Ph. D Qualifying Examination
Complex analysis–Spring 1998

Work all problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

1. Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

by contour integration.

2(a). Prove that

$$f(z) = \sum 2^{-n^2} z^n$$

has infinitely many zeros.

(b). Suppose f is entire, $|f(z)| \leq e^{A|z|}$, and $f(z) = f(-z)$. Prove that either f is a polynomial or f has infinitely many zeros.

3(a). Give an example of a region D in $\mathbb{C}$ whose complement is an infinite set and such that there is no conformal map of D onto a bounded region.

(b). Let Ω be the complement of $\{e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$ with the point at infinity adjoined. Find a function $f(z)$ that maps Ω conformally onto the unit disk.

4(a). Let $|f(z)|$ assume its maximum on $\overline{D} = \{z : |z| \leq 1\}$ at z_0, where f is analytic in a neighborhood of $\overline{D}$. Show that $f'(z_0) \neq 0$.

(b). Assume f is entire, $f(z+1) = f(z)$, and $|f(z)| \leq e^{c|z|}$ for some $c < 2\pi$. Show that f is constant.

5. Assume that f is analytic in a neighborhood of $\overline{D} = \{z : |z| \leq 1\}$. Assume also that $|f(e^{i\theta})| \leq m$ if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and that $|f(e^{i\theta})| \leq M$ if $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. Find the best possible bound for $|f(0)|$.

Hint: First assume that f has no zeros.

6. Let $u(z)$ be a real harmonic function on $\{z : 1 < |z| < 2\}$. Show that for some α and for some function $f(z)$ analytic in $\{z : 1 < |z| < 2\}$,

$$u(z) = \alpha \log r + \operatorname{Re} f(z).$$