Algebra Qualifying Exam, Fall 2000: Part I

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming.

Notation:
$\mathbb{Z}$: Integers
$\mathbb{Q}$: Rational Field
$\mathbb{R}$: Real Field
$\mathbb{C}$: Complex Field
$\mathrm{GL}_n(R)$: Group of invertible $n \times n$ matrices with entries in the ring R
$\mathbb{F}_q$: Finite field with q elements
$\mathbb{Z}/n$: Ring of integers mod n (can also be regarded as the cyclic group of order n)

1. How many distinct isomorphism types are there for groups of order 2525?

2.(a) Suppose K is a field of characteristic zero which contains the p-th roots of 1, where p is a fixed prime. If L/K is a Galois extension of degree p, explain why $L = K[\alpha]$ where $\alpha^p = a \in K$.

(b) If p is odd, show that there is no $\beta \in L$ with $\beta^p = \alpha$. (**Hint**: Use norms.)

(c) Give a counterexample to the assertion in part (b) if $p = 2$.

3. Suppose that W is an even-dimensional real vector space, $T : W \to W$ a linear transformation with $T^m = I$ (the identity transformation) with m odd. Show that there exists a linear transformation $S : W \to W$ with $S^2 = -I$ and $ST = TS$.

4. Let A be a principal ideal domain, M a finitely generated free A-module.

(a) Show that the number of elements in a free basis for M over A is independent of the choice of basis.
(b) Let $N \subset A^m$ be a submodule. Prove that N is free on n generators for some $n \leq m$.
(c) Prove that $n = m$ in part (b) if and only if there is an nonzero $a \in A$ with $aA^m \subset N$.

5. Suppose G is a finite group, K a normal subgroup of G, and that (ρ, V) is an irreducible complex representation of G. Consider the restriction (ρ_K, V) of this representation to K. Show that all K-invariant subspaces of V which are irreducible over K have the same dimension and occur with the same multiplicity in V.

ALGEBRA QUALIFYING EXAM, FALL 2000: PART II

Directions: Work each problem in a separate bluebook. Give reasons for your answers, and make clear which facts you are assuming.

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1. If G is a group, define subgroups $G^{(n)}$ recursively by $G^{(1)} = [G, G]$ (the commutator subgroup) and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. The group G is *solvable* if $G^{(n)} = \{e\}$ for some n.
   (a) If K is a normal subgroup of G such that both K and G/K are solvable, show that G is solvable.
   (b) Show that all groups of order p^n with p prime are solvable.

2. Determine the number of conjugacy classes in the group $\mathrm{GL}_2(\mathbb{F}_q)$ for all finite fields $\mathbb{F}_q$. (**Hint:** Use linear algebra.)

3.(a) If A is a commutative ring with 1 show that the polynomial ring $A[X]$ contains infinitely many distinct maximal ideals.

(b) Describe all maximal ideals in the ring of formal power series $\mathbb{Z}[[X]]$.

4.(a) If G is a non-abelian group of order p^3, show that G has a quotient group isomorphic to $(\mathbb{Z}/p) \times (\mathbb{Z}/p)$. What are the number and dimensions of the irreducible complex representations of G?

(b) If the nonabelian group of order p^3 contains an element x of order p^2 show that G has irreducible p-dimensional representations induced from suitable 1-dimensional representations of $\langle x \rangle \cong \mathbb{Z}/p^2$.

5. Let $\mathbb{Q}[\zeta]$ be the field extension of $\mathbb{Q}$ generated by a primitive 11-th root of unity ζ. The integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\zeta]$ is the ring $\mathbb{Z}[\zeta]$. For each of the following primes $p \in \mathbb{Z}$, describe how the ideal $p\,\mathbb{Z}[\zeta]$ factors in $\mathbb{Z}[\zeta]$.

(a) $p = 11$; (b) $p = 43$; (c) $p = 37$.