Ph. D Qualifying Examination
Complex analysis–Fall 2000

Work all problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

1. Let
$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$
be a polynomial with complex coefficients $a_1, \ldots, a_n$. Let α_k be the real part of a_k. Suppose $f(z)$ has n zeros in the upper-half plane Im $z > 0$. Prove that the polynomial
$$\alpha(x) = x^n + \alpha_1 x^{n-1} + \cdots + \alpha_n$$
has n distinct real roots.

2. Let D be the open unit disk and let $f : D \to \mathbb{C}$ be an odd univalent function. (That is, $f(-z) = -f(z)$ and f is one-one.) Show that there is a univalent analytic function $g : D \to \mathbb{C}$ such that
$$f(z) = \sqrt{g(z^2)}.$$

3. Let Ω be the region $-1 < \mathrm{Re}(z) < 1$ and let $\mathcal{F}$ be the collection of all analytic functions $f(z)$ defined on Ω such that $f(0) = 0$ and $|f(z)| < 1$ for all $z \in \Omega$. Find
$$\sup_{f \in \mathcal{F}} \left\{ \left| f\left(\frac{1}{2}\right) \right| \right\}.$$

4. Define
$$F(z) = \int_0^\infty x^{z-1} e^{-x^2}\, dx$$
for $\mathrm{Re}(z) > 0$.

(a) Prove that F is an analytic function on the region $\mathrm{Re}(z) > 0$.
(b) Prove that F extends to a meromorphic function on the whole complex plane.
(c) Find all poles of F and find the singular parts of F at these poles.

5. Calculate the following integral:
$$\int_0^\infty \frac{\cos x - 1}{x^2}\, dx.$$

6. Let Ω be a connected open subset of $\mathbb{C}$.

(a) Let $h(z)$ be a non-trivial analytic function defined on Ω. Let $\{a_n\}_{n \geq 1}$ be all the (distinct) zeros of $h(z)$ and let $\{c_n\}_{n \geq 1}$ be a sequence of complex numbers. Show that there is an analytic function $H(z)$ defined on Ω such that $H(a_n) = c_n$ for all n.

(b) Let $f(z)$ and $g(z)$ be two analytic functions defined on Ω with no common zeros in Ω. Assume that both $f(z)$ and $g(z)$ have only simple zeros. Prove that there are analytic functions $F(z)$ and $G(z)$ defined on Ω such that over Ω,

$$F(z)f(z) + G(z)g(z) = 1.$$

Hint: One possible approach to (a) is to apply the Mittag-Leffler Theorem for the domain Ω. See below for the exact statement of the theorem. For (b), consider

$$F(z) = \frac{1 - G(z)g(z)}{f(z)}.$$

Mittag-Leffler Theorem: Let $\{b_k\}$ be a sequence of distinct points in Ω without limit points in Ω, and let $\{P_k(z)\}$ be a sequence of polynomials without constant terms. Then there are meromorphic functions ϕ defined on Ω such that the poles of ϕ are the points $\{b_k\}$ and such that (for each k) the singular part of ϕ at $z = b_k$ is $P_k(\frac{1}{z-b_k})$.