Mathematics Department Stanford University
Real Analysis Qualifying Exam, Spring 2003, Paper 1

1. Let f be a continuous function on the unit square $Q \equiv [0,1] \times [0,1]$, and for $s \in [0,1]$ let $g(s) = \max\{f(s,t) : t \in [0,1]\}$.

(a) Show that g is a continuous function on $[0,1]$.

(b) Prove that if $|f(x) - f(y)| \le M|x - y|$ for $x, y \in Q$, then $|g(s_1) - g(s_2)| \le M|s_1 - s_2|$ for $s_1, s_2 \in [0,1]$.

(c) Give an example in which f is C^1 but g is not C^1.

2. Suppose X, d is a metric space without isolated points (i.e. no single point is an open set) such that every continuous function $f : X \to [0,1]$ is uniformly continuous. Prove that X is compact.

3. Suppose X, Y are Banach spaces and $T : X \to Y$ is linear. Prove that T is bounded in each of the following cases:

(a) If there is a family $\mathcal{F}$ of real continuous linear functionals on Y such that $f \circ T$ is continuous for each $f \in \mathcal{F}$ and $\cap_{f \in \mathcal{F}} f^{-1}\{0\} = \{0\}$.

(b) If there are closed sets $A_1, A_2, \ldots \subset X$ with $\cup_{n=1}^{\infty} A_n = X$ and with $T(A_n)$ a bounded subset of Y for each $n = 1, 2, \ldots$.

4. Suppose $T : X \to Y$ is a compact bounded linear operator between Banach spaces (T compact means that the image of each bounded set has compact closure). Prove that the adjoint transformation $T^* : Y^* \to X^*$ (defined by $T^*(f) = f \circ T$ for $f \in Y^*$) is also compact.

5. A sequence $\{\xi_j\}_{j=1,2,\ldots} \subset [0,1]$ is said to be uniformly distributed in the interval $[0,1]$ if $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(\xi_j) = \int_0^1 f(x)\,dx$ for each $f \in C([0,1])$ (i.e. $\frac{1}{n}\sum_{j=1}^{n} \delta_{\xi_j} \to$ Lebesgue measure on $[0,1]$ in the weak* sense).

Prove that $\{\xi_j\}_{j=1,2,\ldots}$ is uniformly distributed in $[0,1]$ if $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i m \xi_j} = 0$ for each integer $m \ne 0$.

Hint: First consider the case when $f(0) = f(1)$.

Mathematics Department Stanford University
Real Analysis Qualifying Exam, Spring 2003, Paper 2

1. If X is a finite dimensional real vector space, prove that all norms on X are equivalent (i.e. for each pair of norms $\| \|_1$, $\| \|_2$ on X there is a constant $C \geq 1$ such that $C^{-1}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$ for every $x \in X$).

2. (a) Prove that a weakly compact subset of a normed space X is bounded.

(b) In the Hilbert space $L^2([0,1])$, give an example of a countable closed bounded subset that is not weakly closed, and justify your answer.

3. Let μ be a finite positive Borel measure on $(0,1)$.

(a) Prove that there is an increasing function α on $(0,1)$ such that $\int_{(0,1)} f \, d\mu = -\int_0^1 f'(t)\alpha(t)\, dt$ for each $f \in C^1((0,1))$ with compact support.

(b) In case μ is non-atomic (i.e. in case $\mu(\{x\}) = 0$ for each point $x \in (0,1)$), prove that α as in (a) is unique up to an additive constant and is also continuous.

4. Prove that the following integrals converge to zero as $n \to \infty$:

(a) $$\int_0^n x^{-1/2}(1+n^2x^2)^{-1/2} \cos nx \, dx.$$

(b) $$\int_0^1 \frac{n(1-x)^2}{(1+nx)(\log x)^2} \cos nx \, dx.$$

5. Prove that if $\alpha \in (0,1)$ and if $f(t)$ is any L^2 function on the circle with Fourier series $\sum_{-\infty}^{\infty} \hat{f}(n)e^{int}$ such that $\sum_{|n|\geq N} |\hat{f}(n)| \leq N^{-\alpha}$ for each $N \geq 1$, then the L^2 class of $f(t)$ has a Hölder continuous representative $f_0(t)$ with exponent α (i.e. $|f_0(t_1) - f_0(t_2)| \leq C|t_1 - t_2|^\alpha$ for each t_1, t_2).

Hint: $\sum_{|n|\leq N} n|\hat{f}(n)| \leq CN^{1-\alpha}$ for each $N \geq 1$, with C a constant depending only on α. (Prove this fact if you make use of it.)