Ph. D Qualifying Examination
Complex analysis–Spring 1999-2000

Work all 6 problems. All problems have equal weight. Write each problem in a separate bluebook.

1. **a**. Let ω_1, ω_2 be two complex numbers such that ω_1/ω_2 is not real and let Λ be the lattice $\{n_1\omega_1 + n_2\omega_2 | \, n_1, n_2 \in \mathbb{Z}\}$. Prove directly that

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right)$$

is a doubly periodic meromorphic function up to constants.
b. Prove that *the necessary and sufficient* condition for the existence of a doubly periodic meromorphic function with (finitely many) prescribed poles and singular parts in a fundamental parallelogram is that the sum of the residues of the poles is zero.

2. Let a_i and b_i be real numbers for $1 \le i \le 3$ such that $0 < b_i < 1$. Let $u = u(z)$ be a holomorphic function in the upper half plane that solves the second order differential equation

$$u'' + u'\left(\frac{b_1}{z-a_1} + \frac{b_2}{z-a_2} + \frac{b_3}{z-a_3}\right) = 0.$$

a. Solve u and express the solution in integral form.
b. Show directly, without quoting the Schwarz-Christoffel theorem, that u maps the real axis to the boundary of a polygon.
c. Determine under what condition on the b_i's does u map the upper half plane conformally onto a triangle? Prove, again without quoting Schwarz-Christoffel theorem, your assertion and determine the angles of this triangle.

3. Let f be a univalent analytic function defined on the unit disk D centered at the origin. Suppose $f(0) = 0$. Show that the function

$$g(z) = \sqrt{f(z^2)}$$

has a single-valued branch, and is also univalent.
(*Hint: Univalent means one-one. Use* $\sqrt{f(z^2)} = z\sqrt{\frac{f(z^2)}{z^2}}$.)

4. *Let f be a nonconstant harmonic function in the unit disk D which restricts smoothly to the boundary. Assume $f(0) = 0$ and $\nabla f(0) = 0$. Show that $f(x) = 0$ for at least four distinct points x on the boundary.*

5. *Let Δ be the half-strip*

$$\text{Re } z > 0, \quad -\pi < \text{Im } z < \pi$$

and let L be the boundary of Δ consisting of three straight pieces. We orient L so that Δ is on the right hand side of L.

a. *Show that the integral*

$$\frac{1}{2\pi i}\int_L \frac{e^{e^\zeta}}{\zeta - z}d\zeta = E(z)$$

defines an analytic function on the complement of Δ*.*
b. *Show that* $E(z)$ *extends to an entire function on* $\mathbf{C}$*.*
c. *Show that* $E(z)$ *assumes real values for real* z*.*

6. *Let* $f(z)$ *be an entire function that satisfies the relation*

$$p_n(z)(f(z))^n + p_{n-1}(z)(f(z))^{n-1} + \cdots + p_0(z) = 0,$$

where $p_0(z), \cdots, p_n(z)$*, not all trivial, are rational functions on* $\mathbf{C}$*. Show that* f *must be a polynomial.*