Ph. D Qualifying Examination
Complex analysis–Spring 2001

Work all six problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

1. Let $P(z)$ be a polynomial in z.

(a) Assume that $P(z) \neq 0$ for $\text{Re}(z) > 0$. Show that $P'(z) \neq 0$ for $\text{Re}(z) > 0$. (Hint: Try taking a logarithmic derivative.)

(b) Show that for any polynomial $P(z)$, the zeroes of P' are contained in the convex hull of the zeroes of P.

2(a). Let $f(z)$ be an entire function with $\text{Re}(f(z)) < c(1+|z|)^p$ for some positive constants c and p. Show that f is a polynomial.

(b). Construct an entire function $f(z)$ that is not a polynomial such that for every $\epsilon > 0$,

$$\lim_{r \to \infty} e^{-\epsilon r} M(r) = 0,$$

where $M(r) = \max_{|z|=r} |f(z)|$. Justify your answer.

3. Let $u(z)$ be a continuous real-valued function on $\mathbb{C}$ that satisfies

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, d\theta$$

for all z and all $r > 0$.

(a) Prove that $\sup_\Omega u = \sup_{\partial\Omega} u$ for every bounded smooth domain Ω.

(b) Show that u is a smooth harmonic function, i.e., that $\Delta u = 0$. (Hint: You may quote the solution for the Dirichlet problem.)

4(a). Find the radius of convergence of the power series $\sum_{n=1}^{\infty} nz^n$. Find the largest domain to which the corresponding analytic function can be analytically extended.

(b). Suppose that the power series $\sum_{n=1}^{\infty} a_n z^n$ converges for $|z| < 1$ and that the coefficients a_n are real and nonnegative. Show that if the corresponding analytic function has an analytic continuation to a neighborhood of $z = 1$, then the radius of convergence of the series is greater than 1.

5. Let u be a subharmonic function defined on $\mathbb{C}$ and let $M(r) = \max_{|z|=r} u(z)$.

(a) Prove that

$$u(z) \leq \frac{\log r_2 - \log|z|}{\log r_2 - \log r_1} M(r_1) + \frac{\log|z| - \log r_1}{\log r_2 - \log r_1} M(r_2)$$

for $0 < r_1 \leq |z| \leq r_2$. (Hint: use the maximum principle.)

(b) Show that $\lim_{r\to\infty} M(r)/\log r$ exists (possibly infinite).

(c) Show that if the the limit $\lambda(u)$ in part (b) is 0, then u is a constant function.

6(a). Let Γ be a smooth simple closed curve in $\mathbb{C}$. Describe the set of points z such that $|P(z)| \leq \max_{w \in \Gamma} |P(w)|$ for all polynomials $P(z)$. Justify your answer.

(b). Let L be a line in $\mathbb{C}$ and let z_0 be a point not on L. Show that there is a sequence of polynomials $P_j(z)$ with $P_j(z_0) = 1$ such that $P_j \to 0$ uniformly on compact subsets of L.