QUALIFYING EXAM – ALGEBRA
SPRING 1999
MORNING SESSION

Do all problems. Use a separate blue book for each.

Notation: $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{F}_q$ denote the ring of integers, and the fields of rational numbers, real numbers, complex numbers, and q elements, respectively.

1. Classify all groups of order 24 containing a normal subgroup which is cyclic of order 4.

2. Describe all similarity classes (conjugacy classes) of 6×6 matrices with minimal polynomial $x^4 + x^2$:

(i) over $\mathbf{Q}$,
(ii) over $\mathbf{F}_5$.

3. Find the Galois group of the splitting field of the polynomial $x^3 - x + 1$:

(i) over $\mathbf{R}$,
(ii) over $\mathbf{Q}$,
(iii) over $\mathbf{F}_2$.

4. Suppose K is a finite extension of $\mathbf{Q}$. Prove that the integral closure of $\mathbf{Z}$ in K is a free $\mathbf{Z}$-module of rank $[K : \mathbf{Q}]$.

5. Suppose G is a *nonabelian* group of order pq, where $p < q$ are primes.

(a) Describe the conjugacy classes in G.
(b) Describe all representations of G (over $\mathbf{C}$).

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AFTERNOON SESSION

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Notation: $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{F}_q$ denote the ring of integers, and the fields of rational numbers, real numbers, complex numbers, and q elements, respectively.

1. Describe all simple left modules over the matrix ring $M_n(\mathbf{Z})$ ($n \times n$ matrices over $\mathbf{Z}$). Recall that a module is simple if it has no proper submodules.

2. Let G be a finite group. Prove that the following are equivalent:

(i) every element of G is conjugate to its inverse,
(ii) every character of G is real-valued.

3. Let R be the ring $\mathbf{C}[x,y]/(y^4 - (x-1)(x-2)(x-3)(x-4))$. (You may assume that $y^4 - (x-1)(x-2)(x-3)(x-4)$ is irreducible.) Let K be the quotient field of R.

(a) Show that K is a Galois extension of $\mathbf{C}(x)$.
(b) Consider R as an extension of $\mathbf{C}[x]$. For every prime $\mathfrak{p}$ of $\mathbf{C}[x]$, find the primes of R above $\mathfrak{p}$ and describe the action of $\mathrm{Gal}(K/\mathbf{C}(x))$ on them.

4. Suppose G is a finite group, F is a field whose characteristic does not divide the order of G, and V is a representation of G over F (i.e., an F-vector space on which G acts F-linearly). Prove that if U is a subspace of V stable under G, then there is a complementary subspace W of V, also stable under G, such that $V = U \oplus W$.

5. Suppose K is an extension of $\mathbf{Q}$ of degree n, and let $\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbf{C}$ be the distinct embeddings of K into $\mathbf{C}$. Let $\alpha \in K$. Regarding K as a vector space over $\mathbf{Q}$, let $\phi : K \to K$ be the linear transformation $\phi(x) = \alpha x$. Show that the eigenvalues of ϕ are $\sigma_1(\alpha), \ldots, \sigma_n(\alpha)$.