Ph. D Qualifying Examination
Complex analysis–Fall 1999

Work all six problems.

1. Show that for $a > 0$,

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2}dx = \frac{\pi(a+1)e^{-a}}{4} \quad \text{if} \quad a > 0.$$

2. Let D be the open unit disk centered at the origin and let $f : \overline{D} \to \mathbb{C}$ be a function. Suppose f is analytic in D, f is continuous in $\overline{D} \setminus \{1\}$ and

$$\lim_{z\to 1} \frac{|f(z)|}{\log|z-1|} = 0.$$

Suppose further that $|f(w)| \le 1$ for $w \in \partial D \setminus \{1\}$. Show that

$$\max_{z\in D}(|f(z)|) \le 1.$$

3. Let $f(z)$ be an analytic function on the punctured disk $0 < |z| < 1$ with 0 an essential singularity. Let ξ be any complex number. Show that, with the possible exception of one value of ξ,

$$\lim_{r\to 0^+} \frac{1}{2\pi\sqrt{-1}} \int_{|z|=r} \frac{f'(z)}{f(z)-\xi}dz = \infty,$$

where the limit is taken for those $r > 0$ for which $f(z) - \xi$ has no zeros on $|z| = r$.

4(a). Let Ω be the (open) ball of radius 1 centered at the origin and E be a compact subset in Ω. Use Poisson's formula for harmonic functions to prove the following version of the Harnack inequality: There is a constant M, depending only on E, such that every positive harmonic function $u(z)$ in Ω satisfies

$$u(z_1) \le Mu(z_2), \quad z_1, z_2 \in E.$$

(Do not quote Harnack's inequality directly.)

4(b). Find the best possible M in case E is the closed disk of radius $\frac{1}{2}$ centered at the origin.

5. Let D be the interior of the triangle whose vertices are $0, 1$ and i.

(a) Prove that there is a unique conformal mapping $w = f(z)$ of D onto the upper half plane such that

$$\lim_{z\to 0} f(z) = 0, \quad \lim_{z\to 1} f(z) = 1 \quad \text{and} \quad \lim_{z\to i} f(z) = \infty.$$

(b) Prove that f extends by reflection to an elliptic function. Find the poles and the periods of this function.

(c) Find the singular part of this function near its poles. Find an explicit formula for f as an infinite sum.

(Hint. For (c), study the rotational symmetry of f near a pole.)

6. Let H be the upper half-plane and let $F(z)$ be the function defined by

$$F(z) = \int_0^z \frac{dw}{(1-w)(1+w)\sqrt{w}}, \quad z \in H$$

where the integral is over a path from 0 to w in H and where $\sqrt{w}$ is the single-value branch of the square root function such that $\sqrt{1} = 1$. Show that F is a one-one conformal map onto the region

$$\Omega = \{(x, y) : x > 0, y > 0, \min(x, y) < \frac{\pi}{2}\}.$$