\mathbf{e}_2) \bullet (B\mathbf{e}_j), \ \dots \ (A\mathbf{e}_n) \bullet (B\mathbf{e}_j) \}$$

is a basis of the vector space $\mathbb{C}^n$. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Find these bases for these matrices.

Problem 6. Let U be an $n \times n$ unitary matrix. Can we conclude that $U \bullet U^*$ is a unitary matrix?

Problem 7. Let $B = (b_{jk})$ be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Thus, there is nonsingular $n \times n$ matrix A such that

$$B = A(\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n))A^{-1}.$$

Show that

$$\begin{pmatrix} b_{11} \\ b_{22} \\ \vdots \\ b_{nn} \end{pmatrix} = (A \bullet (A^{-1})^T) \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$$

where $\bullet$ is the Hadamard product (Schur product, entrywise product). Thus the vector of eigenvalues of B is transformed to the vector of its diagonal entries by the ceofficient matrix $A \bullet (A^{-1})^T$.

Problem 8. Let A, B, C, D be $n \times n$ matrices over $\mathbb{R}$. Let

$$\mathbf{s}^T = (1 \quad 1 \quad \dots \quad 1)$$

be a row vector in $\mathbb{R}^n$. Show that

$$\mathbf{s}^T(A \bullet B)(C^T \bullet D)\mathbf{s} = \mathrm{tr}(C\Gamma D)$$

where $\Gamma = (\gamma_{ij})$ is a diagonal matrix with $\gamma_{jj} = \sum_{i=1}^n a_{ij}b_{ij}$ with $j = 1, 2, \dots, n$.

Problem 9. Given two matrices A and B of the same size. We use $A \bullet B$ to denote the Schur product. If all entries of A are nonzero, then we say that A is *Schur invertible* and define its Schur inverse, $A^{(-)}$ by

$$A_{ij}^{(-)} := \frac{1}{A_{ij}}.$$

Equivalently, we have $A^{(-)} \bullet A = J$, where J is the matrix with all ones. An $n \times n$ matrix W is a *type-II matrix* if

$$WW^{(-)^T} = nI_n$$

where I_n is the $n \times n$ identity matrix. Find such a matrix for $n = 2$.

Problem 10. Let A be an invertible $n \times n$ matrix. Can we conclude that

$$A \bullet A^{-1}$$

is invertible?

Problem 11. The $(n+1) \times (n+1)$ *Hadamard matrix* $H(n)$ of any dimension is generated recursively as follows

$$H(n) = \begin{pmatrix} H(n-1) & H(n-1) \\ H(n-1) & -H(n-1) \end{pmatrix}$$

where $n = 1, 2, \ldots$ and

$$H(0) = (1)\,.$$

Find $H(1)$, $H(2)$ and $H(3)$.

Problem 12. Show that

$$\operatorname{tr}(A(B \bullet C)) \equiv (\operatorname{vec}(A^T \bullet B))^T \operatorname{vec}(C)\,.$$

Problem 13. Let $\circ$ be the Hadamard product. Let A be a positive semidefinite $n \times n$ matrix. Let B be an $n \times n$ matrix with $\|B\| \leq 1$, where $\|\,.\,\|$ denotes the spectral norm. Show that

$$\max\{\, \|A \circ B\| \,:\, \|B\| \leq 1\,\} = \max a_{jj}$$

where $\|\,\|$ denotes the spectral norm.

Chapter 12

Unitary Matrices

Problem 1. (i) Let A, B be $n \times n$ matrices over $\mathbb{R}$. Show that one can find a $2n \times 2n$ unitary matrix U such that

$$U \begin{pmatrix} A & B \\ -B & A \end{pmatrix} U^* = \begin{pmatrix} A+iB & 0_n \\ 0_n & A-iB \end{pmatrix}.$$

Here 0_n denotes the $n \times n$ zero matrix.
(ii) Use the result from (i) to show that

$$\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \det(A+iB)\overline{\det(A+iB)} \geq 0.$$

Problem 2. Let $\mathbf{u}$ be a column vector in $\mathbb{C}^n$ with $\mathbf{u}^*\mathbf{u} = 1$, i.e. the vector is normalized. Consider the matrix

$$U = I_n - 2\mathbf{u}\mathbf{u}^*.$$

(i) Show that U is hermitian.
(ii) Show that U is unitary.

Problem 3. Can one find a 2×2 unitary matrix such that

$$U \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} U^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Problem 4. Let σ_x, σ_y, σ_z be the Pauli spin matrices and

$$U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

be the Hadamard matrix. Find

$$U_H \sigma_x U_H^*, \qquad U_H \sigma_y U_H^*, \qquad U_H \sigma_z U_H^* .$$

Problem 5. Find all 2×2 hermitian and unitary matrices A, B such that

$$AB = e^{i\pi} BA .$$

Problem 6. Find all $(n+1) \times (n+1)$ matrices A such that

$$A^* U A = U$$

where U is the unitary matrix

$$U = \begin{pmatrix} 0 & 0 & i \\ 0 & I_{n-1} & 0 \\ -i & 0 & 0 \end{pmatrix}$$

and $\det(A) = 1$. Consider first the case $n = 2$.

Problem 7. Consider the 2×2 hermitian matrices A and B with $A \neq B$ with the eigenvalues λ_1, λ_2; μ_1, μ_2; and the corresponding normalized eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$; $\mathbf{v}_1$, $\mathbf{v}_2$, respectively. Form from the normalized eigenvectors the 2×2 matrix

$$\begin{pmatrix} \mathbf{u}_1^* \mathbf{v}_1 & \mathbf{u}_1^* \mathbf{v}_2 \\ \mathbf{u}_2^* \mathbf{v}_1 & \mathbf{u}_2^* \mathbf{v}_2 \end{pmatrix} .$$

Is this matrix unitary? Find the eigenvalues of this matrix and the corresponding normalized eigenvectors of the 2×2 matrix. How are the eigenvalues and eigenvectors are linked to the eigenvalues and eigenvectors of A and B?

Problem 8. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $a_j \in \mathbb{R}$ with $j = 0, 1, 2, 3$ and

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 .$$

Show that

$$U = e^{i\phi}(a_0 I_2 + a_1 i\sigma_1 + a_2 i\sigma_2 + a_3 i\sigma_3)$$

is a unitary matrix, where $\phi \in \mathbb{R}$.

Problem 9. Let I_n be the $n \times n$ unit matrix. Is the $2n \times 2n$ matrix

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ I_n & -iI_n \end{pmatrix}$$

unitary?

Problem 10. Consider the two 2×2 unitary matrices

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Can one find a unitary 2×2 matrix V such that

$$U_1 = VU_2V^* \, ?$$

Problem 11. Let U be an $n \times n$ unitary matrix.
(i) Is $U + U^*$ invertible?
(ii) Is $U + U^*$ hermitian?
(iii) Calculate $\exp(\epsilon(U + U^*))$, where $\epsilon \in \mathbb{R}$

Problem 12. Let U be an $n \times n$ unitary matrix. Then $U + U^*$ is a hermitian matrix. Can any hermitian matrix represented in this form?

Problem 13. (i) Find the condition on the $n \times n$ matrix A over $\mathbb{C}$ such that $I_n + A$ is a unitary matrix.
(ii) Let B be an 2×2 matrix over $\mathbb{C}$. Find all solutions of the equation

$$B + B^* + BB^* = 0_2 \, .$$

Problem 14. Find all 2×2 invertible matrices A such that

$$A + A^{-1} = I_2 \, .$$

Problem 15. Let $z_1, z_2, w_1, w_2 \in \mathbb{C}$. Consider the 2×2 matrices

$$U = \begin{pmatrix} 0 & z_1 \\ z_2 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & w_1 \\ w_2 & 0 \end{pmatrix}$$

where $z_1\bar{z}_1 = 1$, $z_2\bar{z}_2 = 1$, $w_1\bar{w}_1 = 1$, $w_2\bar{w}_2 = 1$. This means the matrices U, V are unitary. Find the condition on z_1, z_2, w_1, w_2 such that the commutator $[U, V]$ is again a unitary matrix.

Problem 16. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. Find the conditions on $\alpha_1, \alpha_2, \alpha_3$ such that

$$U = \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3$$

is a unitary matrix.

Problem 17. Consider $n \times n$ unitary matrices. A scalar product of two $n \times n$ matrices U, V can be defined as

$$\langle U, V \rangle := \frac{1}{n}\mathrm{tr}(UV^*) .$$

Find two 2×2 unitary matrices U, V such that

$$\langle U, V \rangle = \frac{1}{2} .$$

Problem 18. Let

$$\{ |a_0\rangle, |a_1\rangle, \ldots .|a_{n-1}\rangle \}$$

be an orthonormal basis in the Hilbert space $\mathbb{C}^n$. The discrete Fourier transform

$$|b_j\rangle = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \omega^{jk} |a_k\rangle, \qquad j = 0, 1, \ldots, n$$

where $\omega := \exp(2\pi i/n)$ is the primitive n-th root of unity.
(i) Apply the discrete Fourier transform to the standard basis in $\mathbb{C}^4$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} .$$

(ii) Apply the discrete Fourier transform to the Bell basis in $\mathbb{C}^4$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} .$$

Problem 19. (i) Consider the Pauli spin matrices $\sigma_0 = I_2$, σ_1, σ_2, σ_3. The matrices are unitary and hermitian. Is the 4×4 matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_0 & \sigma_1 \\ \sigma_2 & \sigma_3 \end{pmatrix}$$

unitary?
(ii) Is the 4×4 matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_1 \\ -i\sigma_2 & \sigma_3 \end{pmatrix}$$

unitary?

Problem 20. Let U be an 2×2 unitary matrix. Is the 4×4 matrix

$$V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U + \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 0 \end{pmatrix} \otimes I_2$$

unitary?

Problem 21. The Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are hermitian and unitary. Together with the 2×2 identity matrix $\sigma_0 = I_2$ they form an orthogonal basis in Hilbert space of the 2×2 matrices over $\mathbb{C}$ with the scalar product $\mathrm{tr}(AB^*)$. Let X be an $n \times n$ hermitian matrix. Then $(X + iI_n)^{-1}$ exists and

$$U = (X - iI_n)(X + iI_n)^{-1}$$

is unitary. This is the so-called Cayley transform of X. Find the Cayley transform of the Pauli spin matrices and the 2×2 identiy matrix. Show that these matrices also form an orthogonal basis in the Hilbert space.

Problem 22. Consider the unitary matrix with determinant $+1$

$$U(r, \phi) = \begin{pmatrix} \cosh(r) & e^{i\phi}\sinh(r) \\ e^{-i\phi}\sinh(r) & \cosh(r) \end{pmatrix}.$$

where $r, \phi \in \mathbb{R}$. Find the eigenvalues and normalized eigenvectors. Construct another unitary matrix using these normalized eigenvectors as columns of this matrix.

Problem 23. Show that the two matrices

$$A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

are conjugate in the Lie group $SU(2)$.

Problem 24. A wave-scattering problem can be described by its scattering matrix U. In a stationary problem, U relates the outgoing-wave to the ingoing-wave amplitudes. The condition of flux conservation implies unitary of U, i.e.

$$UU^\dagger = I$$

where I is the identity operator. If, additionally, the scattering problem is invariant under the operatiom of time reversal, we also have $U = U^T$, i.e. U is symmetric. Find all 2×2 unitary matrices that also satisfy $U = U^T$. Do these matrices form a subgroup of the Lie group $U(2)$?

Problem 25. Let U be an $n \times n$ unitary matrix. Let V be an $n \times n$ unitary matrix such that $V^{-1}UV = D$ is a diagonal matrix D. Is $V^{-1}U^*V$ a diagonal matrix?

Problem 26. Let U be an $n \times n$ unitary matrix. Is $U + U^*$ invertible?

Problem 27. Let U, V be two $n \times n$ unitary matrices. Then we can define a scalar product via

$$\langle U, V \rangle := \frac{1}{n} \mathrm{tr}(UV^*) .$$

Find 2×2 unitary matrices U, V such that $\langle U, V \rangle = 1/2$.

Problem 28. Let $\omega := \exp(2\pi i/4)$. Consider the 3×3 unitary matrices

$$\Sigma = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega^3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} -i/2 & (1+i)/2 & 1/2 \\ (1+i)/2 & 0 & (1-i)/2 \\ 1/2 & (1-i)/2 & i/2 \end{pmatrix} .$$

Do the matrices of the set

$$\Lambda := \{ \Sigma^j C^k \Omega^\ell \, : \, 0 \le j \le 3, \, 0 \le k \le 1, \, 0 \le \ell \le 2 \}$$

form a group under matrix multiplication?

Problem 29. Let $\omega := \exp(2\pi i/4)$. Consider the 4×4 unitary matrices

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

Let $c > 0$. The four-state *Potts quantum chain* is defined by the Hamilton operator

$$\hat{H} = -\frac{1}{\pi\sqrt{c}}\sum_{j=1}^{N}\big((\sigma_j + \sigma_j^2 + \sigma_j^3) + c(\Gamma_j\Gamma_{j+1}^3 + \Gamma_j^2\Gamma_{j+1}^2 + \Gamma_j^3\Gamma_{j+1})\big)$$

where N is the number of sites and one imposes cyclic boundary conditions $N+1 \equiv 1$. Let $N = 2$. Find the eigenvalues and eigenvectors of $\hat{H}$.

Problem 30. Let U be an $n \times n$ unitary matrix and A an arbitrary $n \times n$ matrix. Then we know that

$$Ue^AU^{-1} = e^{UAU^{-1}}.$$

Calculate Ue^AU with $U \neq U^{-1}$.

Problem 31. Consider the Bell matrix

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

which is a unitary matrix. Each column vector of the matrix is a fully entangled state. Are the normalized eigenvectors of B are also fully entangled states?

Problem 32. Consider the unitary matrix

$$U(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\phi_{11}} & e^{i\phi_{12}} \\ e^{i\phi_{21}} & e^{i\phi_{22}} \end{pmatrix}.$$

Calculate the proudct $U(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22})U(\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22})$ and find the conditions on ϕ_{11}, ϕ_{12}, ϕ_{21}, ϕ_{22} and ψ_{11}, ψ_{12}, ψ_{21}, ψ_{22} such that we have again a matrix of this form.

Problem 33. Consider the Hamilton operator $\hat{H} = \hat{H}_0 + \hat{H}_1$, where

$$\hat{H}_0 = \hbar\omega\sigma_z, \qquad \hat{H}_1 = \hbar\omega\sigma_x.$$

Let U and U_0 be the unitary matrices

$$U = \exp(-i\hat{H}t/\hbar), \qquad U_0 = \exp(-i\hat{H}_0t/\hbar).$$

Let n be a positive integer. The *Moller wave operators*

$$\Omega_\pm := \lim_{n\to\pm\infty} U^nU_0^{-n}.$$

Owing to their intertwining property the Moller wave operators transform the eigenvectors of the free dynamics $U_0 = \exp(-i\hat{H}_0 t/\hbar)$ into eigenvectors of the interacting dynamics $U = \exp(-i\hat{H}t/\hbar)$. Find $\Omega_\pm$.

Problem 34. Consider the unitary matrices

$$U_1(\phi_1,\phi_2,\phi_3) = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}, \qquad U_2(\phi_4,\phi_5,\phi_6) = \begin{pmatrix} 0 & 0 & e^{i\phi_4} \\ 0 & e^{i\phi_5} & 0 \\ e^{i\phi_6} & 0 & 0 \end{pmatrix}.$$

What is the condition on $\phi_1,\ldots,\phi_6$ such that $[U_1,U_2] = 0_3$?

Problem 35. Consider the matrices

$$U = \begin{pmatrix} e^{i\alpha}\cos\theta & e^{i\beta}\sin\theta & 0 \\ -e^{-i\beta}\sin\theta & e^{-i\alpha} & 0 \\ 0 & 0 & e^{i\gamma} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} \\ 1 & 1 & 0 \end{pmatrix}.$$

Find the commutator $[U,N]$.

Problem 36. Let $n \geq 2$ and even. Let U be a unitary antisymmetric $n \times n$ matrix. Show that there exists a unitary matrix V such that

$$V^T U V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $\oplus$ denotes the direct sum.

Problem 37. Let U be a unitary and symmetric matrix. Show that there exists a unitary and symmetric matrix V such that $U = V^2$.

Problem 38. Is the matrix

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \exp(i2\pi/3) & \exp(i4\pi/3) \\ 1 & \exp(i4\pi/3) & \exp(i2\pi/3) \end{pmatrix}$$

unitary? Find the eigenvalues and eigenvectors of U.

Problem 39. (i) Let $\tau = (\sqrt{5}-1)/2$ be the golden mean number. Consider the 2×2 matrices

$$B_1 = \begin{pmatrix} e^{-i7\pi/10} & 0 \\ 0 & -e^{-i3\pi/10} \end{pmatrix}, \quad B_2 = \begin{pmatrix} -\tau e^{-i\pi/10} & -i\sqrt{\tau} \\ -i\sqrt{\tau} & -\tau e^{i\pi/10} \end{pmatrix}.$$

The matrices are invertible. Are the matrices unitary? Is $B_1B_2B_1 = B_2B_1B_2$?
(ii) Show that using computer algebra

$$B_2^{-2}B_1^4B_2^{-1}B_1B_2^{-1}B_1B_2B_1^{-2}B_2B_1^{-1}B_2^{-5}B_1B_2^{-1} \approx \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Problem 40. Let U be an $n \times n$ unitary matrix. Show that $|\det(U)| = 1$.

Problem 41. Let A be an $n \times n$ matrix over $\mathbb{R}$. Show that if λ is an eigenvalue of A, then $\bar{\lambda}$ is also an eigenvalue of A. Give an example for a 2×2 matrix, where the eigenvalues are complex.

Problem 42. Let V be an $n \times n$ normal matrix over $\mathbb{C}$. Assume that all its eigenvalues have absolute value of 1, i.e. they are of the form $e^{i\phi}$. Show that V is unitary.

Problem 43. (i) What are the conditions on $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \in \mathbb{R}$ such that

$$U(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\phi_{11}} & e^{i\phi_{12}} \\ e^{i\phi_{21}} & e^{i\phi_{22}} \end{pmatrix}$$

is a unitary matrix?
(ii) What are the condition on $\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22} \in \mathbb{R}$ such that $U(\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22})$ is an element of $SU(2)$?

Chapter 13

Numerical Methods

Problem 1. Let A be an invertible $n \times n$ matrix over $\mathbb{R}$. Consider the system of linear equation $A\mathbf{x} = \mathbf{b}$ or

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \qquad i = 1, 2, \ldots, n\,.$$

Let $A = C - R$. This is called a *splitting* of the matrix A and R is the defect matrix of the splitting. Consider the iteration

$$C\mathbf{x}^{(k+1)} = R\mathbf{x}^{(k)} + \mathbf{b}, \qquad k = 0, 1, 2, \ldots\,.$$

Let

$$A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \qquad \mathbf{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\,.$$

The iteration converges if $\rho(C^{-1}R) < 1$, where $\rho(C^{-1}R)$ denotes the spectral radius of $C^{-1}R$. Show that $\rho(C^{-1}R) < 1$. Perform the iteration.

Problem 2. Let A be an $n \times n$ matrix over $\mathbb{R}$ and let $\mathbf{b} \in \mathbb{R}^n$. Consider the linear equation $A\mathbf{x} = \mathbf{b}$. Assume that $a_{jj} \neq 0$ for $j = 1, 2, \ldots, n$. We define the diagonal matrix $D = \mathrm{diag}(a_{jj})$. Then the linear equation $A\mathbf{x} = \mathbf{b}$ can be written as

$$\mathbf{x} = B\mathbf{x} + \mathbf{c}$$

with $B := -D^{-1}(A - D)$, $\mathbf{c} := D^{-1}\mathbf{b}$. The *Jacobi method* for the solution of the linear equation $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x}^{(k+1)} = B\mathbf{x}^{(k)} + \mathbf{c}, \qquad k = 0, 1, \ldots$$

where $\mathbf{x}^{(0)}$ is any initial vector in $\mathbb{R}^n$. The sequence converges if

$$\rho(B) := \max_{j=1,\ldots,n} |\lambda_j(B)| < 1$$

where $\rho(B)$ is the spectral radius of B. Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(i) Show that the Jacobi method can applied for this matrix.
(ii) Find the solution of the linear equation with $\mathbf{b} = (1\ 1\ 1)^T$.

Problem 3. Let A be an $n \times n$ matrix over $\mathbb{R}$. The (p,q) *Padé approximation* to $\exp(A)$ is defined by

$$R_{pq}(A) := (D_{pq}(A))^{-1} N_{pq}(A)$$

where

$$N_{pq}(A) = \sum_{j=0}^{p} \frac{(p+q-j)!p!}{(p+q)!j!(p-j)!} A^j$$

$$D_{pq}(A) = \sum_{j=0}^{q} \frac{(p+q-j)!q!}{(p+q)!j!(q-j)!} (-A)^j .$$

Nonsingularity of $D_{pq}(A)$ is assured if p and q are large enough or if the eigenvalues of A are negative. Find the Padé approximation for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $p = q = 2$. Compare with the exact solution.

Problem 4. Let A be an $n \times n$ matrix over $\mathbb{R}$. Then we have the Taylor expansion

$$\sin(A) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} A^{2k+1}, \qquad \cos(A) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} A^{2k} .$$

To calculate $\sin(A)$ and $\cos(A)$ from a truncated Taylor series approximation is only worthwhile near the origin. We can use the repeated application of the *double angle formula*

$$\cos(2A) \equiv 2\cos^2(A) - I_n, \qquad \sin(2A) \equiv 2\sin(A)\cos(A).$$

We can find $\sin(A)$ and $\cos(A)$ of a matrix A from a suitably truncated Taylor series approximates as follows

$$S_0 = \text{Taylor approximate to } \sin(A/2^k)$$

$$C_0 = \text{Taylor approximate to } \cos(A/2^k)$$

and the recursion

$$S_j = 2S_{j-1}C_{j-1}, \qquad C_j = 2C_{j-1}^2 - I_n$$

where $j = 1, 2, \ldots$. Here k is a positive integer chosen so that, say $\|A\|_\infty \approx 2^k$. Apply this recursion to calculate sine and cosine of the 2×2 matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Use $k = 2$.

Problem 5. Let A be an $n \times n$ matrix. We define the $j - k$ approximant of $\exp(A)$ by

$$f_{j,k}(A) := \left(\sum_{\ell=0}^{k} \frac{1}{\ell!} \left(\frac{A}{j} \right)^{\ell} \right)^{j}. \tag{1}$$

We have the inequality

$$\|e^A - f_{j,k}(A)\| \le \frac{1}{j^k (k+1)!} \|A\|^{k+1} e^{\|A\|} \tag{2}$$

and $f_{j,k}(A)$ converges to e^A, i.e.

$$\lim_{j \to \infty} f_{j,k}(A) = \lim_{k \to \infty} f_{j,k}(A) = e^A.$$

Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Find $f_{2,2}(A)$ and e^A. Calculate the right-hand side of the inequality (2).

Problem 6. The *power method* is the simplest algorithm for computing eigenvectors and eigenvalues Consider the vector space $\mathbb{R}^n$ with the Euclidean norm $\|\mathbf{x}\|$ of a vector $\mathbf{x} \in \mathbb{R}$. The iteration is as follows: Given a nonsingular $n \times n$ matrix M and a vector $\mathbf{x}_0$ with $\|\mathbf{x}_0\| = 1$. One defines

$$\mathbf{x}_{t+1} = \frac{M\mathbf{x}_t}{\|M\mathbf{x}_t\|}, \quad t = 0, 1, \ldots$$

This defines a dynamical system on the sphere S^{n-1}. Since M is invertible we have

$$\mathbf{x}_t = \frac{M^{-1}\mathbf{x}_{t+1}}{\|M^{-1}\mathbf{x}_{t+1}\|}, \quad t = 0, 1, \ldots$$

(i) Apply the power method to the nonnormal matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(ii) Apply the power method to the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Chapter 14

Binary Matrices

Problem 1. For a 2×2 binary matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{jk} \in \{0, 1\}$$

we define the determinant as

$$\det A = (a_{11} \cdot a_{22}) \oplus (a_{12} \cdot a_{21})$$

where $\cdot$ is the AND-operation and $\oplus$ is the XOR-operation.
(i) Find the determinant for the following 2×2 matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(ii) Find the determinant for the following 2×2 matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Problem 2. The determinant of a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is given by

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\,.$$

For a binary matrix B we replace this expression by

$$\det B = (b_{11}\cdot b_{22}\cdot b_{33}) \oplus (b_{12}\cdot b_{23}\cdot b_{31}) \oplus (b_{13}\cdot b_{21}\cdot b_{32}) \\ \oplus (b_{13}\cdot b_{22}\cdot b_{31}) \oplus (b_{11}\cdot b_{23}\cdot b_{32}) \oplus (b_{12}\cdot b_{21}\cdot b_{33})\,.$$

(i) Calculate the determinant for the binary matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) Calculate the determinant for the binary matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Problem 3. The finite field $GF(2)$ consists of the elements 0 and 1 (bits) which satisfies the following addition (XOR-operation) and multiplication (AND-operation) tables

$\oplus$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Find the determinant of the *binary matrices*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 4. A boolean function $f : \{0,1\}^n \to \{0,1\}$ can be transformed from the domain $\{0,1\}$ into the spectral domain by a linear transformation

$$T\mathbf{y} = \mathbf{s}$$

where T is a $2^n \times 2^n$ orthogonal matrix, $\mathbf{y} = (y_0, y_1, \ldots, y_{2^n-1})^T$, is the two valued ($\{+1,-1\}$ with $0 \leftrightarrow 1$, $1 \leftrightarrow -1$) truth table vector of the boolean function and s_j $(j = 0, 1, \ldots, 7)$ are the spectral coefficients ($\mathbf{s} = (s_0, s_1, \ldots, s_{2^n-1})^T$). Since T is invertible we have

$$T^{-1}\mathbf{s} = \mathbf{y}\,.$$

For T we select the Hadamard matrix. The $2^n \times 2^n$ Hadamard matrix $H(n)$ is recursively defined as

$$H(n) = \begin{pmatrix} H(n-1) & H(n-1) \\ H(n-1) & -H(n-1) \end{pmatrix}, \qquad n = 1, 2, \ldots$$

with $H(0) = (1)$ (1×1 matrix). The inverse of $H(n)$ is given by

$$H^{-1}(n) = \frac{1}{2^n} H(n)\,.$$

Now any boolean function can be expanded as the arithmetical polynomial

$$f(x_1, \ldots, x_n) = \frac{1}{2^{n+1}} \left(2^n - s_0 - s_1(-1)^{x_n} - s_2(-1)^{x_{n-1}} - \cdots - s_{2^n-1}(-1)^{x_1 \oplus x_2 \oplus \cdots \oplus x_n}\right)$$

where $\oplus$ denotes the modulo-2 addition.

Consider the boolean function $f : \{0, 1\}^3 \to \{0, 1\}$ given by

$$f(x_1, x_2, x_3) = \bar{x}_1 \cdot \bar{x}_2 \cdot \bar{x}_3 + \bar{x}_1 \cdot x_2 \cdot \bar{x}_3 + x_1 \cdot x_2 \cdot \bar{x}_3\,.$$

Find the truth table, the vector $\mathbf{y}$ and then calculate, using $H(3)$, the spectral coefficients s_j, $(j = 0, 1, \ldots, 7)$.

Problem 5. Consider the binary matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Calculate the Hadamard product $A \bullet B$.

Problem 6. Consider the two permutation matrices (NOT-gate and XOR-gate)

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Can we generate all other permutation matrices from these two permutation matrices?

Chapter 15

Groups, Lie Groups and Lie Algebras

Problem 1. The Pauli matrix σ_x is not only hermitian, unitary and his own inverse, but also a permutation matrix. Find all 2×2 matrices A such that

$$\sigma_x^{-1} A \sigma_x = A \,.$$

Problem 2. Let $z \in \mathbb{C}$ and $z \neq 0$.
(i) Do the 2×2 matrices

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}$$

form a group under matrix multiplication?
(ii) Do the 3×3 matrices

$$\begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & z \\ 0 & 1 & 0 \\ z^{-1} & 0 & 0 \end{pmatrix}$$

form a group under matrix multiplication?

Problem 3. Find all 3×3 permutation matrices P such that

$$P^{-1} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} P = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix} .$$

Show that these matrices form a group, i.e. a subgroup of the 3×3 permutation matrices.

Problem 4. The generators of the braid group B_3 are given by

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus σ_1 and σ_2 are elements of the Lie group $SL(2,\mathbb{R})$.
(i) Find σ_1^{-1} and σ_2^{-1}. Find $\sigma_1\sigma_2$ and $\sigma_1^{-1}\sigma_2$.
(ii) Is $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$?

Problem 5. Let $\alpha \in \mathbb{R}$. Consider the hermitian matrix which is an element of the noncompact Lie group $SO(1,1)$

$$A(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}.$$

Find the *Cayley transform*

$$B = (A - iI_2)(A + iI_2)^{-1}.$$

Note that B is a unitary matrix and therefore an element of the compact Lie group $U(n)$. Find $B(\alpha \to \infty)$.

Problem 6. Let L be a finite dimensional Lie algebra and $Z(L)$ the center of L. Show that $\text{ad} : L \to g\ell(L)$ is a homomorphisim of the Lie algebra L with kernel $Z(L)$.

Problem 7. Consider the Lie algebra $s\ell(2,\mathbb{R})$ with the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that the Lie algebra $s\ell(2,\mathbb{R})$ has no proper nontrivial ideals.

Problem 8. For the vector space of the $n \times n$ matrices over $\mathbb{R}$ we can introduce a scalar product via

$$\langle A, B \rangle := \text{tr}(AB^T).$$

Consider the Lie group $SL(2,\mathbb{R})$ of the 2×2 matrices with determinant 1. Find $X, Y \in SL(2,\mathbb{R})$ such that

$$\langle X, Y \rangle = 0$$

where neither X nor Y can be 2×2 identity matrix.

Problem 9. The isomorphism of the Lie algebras $s\ell(2, \mathbb{C})$ and $so(3, \mathbb{C})$ has the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & b-c & -i(b+c) \\ c-b & 0 & 2ia \\ i(b+c) & -2ia & 0 \end{pmatrix}.$$

Let $z \in \mathbb{C}$. Find

$$\exp\left(z \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right), \qquad \exp\left(z \begin{pmatrix} 0 & b-c & -i(b+c) \\ c-b & 0 & 2ia \\ i(b+c) & -2ia & 0 \end{pmatrix}\right).$$

Problem 10. If $A \in SL(2, \mathbb{R})$, then it can be uniquely be written in the form

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \exp\begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

Find this decomposition for the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Problem 11. The unit sphere

$$S^3 := \{\, (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,:\, \sum_{j=1}^{4} x_j^2 = 1 \,\}$$

we identitify with the Lie group $SU(2)$

$$(x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

(i) Map the standard basis of $\mathbb{R}^4$ into $SU(2)$ and express these matrices using the Pauli spin matrices and the 2×2 identity matrix.
(ii) Map the Bell basis

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

into $SU(2)$ and express these matrices using the Pauli spin matrices and the 2×2 identity matrix.

Problem 12. In the decomposition of the simple Lie algebra $s\ell(3,\mathbb{R})$ one finds the 3×3 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & -a_{11}-a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}.$$

Find the commutators $[A, A']$, $[B, B']$ and $[A, B]$. Discuss.

Problem 13. Is the 3×3 matrix

$$O(\theta,\phi) = \begin{pmatrix} \sin\theta\cos\phi & -\sin\phi & -\cos\theta\cos\phi \\ \sin\theta\sin\phi & \cos\phi & \cos\theta\sin\phi \\ \cos\theta & 0 & \sin\theta \end{pmatrix}$$

an element of the compact Lie group $SO(3)$?

Problem 14. We know that

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is an ordered basis of the simple Lie algebra $s\ell(2,\mathbb{R})$ with

$$[X, H] = -2X, \quad [X, Y] = H, \quad [Y, H] = 2Y.$$

Consider

$$\widetilde{X} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Find the commutators

$$[\widetilde{X}, \widetilde{H}], \quad [\widetilde{X}, \widetilde{Y}], \quad [\widetilde{Y}, \widetilde{H}].$$

Problem 15. Are the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

elements of $SL(3,\mathbb{R})$ and $SL(5,\mathbb{R})$, respectively? We have to test that $\det(A) = 1$ and $\det(B) = 1$.

Problem 16. (i) Let $\alpha \in \mathbb{R}$. Do the matrices

$$A(\alpha) = \begin{pmatrix} \cos\alpha & i\sin\alpha \\ i\sin\alpha & \cos\alpha \end{pmatrix}$$

form a group under matrix multiplication?
(i) Let $\alpha \in \mathbb{R}$. Do the matrices

$$A(\alpha) = \begin{pmatrix} \cosh\alpha & i\sinh\alpha \\ -i\sinh\alpha & \cosh\alpha \end{pmatrix}$$

form a group under matrix multiplication?

Problem 17. Do the matrices

$$A = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}$$

with $\det(A) \neq 0$ form a group under matrix multiplication?

Problem 18. Consider the Lie group $SL(n,\mathbb{C})$, i.e. the $n \times n$ matrices over $\mathbb{C}$ with determinant 1. Can we find $A, B \in SL(n,\mathbb{C})$ such that $[A, B]$ is an element of $SL(n,\mathbb{C})$?

Problem 19. Consider the 2×2 matrices

$$A(\alpha) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Both are elements of the non-compact Lie group $SL(2,\mathbb{C})$. Can one finds $\alpha \in \mathbb{C}$ such that the commutator $[A(\alpha), B]$ is again an element of $SL(2,\mathbb{C})$?

Problem 20. (i) Let A, B be elements of $SL(n,\mathbb{R})$. Is $A \otimes B$ an element of $SL(n^2,\mathbb{R})$.
(ii) Let A, B be elements of $SL(n,\mathbb{R})$. Is $A \oplus B$ an element of $SL(2n,\mathbb{R})$.
(iii) Let A, B be elements of $SL(2,\mathbb{R})$. Is

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

an element of $SL(4,\mathbb{R})$?

Problem 21. The Lie algebra $s\ell(3,\mathbb{R})$ admits the decomposition

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & -a_{11}-a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}$$

where $a_{jk}, b_{jk} \in \mathbb{R}$. Find the commutator $[A,B]$.

Problem 22. The simple Lie algebra $s\ell(2,\mathbb{R})$ has a basis given by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the commutation relations

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H\,.$$

The *universal enveloping algebra* $U(s\ell(2,\mathbb{R}))$ of the Lie algebra $s\ell(2,\mathbb{R})$ is the associative algebra with generators H, E, F and the relations

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H\,.$$

Find a basis of the Lie algebra $s\ell(2,\mathbb{R})$ so that all matrices are invertible. Find the inverse matrices of these matrices. Give the commutation relations.

Problem 23. A *Chevalley basis* for the semisimple Lie algebra $s\ell(3,\mathbb{R})$ is given by

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $Y_j = X_j^T$ for $j = 1, 2, 3$. The Lie algebra has rank 2 owing to H_1, H_2 and $[H_1, H_2] = 0$. Another basis could be formed by looking at the linear combinations

$$U_j = X_j + Y_j, \qquad V_j = X_j - V_j\,.$$

(i) Find the table of the commutator.
(ii) Calculate the vectors of commutators

$$\begin{pmatrix} [H_1, X_1] \\ [H_2, X_1] \end{pmatrix}, \qquad \begin{pmatrix} [H_1, X_2] \\ [H_2, X_2] \end{pmatrix}, \qquad \begin{pmatrix} [H_1, X_3] \\ [H_2, X_3] \end{pmatrix}$$

and thus find the roots.

Problem 24. Consider the 2×2 matrices over $\mathbb{R}$

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Calculate the commutator $C = [A, B]$ and check whether C can be written as a linear combination of A and B. If so we have a basis of a Lie algebra.

Problem 25. Do the set of 2×2 matrices

$$\begin{pmatrix} e^{i(\alpha+\beta)} \cosh(\tau) & e^{i(\alpha-\beta)} \sinh(\tau) \\ e^{-i(\alpha-\beta)} \sinh(\tau) & e^{-i(\alpha+\beta)} \cosh(\tau) \end{pmatrix}$$

form a group under matrix multiplication, where $\tau, \alpha, \beta \in \mathbb{R}$?

Problem 26. Let $0 \le \alpha < \pi/4$. Consider the transformation

$$X(x, y, \alpha) = \frac{1}{\sqrt{\cos(2\alpha)}} (x \cos(\alpha) + iy \sin(\alpha))$$
$$Y(x, y, \alpha) = \frac{1}{\sqrt{\cos(2\alpha)}} (-ix \sin(\alpha) + y \cos(\alpha)).$$

(i) Show that $X^2 + Y^2 = x^2 + y^2$.
(ii) Do the matrices

$$\frac{1}{\sqrt{\cos(2\alpha)}} \begin{pmatrix} \cos(\alpha) & i \sin(\alpha) \\ -i \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

form a group under matrix multiplication?

Problem 27. In the Lie group $U(N)$ of the $N \times N$ unitary matrices one can find two $N \times N$ matrices U and V such that

$$UV = e^{2\pi i/N} VU.$$

Any $N \times N$ hermitian matrix H can be written in the form

$$H = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} h_{jk} U^j V^k.$$

Using the expansion coefficients h_{jk} one can associate to the hermitian matrix H the function

$$h(q,q) = \sum_{j=0}^{N-1}\sum_{k=1}^{N-1} h_{jk} e^{2\pi i(jp+kq)}$$

where $p = 0, 1, \ldots, N-1$ and $q = 0, 1, \ldots, N-1$. Consider the case $N = 2$ and

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) Consider the hermitian and unitary matrix

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Find $h(p,q)$.
(ii) Consider the hermitian and projection matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find $h(p,q)$.

Problem 28. Given the matrices

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Consider the 4×4 matrices

$$\begin{pmatrix} 0_2 & \sigma_- \\ 0_2 & 0_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & 0_2 \\ \sigma_+ & 0_2 \end{pmatrix}, \quad \begin{pmatrix} \sigma_+ & 0_2 \\ 0_2 & \sigma_+ \end{pmatrix}, \quad \begin{pmatrix} \sigma_- & 0_2 \\ 0_2 & \sigma_- \end{pmatrix}.$$

Calculate the commutators of these matrices and extend the set so that one finds a basis of a Lie algebra.

Problem 29. (i) The standard basis for the vector space of the 2×2 matrices is given by

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We define the star composition of two 2×2 matrices as the 4×4 matrix

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Show that the sixteen 4×4 matrices $E_{jk} \star E_{\ell m}$ $(j,k,\ell,m = 1,2)$ form a basis in the vector space of the 4×4 matrices.
(ii) The matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

form a basis of the Lie algebra $s\ell(2,\mathbb{R})$. Do the nine 4×4 matrices

$$X \star X, \;\; X \star Y, \;\; X \star H, \;\; Y \star X, \;\; Y \star Y, \;\; Y \star H, \;\; H \star X, \;\; H \star Y, \;\; H \star H$$

form a basis of a Lie algebra?

Problem 30. Consider the Lie algebra of real-skew symmetric 3×3 matrices

$$A = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}.$$

Let R be a real orthogonal 3×3 matrix, i.e. $RR^T = I_3$. Show that RAR^T is a real-skew symmetric matrix.

Problem 31. The matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a basis of the simple Lie algebra $s\ell(2,\mathbb{R})$. Define the matrices

$$\Delta(H) = H \otimes I_2 + I_2 \otimes H, \quad \Delta(E) = E \otimes H^{-1} + H \otimes E, \quad \Delta(F) = F \otimes H^{-1} + H \otimes F.$$

Find the commutators

$$[\Delta(H), \Delta(E)], \quad [\Delta(H), \Delta(F)], \quad [\Delta(E), \Delta(F)].$$

Discuss.

Problem 32. Consider the 3×3 permutation matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(i) Find P^2, P^3. Do the matrices P, P^2, P^3 form a group under matrix multiplication?
(ii) Find the eigenvalues and eigenvectors of P. Do the eigenvalues of P form a group under multiplication?

Problem 33. Consider the two 3×3 permutation matrices

$$C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Can the remaining four 3×3 matrices be generated from C_1 and A using matrix multiplication?

Problem 34. Consider the two 4×4 permutation matrices

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Can the remaining twenty-two 4×4 matrices be generated from C_1 and A using matrix multiplication?

Problem 35. Consider the permutation matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Find the condition on a 3×3 matrix A such that

$$CAC^T = A.$$

Note that $C^T = C^{-1}$.

Problem 36. Consider the permutation matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Find the condition on a 4×4 matrix A such that

$$CAC^T = A.$$

Note that $C^T = C^{-1}$.

Problem 37. Let $c, d \in \mathbb{R}$ and $c, d \neq 0$. Do the matrices

$$\begin{pmatrix} c\cos\alpha & d^{-1}\sin\alpha \\ -d\sin\alpha & c^{-1}\cos\alpha \end{pmatrix}$$

form a group under matrix multiplication?

Problem 38. The Lie group 0(2) is generated by a rotation R_1 and a reflection R_2

$$R_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad R_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Find the trace, determinant and eigenvalues of R_1 and R_2.

Problem 39. (i) Consider the group G of all 3×3 permutation matrices. Show that

$$\frac{1}{|G|}\sum_{g\in G} g$$

is a projection matrix. Here $|G|$ denotes the number of elements in the group.
(ii) Consider the subgroup given by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Show that

$$\frac{1}{|G|}\sum_{g\in G} g$$

is a projection matrix.

Problem 40. Let A be an $n\times n$ matrix over $\mathbb{C}$. Show that if $A^T = -A$, then $e^A \in O(n,\mathbb{C})$.

Problem 41. Let A be the 2×2 matrix

$$A = \begin{pmatrix} a_{11}e^{i\phi_{11}} & a_{12}e^{i\phi_{12}} \\ a_{21}e^{i\phi_{21}} & a_{22}e^{i\phi_{22}} \end{pmatrix}$$

where $a_{jk}\in\mathbb{R}$, $a_{jk} > 0$ for $j,k = 1,2$ and $a_{12} = a_{21}$. We also have $\phi_{jk}\in\mathbb{R}$ for $j,k = 1,2$ and impose $\phi_{12} = \phi_{21}$. What are the conditions on a_{jk} and ϕ_{jk} such that $I_2 + iA$ is a unitary matrix?

Problem 42. Let $\alpha,\beta,\phi\in\mathbb{R}$ and $\alpha,\beta\neq 0$. Consider the matrices

$$A(\alpha,\beta,\phi) = \begin{pmatrix} \alpha\cos\phi & -\beta\sin\phi \\ \beta^{-1}\sin\phi & \alpha^{-1}\cos\phi \end{pmatrix}.$$

Do the matrices form a group under matrix multiplication?

Problem 43. Show that the four 2×2 matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

from a group under matrix multiplication. Is the group abelian?

Problem 44. Let $x \in \mathbb{R}$. Is the matrix

$$A(x) = \begin{pmatrix} \cos x & 0 & \sin x & 0 \\ 0 & \cos x & 0 & \sin x \\ -\sin x & 0 & \cos x & 0 \\ 0 & -\sin x & 0 & \cos x \end{pmatrix}$$

an orthogonal matrix?

Problem 45. (i) Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{11} \end{pmatrix}.$$

where $a_{11}, a_{12} \in \mathbb{R}$. Find all invertible 2×2 matrices S over $\mathbb{R}$ such that

$$SAS^{-1} = A\,.$$

Obviously the identity matrix I_2 would be such as matrix.
(ii) Do the matrices S form a group under matrix multiplication? Prove or disprove.
(iii) Use the result form (i) to calculate

$$(S \otimes S)(A \otimes A)(S \otimes S)^{-1}\,.$$

Discuss.

Problem 46. Let $\alpha, \beta, \gamma \in \mathbb{R}$. Do the 3×3 matrices

$$M(\alpha, \beta, \gamma) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & \beta \\ -\sin(\alpha) & \cos(\alpha) & \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

form a group under matrix multiplication?

Problem 47. Do the eight 2×2 matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

form a group under matrix multiplication? If not add the matrices so that one has a group.

Problem 48. The Lie group $SU(2)$ is defined by

$$SU(2) := \{\, U \ 2\times 2\,\text{matrix} \,:\, UU^* = I_2,\ \det U = 1 \,\}\,.$$

Let (3-sphere)

$$S^3 := \{\, (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ :\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \,\}\,.$$

Show that $SU(2)$ can be identified as a real manifold with the 3-sphere S^3.

Problem 49. The *Heisenberg group* is the set of upper 3×3 matrices of the form

$$H = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where a, b, c can be taken from some (arbitrary) commutative ring.
(i) Find the inverse of H.
(ii) Given two elements x, y of a group G, we define the *commutator* of x and y, denoted by $[x, y]$ to be the element $x^{-1}y^{-1}xy$. If a, b, c are integers (in the ring $\mathbb{Z}$ of the integers) we obtain the discrete Heisenberg group H_3. It has two generators

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find

$$z = xyx^{-1}y^{-1}\,.$$

Show that $xz = zx$ and $yz = zy$.
(iii) The derived subgroup (or commutator subgroup) of a group G is the subgroup $[G, G]$ generated by the set of commutators of every pair of elements of G. Find $[G, G]$ for the Heisenberg group.
(iv) Let

$$A = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

and $a, b, c \in \mathbb{R}$. Find $\exp(A)$.

(v) The Heisenberg group is a simple connected Lie group whose Lie algebra consists of matrices

$$L = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} .$$

Find the commutators $[L, L']$ and $[L, L'], L']]$, where $[L, L'] := LL' - L'L$.

Problem 50. Consider the matrices

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

Show that the matrices form a basis of a Lie algebra.

Problem 51. Find all 2×2 matrices S over $\mathbb{C}$ with determinant 1 (i.e. they are elements of $SL(2, \mathbb{C})$) such that

$$S^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Obviously, the 2×2 identity matrix is such an element.

Problem 52. There are six 3×3 permutation matrices which form a group under matrix multiplication.
(i) Can the six elements be generated from the two permutation matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

using matrix multiplication?
(ii) Does A, A^2, A^3 provide a subgroup?

Problem 53. There are twenty-four 4×4 permutation matrices which form a group under matrix multiplication.
(i) Can the 24 elements be generated from the two permutation matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

using matrix multiplication?
(ii) Does A, A^2, A^3, A^4 provide a subgroup?

Problem 54. Both

$$A(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \qquad B(\beta) = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}$$

are elements of the Lie group $SL(2,\mathbb{R})$. Are

$$A(\alpha) \otimes B(\beta), \quad A(\alpha) \oplus B(\beta), \quad A(\alpha) \star B(\beta)$$

elements of $SL(4,\mathbb{R})$?

Chapter 16

Inequalities

Problem 1. Let A be an $n \times n$ positive semidefinite matrix. Let B be an $n \times n$ positive definite matrix. Then we have *Klein's inequality*

$$\mathrm{tr}(A(\ln A - \ln B)) \geq \mathrm{tr}(A - B)\,.$$

(i) Let

$$A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Calculate the left-hand side and the right-hand side of the inequality.
(ii) When is the inequality an equality?

Problem 2. Let A, B be $n \times n$ hermitian matrices. Then (*Golden-Thompson-Symanzik inequality*)

$$\mathrm{tr} e^{A+B} \leq \mathrm{tr}(e^A e^B)\,.$$

Let $A = \sigma_z$ and $B = \sigma_x$. Calculate the left and right-hand side of the inequality.

Problem 3. Let A, B, C be positive definite $n \times n$ matrices. Then (*Lieb inequality*)

$$\mathrm{tr}(e^{\ln A - \ln B + \ln C}) \leq \mathrm{tr} \int_0^\infty A(B + uI_n)^{-1} C (B + uI_n)^{-1} du\,.$$

(i) Let

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}.$$

Calculate the left-hand side and right-hand side of the inequality.
(ii) Give a sufficient condition such that one has an equality.

Problem 4. Let A be an $n \times n$ skew-symmetric matrix over $\mathbb{R}$. Show that

$$\det(I_n + A) \geq 1$$

with equality holding if and only if $A = 0$.

Chapter 17

Braid Group

Let $n \geq 2$. The *braid group* $\mathcal{B}_n$ of n strings has $n-1$ generators $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ satisfying the relations

$$\begin{aligned} \sigma_j \sigma_{j+1} \sigma_j &= \sigma_{j+1} \sigma_j \sigma_{j+1} \quad \text{for} \quad j = 1, 2, \ldots, N-2 \\ \sigma_j \sigma_k &= \sigma_k \sigma_j \quad \text{for} \quad |j-k| \geq 2 \\ \sigma_j \sigma_j^{-1} &= \sigma_j^{-1} \sigma_j = e \end{aligned}$$

where e is the identity element. Thus it is generated by elements σ_j (σ_j interchanges elements j and $j+1$). Thus actually one should write σ_{12}, σ_{23}, ..., σ_{n-1n} instead of $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$. The braid group $\mathcal{B}_n$ is a generalization of the permutation group.

The word written in terms of letters, generators from the set

$$\{ \sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1} \}$$

gives a particular braid. The length of the braid is the total number of used letters, while the minimal irreducible length (referred sometimes as the primitive word) is the shortest non-contractible length of a particular braid which remains after applying all the group relations given above.

Problem 1. Consider the braid group $\mathcal{B}_5$ with the generators σ_1, σ_2, σ_3, σ_4. Simplify

$$\sigma_1^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2\sigma_3\sigma_2\sigma_3^{-1}\sigma_4\sigma_1 \, .$$

Problem 2. Consider the braid group $\mathcal{B}_3$. A faithful representation for the generators σ_1 and σ_2 is

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

Both are elements of $SL(2,\mathbb{Z})$. Find the inverse of σ_1 and σ_2. Do the elements σ_1, σ_2, σ_1^{-1}, σ_2^{-1} and the 2×2 identity matrix form a group under matrix multiplication?

Problem 3. Find all invertible 2×2 matrices A, B such that (*braid-like relation*)

$$ABA = BAB \, .$$

Problem 4. Can one find 2×2 matrices A and B with $[A,B] \neq 0$ and satisfying the braid-like relation

$$ABBA = BAAB \, .$$

Problem 5. (i) Do the 2×2 unitary matrices

$$A = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & ie^{-i\pi/4} \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

satisfy the *braid-like relation*

$$ABA = BAB \, .$$

(ii) Find the smallest $n \in \mathbb{N}$ such that $A^n = I_2$.
(iii) Find the smallest $m \in \mathbb{N}$ such that $B^m = I_2$.

Problem 6. Consider the braid group $\mathcal{B}_n$. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\ldots$, $\mathbf{e}_n$ be the standard basis in $\mathbb{R}^n$. Then $\mathbf{u} \in \mathbb{R}^n$ can be written as

$$\mathbf{u} = \sum_{k=1}^{n} c_k \mathbf{e}_k \, .$$

Consider the linear operators B_j ($j = 1, 2, \ldots, n-1$) in $\mathbb{R}^n$ ($\alpha, \beta, \gamma, \delta \in \mathbb{R}, \alpha, \gamma \neq 0$) defined by

$$B_j \mathbf{u} := c_1\mathbf{e}_1 + \cdots + (\alpha c_{j+1} + \beta)\mathbf{e}_j + (\gamma c_j + \delta)\mathbf{e}_{j+1} + \cdots + c_n \mathbf{e}_n$$

and the corresponding inverse operator B_j^{-1}

$$B_j^{-1}\mathbf{u} = c_1\mathbf{e}_1 + \cdots + \frac{1}{\gamma}(c_{j+1} - \delta)\mathbf{e}_j + \frac{1}{\alpha}(c_j - \beta)\mathbf{e}_{j+1} + \cdots + c_n\mathbf{e}_n \,.$$

Show that the linear operators B_j satisfy the braid condition

$$B_j B_{j+1} B_j = B_{j+1} B_j B_{j+1}$$

if

$$\gamma\beta + \delta = \alpha\delta + \beta \,.$$

Problem 7. If V and W are matrices of the same order, then their Schur product $V \bullet W$ is defined by (entrywise multiplication)

$$(V \bullet W)_{j,k} := V_{j,k} W_{j,k} \,.$$

If all entries of V are nonzero, then we say that X is Schur invertible and define its Schur inverse, $V^{(-)}$, by $V^{(-)} \bullet V = J$, where J is the matrix with all 1's.

The vector space $M_n(\mathbb{F})$ of $n \times n$ matrices acts on itself in three distinct ways: if $C \in M_n(\mathbb{F})$ we can define endomorphisms X_C, Δ_C and Y_C by

$$X_C M := CM, \qquad \Delta_C M := C \bullet M, \qquad Y_C := MC^T \,.$$

Let A, B be $n \times n$ matrices. Assume that X_A is invertible and Δ_B is invertible in the sense of Schur. Note that X_A is invertible if and only if A is, and Δ_B is invertible if and only if the Schur inverse $B^{(-)}$ is defined. We say that (A, B) is a *one-sided Jones pair* if

$$X_A \Delta_B X_A = \Delta_B X_A \Delta_B \,.$$

We call this the *braid relation*. Give an example for a one-sided Jones pair.

Problem 8. The braid linking matrix B is a square symmetric $k \times k$ matrix defined by $B = (b_{ij})$ with b_{ii} the sum of half-twists in the i-th branch, b_{ij} the sum of the crossings between the i-th and the j-th branches of the ribbon graph with standard insertion. Thus the i-th diagonal element of B is the local torsion of the i-th branch. The off-diagonal elements of B are twice the linking numbers of the ribbon graph for the i-th and j-th branches. Consider the braid linking matrix

$$B = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix} .$$

Discuss. Draw a graph.

Problem 9. Consider the five 4×4 matrices

$$B_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad B_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & 1 \end{pmatrix},$$

$$B_5 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Is

$$B_1B_2B_1 = B_2B_1B_2, \quad B_2B_3B_2 = B_3B_2B_3, \quad B_3B_4B_3 = B_4B_3B_4, \quad B_4B_5B_4 = B_5B_4B_5\,?$$

Problem 10. Let $n \geq 3$ and let $\sigma_1, \ldots, \sigma_{n-1}$ be the generators. The *braid group* B_n on n-strings where $n \geq 3$ has a finite presentation of B_n given by

$$\langle \sigma_1, \ldots, \sigma_{n-1} : \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i \rangle$$

where $1 \leq i, j < n-1$, $|i-j| > 1$ or $j = n-1$. Here $\sigma_i\sigma_j = \sigma_j\sigma_i$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ are called the braid relations. The second one is also called the Yang-Baxter equation.
(i) Consider B_3, $a = \sigma_1\sigma_2\sigma_1$ and $b = \sigma_1\sigma_2$. Show that $a^2 = b^3$.
(ii) Consider B_3. The cosets $[\sigma_1]$ of σ_1 and $[\sigma_2]$ of σ_2 map to the 2×2 matrices

$$[\sigma_1] \mapsto R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad [\sigma_2] \mapsto L^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

where $L, R \in SL(2,\mathbb{Z})$. Thus $L^{-1}, R^{-1} \in SL(2,\mathbb{Z})$. Show that

$$RL^{-1}R = L^{-1}RL^{-1}.$$

Problem 11. (i) Do the matrices

$$S_1 = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}$$

satisfy the braid-like relation $S_1S_2S_1 = S_2S_1S_2$.
(ii) Do the matrices $S_1 \otimes S_1$ and $S_2 \otimes S_2$ satisfy the braid-like relation

$$(S_1 \otimes S_1)(S_2 \otimes S_2)(S_1 \otimes S_1) = (S_2 \otimes S_2)(S_1 \otimes S_1)(S_2 \otimes S_2)\,?$$

Problem 12. Consider the five 4×4 matrices

$$B_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

$$B_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & 1 \end{pmatrix}, \quad B_5 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Are the matrices unitary? Is (braid-like relation)

$$B_jB_{j+1}B_j = B_{j+1}B_jB_{j+1}, \quad j = 1, 2, 3, 4$$

Chapter 18

vec Operator

Problem 1. Consider the 2×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} .$$

Let $B = A^T$. Thus B is a 3×2 matrix. Find the 6×6 permutation matrix P such that

$$\mathrm{vec}(B) = P\mathrm{vec}(A) .$$

Problem 2. Let A be an $m \times n$ matrix over $\mathbb{C}$ and B be a $s \times t$ matrix over $\mathbb{C}$. Find the permutation matrix P such that

$$\mathrm{vec}(A \otimes B) = P(\mathrm{vec}(A) \otimes \mathrm{vec}(B)).$$

Problem 3. Let A be an $m \times n$ matrix over $\mathbb{C}$. Using

$$\mathrm{vec}_{m\times n} A := \sum_{j=1}^{n} \mathbf{e}_{j,n} \otimes (A\mathbf{e}_{j,n}) = (I_n \otimes A) \sum_{j=1}^{n} \mathbf{e}_{j,n} \otimes \mathbf{e}_{j,n}$$

and

$$\mathrm{vec}_{m\times n}^{-1}\mathbf{x} = \sum_{i=1}^{m}\sum_{j=1}^{n} \left((\mathbf{e}_{j,n} \otimes \mathbf{e}_{i,m})^* \, \mathbf{x} \right) \mathbf{e}_{i,m} \otimes \mathbf{e}_{j,n}^* .$$

Show that

$$\mathrm{vec}_{m\times n}^{-1}(\mathrm{vec}_{m\times n} A) = A.$$

Problem 4. Let A be an $m \times n$ matrix over $\mathbb{C}$ and B be a $s \times t$ matrix over $\mathbb{C}$. Show that

$$A \otimes B = \mathrm{vec}^{-1}_{ms\times nt}\left(L_{A,s\times t}\left(\mathrm{vec}_{s\times t} B\right)\right)$$

where

$$L_{A,s\times t} := (I_n \otimes I_t \otimes A \otimes I_s) \sum_{j=1}^{n} \mathbf{e}_{j,n} \otimes I_t \otimes \mathbf{e}_{j,n} \otimes I_s.$$

Problem 5. (i) Let

$$AX + XB = C$$

where C is an $m \times n$ matrix over $\mathbb{R}$. What are the dimensions of A, B, and X?
(ii) Solve the equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X + X \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

for the real valued matrix X.

Problem 6. Let A be an $m \times n$ matrix over $\mathbb{C}$ and B be an $s \times t$ matrix over $\mathbb{C}$. Define

$$R(A \otimes B) := \mathrm{vec}A\,(\mathrm{vec}B)^T.$$

Find an algebraic expression for R. Find

$$R\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right).$$

R is the reshaping operator.

Problem 7. Show that

$$\mathrm{tr}(ABCD) \equiv (\mathrm{vec}(D^T))(A \otimes C^T)\mathrm{vec}(B^T)\,.$$

Chapter 19

Star Product

Problem 1. Consider the 2×2 matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

and the composition (star product)

$$A \star B := \begin{pmatrix} b_{11} & 0 & 0 & b_{12} \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ b_{21} & 0 & 0 & b_{22} \end{pmatrix}.$$

(i) What can be said about the trace of $A \star B$? What can be said about the determinant of $A \star B$?
(ii) Let A_1, A_2, A_3, A_4 be a basis in the vector space of 2×2 matrices over $\mathbb{C}$. Let B_1, B_2, B_3, B_4 be a basis in the vector space of 2×2 matrices over $\mathbb{C}$. Do the 16 matrices $A_j \star B_k$ $(j, k = 1, 2, 3, 4)$ form a basis in the vector space of 4×4 matrices?
(iii) Given the eigenvalues of A and B. What can be said about the eigenvalues of $A \star B$?
(iv) Can one find 4×4 permutation matrices P and Q such that

$$P(A \star B)Q = A \oplus B\,?$$

Here $\oplus$ denotes the direct sum

Problem 2. Consider the 2×2 matrices A, B over $\mathbb{C}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

We define the following product

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

(i) Answer the following questions: Let A and B be normal matrices. Is $A \star B$ normal. Let A and B be invertible matrices. Is $A \star B$ an invertible matrix? Let A and B be unitary matrices. Is $A \star B$ a unitary matrix? Let A and B be nilpotent matrices. Is $A \star B$ a nilpotent matrix? Answer these questions also for $A \star A$.
(ii) What is the conditions on A and B such that

$$A \star B = A \otimes B\,?$$

Problem 3. Let A, B be 2×2 matrices. We define

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Can one find a permutation matrix such that

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix} = P(A \star B)P^T.$$

Problem 4. (i) Let A, B be 2×2 matrices. We define

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

The 2×2 matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

form a group under matrix multiplication. Do the four 4×4 matrices $A \star A$, $A \star B$, $B \star A$, $B \star B$ form a group under matrix multiplication?

(ii) Let G be a finite group represented by 2×2 matrices. Let the order be n with the group elements $g_1 = e$, g_2, ..., g_n. Do the 4×4 matrices $g_j \star g_k$ $(j, k = 1, \ldots, n)$ form a group under matrix multiplication.

Problem 5. Let A, B be 2×2 matrices. We define

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Show that one can find a 4×4 permutation matrix P such that

$$P(A \star B)P^T = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix}.$$

Problem 6. Let A, B be invertible 2×2 matrices. We define

$$A \star B := \begin{pmatrix} b_{11} & 0 & 0 & b_{12} \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ b_{21} & 0 & 0 & b_{22} \end{pmatrix}.$$

Is $A \star B$ invertible?

Problem 7. (i) The 2×2 matrices

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

form a group under matrix multiplication. Do the four 4×4 matrices $A \star A$, $A \star B$, $B \star A$, $B \star B$ form a group under matrix multiplication?
(ii) Let G be a finite group represented by 2×2 matrices. Let the order be n with the group elements $g_1 = e$, g_2, ..., g_n. Do the 4×4 matrices $g_j \star g_k$ $(j, k = 1, \ldots, n)$ form a group under matrix multiplication.

Problem 8. Let A, B be 2×2 matrices. We define

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Show that one can find a 4×4 permutation matrix P such that

$$P(A \star B)P^T = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{pmatrix}.$$

Problem 9. Among others one can form a 4×4 matrix from two 2×2 matrices A and B using the direct sum $A \oplus B$, the Kronecker product $A \otimes B$ and the star product

$$A \star B := \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & b_{11} & b_{12} & 0 \\ 0 & b_{21} & b_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Given the eigenvalues and eigenvectors of A and B. What can be said about the eigenvalues and eigenvectors of $A \oplus B$, $A \otimes B$, $A \star B$?

Problem 10. (i) Let A, B be invertible 2×2 matrices. Is $A \star B$ invertible?
(ii) Let U and V be elements of $SU(2)$. Is $U \star V$ an element of $SU(4)$?
(iii) Let X and Y be elements of $SL(2,\mathbb{R})$. Is $X \star Y$ an element of $SL(4,\mathbb{R})$?

Problem 11. Let A, B be normal 2×2 matrices with eigenvalues λ_1, λ_2 and μ_1, μ_2, respectively. What can be said about the eigenvalues of $A \star B - B \star A$?

Problem 12. (i) Given the eigenvalues of A and B. What can be said about the eigenvalues of $A \star B$?
(ii) Can one find 4×4 permutation matrices P and Q such that

$$P(A \star B)Q = A \oplus B\,?$$

Here $\oplus$ denotes the direct sum

Problem 13. Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where A_j, B_j ($j = 1,2,3,4$) are 2×2 matrices. We define the product

$$A \star B := \begin{pmatrix} A_1 & 0_2 & 0_2 & A_2 \\ 0_2 & B_1 & B_2 & 0_2 \\ 0_2 & B_3 & B_4 & 0_2 \\ A_3 & 0_2 & 0_2 & A_4 \end{pmatrix}$$

where 0_2 is the 2×2 zero matrix. Thus $A \star B$ is an 8×8 matrix.
(i) Assume that A and B are invertible. Is $A \star B$ invertible?
(ii) Assume that A, B are unitary. Is $A \star B$ unitary?

Problem 14. Let A, B be the 4×4 matrices

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

where A_j, B_j ($j = 1, 2, 3, 4$) are 2×2 matrices. We define the product

$$A \star B := \begin{pmatrix} A_1 & 0_2 & 0_2 & A_2 \\ 0_2 & B_1 & B_2 & 0_2 \\ 0_2 & B_3 & B_4 & 0_2 \\ A_3 & 0_2 & 0_2 & A_4 \end{pmatrix}$$

where 0_2 is the 2×2 zero matrix. Thus $A \star B$ is an 8×8 matrix.
(i) Assume that A and B are invertible. Is $A \star B$ invertible?
(ii) Assume that A, B are unitary. Is $A \star B$ unitary?

Problem 15. Let A, B be 3×3 matrices. We define the composition

$$A \Diamond B := \begin{pmatrix} a_{11} & 0 & a_{12} & 0 & a_{13} \\ 0 & b_{11} & b_{12} & b_{13} & 0 \\ a_{21} & b_{21} & a_{22}b_{22} & b_{23} & a_{23} \\ 0 & b_{31} & b_{32} & b_{33} & 0 \\ a_{31} & 0 & a_{32} & 0 & a_{33} \end{pmatrix}.$$

Let

$$M = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}.$$

Find the eigenvalues of M and $M \Diamond M$.

Problem 16. Let P and Q be 2×2 projection matrices. Is the 4×4 matrix $P \star Q$ a projection matrix? Apply it to $P \star P$ where

$$P = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Chapter 20

Nonnormal Matrices

Problem 1. A square matrix M over $\mathbb{C}$ is called nonnormal if $MM^* = M^*M$. Let $s_j := 2\sin(2\pi j/5)$ with $j = 1, 2, \ldots, 5$. Consider the 5×5 matrix

$$M = \begin{pmatrix} s_1 & 1 & 0 & 0 & -1 \\ -1 & s_2 & 1 & 0 & 0 \\ 0 & -1 & s_3 & 1 & 0 \\ 0 & 0 & -1 & s_4 & 1 \\ 1 & 0 & 0 & -1 & s_5 \end{pmatrix}.$$

Show that the matrix is nonnormal. Find the eigenvalues and eigenvectors. Is the matrix diagonalizable?

Problem 2. Let $\epsilon \neq 0$. Show that the matrix

$$A = \begin{pmatrix} 1 & \epsilon \\ 0 & -1 \end{pmatrix}$$

is nonnormal. Give the eigenvalues and eigenvectors.

Problem 3. Let $a > 0$, $b \geq 0$ and $\phi \in [0, \pi]$. What are the conditions a, b, ϕ such that

$$A(a, b, \phi) = \begin{pmatrix} 0 & a \\ e^{i\phi}b & 0 \end{pmatrix}$$

is a normal matrix?

Problem 4. Let A, B be nonnormal matrices. Is $A \otimes B$ nonnormal? Is $A \oplus B$ nonnormal?

Problem 5. Can we conclude that an invertible matrix is normal?

Problem 6. Show that not all nonnormal matrices are non-diagonalizable, but, vice versa all non-diagonalizable matrices are non-normal.

Problem 7. Find all 2×2 matrices over $\mathbb{C}$ which are nonnormal but diagonalizable.

Chapter 21

Miscellaneous

Problem 1. (i) For $n = 4$ the transform matrix for the *Daubechies wavelet* is given by

$$D_4 = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & -c_2 & c_1 & -c_0 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & -c_0 & c_3 & -c_2 \end{pmatrix}, \qquad \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{4\sqrt{2}} \begin{pmatrix} 1+\sqrt{3} \\ 3+\sqrt{3} \\ 3-\sqrt{3} \\ 1-\sqrt{3} \end{pmatrix}.$$

Is D_4 orthogonal? Prove or disprove.
(ii) For $n = 8$ the transform matrix for the Daubechies wavelet is given by

$$D_8 = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & 0 & 0 & 0 & 0 \\ c_3 & -c_2 & c_1 & -c_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_0 & c_1 & c_2 & c_3 & 0 & 0 \\ 0 & 0 & c_3 & -c_2 & c_1 & -c_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & 0 & c_3 & -c_2 & c_1 & -c_0 \\ c_2 & c_3 & 0 & 0 & 0 & 0 & c_0 & c_1 \\ c_1 & -c_0 & 0 & 0 & 0 & 0 & c_3 & -c_2 \end{pmatrix}.$$

Is D_8 orthogonal? Prove or disprove.

Problem 2. Consider the $2n \times 2n$ matrix

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}.$$

We define that the $2n \times 2n$ matrix H over $\mathbb{R}$ is *Hamiltonian* if $(JH)^T = JH$. We define that the $2n \times 2n$ matrix S over $\mathbb{R}$ is *symplectic* if $S^T JS = J$. Show that if H is Hamiltonian and S is symplectic, then the matrix $S^{-1}HS$ is Hamiltonian.

Problem 3. Let A be an $n \times n$ matrix over $\mathbb{R}$. Consider the $2n \times 2n$ matrix

$$S = \begin{pmatrix} I_n & I_n \\ A & I_n + A \end{pmatrix} .$$

Let

$$\widetilde{S} = \begin{pmatrix} A + 2I_n & -I_n \\ I_n & 0_n \end{pmatrix} .$$

Can we find an invertible $2n \times 2n$ matrix T such that

$$\widetilde{S} = T^{-1} S T \,?$$

Problem 4. Let

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix} .$$

Find P^*P. Show that P^*JP is a diagonal matrix.

Problem 5. Let J be the $2n \times 2n$ matrix

$$J := \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} .$$

We define symplectic **G**-reflectors to be those $2n \times 2n$ symplectic matrices that have a $(2n-1)$-dimensional fixed-point subspace. It can be shown that any symplectic **G**-reflector can be expressed in the form

$$G = I_{2n} + \beta \mathbf{u}\mathbf{u}^T J \tag{1}$$

for some $0 \neq \beta \in \mathbb{F}$, $\mathbf{0} \neq \mathbf{u} \in \mathbb{F}^{2n}$ and $\mathbf{u}$ is considered as a column vector. The underlying field is $\mathbb{F}$. Conversely, any G given by (1) is always a symplectic **G**-reflector. Show that $\det G = +1$.

Problem 6. Consider the two polynomials

$$p_1(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad p_2(x) = b_0 + b_1 x + \cdots + b_m x^m$$

where $n = \deg(p_1)$ and $m = \deg(p_2)$. Assume that $n > m$. Let

$$r(x) = \frac{p_2(x)}{p_1(x)}.$$

We expand $r(x)$ in powers of $1/x$, i.e.

$$r(x) = \frac{c_1}{x} + \frac{c_2}{x^2} + \cdots$$

From the coefficients c_1, c_2, $\dots$,c_{2n-1} we can form an $n \times n$ *Hankel matrix*

$$H_n = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{pmatrix}.$$

The determinant of this matrix is proportional to the *resultant* of the two polynomials. If the resultant vanishes, then the two polynomials have a non-trivial greatest common divisor. Apply this theorem to the polynomials

$$p_1(x) = x^3 + 6x^2 + 11x + 6, \qquad p_2(x) = x^2 + 4x + 3.$$

Problem 7. Consider the 3×3 matrix

$$A = \begin{pmatrix} -1/2 & -\sqrt{3}/6 & \sqrt{6}/3 \\ -\sqrt{3}/6 & -5/6 & -\sqrt{2}/3 \\ \sqrt{6}/3 & -\sqrt{2}/3 & 1/3 \end{pmatrix}.$$

Show that $A^T = A^{-1}$ by showing that the column of the matrix are normalized and pairwise orthonormal.

Problem 8. Let P_j $(j = 0, 1, 2, \dots)$ be the Legendre polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \dots.$$

Calculate the infinite dimensional matrix $A = (A_{jk})$

$$A_{jk} = \int_{-1}^{+1} P_j(x) \frac{dP_k(x)}{dx} dx$$

where $j, k = 0, 1, \dots$. Consider the matrix A as a linear operator in the Hilbert space $\ell_2(\mathbb{N}_0)$. Is $\|A\| < \infty$?

Problem 9. Consider the vector space $\mathbb{R}^3$ and the vector product $\times$. The vector product is not associative. The *associator* of three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is defined by

$$\mathrm{ass}(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) := (\mathbf{u} \times \mathbf{v}) \times \mathbf{w} - \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

The associator measures the failure of associativity.
(i) Consider the unit vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find the associator.
(ii) Consider the normalized vectors

$$\mathbf{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Find the associator.

Problem 10. Find the Moore-Penrose pseudo inverses of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (1 \quad 1).$$

Problem 11. Let j be a positive integer. Let A, B be $n \times n$ matrices over $\mathbb{R}$. Calculate

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}((A + \epsilon B)^j - A^j).$$

Calculate

$$\frac{d}{d\epsilon} \operatorname{tr}(A + \epsilon B)^j \Big|_{\epsilon = 0}.$$

Problem 12. Let A be an $n \times n$ matrix over $\mathbb{R}$. Show that there exists nonnull vectors $\mathbf{x}_1$, $\mathbf{x}_2$ in $\mathbb{R}^n$ such that

$$\frac{\mathbf{x}_1^T A \mathbf{x}_1}{\mathbf{x}_1^T \mathbf{x}_1} \le \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \le \frac{\mathbf{x}_2^T A \mathbf{x}_2}{\mathbf{x}_2^T \mathbf{x}_2}$$

for every nonnull vector $\mathbf{x}$ in $\mathbb{R}^n$.

Problem 13. A generalized Kronecker delta can be be defined as follows

$$\delta_{I,J} := \begin{cases} 1 & \text{if } J=(j_1,\dots,j_r) \text{ is an even permutation of } I=(i_1,\dots,i_r) \\ -1 & \text{if } J \text{ is an odd permutation of } I \\ 0 & \text{if } J \text{ is not a permutation of } I \end{cases}$$

Find $\delta_{126,621}$, $\delta_{126,651}$, $\delta_{125,512}$.

Problem 14. Let $c_j > 0$ for $j = 1, \dots, n$. Show that the $n \times n$ matrices

$$\left(\frac{\sqrt{c_j c_k}}{c_j + c_k} \right), \qquad \left(\frac{1/c_j + 1/c_k}{\sqrt{c_j c_k}} \right)$$

$(k = 1, \dots, n)$ are positive definite.

Problem 15. Let $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ be nontrivial involutions. This means that $R = R^{-1} \neq \pm I_m$ and $S = S^{-1} \neq I_n$. A matrix $A \in \mathbb{C}^{m\times n}$ is called (R,S)-symmetric if $RAS = A$. Consider the case $m = n = 2$ and the Pauli spin matrices

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Find all 2×2 matrices A over $\mathbb{C}$ such that $RAS = A$.

Problem 16. Let $X \in \mathbb{R}^{n\times n}$. Show that X can be written as

$$X = A + S + cI_n$$

where A is antisymmetric ($A^T = -A$), S is symmetric ($S^T = S$) with $\mathrm{tr}(S) = 0$ and $c \in \mathbb{R}$.

Problem 17. Find the partial differential equation given by the condition

$$\det \begin{pmatrix} 0 & \partial u/\partial x_1 & \partial u/\partial x_2 \\ \partial u/\partial x_1 & \partial^2 u/\partial x_1^2 & \partial^2 u/\partial x_1 \partial x_2 \\ \partial u/\partial x_2 & \partial^2 u/\partial x_2 \partial x_1 & \partial^2 u/\partial x_2^2 \end{pmatrix}.$$

Find a nontrivial solution of the partial differential equation.

Problem 18. Consider the 2×2 matrix

$$A(\epsilon) = \begin{pmatrix} f_1(\epsilon) & f_2(\epsilon) \\ f_3(\epsilon) & f_4(\epsilon) \end{pmatrix}$$

where f_j $(j = 1, 2, 3, 4)$ are smooth functions and $\det(A(\epsilon)) > 0$ for all ϵ. Show that

$$\operatorname{tr}((dA(\epsilon))A(\epsilon)^{-1}) = d(\ln(\det(A(\epsilon))))$$

where d is the exterior derivative.

Problem 19. Consider vectors in the vector space $\mathbb{R}^3$ and the vector product. Consider the mapping of the vectors into 3×3 skew-symmetric matrices

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}.$$

Calculate

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \times \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$$

and $[M_1, M_2]$, where

$$M_1 = \begin{pmatrix} 0 & c_1 & -b_1 \\ -c_1 & 0 & a_1 \\ b_1 & -a_1 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & c_2 & -b_2 \\ -c_2 & 0 & a_2 \\ b_2 & -a_2 & 0 \end{pmatrix}.$$

Discuss.

Problem 20. Let $\mathbf{u}_1(t), \mathbf{u}_2(t), \mathbf{u}_3(t) \in \mathbb{R}^3$. Solve the initial value problem of the nonlinear autonomous system of first order differential equations

$$\frac{d\mathbf{u}_1}{dt} = \mathbf{u}_2 \times \mathbf{u}_3, \quad \frac{d\mathbf{u}_2}{dt} = \mathbf{u}_3 \times \mathbf{u}_1, \quad \frac{d\mathbf{u}_3}{dt} = \mathbf{u}_1 \times \mathbf{u}_2$$

where $\times$ denotes the vector product.

Problem 21. Let $\mathbf{u}(t) \in \mathbb{R}^3$. Solve the initial value problem for the differential equation

$$\frac{d^2\mathbf{u}}{dt^2} = \mathbf{u} \times \frac{d\mathbf{u}}{dt}$$

where $\times$ denotes the vector product.

Problem 22. Let A_n be the $n \times n$ matrices of the form

$$A_1 = 1, \quad A_2 = \begin{pmatrix} 0 & t \\ t & r \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & t \\ 0 & 1 & 0 \\ t & 0 & r \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & 0 & t & 0 \\ 0 & t & r & 0 \\ t & 0 & 0 & r \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & t & 0 & r & 0 \\ t & 0 & 0 & 0 & r \end{pmatrix}.$$

Thus the even dimensional matrix A_{2n} has t along the skew-diagonal and r along the lower main diagonal. Otherwise the entries are 0. The odd dimensional matrix A_{2n+1} has t along the skew-diagonal except 1 at the centre and r along the lower main diagonal. Otherwise the entries are 0. Find the eigenvalues of these matrices.

Problem 23. Let

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

be elements of $\mathbb{C}^2$. Solve the equation $\mathbf{z}^* \mathbf{w} = \mathbf{w}^* \mathbf{z}$.

Problem 24. Let A, B be $n \times n$ matrices over $\mathbb{C}$. We define the quasi-multiplication

$$A \bullet B := \frac{1}{2}(AB + BA).$$

Obviously $A \bullet B = B \bullet A$. Show that

$$(A^2 \bullet B) \bullet A = A^2 \bullet (B \bullet A).$$

This is called the *Jordan identity.*

Problem 25. Let $\mathbb{C}^{n \times N}$ be the vector space of all $n \times N$ complex matrices. Let $Z \in \mathbb{C}^{n \times N}$. Then $Z^* \equiv \bar{Z}^T$, where ^{T} denotes transpose. One defines a *Gaussian measure* μ on $\mathbb{C}^{n \times N}$ by

$$d\mu(Z) := \frac{1}{\pi^{nN}} \exp(-\mathrm{tr}(ZZ^*)) dZ$$

where dZ denotes the Lebesgue measure on $\mathbb{C}^{n \times N}$. The *Fock space* $\mathcal{F}(\mathbb{C}^{n \times N})$ consists of all entire functions on $\mathbb{C}^{n \times N}$ which are square integrable with respect to the Gaussian measure $d\mu(Z)$. With the scalar product

$$\langle f | g \rangle := \int_{\mathbb{C}^{n \times N}} f(Z) \overline{g(Z)} d\mu(Z), \qquad f, g \in \mathcal{F}(\mathbb{C}^{n \times N})$$

one has a Hilbert space. Show that this Hilbert space has a *reproducing kernel* K. This means a continuous function $K(Z, Z') : \mathbb{C}^{n \times N} \times \mathbb{C}^{n \times N} \to \mathbb{C}$ such that

$$f(Z) = \int_{\mathbb{C}^{n \times N}} K(Z, Z') f(Z') d\mu(Z')$$

for all $Z \in \mathbb{C}^{n \times N}$ and $f \in \mathcal{F}(\mathbb{C}^{n \times N})$.

Problem 26. The vector space of all $n \times n$ matrices over $\mathbb{C}$ form a Hilbert space with the scalar product defined by

$$\langle A, B \rangle := \mathrm{tr}(AB^*) \,.$$

This implies a norm $\|A\|^2 = \mathrm{tr}(AA^*)$.
(i) Consider the Lie group $U(n)$. Find two unitary 2×2 matrices U_1, U_2 such that $\|U_1 - U_2\|$ takes a maximum.
(ii) Are the matrices

$$U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

such a pair?

Problem 27. Let σ_j $(j = 0, 1, 2, 3)$ be the Pauli spin matrices, where $\sigma_0 = I_2$. Does the set of matrices

$$\begin{pmatrix} \sigma_j & 0_2 \\ 0_2 & \sigma_k \end{pmatrix}, \qquad \begin{pmatrix} 0_2 & \sigma_j \\ \sigma_k & 0_2 \end{pmatrix}, \qquad j, k = 0, 1, 2, 3$$

form a group under matrix multiplication. If not add the elements to find a group. Here 0_2 is the 2×2 zero matrix.

Problem 28. Let A be a symmetric 2×2 matrix over $\mathbb{R}$

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}.$$

Thus $a_{01} = a_{10}$. Assume that

$$a_{00}a_{01} = a_{01}^2, \qquad a_{00}a_{11} = a_{01}a_{11} \,.$$

Find all matrices A that satisfy these conditions.

Problem 29. Let A, B be 3×3 matrices. We define the composition

$$A \Diamond B := \begin{pmatrix} a_{11} & 0 & a_{12} & 0 & a_{13} \\ 0 & b_{11} & b_{12} & b_{13} & 0 \\ a_{21} & b_{21} & a_{22}b_{22} & b_{23} & a_{23} \\ 0 & b_{31} & b_{32} & b_{33} & 0 \\ a_{31} & 0 & a_{32} & 0 & a_{33} \end{pmatrix}.$$

Let

$$M = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}.$$

Find the eigenvalues of M and $M \Diamond M$.

Problem 30. Find a 3×3 matrix A over $\mathbb{R}$ which satisfies

$$A^2 A^T + A^T A^2 = 2A, \qquad AA^T A = 2A, \qquad A^3 = 0$$

Thus the matrix is nilpotent.

Problem 31. Consider the skew-symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{R}$. Find the eigenvalues. Let 0_3 be the 3×3 zero matrix. Let A_1, A_2, A_3 be skew-symmetric 3×3 matrices over $\mathbb{R}$. Find the eigenvalues of the 9×9 matrix

$$B = \begin{pmatrix} 0_3 & -A_3 & A_2 \\ A_3 & 0_3 & -A_1 \\ -A_2 & A_1 & 0_3 \end{pmatrix}.$$

Problem 32. Consider the four 2×2 matrices

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \quad R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

(i) Show that they form an orthonormal basis in the Hilbert space of the 2×2 matrices with the scalar product $\langle X, Y \rangle = \mathrm{tr}(XY^*)$.
(ii) Find the multiplication table.

Problem 33. Find the Cayley transform of the Hermitian matrix

$$H = \begin{pmatrix} h_{11} & h_{12} \\ \bar{h}_{12} & h_{22} \end{pmatrix}, \qquad h_{11}, h_{22} \in \mathbb{R}, \quad h_{12} \in \mathbb{C}.$$

Problem 34. Let S be an invertible $n \times n$ matrix. Find the inverse of the $2n \times 2n$ matrix

$$\begin{pmatrix} 0_n & S^{-1} \\ S & 0_n \end{pmatrix}$$

where 0_n is the $n \times n$ zero matrix.

Problem 35. Consider the $n \times n$ matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad B = \text{diag}(1 \ \omega \ \omega^2 \ \dots \ \omega^{n-1})$$

where ω is the n-th primitive root of unity. We have $A^n = B^n = I_n$ and $\omega^n = 1$. We have

$$AB = \omega BA .$$

Let $R = A \otimes I_n$ and $S = B \otimes I_n$. Find RS. Let $X = A \otimes A$ and $Y = B \otimes B$. Find XY. Find the commutator $[X, Y]$.

Problem 36. Find all 2×2 matrices C over $\mathbb{R}$ such that

$$C^T C + C C^T = I_2, \qquad C^2 = 0_2 .$$

Problem 37. Let $\delta_j, \eta_j \in \mathbb{R}$ with $j = 1, 2, 3$. Any 3×3 unitary symmetric matrix U can be written in the product form

$$U = \begin{pmatrix} e^{i\delta_1} & 0 & 0 \\ 0 & e^{i\delta_2} & 0 \\ 0 & 0 & e^{i\delta_3} \end{pmatrix} \begin{pmatrix} \eta_1 & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & \eta_2 & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & \eta_3 \end{pmatrix} \begin{pmatrix} e^{i\delta_1} & 0 & 0 \\ 0 & e^{i\delta_2} & 0 \\ 0 & 0 & e^{i\delta_3} \end{pmatrix}$$

where $\gamma_{jk} = N_{jk} \exp(i\beta_{jk})$ with $N_{jk}, \beta_{jk} \in \mathbb{R}$. It follows that

$$U_{jj} = \eta_j \exp(2i\delta_j), \qquad U_{jk} = N_{jk} \exp(i(\delta_j + \delta_k + \beta_{jk})) .$$

The unitary condition $UU^* = I_3$ provides

$$\sum_{k \neq j}^{3} N_{jk}^2 + \eta_j = 1, \quad j = 1, 2, 3$$

and

$$N_{12}(\eta_1 \exp(i\beta_{12}) + \eta_2 \exp(-i\beta_{12})) = N_{13} N_{23} \exp(i(\pi + \beta_{23} - \beta_{13}))$$

and cyclic $(1 \to 2 \to 3 \to 1)$. Write the unitary symmetric matrix

$$W = \begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{pmatrix}$$

in this form.

Problem 38. Consider the map $\mathbf{f} : \mathbb{C}^2 \mapsto \mathbb{R}^3$

$$\begin{pmatrix} \cos(\theta) \\ e^{i\phi}\sin(\theta) \end{pmatrix} \mapsto \begin{pmatrix} \sin(2\theta)\cos(\phi) \\ \sin(2\theta)\sin(\phi) \\ \cos(2\theta) \end{pmatrix} .$$

Consider the map for the special cases $\theta = 0$, $\phi = 0$ and $\theta = \pi/4$, $\phi = \pi/4$.

Problem 39. (i) Consider the hermitian 3×3 matrices to describe a particle with *spin-1*

$$S_1 := \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 := \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_3 := \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

With $S_+ := S_1 + iS_2$, $S_- := S_1 - iS_2$ we find

$$S_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3}$$

An example of a spin-1 particle is the photon. Let $\mathbf{m}$, $\mathbf{n}$ be normalized vectors in $\mathbb{R}^3$ which are orthogonal, i.e. $\mathbf{m}^T\mathbf{n} = 0$. Find the eigenvalues of the 3×3 matrix

$$K = (\mathbf{m} \cdot \mathbf{S})^2 - (\mathbf{n} \cdot \mathbf{S})^2$$

where $\mathbf{m} \cdot \mathbf{S} = m_1S_1 + m_2S_2 + m_3S_3$.
(ii) Show that

$$P_{\mathbf{m}} = I_3 - (\mathbf{m} \cdot \mathbf{S})^2$$

is a projection operator.

Problem 40. Let A, B be 2×2 matrices over $\mathbb{R}$. Find A, B such that

$$\min \|[A, B] - I_2\|$$

where $[\, ,\,]$ denotes the commutator and for the norm $\| \, \|$ consider the Frobenius norm and max-norm.

Problem 41. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that A^{-1} exists. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, where $\mathbf{u}, \mathbf{v}$ are considered as column vectors. (i) Show that if

$$\mathbf{v}^T A^{-1} \mathbf{u} = -1$$

then $A + \mathbf{u}\mathbf{v}^T$ is not invertible.
(ii) Assume that $\mathbf{v}^T A^{-1}\mathbf{u} \neq -1$. Show that

$$(A + \mathbf{u}\mathbf{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{u}}.$$

Problem 42. Can we find an invertible 2×2 matrix S over the real numbers such that

$$S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ?$$

Problem 43. Let A and B be $n \times n$ matrices over $\mathbb{R}$. Assume that $A \neq B$, $A^3 = B^3$ and $A^2 B = B^2 A$. Is $A^2 + B^2$ invertible?

Problem 44. Let

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$$

and I_2 be the 2×2 identity matrix. For $j \geq 1$, let d_j be the greatest common divisor of the entries of $A^j - I_2$. Show that

$$\lim_{j \to \infty} d_j = \infty .$$

Problem 45. Let $a, d \in \mathbb{R}$ and $b \in \mathbb{C}$. Consider the hermitian matrix

$$K = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix} .$$

Show that the matrix can be written as linear combination of the 2×2 identity matrix and the Pauli spin matrices.

Problem 46. (i) Consider the polynomial

$$p(x) = x^2 - sx + d, \quad s, d \in \mathbb{C} .$$

Find a 2×2 matrix A such that its characteristic polynomial is p.
(ii) Consider the polynomial

$$q(x) = -x^3 + sx^2 - qx + d, \quad s, q, d \in \mathbb{C} .$$

Find a 3×3 matrix B such that its characteristic polynomial is q.

Problem 47. Let A be an $n \times n$ positive definite matrix over $\mathbb{R}$, i.e. $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Calculate

$$\int_{\mathbb{R}^n} \exp(-\mathbf{x}^T A\mathbf{x}) d\mathbf{x} .$$

Problem 48. Let A, B be $n \times n$ matrices over $\mathbb{C}$. The matrix A is called *similar* to the matrix B if there is a $n \times n$ invertible matrix S such that

$$A = S^{-1}BS .$$

If A is similar to B, then B is also similar to A, since $B = SAS^{-1}$.
(i) Consider the two matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Are the matrices similar?
(ii) Consider the two matrices

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Are the matrices similar?
(iii) Consider the two matrices

$$X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} .$$

Are the two matrices normal? Are the matrices similar? Are the matrices $X \otimes Y$ and $Y \otimes X$ similar?

Problem 49. (i) Consider the matrix

$$R = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} .$$

Show that $R^{-1} = R^* = R$. Use these properties and $\text{tr}(R)$ to find all the eigenvalues of R. Find the eigenvectors.
(ii) Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

Calculate RA_1R^{-1} and RA_2R^{-1}. Discuss.

Problem 50. (i) Are the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

similar? Prove or disprove.
(ii) Are the matrices $A \otimes B$ and $B \otimes A$ similar? Prove or disprove.

Problem 51. (i) Find the conditions on the 2×2 matrices over $\mathbb{C}$ such that

$$ABA = BAB\,.$$

Find solutions where $AB \neq BA$, i.e. $[A, B] \neq 0_2$.
(ii) Find the conditions on the 2×2 matrices A and B such that

$$A \otimes B \otimes A = B \otimes A \otimes B\,.$$

Find solutions where $AB \neq BA$, i.e. $[A, B] \neq 0_2$.

Problem 52. Is every invertible matrix normal? Prove or disprove.

Problem 53. Consider the Hilbert space $M_d(\mathbb{C})$ of $d \times d$ matrices with scalar product $\langle A, B\rangle := \mathrm{tr}(AB^*)$, $A, B \in M_d(\mathbb{C})$. Consider an orthogonal basis of d^2 $d \times d$ hermitian matrices B_1, B_2, $\ldots$, B_{d^2}, i.e.

$$\langle B_j, B_k\rangle = \mathrm{tr}(B_j B_k) = d\delta_{jk}$$

since $B_k^* = B_k$ for a hermitian matrix. Let M be a $d \times d$ hermitian matrix. Let

$$m_j = \mathrm{tr}(B_j M) \qquad j = 1, \ldots, d^2\,.$$

Given m_j and B_j $(j = 1, \ldots, d^2)$. Find M.

Problem 54. Let A be an $n \times n$ matrix over $\mathbb{R}$. Assume that A^{-1} exists. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, where $\mathbf{u}, \mathbf{v}$ are considered as column vectors. (i) Show that if

$$\mathbf{v}^T A^{-1} \mathbf{u} = -1$$

then $A + \mathbf{u}\mathbf{v}^T$ is not invertible.
(ii) Assume that $\mathbf{v}^T A^{-1} \mathbf{u} \neq -1$. Show that

$$(A + \mathbf{u}\mathbf{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{u}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{u}}\,.$$

Problem 55. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let

$$U(\alpha, \beta, \gamma) = e^{-i\alpha\sigma_3/2} e^{-i\beta\sigma_2/2} e^{-i\gamma\sigma_3/2}$$

where α, β, γ are the three *Euler angles* with the range $0 \leq \alpha < 2\pi$, $0 \leq \beta \leq \pi$ and $0 \leq \gamma < 2\pi$. Show that

$$U(\alpha,\beta,\gamma) = \begin{pmatrix} e^{-i\alpha/2}\cos(\beta/2)e^{-i\gamma/2} & -e^{-i\alpha/2}\sin(\beta/2)e^{i\gamma/2} \\ e^{-i\alpha/2}\sin(\beta/2)e^{-i\gamma/2} & e^{i\alpha/2}\cos(\beta/2)e^{i\gamma/2} \end{pmatrix}. \qquad (1)$$

Problem 56. Consider the Hilbert space $\mathcal{H}$ of the 2×2 matrices over the complex numbers with the scalar product

$$\langle A, B \rangle := \mathrm{tr}(AB^*), \qquad A, B \in \mathcal{H}.$$

Show that the rescaled Pauli matrices $\mu_j = \frac{1}{\sqrt{2}}\sigma_j$, $j = 1, 2, 3$

$$\mu_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mu_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

plus the rescaled 2×2 identity matrix

$$\mu_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

form an orthonormal basis in the Hilbert space $\mathcal{H}$.

Problem 57. Can we find an invertible 2×2 matrix S over the real numbers such that

$$S\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ?$$

Problem 58. Let A and B be $n \times n$ matrices over $\mathbb{R}$. Assume that $A \neq B$, $A^3 = B^3$ and $A^2B = B^2A$. Is $A^2 + B^2$ invertible?

Problem 59. Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that A and $A + B$ are invertible. Show that

$$(A+B)^{-1} \equiv A^{-1} - A^{-1}B(A+B)^{-1}.$$

Apply the identity to $A = \sigma_x$, $B = \sigma_z$.

Problem 60. Let A be an $n \times n$ matrix over $\mathbb{R}$ and let $\mathbf{u}$ be an n-vector in $\mathbb{R}^n$ (column vector) with $\mathbf{u} \neq \mathbf{u}$. In numerical linear algebra we often have to compute

$$\left(I_n - \frac{2\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}}\right) \qquad (1)$$

where I_n is the $n \times n$ identity matrix. Naively we would form the matrix $(I_n - 2\mathbf{u}\mathbf{u}^T/\mathbf{u}^T\mathbf{u})$ from the vector $\mathbf{u}$ and then form the matrix product explicitly with A. This would require $O(m^3)$ flops. Provide a faster computation for expression (1).

Problem 61. Let $z \in \mathbb{C}$. Construct all 2×2 matrices A and B over $\mathbb{C}$ such that

$$\exp(zA)B\exp(-zA) = e^{-z}B\,.$$

Problem 62. Let $f_{jk} : \mathbb{R} \to \mathbb{R}$ be analytic functions, where $j, k = 1, 2$. Find the differential equations for f_{jk} such that

$$\begin{pmatrix} f_{11}(\epsilon) & f_{12}(\epsilon) \\ f_{21}(\epsilon) & f_{22}(\epsilon) \end{pmatrix} \frac{d}{d\epsilon} \begin{pmatrix} f_{11}(\epsilon) & f_{12}(\epsilon) \\ f_{21}(\epsilon) & f_{22}(\epsilon) \end{pmatrix} = \left(\frac{d}{d\epsilon} \begin{pmatrix} f_{11}(\epsilon) & f_{12}(\epsilon) \\ f_{21}(\epsilon) & f_{22}(\epsilon) \end{pmatrix} \right) \begin{pmatrix} f_{11}(\epsilon) & f_{12}(\epsilon) \\ f_{21}(\epsilon) & f_{22}(\epsilon) \end{pmatrix}.$$

Problem 63. Consider the matrices

$$A(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \quad B(t) = \int_0^t A(s)ds\,.$$

Find the commutator $[A(t), B(t)]$. Discuss. What is the condition such that $[A(t), B(t)] = 0_2$.

Problem 64. Find all 2×2 matrices A_1, A_2, A_3 such that

$$A_1A_2 = A_2A_3, \qquad A_3A_1 = A_2A_3\,.$$

Problem 65. Let A be an $n \times n$ normal matrix with pairwise different eigenvalues. Are the matrices

$$P_j = \prod_{k=1, j\neq k}^{n} \frac{A - \lambda_k I_n}{\lambda_j - \lambda_k}$$

projection matrices?

Problem 66. Let $n \geq 2$ and $\omega = \exp(2\pi i/n)$. Consider the diagonal and permutation matrices, respectively

$$D = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \omega^{n-1} \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

(i) Show that $D^n = P^n = I_n$.
(ii) Show that the set of matrices

$$\{ D^j P^k \; : \; j, k = 0, 1, 2, \dots, n-1 \}$$

form a basis of the vector space of $n \times n$ matrices.
(iii) Show that

$$PD = \omega DP, \qquad P^j D^k = \omega^{jk} D^k P^j \, .$$

(iv) Find the matrix

$$X = \zeta P + \zeta^{-1} P^{-1} + \eta D + \eta^{-1} D^{-1}$$

and calculate the eigenvalues.

Problem 67. Let $z \in \mathbb{C}$. Consider the 2×2 matrices

$$A = \begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & z \\ z & -1 \end{pmatrix}, \quad C = \begin{pmatrix} z & 1 \\ 1 & z \end{pmatrix}, \quad D = \begin{pmatrix} z & 1 \\ -1 & z \end{pmatrix}.$$

Find the condition on z such that A, B, C, D are invertible.

Problem 68. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an analytic functions. Consider the matrices

$$A(t) = \begin{pmatrix} e^{i\phi(t)} & 1 \\ 1 & e^{id\phi(t)/dt} \end{pmatrix}, \quad B(t) = \begin{pmatrix} 1 & e^{i\phi(t)} \\ e^{id\phi(t)/dt} & 1 \end{pmatrix}.$$

(i) Find the differential equation for ϕ from the condition

$$\mathrm{tr}(AB) = 0 \, .$$

(ii) Find the differential equation for ϕ from the condition

$$\det(AB) = 0 \, .$$

Problem 69. How many 3×3 binary matrices can one form which contain three 1's? Write down these matrices. Which of them are invertible?

Problem 70. Let $s = 1/2, 1, 3/2, 2, \dots$ be the spin. Let $n = 2s+1$, i.e. for $s = 1/2$ we have $n = 2$, for $s = 1$ we have $n = 3$ etc. Consider the $n \times n$ matrix $V_s = (V_{jk})$ with

$$V_{jk} = \exp(c(s-j+1)(s-k+1))$$

where $j, k = 1, 2, \ldots, n$ and c is a positive constant.
(i) Let $s = 1/2$, i.e. $n = 2$. Let

$$R_{1/2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

Find $R_{1/2}^{-1} V_{1/2} R_{1/2}$.
(ii) Let s be positive integer with $n = 2s+1$ and the $n \times n$ matrix

$$R_s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 0 & 1 & & \\ & & 0 & \sqrt{2} & 0 & & \\ & & 1 & 0 & -1 & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ 1 & & & & & & -1 \end{pmatrix}$$

Find $R_s V_s R_s$.
(iii) Let s be $1/2, 3/2, \ldots$ with $n = 2s+1$ and the $n \times n$ matrix

$$R_s = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & & & & 1 \\ & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 1 & & \\ & & 1 & -1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & \\ 1 & & & & & -1 \end{pmatrix} .$$

Find $R_s V_s R_s$.

Problem 71. Let A be an $n \times n$ matrix over $\mathbb{C}$. Consider the Taylor series

$$(I_n + A)^{1/2} = I_n + \frac{1}{2}A - \frac{1}{2 \cdot 4}A^2 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}A^3 - \cdots$$

and

$$(I_n + A)^{-1/2} = I_n - \frac{1}{2}A + \frac{1 \cdot 3}{2 \cdot 4}A^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}A^3 + \cdots$$

What is the condition (the norm) on A such that the Taylor series exist? Can it be applied to the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} ?$$

Note that for $n = 1$ we have the condition $-1 < A \leq +1$.

Problem 72. Let A be a normal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and pairwise orthonormal eigenvectors $\mathbf{a}_j$ (column vectors), i.e. $\mathbf{a}_j^*\mathbf{a}_k = \delta_{jk}$. Then we can write A as (spectral decomposition)

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{a}_j \mathbf{a}_j^* \,.$$

Analogously for a normal matrix B we have

$$B = \sum_{k=1}^{n} \mu_k \mathbf{b}_k \mathbf{b}_k^* \,.$$

(i) Find the condition on λ_j, $\mathbf{a}_j$ and μ_k, $\mathbf{b}_k$ such that $\mathrm{tr}(AB^*) = 0$, i.e. the two matrices are orthogonal to each other.
(ii) Find the condition on λ_j, $\mathbf{a}_j$ and μ_k, $\mathbf{b}_k$ such that $[A, B] = 0_n$, i.e. the the commutator of the matrices vanishes.

Problem 73. Let A be a normal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and pairwise orthonormal eigenvectors $\mathbf{a}_j$ (column vectors), i.e. $\mathbf{a}_j^*\mathbf{a}_k = \delta_{jk}$. Then we can write A as (spectral decomposition)

$$A = \sum_{j=1}^{n} \lambda_j \mathbf{a}_j \mathbf{a}_j^* \,.$$

Analogously for a normal matrix B we have

$$B = \sum_{k=1}^{n} \mu_k \mathbf{b}_k \mathbf{b}_k^* \,.$$

Let $z \in \mathbb{C}$. Use the spectral decomposition to calculate

$$e^{zA} B e^{-zA} \,.$$

Problem 74. Let U be an $n \times n$ unitary matrix. Let $H = U + U^*$. Calculate

$$\exp(zH) \,.$$

Problem 75. Consider the Bell matrix

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} .$$

(i) Find all matrices A such that $BAB^* = A$.
(ii) Find all matrices A such that BAB^* is a diagonal matrix.

Problem 76. Consider the invertible 2×2 matrix

$$A(\theta) = \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}.$$

Show that

$$d(\ln(\det(A))) = \mathrm{tr}(A^{-1}dA)$$

where d denotes the exterior derivative.

Problem 77. Let $n \geq 2$. An invertible integer matrix, $A \in GL_n(\mathbb{Z})$, generates a toral automorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ via the formula

$$f \circ \pi = \pi \circ A, \qquad \pi : \mathbb{R}^n \to \mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n.$$

The set of fixed points of f is given by

$$\#\mathrm{Fix}(f) := \{\, x^* \in \mathbb{T}^n \; : \; f(x^*) = x^* \,\}$$

Now we have: if $\det(I_n - A) \neq 0$, then

$$\#\mathrm{Fix}(f) = |\det(I_n - A)|.$$

Let $n = 2$ and

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Show that $\det(I_2 - A) \neq 0$ and find $\#\mathrm{Fix}(f)$.

Problem 78. Consider the Hamilton operator

$$\hat{H} = \hbar\omega \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \equiv \hbar\omega(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z).$$

Find the eigenvalues and normalized eigenvectors of $\hat{H}$.

Problem 79. Consider the symmetric matrix over $\mathbb{R}$

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Find a invertible matrix B such that $B^{-1}AB$ is a diagonal matrix.

Problem 80. Let V_1 be a hermitian $n \times n$ matrix. Let V_2 be a positive semidefinite $n \times n$ matrix. Let k be a positive integer. Show that

$$\mathrm{tr}((V_2 V_1)^k)$$

can be written as $\mathrm{tr}(V^k)$, where $V := V_2^{1/2} V_1 V_2^{1/2}$.

Problem 81. Can one find a 2×2 unitary matrix such that

$$U \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} U^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Problem 82. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Consider the 4×4 gamma matrices

$$\gamma_1 = \begin{pmatrix} 0_2 & \sigma_1 \\ -\sigma_1 & 0_2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0_2 & \sigma_2 \\ -\sigma_2 & 0_2 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0_2 & \sigma_3 \\ -\sigma_3 & 0_2 \end{pmatrix}$$

and

$$\gamma_0 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}.$$

Find $\gamma_1\gamma_2\gamma_3\gamma_0$ and $\mathrm{tr}(\gamma_1\gamma_2\gamma_3\gamma_0)$.

Problem 83. Let j be a positive integer. Let A, B be $n \times n$ matrices over $\mathbb{R}$. Calculate

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}((A + \epsilon B)^j - A^j).$$

Calculate

$$\frac{d}{d\epsilon} \mathrm{tr}(A + \epsilon B)^j \Big|_{\epsilon=0}.$$

Problem 84. Let σ_1, σ_2, σ_3 be the Pauli spin matrices. Let A, B be two arbitrary 2×2 matrices. Is

$$\frac{1}{2}\mathrm{tr}(AB) \equiv \sum_{j=1}^{3} \left(\frac{1}{2}\mathrm{tr}(\sigma_j A)\right)\left(\frac{1}{2}\mathrm{tr}(\sigma_j B)\right)?$$

Problem 85. In the following we count from $(0,0)$ to $(n-1, n-1)$ for $n \times n$ matrices. Let $\omega := \exp(2\pi i/n)$. Consider the $n \times n$ matrices

$$H = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & & \\ 0 & & & & 1 \\ 1 & 0 & & & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \ddots & \\ & & & \omega^{n-1} \end{pmatrix}.$$

Then $H^n = G^n = I_n$, $HH^* = GG^* = I_n$, $HG = \omega GH$. Let U be the unitary matrix

$$U = \frac{1}{\sqrt{n}}(\omega^{jk})$$

where $j, k = 0, 1, \ldots, n-1$. Show that

$$UHU^{-1} = G\,.$$

Problem 86. The *standard simplex* Δ_n is defined by the set in $\mathbb{R}^n$

$$\Delta_n := \{\,(x_1, \ldots, x_n)^T \,:\, x_j \geq 0,\ \sum_{j=1}^{n} x_j = 1\,\}$$

Consider n affinely independent points $B_1, \ldots, B_n \in \Delta_n$. They span an $(n-1)$-simplex denoted by $\Lambda = \mathrm{Con}(B_1, \ldots, B_n)$, that is

$$\Lambda = \mathrm{Con}(B_1, \ldots, B_n) = \{\,\lambda_1 B_1 + \cdots + \lambda_n B_n \,:\, \sum_{j=1}^{n} \lambda_j = 1,\ \lambda_1, \ldots, \lambda_n \geq 0\,\}\,.$$

The set corresponds to an invertible $n \times n$ matrix whose columns are $B_1, \ldots, B_n$. Conversely, consider the matrix $C = (b_{jk})$, where $C_k = (b_{1k}, \ldots, b_{nk})^T$ $(k = 1, \ldots, n)$. If $\det(C) \neq 0$ and the sum of the entries in each column is 1, then the matrix C corresponds to an $(n-1)$-simplex $\mathrm{Con}(B_1, \ldots, B_n)$ in Δ_n. Let C_1 and C_2 be $n \times n$ matrices with nonegative entries and all the columns of each matrix add up to 1. Show that $C_1 C_2$ and $C_2 C_1$ are also such matrices. Are the $n^2 \times n^2$ matrices $C_1 \otimes C_2$, $C_2 \otimes C_1$ such matrices?

Problem 87. (i) Consider the analytic function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$f_1(x_1, x_2) = \sinh(x_2), \quad f_2(x_1, x_2) = \sinh(x_1)\,.$$

Show that this function admits the (only) fixed point $(0, 0)$. Find the functional matrix at the fixed point

$$\left.\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}\,.$$

(ii) Consider the analytic function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$

$$g_1(x_1, x_2) = \sinh(x_1), \quad g_2(x_1, x_2) = -\sinh(x_2)\,.$$

Show that this function admits the (only) fixed point $(0, 0)$. Find the functional matrix at the fixed point

$$\left.\begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}\,.$$

(iii) Multiply the two matrices found in (i) and (ii).
(iv) Find the composite function $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{h}(\mathbf{x}) = (\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x}))\,.$$

Show that this function also admits the fixed point $(0,0)$. Find the functional matrix at this fixed point

$$\left.\begin{pmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 \end{pmatrix}\right|_{(0,0)}\,.$$

Compare this matrix with the matrix found in (iii).

Problem 88. Is the matrix

$$U = \frac{1}{\sqrt{3}}(I_2 \otimes I_2 \otimes I_2 + i\sigma_1 \otimes \sigma_1 \otimes \sigma_1 + i\sigma_3 \otimes \sigma_3 \otimes \sigma_3$$

unitary?

Problem 89. Consider the matrices

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Let $n = 0, 1, 2, \ldots$. Study the sequence of matrices

$$A_{n+1} = A_n B_n, \qquad B_{n+1} = A_n\,.$$

Discuss. Is the sequence of matrices periodic?

Problem 90. Let S, T be $n \times n$ matrices over $\mathbb{C}$ with

$$S^2 = I_n, \qquad (TS)^2 = I_n\,.$$

Show that

$$STS^{-1} = T^{-1}, \qquad ST^{-1}S = T\,.$$

Problem 91. Consider the alphabet $\Sigma = \{U, V, W\}$, axiom: $\omega = U$ and the set of production rules

$$U \mapsto UVW, \quad V \mapsto UV, \quad W \mapsto U\,.$$

(i) Apply it to $U = \sigma_1$, $V = \sigma_2$, $W = \sigma_3$ and matrix multiplication. Is the sequence periodic?
(ii) Apply it to $U = \sigma_1$, $V = \sigma_2$, $W = \sigma_3$ and the Kronecker product.

Problem 92. (i) Consider the 2×2 matrix

$$V(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

Calculate $dV(t)/dt$ and then find the commutator $[dV(t)/dt, V(t)]$.
(ii) Let $V(t)$ be a 2×2 matrix where all the entries are smooth functions of t. Calculate $dV(t)/dt$ and then find the conditions on the entries such that $[dV(t)/dt, V(t)] = 0_2$.

Problem 93. Let $f_j(x_1, x_2)$ ($j = 1, 2, 3$) be realvalued smooth functions. Consider the matrix

$$N(x_1, x_2) = f_1\sigma_1 + f_2\sigma_2 + \sigma_3 \equiv \begin{pmatrix} f_3 & -if_2 + f_1 \\ if_2 + f_1 & -f_3 \end{pmatrix}.$$

Find dN, N^*. Then calculate $d(N^* dN)$. Find the conditions of f_1, f_2, f_3 such that

$$d(N^* dN) = 0_2$$

where 0_2 is the 2×2 zero matrix.

Problem 94. (i) Find all invertible 2×2 matrices over $\mathbb{R}$ such that

$$T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} T^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(ii) Do these matrices form a group?

Problem 95. (i) Let A, B be $n \times n$ matrices over $\mathbb{C}$. Assume that the inverse of A and $A + B$ exists. Show that

$$(A + B)^{-1} \equiv A^{-1} - A^{-1}B(A + B)^{-1}.$$

Apply the identity to

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Index