

Chứng minh rằng :

$$1. \frac{\pi}{4} \leq \int_{\pi/4}^{3\pi/4} \frac{1}{3-2\sin^2 x} dx \leq \frac{\pi}{2}$$

$$2. \frac{\sqrt{3}}{12} \leq \int_{\pi/4}^{\pi/3} \frac{\cot x}{x} dx \leq \frac{1}{3}$$

$$3. \frac{1}{2} \leq \int_0^{1/2} \frac{1}{\sqrt{1-x^6}} dx \leq \frac{\pi}{6}$$

$$4. \ln 2 < \int_0^1 \frac{1}{1+x\sqrt{x}} dx < \frac{\pi}{4}$$

$$5. \int_0^1 \frac{1}{x^2+x+1} dx \leq \frac{\pi}{8}$$

$$6. \frac{\pi}{18} \leq \int_0^1 \frac{\sqrt{x}}{x^5+x^4+x^3+3} dx \leq \frac{\pi}{9\sqrt{3}}$$

Bài giải :

$$1. \frac{\pi}{4} \leq x \leq \frac{3\pi}{4} \Rightarrow \frac{1}{\sqrt{2}} \leq \sin x \leq 1 \Rightarrow \frac{1}{2} \leq \sin^2 x \leq 1 \Rightarrow 1 \leq 2\sin^2 x \leq 2 \Rightarrow 1 \leq 3-2\sin^2 x \leq 2 \Rightarrow \frac{1}{2} \leq \frac{1}{3-2\sin^2 x} \leq 1$$

$$\Rightarrow \frac{1}{2} \int_{\pi/4}^{3\pi/4} dx \leq \int_{\pi/4}^{3\pi/4} \frac{1}{3-2\sin^2 x} dx \leq \int_{\pi/4}^{3\pi/4} dx \Rightarrow \frac{\pi}{4} \leq \int_{\pi/4}^{3\pi/4} \frac{1}{3-2\sin^2 x} dx \leq \frac{\pi}{2}$$

$$2. \frac{\pi}{4} \leq x \leq \frac{\pi}{3} \Rightarrow \begin{cases} \frac{1}{\sqrt{3}} \leq \cot x \leq 1 \\ \frac{3}{\pi} \leq \frac{1}{x} \leq \frac{4}{\pi} \end{cases} \Rightarrow \frac{\sqrt{3}}{\pi} \leq \frac{\cot x}{x} \leq \frac{4}{\pi} \Rightarrow \frac{\sqrt{3}}{\pi} \int_{\pi/4}^{\pi/3} dx \leq \int_{\pi/4}^{\pi/3} \frac{\cot x}{x} dx \leq \frac{4}{\pi} \int_{\pi/4}^{\pi/3} dx$$

$$\Rightarrow \frac{\sqrt{3}}{12} \leq \int_{\pi/4}^{\pi/3} \frac{\cot x}{x} dx \leq \frac{1}{3}$$

Bài toán này có thể giải theo phương pháp nào khác.

$$3. 0 \leq x \leq \frac{1}{2} < 1 \Rightarrow 0 \leq x^6 \leq \dots \leq x^2 < 1 \Rightarrow -1 \leq -x^2 \leq -x^6 \leq 0 \Rightarrow 0 \leq 1-x^2 \leq 1-x^6 \leq 1 \Rightarrow \sqrt{1-x^2} \leq \sqrt{1-x^6} \leq 1$$

$$\Rightarrow 1 \leq \frac{1}{\sqrt{1-x^6}} \leq \frac{1}{\sqrt{1-x^2}} \Rightarrow \int_0^{1/2} dx \leq \int_0^{1/2} \frac{1}{\sqrt{1-x^6}} dx \leq I$$

$$\text{Với } I = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx \text{ Đặt } x = \sin t; t \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right] \Rightarrow dx = \cos t dt$$

$$\begin{array}{c|cc} x & 0 & 1/2 \\ t & 0 & \pi/6 \end{array} \Rightarrow I = \int_0^{1/2} \frac{\cos t dt}{\sqrt{1-\sin^2 t}} = \int_0^{1/2} dt = \frac{\pi}{6} \text{ Vậy } \frac{1}{2} \leq \int_0^{1/2} \frac{1}{\sqrt{1-x^6}} dx \leq \frac{\pi}{6}$$

$$4. 0 \leq x \leq 1 \Rightarrow x \leq \sqrt{x} \leq 1 \Rightarrow x^2 \leq x\sqrt{x} \leq x \Rightarrow 1+x^2 \leq 1+x\sqrt{x} \leq 1+x$$

$$\Rightarrow \frac{1}{x+1} \leq \frac{1}{1+x\sqrt{x}} \leq \frac{1}{1+x^2} (1); \forall x \in [0,1]$$

Đầu thức trong (1) xảy ra khi :

$$\begin{cases} x=0 & VT_{(1)} \leq VG_{(1)} \\ x=1 & VG_{(1)} \leq VP_{(1)} \end{cases} \Rightarrow x \in \emptyset$$

$$\text{Do đó: } \int_0^1 \frac{1}{1+x} dx < \int_0^1 \frac{1}{1+x\sqrt{x}} dx < \int_0^1 \frac{1}{1+x^2} dx \Rightarrow \ln 2 < \int_0^1 \frac{1}{1+x\sqrt{x}} dx < \frac{\pi}{4}$$

$$\text{Chứng minh: } \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4} \text{ Xem bài tập 5.}$$

$$5. \quad 0 \leq x \leq 1 \Rightarrow x^2 \leq x \Rightarrow x^2 + x^2 \leq x^2 + x \Rightarrow 2 + 2x^2 \leq x^2 + x + 2 \Rightarrow \frac{1}{x^2 + x + 2} \leq \frac{1}{2(x^2 + 1)}$$

$$\Rightarrow \int_0^1 \frac{1}{x^2 + x + 2} dx \leq \frac{1}{2} \int_0^1 \frac{1}{x^2 + 1} dx \quad ; \quad I = \int_0^1 \frac{1}{1 + x^2} dx$$

$$\text{Đặt } x = \tan t \Rightarrow dx = \frac{1}{\cos^2 t} dt = (1 + \tan^2 t) dt$$

$$\begin{array}{c|ccc} x & 0 & & 1 \\ t & 0 & & \pi/4 \end{array} \Rightarrow I = \int_0^{\pi/4} \frac{1 + \tan^2 t}{1 + \tan^2 t} dt = \int_0^{\pi/4} dt = \frac{\pi}{4} \Rightarrow I = \frac{\pi}{4} \quad \text{Vậy } \int_0^1 \frac{1}{x^2 + x + 2} dx \leq \frac{\pi}{8}$$

$$6. \quad 0 \leq x \leq 1 \Rightarrow \begin{cases} 0 \leq x^5 \leq x^3 \\ 0 \leq x^4 \leq x^3 \end{cases} \Rightarrow 0 \leq x^5 + x^4 \leq 2x^3 \Rightarrow x^3 + 3 \leq x^5 + x^4 + x^3 + 3 \leq 3x^3 + 3$$

$$\Rightarrow \frac{1}{3x^3 + 3} \leq \frac{1}{x^5 + x^4 + x^3 + 3} \leq \frac{1}{x^3 + 3} \Rightarrow \frac{\sqrt{x}}{3x^3 + 3} \leq \frac{\sqrt{x}}{x^5 + x^4 + x^3 + 3} \leq \frac{\sqrt{x}}{x^3 + 3}$$

$$\Rightarrow \int_0^1 \frac{\sqrt{x}}{3x^3 + 3} dx \leq \int_0^1 \frac{\sqrt{x}}{x^5 + x^4 + x^3 + 3} dx \leq \int_0^1 \frac{\sqrt{x}}{x^3 + 3} dx \quad (1)$$

$$\circ \quad I_1 = \int_0^1 \frac{\sqrt{x}}{3x^3 + 3} dx = \frac{1}{3} \int_0^1 \frac{\sqrt{x}}{x^3 + 1} dx \quad ; \quad \text{Đặt } x = t^2; (t \geq 0) \Rightarrow dx = 2t dt \quad \begin{array}{c|ccc} x & 0 & & 1 \\ t & 0 & & 1 \end{array}$$

$$I_1 = \frac{1}{3} \int_0^1 \frac{2t}{t^6 + 1} dt = \frac{2}{9} \int_0^1 \frac{3t^2 \cdot dt}{(t^3)^2 + 1} \quad \text{Đặt } u = t^3 \Rightarrow du = 3t^2 dt \quad \begin{array}{c|ccc} t & 0 & & 1 \\ u & 0 & & 1 \end{array} \Rightarrow I_1 = \frac{2}{9} \int_0^1 \frac{du}{u^2 + 1} = \frac{\pi}{18}$$

$$\text{Kết quả : } I = \frac{\pi}{4} \text{ (bài tập 5)}$$

$$\circ \quad I_2 = \int_0^1 \frac{\sqrt{x}}{x^3 + 3} = \frac{\pi}{9\sqrt{3}} \text{ (đồng thời) } \text{ Vậy } (1) \Leftrightarrow I_1 \leq \int_0^1 \frac{\sqrt{x}}{x^5 + x^4 + x^3 + 3} dx \leq I_2$$

$$\frac{\pi}{18} \leq \int_0^1 \frac{\sqrt{x}}{x^5 + x^4 + x^3 + 3} dx \leq \frac{\pi}{9\sqrt{3}}$$

$$\underline{1. Chứng minh rằng} : \int_0^{\pi/2} \frac{\sin x \cdot \cos x}{(1 + \sin^4 x)(1 + \cos^4 x)} dx \geq \frac{\pi}{12}$$

$$\underline{2. Nếu} : I_{(t)} = \int_0^t \frac{tg^4 x}{\cos 2x} dx > 0, \forall t \in \left(0, \frac{\pi}{4}\right); \text{ thì } : tg\left(t + \frac{\pi}{4}\right) > e^{\frac{2}{3}(tg^3 t + 3tg t)}$$

Bài giải :

$$1. \text{ Ta có : } \frac{3}{(1 + \sin^4 x)(1 + \cos^4 x)} = \frac{2 + \cos^2 x + \sin^2 x}{(1 + \sin^4 x)(1 + \cos^4 x)} \geq \frac{2 + \sin^4 x + \cos^4 x}{(1 + \sin^4 x)(1 + \cos^4 x)}$$

$$\Rightarrow \frac{3}{(1 + \sin^4 x)(1 + \cos^4 x)} \geq \frac{1 + \sin^4 x + 1 + \cos^4 x}{(1 + \sin^4 x)(1 + \cos^4 x)} = \frac{1}{1 + \sin^4 x} + \frac{1}{1 + \cos^4 x}$$

$$\Rightarrow \frac{3 \sin x \cdot \cos x}{(1 + \sin^4 x)(1 + \cos^4 x)} \geq \frac{\sin x \cdot \cos x}{1 + \sin^4 x} + \frac{\sin x \cdot \cos x}{1 + \cos^4 x} \Rightarrow \frac{\sin x \cdot \cos x}{(1 + \sin^4 x)(1 + \cos^4 x)} \geq \frac{1}{6} \left(\frac{\sin 2x}{1 + \sin^4 x} + \frac{\sin 2x}{1 + \cos^4 x} \right)$$

$$\Rightarrow \int_0^{\pi/2} \frac{3 \sin x \cdot \cos x}{(1 + \sin^4 x)(1 + \cos^4 x)} dx \geq \frac{1}{6} \left(\int_0^{\pi/2} \frac{\sin 2x}{1 + \sin^4 x} dx + \int_0^{\pi/2} \frac{\sin 2x}{1 + \cos^4 x} dx \right)$$

$$\circ J_1 = \int_0^{\pi/2} \frac{\sin 2x}{1 + \sin^4 x} dx \quad \text{Đặt } t = \sin^2 x \Rightarrow dt = \sin 2x dx$$

$$\begin{array}{c|ccc} x & 0 & & \pi/2 \\ t & 0 & & 1 \end{array} \Rightarrow J_1 = \int_0^1 \frac{dt}{t^2 + 1} = \frac{\pi}{4} \quad (\text{kết quả } = \frac{\pi}{4} \text{ bài tập 5})$$

$$\circ J_2 = \int_0^{\pi/2} \frac{\sin 2x}{1 + \cos^4 x} dx \quad \text{Đặt } u = \cos^2 x \Rightarrow du = -\sin 2x dx$$

$$\begin{array}{c|ccc} x & 0 & & \pi/2 \\ u & 1 & & 0 \end{array} \Rightarrow J_2 = \int_0^1 \frac{du}{u^2 + 1} = \frac{\pi}{4} \quad (\text{kết quả } = \frac{\pi}{4} \text{ bài tập 5})$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin x \cdot \cos x}{(1 + \sin^4 x)(1 + \cos^4 x)} dx \geq \frac{1}{6} (I + J) \quad \text{Vậy } \int_0^{\pi/2} \frac{\sin x \cdot \cos x}{(1 + \sin^4 x)(1 + \cos^4 x)} dx \geq \frac{\pi}{12}$$

$$2. \text{Đặt } t = \tan x \Rightarrow dt = (1 + \tan^2 x) dx \Rightarrow dx = \frac{dt}{1 + t^2}$$

$$I_t = \int_0^{\tan t} \frac{t^4}{1 + t^2} \cdot \frac{dt}{1 + t^2} = \int_0^{\tan t} \frac{t^4 dt}{1 + t^2} = \int_0^{\tan t} \left(-t^2 - 1 + \frac{1}{1 + t^2} \right) dt = \left(-\frac{1}{3} t^3 - t - \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| \right) \Big|_0^{\tan t} = -\frac{1}{3} \tan^3 t - \tan t - \frac{1}{2} \ln \left| \frac{\tan t - 1}{\tan t + 1} \right|$$

Vì

$$I_t > 0 \text{ nên } : -\frac{1}{3} \tan^3 t - \tan t - \frac{1}{2} \ln \left| \frac{\tan t - 1}{\tan t + 1} \right| > 0$$

$$\Leftrightarrow \frac{1}{2} \ln \left| \frac{\tan t - 1}{\tan t + 1} \right| = \frac{1}{2} \ln \left| \tan \left(t + \frac{\pi}{4} \right) \right| > \frac{1}{3} \tan^3 t + \tan t \Rightarrow \tan \left(t + \frac{\pi}{4} \right) > e^{\frac{2}{3}(\tan^3 t + 3 \tan t)}$$

$$1. I_n = \frac{x^2}{x+1} \quad \text{Chứng minh : } \frac{1}{2(n+1)} \leq \int_0^1 I_n dx \leq \frac{1}{n+1} \text{ và } \lim_{n \rightarrow +\infty} I_n dx = 0$$

$$2. J_n = x^n (1 + e^{-x}) \quad \text{Chứng minh : } 0 < \int_0^1 J_n dx \leq \frac{2}{n+1} \text{ và } \lim_{n \rightarrow +\infty} J_n dx = 0$$

Bài giải:

$$1. 0 \leq x \leq 1 \Rightarrow 1 \leq x+1 \leq 2 \Rightarrow \frac{1}{2} \leq \frac{1}{x+1} \leq 1 \quad ; \quad \frac{x^n}{2} \leq \frac{x^n}{x+1} \leq x^n \Rightarrow \frac{1}{2} \int_0^1 x^n dx \leq \int_0^1 \frac{x^n}{x+1} dx \leq \int_0^1 x^n dx$$

$$\Rightarrow \frac{x^{n+1}}{2(n+1)} \Big|_0^1 \leq \int_0^1 \frac{x^n}{x+1} dx \leq \frac{x^{n+1}}{n+1} \Big|_0^1 \Rightarrow \frac{1}{2(n+1)} \leq \int_0^1 \frac{x^n}{x+1} dx \leq \frac{1}{n+1}$$

Ta có:
$$\begin{cases} \lim_{n \rightarrow \infty} \frac{1}{2(n+1)} = 0 \\ \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 \end{cases} \Rightarrow \lim_{n \rightarrow \infty} \frac{x^n}{x+1} = 0$$

2. $0 \leq x \leq 1 \Rightarrow 0 \leq e^{-x} \leq e^0 = 1 \Rightarrow 1 \leq 1 + e^{-x} \leq 2 \Rightarrow x^n \leq x^n(1 + e^{-x}) \leq 2x^n$ hay $0 \leq x^n(1 + e^{-x}) \leq 2x^n$
 $\Rightarrow 0 \leq \int_0^1 x^n(1 + e^{-x}) dx \leq 2 \int_0^1 x^n dx \Rightarrow 0 \leq \int_0^1 x^n(1 + e^{-x}) dx \leq \frac{2}{n+1}$

Ta có: $\lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 \Rightarrow \lim_{n \rightarrow \infty} \int_0^1 x^n(1 + e^{-x}) dx = 0$

Chứng minh rằng:

1. $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x}(4 - 3\sqrt{\cos x})(2\sqrt{\cos x} + 2) dx \leq 8\pi$ 2. $\int_1^2 \sqrt{\ln x}(9 - 3\sqrt{\ln x} - 2\ln x) dx \leq 8(e-1)$

3. $\int_{\pi/4}^{\pi/3} \sqrt{\sin x}(1 + 2\sqrt{\sin x})(5 - 3\sqrt{\sin x}) dx < \frac{2\pi}{3}$ 4. $\int_0^{\pi/4} \sqrt{\tan x}(7 - 4\sqrt{\tan x}) dx \leq \frac{49\pi}{64}$

5. $\int_0^{\pi} \sin^4 x \cdot \cos^6 x dx \leq \frac{243\pi}{6250}$

Bài giải:

1. Xét $f(x) = \sqrt{\cos x}(4 - 3\sqrt{\cos x})(2\sqrt{\cos x} + 2)$

$f(x) \leq \left(\frac{\sqrt{\cos x} + 4 - 3\sqrt{\cos x} + 2\sqrt{\cos x} + 2}{3} \right)^3 = 8$

$\Rightarrow \int_{-\pi/2}^{\pi/2} f(x) dx \leq 8 \int_{-\pi/2}^{\pi/2} dx \Rightarrow \int_{-\pi/2}^{\pi/2} \sqrt{\cos x}(4 - 3\sqrt{\cos x})(2\sqrt{\cos x} + 2) dx \leq 8\pi$

2. Xét $f(x) = \sqrt{\ln x}(9 - 3\sqrt{\ln x} - 2\ln x) = \sqrt{\ln x}(3 + \sqrt{\ln x})(3 - 2\sqrt{\ln x})$

$f(x) \leq \left(\frac{\sqrt{\ln x} + 3 + \sqrt{\ln x} + 3 - 2\sqrt{\ln x}}{3} \right)^3 = 8$

$\Rightarrow \int_1^e f(x) dx \leq 8 \int_1^e dx \Rightarrow \int_1^e \sqrt{\ln x}(9 - 3\sqrt{\ln x} - 2\ln x) dx \leq 8(e-1)$

3. Xét $f(x) = \sqrt{\sin x}(1 + 2\sqrt{\sin x})(5 - 3\sqrt{\sin x})$; $f(x) \leq \left(\frac{\sqrt{\sin x} + 1 + 2\sqrt{\sin x} + 5 - 3\sqrt{\sin x}}{3} \right)^3 \leq 8$

Áng thuộc $\Leftrightarrow \begin{cases} \sqrt{\sin x} = 1 + 2\sqrt{\sin x} \\ \sqrt{\sin x} = 5 - 3\sqrt{\sin x} \end{cases} \Leftrightarrow \begin{cases} \sqrt{\sin x} = -1 \\ 4\sqrt{\sin x} = 5 \end{cases} \Leftrightarrow x \in \emptyset$

$\Rightarrow f(x) < 8 \Rightarrow \int_{\pi/4}^{\pi/3} f(x) dx < 8 \int_{\pi/4}^{\pi/3} dx \Rightarrow \int_{\pi/4}^{\pi/3} \sqrt{\sin x}(1 + 2\sqrt{\sin x})(5 - 3\sqrt{\sin x}) dx < \frac{2\pi}{3}$

4. Xét $f(x) = \sqrt{\tan x}(7 - 4\sqrt{\tan x}) = \frac{1}{4} \cdot 4\sqrt{\tan x}(7 - 4\sqrt{\tan x})$

$$f(x) \leq \frac{1}{4} \left(\frac{4\sqrt{tgx} + 7 - 4\sqrt{tgx}}{2} \right)^2 = \frac{49}{16}$$

$$\Rightarrow \int_0^{\pi/4} f(x) dx \leq \frac{49}{16} \int_0^{\pi/4} dx \Rightarrow \int_0^{\pi/4} \sqrt{tgx} (7 - 4\sqrt{tgx}) dx \leq \frac{49\pi}{16}$$

$$\begin{aligned} 5. \sin^4 x \cdot \cos^6 x &= (1 - \cos^2 x) \cdot (1 - \cos^2 x) \cdot \cos^2 x \cdot \cos^2 x \cdot \cos^2 x \\ &= \frac{1}{2} (2 - 2\cos^2 x)(1 - \cos^2 x) \cdot \cos^2 x \cdot \cos^2 x \cdot \cos^2 x \\ &\leq \frac{1}{2} \left(\frac{2 - 2\cos^2 x + 1 - \cos^2 x + \cos^2 x + \cos^2 x + \cos^2 x}{5} \right)^5 \\ \Rightarrow \sin^4 x \cdot \cos^6 x &\leq \frac{243}{6250} \Rightarrow \int_0^{\pi} \sin^4 x \cdot \cos^6 x dx \leq \frac{243\pi}{6250} \end{aligned}$$

Chứng minh rằng :

$$1. \int_{-\pi/3}^{\pi/2} (\sqrt{\cos^2 x + 3\sin^2 x} + \sqrt{\sin^2 x + 3\cos^2 x}) dx \leq \frac{5\pi\sqrt{2}}{3}$$

$$2. \int_1^e (\sqrt{3 + 2\ln^2 x} + \sqrt{5 - 2\ln^2 x}) dx \leq 4(e - 1)$$

$$3. -\frac{\pi}{4} \leq \int \frac{\sqrt{3}\cos x + \sin x}{x^2 + 4} dx \leq \frac{\pi}{4}$$

Bài giải :

$$1. \text{Ñặt } f(x) = 1\sqrt{\cos^2 x + 3\sin^2 x} + 1\sqrt{\sin^2 x + 3\cos^2 x}$$

$$f^2(x) \leq 2(\cos^2 x + 3\sin^2 x + 3\cos^2 x + \sin^2 x) \Rightarrow f(x) \leq 2\sqrt{2}$$

$$\Rightarrow \int_{-\pi/3}^{\pi/2} f(x) dx \leq 2\sqrt{2} \int_{-\pi/3}^{\pi/2} dx \Rightarrow \int_{-\pi/3}^{\pi/2} (\sqrt{\cos^2 x + 3\sin^2 x} + \sqrt{\sin^2 x + 3\cos^2 x}) dx \leq \frac{5\pi\sqrt{2}}{3}$$

$$2. \text{Ñặt } f(x) = 1\sqrt{3 + 2\ln^2 x} + 1\sqrt{5 - 2\ln^2 x}$$

$$f^2(x) \leq 2(3 + 2\ln^2 x + 5 - 2\ln^2 x) \Rightarrow f(x) \leq 4$$

$$\Rightarrow \int_1^e f(x) dx \leq 4 \int_1^e dx \Rightarrow \int_1^e (\sqrt{3 + 2\ln^2 x} + \sqrt{5 - 2\ln^2 x}) dx \leq 4(e - 1)$$

$$3. |\sqrt{3}\cos x + \sin x| \leq \sqrt{[(\sqrt{3})^2 + 1](\cos^2 x + \sin^2 x)}$$

$$\Rightarrow \frac{|\sqrt{3}\cos x + \sin x|}{x^2 + 4} \leq \frac{2}{x^2 + 4} \Rightarrow \int_0^2 \frac{|\sqrt{3}\cos x + \sin x|}{x^2 + 4} dx \leq 2 \int_0^2 \frac{dx}{x^2 + 4}$$

Đặt $x = 2tg t \Rightarrow dx = 2(1 + tg^2 t) dt$

$$\frac{x}{t} \quad \begin{matrix} 0 & 1 \\ 0 & \frac{\pi}{4} \end{matrix} \Rightarrow \int_0^2 \frac{dx}{x^2 + 4} = \int_0^{\pi/4} \frac{2(1 + tg^2 t)}{4(1 + tg^2 t)} dt = \frac{1}{2} \int_0^{\pi/4} dt = \frac{\pi}{8}$$

$$\Rightarrow \int_0^2 \frac{|\sqrt{3} \cos x + \sin x|}{x^2 + 4} dx \leq \frac{\pi}{4} \Rightarrow -\frac{\pi}{4} \leq \int_0^2 \frac{\sqrt{3} \cos x + \sin x}{x^2 + 4} dx \leq \frac{\pi}{4}$$

PHẦN GIẢI TÍCH PHÂN DẪN VÀO TẬP GIẢI TRÒ CỦA HÀM DẪN ĐÁU TÍCH PHÂN

Chứng minh rằng :

$$1. \int_0^{\pi/4} \sin 2x dx \leq 2 \int_0^{\pi/4} \cos x dx$$

$$2. \int_0^{\pi/2} \sin 2x dx \leq 2 \int_0^{\pi/2} \sin x dx$$

$$3. \int_1^2 \frac{x-1}{x} dx < \int_1^2 \frac{2x-1}{x+1} dx$$

$$4. \int_0^{\pi/2} \frac{\sin x}{x} dx > \int_{\pi/2}^{\pi} \frac{\sin x}{x} dx$$

$$5. \int_1^2 (\ln x)^2 dx < \int_1^2 \ln x dx$$

$$6. \int_0^{\pi/4} \sin x dx < \int_0^{\pi/4} \cos x dx$$

Bài giải :

$$1. \forall x \in \left[0; \frac{\pi}{4}\right] \Rightarrow \begin{cases} 0 \leq \sin x \leq 1 \\ 0 \leq \cos x \leq 1 \end{cases} \Rightarrow 2 \sin x \cdot \cos x \leq 2 \cos x$$

$$\Leftrightarrow \sin 2x \leq 2 \cos x \Rightarrow \int_0^{\pi/4} \sin 2x dx \leq 2 \int_0^{\pi/4} \cos x dx$$

$$2. \forall x \in \left[0; \frac{\pi}{2}\right] \Rightarrow \begin{cases} \cos x \leq 1 \\ 0 \leq \sin x \end{cases} \Rightarrow 2 \sin 2x \cdot \cos x \leq 2 \sin x$$

$$\Leftrightarrow \sin 2x \leq 2 \sin x \Rightarrow \int_0^{\pi/2} \sin 2x dx \leq 2 \int_0^{\pi/2} \sin x dx$$

$$3. \forall x \in [1; 2] \text{ Xét hiệu: } \frac{x-1}{x} - \frac{2x-1}{x+1} = \frac{-x^2 + x - 1}{x(x+1)} < 0$$

$$\Rightarrow \frac{x-1}{x} < \frac{2x-1}{x+1} \Rightarrow \int_1^2 \frac{x-1}{x} dx < \int_1^2 \frac{2x-1}{x+1} dx$$

$$4. \text{ Đặt } x = \pi - u \Rightarrow dx = -du$$

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{x} dx = \int_{\pi/2}^0 \frac{\sin(\pi-u)}{\pi-u} (-du) = \int_0^{\pi/2} \frac{\sin x}{\pi-x} dx$$

$$0 < x < \frac{\pi}{2} \Rightarrow 0 < x < \pi - x \Rightarrow \frac{1}{\pi-x} < \frac{1}{x}$$

$$\text{Vì: } \sin x > 0 \Rightarrow \frac{\sin x}{\pi-x} < \frac{\sin x}{x} \Rightarrow \int_0^{\pi/2} \frac{\sin x}{\pi-x} dx < \int_0^{\pi/2} \frac{\sin x}{x} dx$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin x}{x} dx > \int_{\pi/2}^{\pi} \frac{\sin x}{x} dx$$

$$5. \text{ Hàm số } y = f(x) = \ln x \text{ liên tục trên } [1, 2] \text{ nên } y = g(x) = (\ln x)^2 \text{ cũng liên tục trên } [1, 2]$$

$$1 \leq x \leq 2 \Rightarrow 0 \leq \ln x \leq \ln 2 < 1 (*) \Rightarrow 0 \leq (\ln x)^2 < \ln x$$

$$\forall x \in [1, 2] \Rightarrow \int_1^2 (\ln x)^2 dx < \int_1^2 \ln x dx$$

Chú ý: dấu bất đẳng thức (*) xảy ra tại $x_0 = 1 \in [1, 2]$

$$6. 0 < x < \frac{\pi}{4} \Rightarrow 0 < \tan x < \tan \frac{\pi}{4} = 1 \Leftrightarrow \frac{\sin x}{\cos x} < 1$$

$$\Leftrightarrow \sin x < \cos x \Leftrightarrow \int_0^{\pi/4} \sin x dx < \int_0^{\pi/4} \cos x dx$$

Chứng minh rằng :

$$1. 2 \leq \int_0^1 \sqrt{x^2 + 4} \, dx \leq \sqrt{5}$$

$$2. \frac{1}{\sqrt{2}} \leq \int_0^1 \frac{1}{\sqrt{x^8 + 1}} \, dx \leq 1$$

$$3. \frac{1}{26\sqrt[3]{2}} \leq \int_0^1 \frac{x^{25}}{\sqrt[3]{x^{10} + 1}} \, dx \leq \frac{1}{26}$$

$$4. \int_0^1 \frac{x \sin x}{1 + x \sin x} \, dx \leq 1 - \ln 2$$

$$5. 0 < \int_1^{\sqrt{3}} \frac{e^{-x} \sin x}{x^2 + 1} \, dx \leq \frac{\pi}{12e}$$

$$6. \frac{\pi}{6} \leq \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \leq \frac{\pi\sqrt{2}}{8}$$

Bài Giải

$$1. 0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq 1 \Rightarrow 4 \leq x^2 + 4 \leq 5 \Rightarrow 2 \leq \sqrt{x^2 + 4} \leq \sqrt{5}$$

$$\Rightarrow 2 \int_0^1 dx \leq \int_0^1 \sqrt{x^2 + 4} \, dx \leq \sqrt{5} \int_0^1 dx \Rightarrow 2 \leq \int_0^1 \sqrt{x^2 + 4} \, dx \leq \sqrt{5}$$

$$2. 0 \leq x \leq 1 \Rightarrow 0 \leq x^8 \leq 1 \Rightarrow 1 \leq x^8 + 1 \leq 2$$

$$\Rightarrow 0 \leq \sqrt{x^8 + 1} \leq \sqrt{2} \Rightarrow \frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{x^8 + 1}} \leq 1$$

$$\Rightarrow \frac{1}{\sqrt{2}} \int_0^1 dx \leq \int_0^1 \frac{dx}{\sqrt{x^8 + 1}} \leq \int_0^1 dx \Rightarrow \frac{1}{\sqrt{2}} \leq \int_0^1 \frac{dx}{\sqrt{x^8 + 1}} \leq 1$$

$$3. 0 \leq x \leq 1 \Rightarrow 1 \leq x^{10} + 1 \leq 2 \Rightarrow 1 \leq \sqrt[3]{x^{10} + 1} \leq \sqrt[3]{2}$$

$$\Rightarrow \frac{1}{\sqrt[3]{2}} \leq \frac{1}{\sqrt[3]{x^{10} + 1}} \leq 1 \Leftrightarrow \frac{x^{25}}{\sqrt[3]{2}} \leq \frac{x^{25}}{\sqrt[3]{x^{10} + 1}} \leq x^{25}$$

$$\Rightarrow \frac{1}{\sqrt[3]{2}} \int_0^1 x^{25} \, dx \leq \int_0^1 \frac{x^{25}}{\sqrt[3]{x^{10} + 1}} \, dx \leq \int_0^1 x^{25} \, dx \Rightarrow \frac{1}{26\sqrt[3]{2}} \leq \int_0^1 \frac{x^{25}}{\sqrt[3]{x^{10} + 1}} \, dx \leq \frac{1}{26}$$

$$4. \text{Trước hết ta chứng minh: } \frac{x \sin x}{1 + x \sin x} \leq \frac{x}{1 + x}; (1) \forall x \in [0, 1].$$

Gia sử ta có: (1).

$$(1) \Leftrightarrow 1 - \frac{1}{1 + x \sin x} \leq 1 - \frac{1}{1 + x}; \forall x \in [0, 1] \Leftrightarrow \frac{1}{1 + x \sin x} \geq \frac{1}{1 + x}$$

$$\Leftrightarrow 1 + x \geq 1 + x \sin x \Leftrightarrow x(1 - \sin x) \geq 0 \text{ đúng } \forall x \in [0, 1]$$

$$(1) \Leftrightarrow \int_0^1 \frac{x \sin x}{1 + x \sin x} \, dx \leq \int_0^1 \frac{x}{1 + x} \, dx = \int_0^1 \left(1 - \frac{1}{1 + x}\right) \, dx$$

Vậy (1) đúng theo đúng, khi đó:

$$\Leftrightarrow \int_0^1 \frac{x \sin x}{1 + x \sin x} \, dx \leq (x - \ln|1 + x|) \Big|_0^1 = 1 - \ln 2$$

$$\Rightarrow \int_0^1 \frac{x \sin x}{1 + x \sin x} \, dx \leq 1 - \ln 2.$$

$$5. x \in [1, \sqrt{3}] \subset (0, \Pi) \Rightarrow \begin{cases} 0 < e^{-x} = \frac{1}{e^x} \leq \frac{1}{e} \Rightarrow 0 < \frac{e^{-x} \sin x}{x^2 + 1} < \frac{1}{e(x^2 + 1)} \\ 0 < \sin x < 1 \end{cases}$$

$$\Rightarrow 0 < \int_1^{\sqrt{3}} \frac{e^{-x} \sin x}{x^2 + 1} dx < \frac{1}{e} \int_1^{\sqrt{3}} \frac{dx}{x^2 + 1} = \frac{1}{e} I \quad ; I = \int_1^{\sqrt{3}} \frac{dx}{x^2 + 1}$$

Đặt $x = \tan t \Rightarrow dx = \frac{1}{\cos^2 t} dt = (1 + \tan^2 t) dt$

$$\frac{x}{t} \quad \frac{1}{\frac{\pi}{4}} \quad \frac{\sqrt{3}}{\frac{\pi}{4}} \quad \Rightarrow I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(1 + \tan^2 t)}{1 + \tan^2 t} dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} dt = t \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{12}$$

Vậy $0 < \int_1^{\sqrt{3}} \frac{e^{-x} \sin x}{x^2 + 1} dx < \frac{\pi}{12e}$

$$6. 0 \leq x \leq 1 \Rightarrow 0 \leq x^3 \leq x^2 \Rightarrow -x^2 \leq -x^3 \leq 0$$

$$\Rightarrow 4 - 2x^2 \leq 4 - x^2 - x^3 \leq 4 - x^2$$

$$\Rightarrow \sqrt{4 - 2x^2} \leq \sqrt{4 - x^2 - x^3} \leq \sqrt{4 - x^2}$$

$$\Rightarrow \frac{1}{\sqrt{4 - 2x^2}} \geq \frac{1}{\sqrt{4 - x^2 - x^3}} \geq \frac{1}{\sqrt{4 - x^2}}$$

$$\Rightarrow I = \int_0^1 \frac{1}{\sqrt{4 - x^2}} dx \leq \int_0^1 \frac{1}{\sqrt{4 - x^2 - x^3}} dx \leq \int_0^1 \frac{1}{\sqrt{4 - 2x^2}} dx = J$$

Đặt $x = 2 \sin t \Rightarrow dx = 2 \cos t dt$

$$\frac{x}{t} \quad \frac{0}{0} \quad \frac{1}{\frac{\pi}{6}} \quad \Rightarrow I = \int_0^{\frac{\pi}{6}} \frac{2 \cos t dt}{\sqrt{4 - (2 \sin t)^2}} = \int_0^{\frac{\pi}{6}} dt = \frac{\pi}{6}$$

Đặt $x = \sqrt{2} \sin t \Rightarrow dx = \sqrt{2} \cos t dt$

$$\frac{x}{t} \quad \frac{0}{0} \quad \frac{1}{\frac{\pi}{4}}$$

$$\Rightarrow J = \int_0^{\frac{\pi}{4}} \frac{\sqrt{2} \cos t dt}{\sqrt{4 - 2(\sqrt{2} \sin t)^2}} = \frac{\sqrt{2}}{2} \Big|_0^{\frac{\pi}{4}} = \frac{\pi \sqrt{2}}{8}$$

$$\Rightarrow \frac{\pi}{6} \leq \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} \leq \frac{\pi \sqrt{2}}{8}$$

Chứng minh rằng :

$$1. \frac{e-1}{e} \leq \int_0^1 e^{-x^2} dx \leq 1$$

$$3. \frac{\pi}{2} \leq \int_0^{\pi/2} \sqrt{1 + \frac{1}{2} \sin^2 x} \cdot dx \leq \frac{\pi\sqrt{6}}{4}$$

$$2. \frac{\pi}{2} \leq \int_0^{\pi/2} e^{\sin^2 x} dx \leq \frac{\pi}{2} e$$

$$4. 0.88 < \int_0^1 \frac{1}{\sqrt{1+x^4}} dx < 1$$

Bài giải :

$$1.^\circ 0 \leq x \leq 1 \Rightarrow 0 \leq x^2 \leq x \leq 1 \Rightarrow 0 < e^{-x^2} \leq e^{-x}$$

$$\Rightarrow \frac{1}{e^{x^2}} \geq \frac{1}{e^x} \Leftrightarrow e^{-x^2} \geq e^{-x} \quad (1)$$

$$^\circ x^2 \geq 0 \Rightarrow e^{-x^2} \geq e^0 = 1 \Rightarrow e^{-x^2} \leq 1 \quad (2)$$

Từ (1) và (2) suy ra : $e^{-x} \leq e^{-x^2} \leq 1$

$$\Rightarrow \int_0^1 e^{-x^2} dx \leq \int_0^1 e^{-x} dx \leq \int_0^1 1 dx \Rightarrow \frac{e-1}{e} \leq \int_0^1 e^{-x^2} dx \leq 1$$

$$2. 0 \leq \sin^2 x \leq 1 \Rightarrow 1 \leq e^{\sin^2 x} \leq e$$

$$\Rightarrow \int_0^{\pi/2} 1 dx \leq \int_0^{\pi/2} e^{\sin^2 x} dx \leq e \cdot \int_0^{\pi/2} 1 dx \Rightarrow \frac{\pi}{2} \leq \int_0^{\pi/2} e^{\sin^2 x} dx \leq \frac{\pi}{2} e$$

$$3. 0 \leq \sin^2 x \leq 1 \Rightarrow 0 \leq \frac{1}{2} \sin^2 x \leq \frac{1}{2} \Rightarrow 1 \leq \sqrt{1 + \frac{1}{2} \sin^2 x} \leq \sqrt{\frac{3}{2}}$$

$$\Rightarrow \int_0^{\pi/2} 1 dx \leq \int_0^{\pi/2} \sqrt{1 + \frac{1}{2} \sin^2 x} dx \leq \sqrt{\frac{3}{2}} \int_0^{\pi/2} 1 dx \Rightarrow \frac{\pi}{2} \leq \int_0^{\pi/2} \sqrt{1 + \frac{1}{2} \sin^2 x} \cdot dx \leq \frac{\pi\sqrt{6}}{4}$$

4. Cách 1:

$$\forall x \in (0,1) \text{ thì } x^4 < x^2 \Rightarrow 1+x^4 < 1+x^2 \Rightarrow \frac{1}{\sqrt{1+x^4}} > \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1+x^4}} dx > \int_0^1 \frac{1}{\sqrt{1+x^2}} dx = \ln \left| x + \sqrt{1+x^2} \right| \Big|_0^1 = \ln(1+\sqrt{2}) > 0,88$$

$$\text{Mặt khác : } 1+x^4 > 1 \Rightarrow \frac{1}{\sqrt{1+x^4}} < 1 \Rightarrow \int_0^1 \frac{1}{\sqrt{1+x^4}} dx < 1$$

$$\text{Vậy : } 0,88 < \int_0^1 \frac{1}{\sqrt{1+x^4}} dx < 1$$

Chú ý : hoặc sinh viên chứng minh $\int \frac{1}{\sqrt{a^2+x^2}} dx = \ln \left| x + \sqrt{x^2+a^2} \right| + C$ bằng phương pháp tích phân từng phần.

Cách 2 :

$$x \in (0,1) \Rightarrow x^4 < x^2 \Rightarrow 1+x^4 < 1+x^2$$

$$\Rightarrow \frac{1}{\sqrt{1+x^4}} > \frac{1}{\sqrt{1+x^2}} \Rightarrow \int_0^1 \frac{1}{\sqrt{1+x^4}} dx > I$$

$$\text{Vậy: } I = \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$$

$$\text{Đặt } x = \tan t \Rightarrow dx = \frac{1}{\cos^2 t} dt = (1+\tan^2 t) dt$$

$$\begin{array}{ccc} x & 0 & 1 \\ t & 0 & \frac{\pi}{4} \end{array} \quad I = \int_0^{\pi/4} \frac{(1+\tan^2 t)}{\sqrt{(1+\tan^2 t)}} dt = \int_0^{\pi/4} \frac{1}{\cos t} dt$$

$$I = \int_0^{\pi/4} \frac{\cos t}{1-\sin^2 t} dt$$

$$\text{Đặt } u = \sin t \Rightarrow du = \cos t dt \quad \begin{array}{ccc} t & 0 & \frac{\pi}{4} \\ u & 0 & \frac{1}{\sqrt{2}} \end{array}$$

$$I = \int_0^{1/\sqrt{2}} \frac{du}{1-u^2} = \frac{1}{2} \int_0^{1/\sqrt{2}} \frac{1-u+u+1}{(1-u)(1+u)} du = \frac{1}{2} \int_0^{1/\sqrt{2}} \left(\frac{1}{1+u} + \frac{1}{1-u} \right) du$$

$$= \frac{1}{2} \int_0^{1/\sqrt{2}} \frac{1}{1+u} du + \frac{1}{2} \int_0^{1/\sqrt{2}} \frac{1}{1-u} du = \frac{1}{2} \ln \left| \frac{1+u}{1-u} \right|_0^{1/\sqrt{2}}$$

$$I = \frac{1}{2} \ln \frac{2+\sqrt{2}}{2-\sqrt{2}} > 0,88 \Rightarrow \int_0^1 \frac{1}{\sqrt{1+x^4}} dx > 0,88$$

$$\text{Mặt khác: } 1+x^4 > 1 \Rightarrow \frac{1}{\sqrt{1+x^4}} < 1$$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1+x^4}} dx < \int_0^1 dx = 1 \quad (2)$$

$$\text{Từ (1) và (2) suy ra: } 0,88 < \int_0^1 \frac{1}{\sqrt{1+x^4}} dx < 1$$

Chứng minh rằng:

$$1. 0 < \int_0^{\pi/4} x \sqrt{\tan x} dx < \frac{\pi^2}{32}$$

$$2. \left| \int_0^1 \frac{\cos nx}{1+x} dx \right| \leq \ln 2$$

$$3. \left| \int_1^{\sqrt{3}} \frac{e^{-x} \cdot \sin x}{1+x^2} dx \right| < \frac{\pi}{12e}$$

$$4. \left| \int_1^{\sqrt{3}} \frac{e^{-x} \cos x}{1+x^2} dx \right| < \frac{\pi}{12e}$$

$$5. \int_{100\pi}^{200\pi} \frac{\cos x}{x} dx \leq \frac{1}{200\pi}$$

$$6. \frac{1}{n-1} \left(1 - \frac{1}{2^{n-1}} \right) \leq \int_0^1 \frac{e^x}{(1+x)^n} dx \leq \frac{e}{n-1} \left(1 - \frac{1}{2^{n-1}} \right)$$

Bài giải:

$$1. 0 \leq x \leq \frac{\pi}{4} \Rightarrow 0 \leq \tan x \leq 1 \Rightarrow 0 \leq \sqrt{\tan x} \leq 1 \Rightarrow 0 \leq x\sqrt{\tan x} \leq x$$

Chọn : $0 < \alpha < x < \beta < \frac{\pi}{4}$ ta có :

$$\left. \begin{array}{l} 0 < \tan x < 1 \\ 0 < x < \frac{\pi}{4} \end{array} \right\} \Rightarrow 0 < x\sqrt{\tan x} \leq x$$

$$I = \int_0^{\pi/4} x\sqrt{\tan x} dx = \int_0^{\alpha} x\sqrt{\tan x} dx + \int_{\alpha}^{\beta} x\sqrt{\tan x} dx + \int_{\beta}^{\pi/4} x\sqrt{\tan x} dx$$

Ta có :

$$\left. \begin{array}{l} 0 \leq \int_0^{\alpha} x\sqrt{\tan x} dx \leq \int_0^{\alpha} x dx \\ 0 < \int_{\alpha}^{\beta} x\sqrt{\tan x} dx < \int_{\alpha}^{\beta} x dx \\ 0 \leq \int_{\beta}^{\pi/4} x\sqrt{\tan x} dx \leq \int_{\beta}^{\pi/4} x dx \end{array} \right\} \Rightarrow 0 \leq \int_0^{\pi/4} x\sqrt{\tan x} dx < \int_0^{\pi/4} x dx$$

$$\Rightarrow 0 < \int_0^{\pi/4} x\sqrt{\tan x} dx < \frac{\pi^2}{32}$$

Chọn : $(\alpha, \beta) \subset [a, b]$ thì $\int_a^b f(x) dx = \int_a^{\alpha} f(x) dx + \int_{\alpha}^{\beta} f(x) dx + \int_{\beta}^b f(x) dx$

Tuy nhiên nếu : $m \leq f(x) \leq M$ thì :

$$m \int_a^b dx \leq \int_a^b f(x) dx \leq M \int_a^b dx \Rightarrow m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

Nhưng $(\alpha, \beta) \subset [a, b]$ thì $m \int_a^b dx < \int_a^b f(x) dx < M \int_a^b dx$

(Nhảy lạc phần maéc phải sai lầm phải biến nhất) Do chớ hiểu hết ý nghĩa hàm số $f(x)$ chớ (α, β) liên tục $[a, b]$ mà $(\alpha, \beta) \subset [a, b]$)

$$2. \left| \int_0^1 \frac{\cos nx}{1+x} dx \right| \leq \int_0^1 \left| \frac{\cos nx}{1+x} \right| dx = \int_0^1 \frac{|\cos nx|}{1+x} dx \leq \int_0^1 \frac{1}{1+x} dx = \ln|1+x|_0^1 = \ln 2$$

$$\Rightarrow \left| \int_0^1 \frac{\cos nx}{1+x} dx \right| \leq \ln 2$$

$$3. 1 \leq x \leq \sqrt{3} \Rightarrow \begin{cases} e^{-x} \leq e^{-1} = \frac{1}{e} \\ |\sin x| \leq 1 \end{cases}$$

$$\Rightarrow \left| \int_1^{\sqrt{3}} \frac{e^{-x} \cdot \sin x}{1+x^2} dx \right| \leq \int_1^{\sqrt{3}} \left| \frac{e^{-x} \cdot \sin x}{1+x^2} \right| dx \leq \int_1^{\sqrt{3}} \frac{\frac{1}{e}}{1+x^2} dx$$

$$\Rightarrow \left| \int_1^{\sqrt{3}} \frac{e^{-x} \cdot \sin x}{1+x^2} dx \right| \leq \frac{1}{e} \cdot I \quad \text{với } I = \int_1^{\sqrt{3}} \frac{1}{1+x^2} dx$$

$$\text{Đặt } x = \tan t \Rightarrow dx = (1 + \tan^2 t) dt$$

$$\frac{x}{t} \quad \frac{1}{\frac{\pi}{4}} \quad \frac{\sqrt{3}}{\frac{\pi}{3}} \Rightarrow I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{(1 + \tan^2 t)}{1 + \tan^2 t} dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} dt = \frac{\pi}{12}$$

$$\Rightarrow \left| \int_1^{\sqrt{3}} \frac{e^{-x} \cdot \sin x}{1+x^2} dx \right| \leq \frac{\pi}{12e} (*) \quad (\text{Cách 2 xem bài 4 dưới đây})$$

Đánh dấu ra khi :

$$\begin{cases} e^{-x} = e^{-1} \\ \sin x = 1 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ \sin x = 1 \end{cases} \Rightarrow x \in \emptyset, \forall x \in [1, \sqrt{3}]$$

$$\text{Vậy : } \left| \int_1^{\sqrt{3}} \frac{e^{-x} \cdot \sin x}{1+x^2} dx \right| < \frac{\pi}{12e}$$

Xem lại chuỗi y ở trên, đây là phản sai làm thông maéc phải không ít người nào vội kết luận đánh dấu (*) đúng. Thật vô lý

$$4. \left| \int_1^{\sqrt{3}} \frac{e^{-x} \cos x}{1+x^2} dx \right| \leq \int_1^{\sqrt{3}} \left| \frac{e^{-x} \cos x}{1+x^2} \right| dx \leq \int_1^{\sqrt{3}} \frac{e^{-x}}{1+x^2} dx$$

$$\text{Do } y = e^{-x} \text{ giảm} \Rightarrow \max(e^{-x}) = e^{-1} = \frac{1}{e}$$

$$\Rightarrow \left| \int_1^{\sqrt{3}} \frac{e^{-x} \cos x}{1+x^2} dx \right| \leq \frac{1}{e} \int_1^{\sqrt{3}} \frac{1}{1+x^2} dx = \frac{\pi}{12e} \quad ; \text{do I bài 3}$$

Đánh dấu ra khi :

$$\begin{cases} e^{-x} = e^{-1} \\ \cos x = 1 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ \cos x = 1 \end{cases} \Leftrightarrow x \in \emptyset, \forall x \in [1, \sqrt{3}]$$

$$\text{Vậy } \left| \int_1^{\sqrt{3}} \frac{e^{-x} \cos x}{1+x^2} dx \right| < \frac{\pi}{12e}$$

$$5. \text{Đặt } \begin{cases} u = \frac{1}{x} \\ dv = \cos x dx \end{cases} \Rightarrow \begin{cases} du = -\frac{1}{x^2} dx \\ v = \sin x \end{cases}$$

$$\Rightarrow \int_{100\pi}^{200\pi} \frac{\cos x}{x} dx = \frac{1}{x} \sin x \Big|_{100\pi}^{200\pi} + \int_{100\pi}^{200\pi} \frac{\sin x}{x^2} dx$$

$$\Rightarrow \int_{100\pi}^{200\pi} \frac{\cos x}{x} dx \leq \int_{100\pi}^{200\pi} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{100\pi}^{200\pi} = \frac{1}{200\pi}$$

$$\text{Vậy } \int_{100\pi}^{200\pi} \frac{\cos x}{x} dx \leq \frac{1}{200\pi}$$

Bài toán này có thể giải theo phương pháp hàm.

$$6. 0 \leq x \leq 1 \Rightarrow 1 \leq e^x \leq e \Rightarrow \frac{1}{(1+x)^n} \leq \frac{e^x}{(1+x)^n} \leq \frac{e}{(1+x)^n}$$

$$\Rightarrow \int_0^1 \frac{1}{(1+x)^n} dx \leq \int_0^1 \frac{e^x}{(1+x)^n} dx \leq e \int_0^1 \frac{1}{(1+x)^n} dx$$

$$\Leftrightarrow \frac{(x+1)^{1-n}}{1-n} \Big|_0^1 \leq \int_0^1 \frac{e^x}{(1+x)^n} dx \leq e \cdot \frac{(x+1)^{1-n}}{1-n} \Big|_0^1$$

$$\text{Vậy : } \frac{1}{n-1} \left(1 - \frac{1}{2^{n-1}} \right) \leq \int_0^1 \frac{e^x}{(1+x)^n} dx \leq \frac{e}{n-1} \left(1 - \frac{1}{2^{n-1}} \right); n > 1$$

Bài toán này có thể giải theo phương pháp nhò thòc Newton.

Chòùng minh rằng : nếu $f(x)$ và $g(x)$ là hai hàm số liên tục và xác định trên $[a, b]$, thì ta có

$$\left(\int_a^b f(x) \cdot g(x) \cdot dx \right)^2 \leq \int_a^b f^2(x) \cdot dx \cdot \int_a^b g^2(x) \cdot dx$$

Cách 1 :

Cho các số α_i , tùy ý ($i \in \overline{1, n}$) ta có

$$(\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2)(\beta_1^2 + \beta_2^2 + \dots + \beta_n^2) \geq (\alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_n\beta_n)^2 \quad (1)$$

Áp dụng (1) xảy ra khi : $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \dots = \frac{\alpha_n}{\beta_n}$

Thật vậy : phân hoạch $[a, b]$ thành n đoạn nhỏ bằng nhau đều chia :
 $a = x_0 < x_1 < x_2 < \dots < x_n = b$ và chọn :

$$\xi_i \in [x_{i-1}, x_i] = \frac{b-a}{n} \quad \forall i \in \overline{1, n}$$

Do f và g liên tục, ta có :

$$\int_a^b f^2(x) dx = \lim_{n \rightarrow +\infty} \sum_{i=1}^n f^2(\xi_i) \frac{b-a}{n} \quad (2)$$

$$\int_a^b g^2(x) dx = \lim_{n \rightarrow +\infty} \sum_{i=1}^n g^2(\xi_i) \frac{b-a}{n} \quad (3)$$

Khi đó (1)

$$\Leftrightarrow \lim_{n \rightarrow +\infty} \sum_{i=1}^n f^2(\xi_i) \frac{b-a}{n} \cdot \lim_{n \rightarrow +\infty} \sum_{i=1}^n g^2(\xi_i) \frac{b-a}{n} \\ \geq \left[\lim_{n \rightarrow +\infty} \sum_{i=1}^n f(\xi_i) \cdot g(\xi_i) \frac{b-a}{n} \right]^2 \quad (4)$$

Tổ (4) ta có:

$$\sum_{i=1}^n f^2(\xi_i) \sum_{i=1}^n g^2(\xi_i) \geq \left[\sum_{i=1}^n f(\xi_i) \sum_{i=1}^n g(\xi_i) \right]^2 \quad (5)$$

Định lý tích phân: $f(x) \cdot g(x) = k$ hay $f(x) = k \cdot g(x)$

$$\text{Tổ (5)} \Rightarrow \left(\int_a^b f(x) \cdot g(x) dx \right)^2 \leq \int_a^b f^2(x) dx \cdot \int_a^b g^2(x) dx$$

Chứng minh: $\forall t \in \mathbb{R}^+$ ta có:

$$0 \leq [tf(x) - g(x)]^2 = t^2 f^2(x) - 2t \cdot f(x) \cdot g(x) + g^2(x) \\ \Rightarrow h(t) = t^2 \int_a^b f^2(x) dx - 2t \int_a^b f(x) \cdot g(x) dx + \int_a^b g^2(x) dx \geq 0$$

h(t) là 1 tam thức bậc 2 luôn không âm nên cần phải có điều kiện:

$$\begin{cases} a_h = t^2 > 0 \\ \Delta_h \leq 0 \end{cases} \Leftrightarrow \Delta'_h \leq 0$$

$$\Leftrightarrow \left[\int_a^b f(x) \cdot g(x) dx \right]^2 - \int_a^b f^2(x) dx \cdot \int_a^b g^2(x) dx \leq 0$$

$$\Rightarrow \left(\int_a^b f(x) \cdot g(x) dx \right)^2 \leq \int_a^b f^2(x) dx \cdot \int_a^b g^2(x) dx$$

Chứng minh:

$$1. \int_0^1 \sqrt{1+x^3} dx < \frac{\sqrt{5}}{2}$$

$$3. e^x - 1 < \int_0^x \sqrt{e^{2t} + e^{-t}} dt < \sqrt{(e^x - 1) \left(e^x - \frac{1}{2} \right)}$$

$$2. \int_0^1 e^{\sin^2 x} dx > \frac{3\pi}{2}$$

$$4. \left| \int_0^1 \frac{3\cos x - 4\sin x}{1+x^2} dx \right| \leq \frac{5\pi}{4}$$

Bài tập:

1. Chứng minh: $\left(\int_a^b f(x) \cdot g(x) dx \right)^2 \leq \int_a^b f^2(x) dx \cdot \int_a^b g^2(x) dx$ (định lý tích phân)

$$\Rightarrow \left| \int_a^b f(x) \cdot g(x) dx \right| \leq \sqrt{\int_a^b f^2(x) dx} \cdot \sqrt{\int_a^b g^2(x) dx}$$

$$\sqrt{1+x^3} = \sqrt{(1+x) \cdot (1-x+x^2)} = \sqrt{1+x} \cdot \sqrt{1-x+x^2}$$

$$\Rightarrow \int_0^1 \sqrt{1+x^3} dx = \int_0^1 \sqrt{1+x} \sqrt{1-x+x^2} dx < \sqrt{\int_0^1 (1+x) dx} \sqrt{\int_0^1 (x^2 - x + 1) dx}$$

$$\int_0^1 \sqrt{1+x^3} dx < \sqrt{\left(\frac{x^2}{2} + x\right)} \bigg|_0^1 \sqrt{\left(\frac{x^3}{3} - \frac{x^2}{2} + x\right)} \bigg|_0^1 = \frac{\sqrt{5}}{2}$$

$$\Rightarrow \int_0^1 \sqrt{1+x^3} dx < \frac{\sqrt{5}}{2}$$

$$2. \int_0^\pi e^{\sin^2 x} dx = \int_0^{\pi/2} e^{\sin^2 x} dx + \int_{\pi/2}^\pi e^{\sin^2 x} dx$$

Đặt $t = \frac{x}{2} + t \Rightarrow dx = dt$

x	$\frac{\pi}{2}$	π
t	0	$\frac{\pi}{2}$

$$\Rightarrow \int_0^\pi e^{\sin^2 x} dx = \int_0^{\pi/2} e^{\sin^2 x} dx + \int_0^{\pi/2} e^{\sin^2(\frac{\pi}{2}+t)} dt$$

$$= \int_0^{\pi/2} e^{\sin^2 x} dx + \int_0^{\pi/2} e^{\cos^2 x} dx = 2 \int_0^{\pi/2} e^{\sin^2 x} dx$$

Ta lại có $\left(\int_0^{\pi/2} \sqrt{e} dx \right)^2 = \left(\int_0^{\pi/2} e^{\sin^2 x/2} \cdot e^{\cos^2 x/2} dx \right)^2$

$$< \int_0^{\pi/2} e^{\sin^2 x} dx \cdot \int_0^{\pi/2} e^{\cos^2 x} dx$$

hay $\left(\int_0^{\pi/2} \sqrt{e} dx \right)^2 < \left(\int_0^{\pi/2} e^{\sin^2 x} dx \right)^2 \Rightarrow \int_0^{\pi/2} \sqrt{e} dx < \int_0^{\pi/2} e^{\sin^2 x} dx$

$$\Rightarrow \int_0^\pi e^{\sin^2 x} dx > \frac{1}{2} \sqrt{e} \bigg|_0^\pi = \pi \sqrt{e}; \left(\sqrt{e} > \frac{3}{2} \right)$$

$$\Rightarrow \int_0^\pi e^{\sin^2 x} dx > \frac{3}{2}$$

Chú ý : bài này có thể giải theo phương pháp nhân đôi.

$$3. \int_0^x \sqrt{e^{2t} + e^{-t}} dt = \int_0^x e^{t/2} \sqrt{e^t + e^{-2t}} dt$$

$$\left(\int_0^x e^{t/2} \sqrt{e^t + e^{-2t}} dt \right)^2 \leq \int_0^x e^t dt \int_0^x (e^t + e^{-2t}) dt$$

$$\text{vì } \left(\int_a^b f(x) \cdot g(x) dx \right)^2 \leq \int_a^b f^2(x) dx \cdot \int_a^b g^2(x) dx$$

$$\Rightarrow \left(\int_0^x \sqrt{e^{2t} + e^{-t}} dt \right)^2 \leq (e^x - 1) \left(e^x - \frac{1}{2} - \frac{1}{e^{2x}} \right) < (e^x - 1) \left(e^x - \frac{1}{2} \right)$$

$$\Rightarrow \int_0^1 \sqrt{e^{2t} + e^{-t}} dt \leq \sqrt{(e^x - 1) \left(e^x - \frac{1}{2} \right)} \quad (1)$$

Mặt khác : $\sqrt{e^{2t} + e^{-t}} > e^t ; \forall 0 < t < x$

$$\Rightarrow \int_0^x \sqrt{e^{2t} + e^{-t}} dt > \int_0^x e^t dt = e^x - 1 \quad (2)$$

$$\text{Tổ hợp (1) và (2) suy ra : } e^x - 1 < \int_0^x \sqrt{e^{2t} + e^{-t}} dt < \sqrt{(e^x - 1) \left(e^x - \frac{1}{2} \right)}$$

$$4. \left| \frac{3 \cos x - 4 \sin x}{1 + x^2} \right| \leq \frac{1}{1 + x^2} \sqrt{[3^2 + (-4)^2] [\sin^2 x + \cos^2 x]} = \frac{5}{x^2 + 1}$$

$$\Rightarrow \left| \int_0^1 \frac{3 \cos x - 4 \sin x}{1 + x^2} dx \right| \leq \int_0^1 \left| \frac{3 \cos x - 4 \sin x}{1 + x^2} \right| dx \leq 5 \int_0^1 \frac{1}{1 + x^2} dx$$

$$\text{Đặt } x = \tan t \Rightarrow dx = (1 + \tan^2 t) dt$$

$$\begin{array}{ccc} x & 0 & 1 \\ t & 0 & \frac{\pi}{4} \end{array} \Rightarrow \int_0^1 \frac{1}{1 + x^2} dx = \int_0^{\frac{\pi}{4}} \frac{(1 + \tan^2 t)}{1 + \tan^2 t} dt = \int_0^{\frac{\pi}{4}} dt = \frac{\pi}{4}$$

$$\Rightarrow 4. \left| \int_0^1 \frac{3 \cos x - 4 \sin x}{1 + x^2} dx \right| \leq \frac{5\pi}{4}$$

Chứng minh bất đẳng thức tích phân bằng phương pháp hàm.

Chứng minh rằng :

$$1. 54\sqrt{2} \leq \int_{-7}^{11} (\sqrt{x+7}) + (\sqrt{11-x}) dx \leq 108 \quad \frac{\pi}{4} \leq \int_0^{\pi/4} (\sin x + \cos x) dx \leq \frac{\pi\sqrt{2}}{4}$$

$$2. 0 < \int_0^1 x(1-x^2) dx < \frac{4}{27} \quad 4. \int_0^e e^{\sin^2 x} dx > \frac{3\pi}{2}$$

Bài giải :

$$1. \text{ Xét } f(x) = (\sqrt{x+7}) + (\sqrt{11-x}); x \in [-7, 11]$$

$$f'(x) = \frac{\sqrt{11-x} - \sqrt{x+7}}{2\sqrt{11-x}\sqrt{x+7}} \Rightarrow f'(x) = 0 \Leftrightarrow x = 2$$

x	-7	2	11
f'(x)	+	0	-
f(x)		6	
	$3\sqrt{2}$		$3\sqrt{2}$

$$\Rightarrow 3\sqrt{2} \leq f(x) \leq 6 \Rightarrow 3\sqrt{2} \int_{-7}^{11} dx \leq \int_{-7}^{11} f(x) dx \leq 6 \int_{-7}^{11} dx$$

$$\Rightarrow 54\sqrt{2} \leq \int_{-7}^{11} (\sqrt{x+7} + \sqrt{11-x}) dx \leq 108$$

2. Xét hàm số: $f(x) = x(1-x^2)$; $\forall x \in [0,1]$ $\Rightarrow f'(x) = 3x^2 - 4x + 1$

$$\Rightarrow f'(x) = 0 \Leftrightarrow x = \frac{1}{3} \vee x = 1$$

x	$-\infty$	0	$\frac{1}{3}$	1	$+\infty$
$f'(x)$		+	0	-	
$f(x)$			$\frac{4}{27}$	0	

$$\Rightarrow 0 \leq f(x) \leq \frac{4}{27}$$

$$\text{và } \begin{cases} \exists x \in (0, \frac{1}{3}); (\frac{1}{3}, 1) \Rightarrow 0 < f(x) < \frac{4}{27} \\ f(0) = f(1) = 0 \end{cases}$$

$$\Rightarrow 0 < \int_0^1 f(x) dx < \frac{4}{27} \int_0^1 dx \Rightarrow 0 < \int_0^1 f(x) dx < \frac{4}{27}$$

3. Xét hàm số:

$$f(x) = \sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right); x \in \left[0, \frac{\pi}{4}\right]$$

$$f'(x) = \sqrt{2} \cos\left(x + \frac{\pi}{4}\right) \geq 0, \forall x \in \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow f(x) \text{ là hàm số tăng } \forall x \in \left[0, \frac{\pi}{4}\right] \Rightarrow f(0) \leq f(x) \leq f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow 1 \leq \sin x + \cos x \leq \sqrt{2} \Rightarrow \frac{\pi}{4} \leq \int_0^{\pi/4} (\sin x + \cos x) dx \leq \frac{\pi\sqrt{2}}{4}$$

4. Nhận xét $\forall x > 0$ thì $e^x > 1 + x$ (chứng minh bất đẳng thức bằng pp đạo hàm)

$$\text{Xét } f(t) = e^t - 1 - t; t \geq 0 \Rightarrow f'(t) = e^t - 1 > 0; \forall t > 0$$

$$\Rightarrow \text{hàm số } f(t) \text{ đồng biến } \forall t \geq 0$$

$$\text{Vì } x > 0 \text{ nên } f(x) > f(0) = 0 \Rightarrow e^x - 1 - x > 0 \Leftrightarrow e^x > 1 + x \quad (1)$$

$$\text{Do vậy: } \forall x \in (0, \pi) \text{ thì } e^{\sin^2 x} > 1 + \sin^2 x \quad (\text{do (1)})$$

$$\Rightarrow \int_0^{\pi} e^{\sin^2 x} dx > \int_0^{\pi} (1 + \sin^2 x) dx = \pi + \int_0^{\pi} \frac{1 - \cos 2x}{2} dx$$

$$\Rightarrow \int_0^{\pi} e^{\sin^2 x} dx > \frac{3\pi}{2}$$

Chứng minh rằng :

$$1. \frac{2}{5} \leq \int_1^2 \frac{x}{x^2+1} dx \leq \frac{1}{2}$$

$$4. \frac{\sqrt{3}}{12} \leq \int_{\pi/6}^{\pi/3} \frac{\cot gx}{x} dx \leq \frac{1}{3}$$

$$2. \frac{\sqrt{3}}{4} \leq \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx \leq \frac{1}{2}$$

$$5. \frac{2}{3} < \int_0^1 \frac{1}{\sqrt{2+x-x^2}} dx < \frac{1}{\sqrt{2}}$$

$$3. \frac{\pi\sqrt{3}}{3} \leq \int_0^{\pi} \frac{1}{\sqrt{\cos^2 x + \cos x + 1}} dx \leq \frac{2\pi\sqrt{3}}{3}$$

$$6. 2\sqrt[4]{2} < \int_{-1}^1 (\sqrt[4]{1+x} + \sqrt[4]{1-x}) dx < 4$$

Bài giải :

1. Xét: $f(x) = \frac{x}{x^2+1}$; $x \in [1, 2]$. có $f'(x) = \frac{1-x^2}{(1+x^2)^2} \leq 0$; $\forall x \in [1, 2]$

$\Rightarrow$ hàm số nghịch biến $\forall x \in [1, 2] \Rightarrow f_{(2)} \leq f_{(x)} \leq f_{(1)}$

$$\Rightarrow \frac{2}{5} \leq \frac{x}{x^2+1} \leq \frac{1}{2} \Rightarrow \frac{2}{5} \int_1^2 dx \leq \int_1^2 \frac{x}{x^2+1} dx \leq \frac{1}{2} \int_1^2 dx$$

$$\Rightarrow \frac{2}{5} \leq \int_1^2 \frac{x}{x^2+1} dx \leq \frac{1}{2}$$

2. Xét $f(x) = \frac{\sin x}{x}$; $\forall x \in \left[\frac{\pi}{6}; \frac{\pi}{3}\right] \Rightarrow f'(x) = \frac{x \cdot \cos x - \sin x}{x^2}$

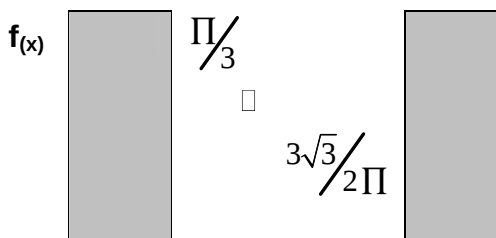
Đặt $Z = x \cdot \cos x - \sin x \Rightarrow Z' = -x$; $x < 0$; $\forall x \in \left[\frac{\pi}{6}; \frac{\pi}{3}\right]$

$\Rightarrow Z$ đồng biến trên $\forall x \in \left[\frac{\pi}{6}; \frac{\pi}{3}\right]$ **ta có**

$$Z \leq Z_{(\pi/3)} = \frac{\pi - 3\sqrt{3}}{6} < 0; \forall x \in \left[\frac{\pi}{6}; \frac{\pi}{3}\right]$$

$$\Rightarrow f'(x) < 0; \forall x \in \left[\frac{\pi}{6}; \frac{\pi}{3}\right]$$

x	$-\infty$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$+\infty$
f'(x)			-	



$$\Rightarrow \frac{3\sqrt{3}}{2\Pi} \leq f_{(x)} \leq \frac{3}{\Pi}$$

$$\text{hay: } \frac{3\sqrt{3}}{2\Pi} \leq \frac{\sin x}{x} \leq \frac{3}{\Pi}$$

$$\Rightarrow \frac{3\sqrt{3}}{2\Pi} \int_{\Pi/6}^{\Pi/3} dx \leq \int_{\Pi/6}^{\Pi/3} \frac{\sin x}{x} dx \leq \frac{3}{\Pi} \int_{\Pi/6}^{\Pi/3} dx \Rightarrow \frac{\sqrt{3}}{4} \leq \int_{\Pi/6}^{\Pi/3} \frac{\sin x}{x} dx \leq \frac{1}{2}$$

3. Xét $t = \cos x; x \in [0, \Pi] \Rightarrow t \in [-1, 1]$

$$\text{vào } f_{(t)} = t^2 + t + 1; t \in [-1, 1]$$

$$f'_{(t)} = 2t + 1; f'_{(t)} = 0 \Leftrightarrow t = -\frac{1}{2}$$

t	$-\infty$	-1	$-\frac{1}{2}$	1	$+\infty$
$f'_{(t)}$		-	0	+	
$f_{(t)}$		1	$\frac{3}{4}$	3	

$$\Rightarrow \frac{3}{4} \leq f_{(t)} \leq 3; \forall t \in [-1, 1]$$

$$\Rightarrow \frac{3}{4} \leq \cos^2 x + \cos x + 1 \leq 3; \forall x \in [0, \Pi]$$

$$\text{hay } \frac{\sqrt{3}}{2} \leq \sqrt{\cos^2 x + \cos x + 1} \leq \sqrt{3} \Rightarrow \frac{1}{\sqrt{3}} \leq \frac{1}{\sqrt{\cos^2 x + \cos x + 1}} \leq \frac{2}{\sqrt{3}}$$

$$\Rightarrow \frac{1}{\sqrt{3}} \int_0^{\Pi} dx \leq \int_0^{\Pi} \frac{1}{\sqrt{\cos^2 x + \cos x + 1}} dx \leq \frac{2}{\sqrt{3}} \int_0^{\Pi} dx$$

$$\Rightarrow \frac{\Pi\sqrt{3}}{3} \leq \int_0^{\Pi} \frac{1}{\sqrt{\cos^2 x + \cos x + 1}} dx \leq \frac{2\Pi\sqrt{3}}{3}$$

Chú ý : thối chất bất đẳng thức trên phải là :

$$\frac{\Pi\sqrt{3}}{3} < \int_0^{\Pi} \frac{1}{\sqrt{\cos^2 x + \cos x + 1}} dx < \frac{2\Pi\sqrt{3}}{3} \text{ (hoặc sinh tởi giới thích vì sao)}$$

$$4. f(x) = \frac{\cot gx}{x}; \text{ liên tục } \forall x \in \left[\frac{\pi}{4}; \frac{\pi}{3} \right]$$

$$\text{còn } f'(x) = \frac{-(2x + \sin 2x)}{2x^2 \sin^2 x} < 0; \forall x \in \left(\frac{\pi}{4}; \frac{\pi}{3} \right) \Rightarrow f(x) : \text{nghịch biến trên } \left[\frac{\pi}{4}; \frac{\pi}{3} \right]$$

$$\Rightarrow f\left(\frac{\pi}{3}\right) \leq f(x) \leq f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\sqrt{3}}{\pi} \leq \frac{\cot gx}{x} \leq \frac{4}{\pi} \Rightarrow \frac{\sqrt{3}}{\pi} \int_{\pi/4}^{\pi/3} dx \leq \int_{\pi/4}^{\pi/3} \frac{\cot gx}{x} dx \leq \frac{4}{\pi} \int_{\pi/4}^{\pi/3} dx$$

$$\Rightarrow \frac{\sqrt{3}}{12} \leq \int_{\pi/4}^{\pi/3} \frac{\cot gx}{x} dx \leq \frac{1}{3}$$

$$5. f(x) = 2 + x - x^2; \forall x \in [0, 1] \text{ còn } f'(x) = 1 - 2x$$

$$\Rightarrow f'(x) = 0 \Leftrightarrow x = \frac{1}{2}$$

x	$-\infty$	0	$\frac{1}{2}$	1	$+\infty$
$f'(x)$		+	0	-	
$f(x)$			$\frac{9}{4}$		
			$\square$	$\square$	
		2		2	

$$\Rightarrow 2 \leq f(x) \leq \frac{9}{4}$$

$$\text{và } \begin{cases} \exists x \in (0, \frac{1}{2}); (\frac{1}{2}, 1) \Rightarrow 2 < f(x) < \frac{9}{4} \\ f(0) = f(1) = 2 \end{cases}$$

$$\Rightarrow 2 < 2 + x - x^2 < \frac{9}{4} \Rightarrow \frac{2}{3} < \frac{1}{\sqrt{2+x-x^2}} < \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{2}{3} \int_0^1 dx < \int_0^1 \frac{1}{\sqrt{2+x-x^2}} dx < \frac{1}{\sqrt{2}} \int_0^1 dx$$

$$\Rightarrow \frac{2}{3} < \int_0^1 \frac{1}{\sqrt{2+x-x^2}} dx < \frac{1}{\sqrt{2}}$$

6. Xét :

$$f(x) = \sqrt[4]{1+x} + \sqrt[4]{1-x} \quad ; x \in [-1, 1]$$

$$f'(x) = \frac{1}{4} \left(\frac{1}{\sqrt[4]{(1+x)^3}} - \frac{1}{\sqrt[4]{(1-x)^3}} \right)$$

$$f'(x) = 0 \Leftrightarrow \sqrt[4]{(1-x)^3} = \sqrt[4]{(1+x)^3} \Leftrightarrow x = 0$$

Ma trận biến thiên : $f'(x) > 0 \Leftrightarrow \frac{1}{\sqrt[4]{(1+x)^3}} > \frac{1}{\sqrt[4]{(1-x)^3}} \Leftrightarrow -1 < x < 0$

x	$-\infty$	-1	0	1	$+\infty$
$f'(x)$		+	0	-	
$f(x)$			2		
			$\square$	$\square$	
		$\sqrt[4]{2}$		$\sqrt[4]{2}$	

$$\Rightarrow \sqrt[4]{2} \leq f(x) \leq 2$$

$$va \begin{cases} \exists x \in (-1, 0); (0, 1) \\ f_{(-1)} = f_{(1)} = \sqrt[4]{2} \end{cases} \Rightarrow \sqrt[4]{2} < f(x) < 2$$

$$\Rightarrow \sqrt[4]{2} \int_{-1}^1 dx < \int_{-1}^1 (\sqrt[4]{1+x} + \sqrt[4]{1-x}) dx < 2 \int_{-1}^1 dx \Rightarrow 2\sqrt[4]{2} < \int_{-1}^1 (\sqrt[4]{1+x} + \sqrt[4]{1-x}) dx < 4$$

Chứng minh rằng :

$$1. 2e^{-2} \leq \int_0^2 e^{x-x^2} dx \leq 2\sqrt[4]{e}$$

$$2. \int_{100}^{200} e^{-x^2} dx < 0,005$$

$$3. 90 - \ln 10 \leq \int_{10}^{100} e^{\frac{1}{x}} dx < 90 + \frac{9}{200} + \ln 10$$

$$4. 9 \leq \int_0^{\pi/3} \left(\frac{3}{\cos^4 x} - 2 \tan^4 x \right) dx \leq 90$$

$$5. \int_0^1 e^{\frac{1}{x^2+1}} dx \geq 1 + \frac{\pi}{4}$$

$$6. \int_0^{\pi/2} \frac{\tan^{x/2} x}{x} dx < 1$$

Bài giải :

$$1. \text{ Xét } f(x) = x - x^2 \quad ; x \in [0, 2] \text{ có } f'(x) = 1 - 2x$$

$$\text{có } f'(x) = 0 \Leftrightarrow x = \frac{1}{2}$$

x	$-\infty$	0	$\frac{1}{2}$	$2+\infty$
$f'(x)$		+	0	-
$f(x)$			$\frac{1}{4}$	
		□	□	
		0	-2	

$$\Rightarrow -2 \leq f(x) \leq \frac{1}{4}$$

$$\text{hay } -2 \leq x - x^2 \leq \frac{1}{4}$$

$$\Rightarrow e^{-2} \leq e^{x-x^2} \leq e^{1/4} = \sqrt[4]{e} \Rightarrow e^{-2} \leq \int_0^2 dx \leq \int_0^2 e^{x-x^2} dx \leq \sqrt[4]{e} \int_0^2 dx$$

$$2.e^{-2} \leq \int_0^2 e^{x-x^2} dx \leq 2.\sqrt[4]{e}$$

$$\text{Chuẩn y: } \text{thời gian bất đẳng thức trên là: } 2.e^{-2} < \int_0^2 e^{x-x^2} dx < 2.\sqrt[4]{e}$$

$$2. \text{ Tröôùc heát ta chöùng minh : } e^{-x^2} \leq \frac{1}{x^2}; (1) x \neq 0$$

$$\text{Ñaët } t = x^2; x \neq 0 \Rightarrow t > 0$$

$$\text{Giaû söù ta coù (1) vaø (1)} \Leftrightarrow e^{-t} \leq \frac{1}{t}; t > 0 \Leftrightarrow e^t \geq t; t > 0$$

$$\Leftrightarrow e^t - t \geq 0 (2); t > 0$$

$$\text{Ñaët } f(x) = e^t - t \text{ co } f'(t) = e^t - 1 > 0, t > 0$$

$$\Rightarrow f(t) \text{ luôn ñoàng bieán } \forall t > 0 \text{ vaø } f(t) \geq f(0) = 1 > 0$$

$$\Rightarrow f(t) \geq 0, t > 0 \Rightarrow e^{-x^2} \leq \frac{1}{x^2} \Rightarrow \int_{100}^{200} e^{-x^2} dx \leq \int_{100}^{200} \frac{1}{x^2} dx$$

$$\Rightarrow \int_{100}^{200} e^{-x^2} dx < 0,005$$

$$3. \text{ Tröôùc heát ta chöùng minh : } 1 - \frac{1}{x} \leq e^{-1/x} \leq 1 - \frac{1}{x} + \frac{1}{2x^2}; (1) \forall x > 0$$

$$\text{Ñaët } t = -\frac{1}{x}; x > 0 \Rightarrow t < 0$$

$$(1) \Leftrightarrow 1+t \leq e^t \leq 1+t+\frac{1}{2}t^2; (2) t < 0$$

$$\text{Xeùt haøm soá } f(t) = e^t - t - 1; h(t) = e^t - 1 - t - \frac{1}{2}t^2; t < 0$$

$$\circ f'(t) = e^t - 1$$

$$t \quad | \quad -\infty \quad 0 \quad +\infty$$

$f'(t)$	—	
$f(t)$	$+\infty$	
	$\square$	
	0	

$$\Rightarrow f(t) > 0 \quad ; \forall t < 0$$

$$\text{hay } e^t - 1 - t > 0 \quad ; \forall t < 0$$

$$\Rightarrow 1 + t < e^t \quad ; \forall t < 0 \quad (3)$$

$$\bullet h'(t) = e^t - 1 - t$$

x	$-\infty$	0	$+\infty$
$h'(t)$			+
$h(t)$			0
		$\square$	

$$\Rightarrow h(t) < 0 \quad ; \forall t < 0$$

$$\text{hay } e^t < 1 + t + \frac{1}{2} > 0 \quad ; \forall t < 0 \quad (4)$$

Tõ (3) và (4) suy ra :

$$1 + t \leq e^t \leq 1 + t + \frac{1}{2}t^2 \quad ; \forall t < 0$$

$$\text{hay } 1 - \frac{1}{x} \leq e^{-1/x} \leq 1 - \frac{1}{x} + \frac{1}{2x^2} \quad ; x > 0$$

$$\Rightarrow \int_{10}^{100} \left(1 - \frac{1}{x} \right) dx \leq \int_{10}^{100} e^{-1/x} dx \leq \int_{10}^{100} \left(1 - \frac{1}{x} + \frac{1}{2x^2} \right) dx$$

$$90 - \ln 10 \leq \int_{10}^{100} e^{-1/x} dx < 90 + \frac{9}{200} + \ln 10$$

* Làm bài toán khó, hi vọng các em tìm nhiều thủ thuật trong bài toán trên – chúc thành công.

$$4. \text{ Xét } f(x) = \frac{3}{\cos^4 x} - 2tg^4 x \quad ; x \in \left[0, \frac{\pi}{3} \right]$$

$$\text{Nhật } t = \frac{1}{\cos^2 x} = 1 + tg^2 x \quad ; x \in x \in \left[0, \frac{\pi}{3} \right] \Rightarrow t \in [1; 4]$$

$$\Rightarrow f_{(t)} = t^2 + 4t - 2 \Rightarrow f'_{(t)} = 4t + 4 > 0 ; \forall t \in [1, 4]$$

$$\Rightarrow f_{(1)} \leq f_{(t)} \leq f_{(4)} \Rightarrow 3 \leq f_{(t)} \leq 30$$

$$\Rightarrow 3 \int_1^4 dt \leq \int_1^4 f_{(t)} dt \leq 30 \int_1^4 dt$$

$$\Rightarrow 9 \leq \int_0^{\pi/3} \left(\frac{3}{\cos^4} - 2 \tan^4 x \right) dx \leq 90$$

5. Xét hàm số $f_{(x)} = e^x - 1 - x ; \forall x \geq 0$

ta có $f'_{(x)} = e^x - 1 > 0, \forall x \geq 0 \Rightarrow f_{(x)}$ đồng biến $\forall x \in [0, +\infty)$

$$\Rightarrow f_{(x)} \geq f_{(0)} = 0 \Rightarrow e^x - 1 - x \geq 0 \Rightarrow e^x \geq 1 + x ; \forall x \geq 0$$

$$\Rightarrow e^{\frac{1}{1+x^2}} \geq 1 + \frac{1}{1+x^2} ; \forall x \geq 0$$

$$\Rightarrow \int_0^1 e^{\frac{1}{1+x^2}} dx \geq \int_0^1 \left(1 + \frac{1}{1+x^2} \right) dx = 1 + \int_0^1 \frac{1}{1+x^2} dx \quad (*)$$

Đặt $x = \tan t \Rightarrow dx = (1 + \tan^2 t) dt$

$$\begin{cases} x=0 \\ x=1 \end{cases} \Rightarrow \begin{cases} t=0 \\ t=\pi/4 \end{cases} \Rightarrow \int_0^1 \frac{1}{1+x^2} dx = \int_0^{\pi/4} \frac{(1+\tan^2 t) dt}{1+\tan^2 t} = \frac{\pi}{4}$$

Tõ (*) suy ra : $\int e^{\frac{1}{x^2+1}} dx \geq 1 + \frac{\pi}{4}$

6. Chứng minh : $\frac{\tan \frac{x}{2}}{x} < \frac{2}{\pi} ; x \in \left(0, \frac{\pi}{2} \right)$

Xét hàm số $f_{(x)} = \frac{1}{x} \cdot \tan \frac{x}{2} ; x \in \left(0, \frac{\pi}{2} \right)$

$$f'_{(x)} = \frac{x - \sin x}{2x^2 \cdot \cos^2 \frac{x}{2}}$$

Đặt $Z = x - \sin x \Rightarrow Z' = 1 - \cos x > 0, \forall x \in \left(0, \frac{\pi}{2} \right)$

$$\Rightarrow Z > Z_{(0)} = 0 \Rightarrow f'_{(x)} > 0, \forall x \in \left(0, \frac{\pi}{2} \right)$$

x	$-\infty$	0	$\frac{\pi}{2}$	$+\infty$
$f'_{(x)}$			+	
$f_{(x)}$			$\frac{2}{\pi}$	
		$-\infty$		

$$\Rightarrow f_{(x)} < \frac{2}{\pi} \Rightarrow \frac{\tan \frac{x}{2}}{x} < \frac{2}{\pi}$$

$$\Rightarrow \int_0^{\pi/2} \frac{\tan \frac{x}{2}}{x} dx < \int_0^{\pi/2} \frac{2}{\pi} dx \Rightarrow \int_0^{\pi/2} \frac{\tan \frac{x}{2}}{x} dx < 1$$

Chứng minh rằng :

$$1. \frac{1}{2} \int_0^{\pi} x^{1999} \cdot e^{2x} \cdot dx > \frac{\pi^{2001}}{2001} + \frac{\pi^{2001}}{2002}$$

$$2. \int_0^1 x \ln(x + \sqrt{1+x^2}) dx \geq \frac{1}{2} \ln(1 + \sqrt{2}) + \frac{\sqrt{2}}{2} - 1$$

$$3. \int_0^{\pi/4} x \tan^n x dx \geq \frac{1}{n+2} \left(\frac{\pi}{4} \right)^{n+2}$$

Bài giải :

1. Trước hết ta chứng minh : $e^{2x} > 2(x^2 + x) ; \forall x > 0$

Xét hàm số:

$$f_{(x)} = e^{2x} - 2(x^2 + x) ; \forall x > 0$$

$$f'_{(x)} = 2e^{2x} - 4x - 2 ; f''_{(x)} = 4e^{2x} - 4 > 0 ; \forall x > 0$$

$$\Rightarrow f'_{(x)} \text{ là hàm tăng ; } \forall x > 0 \Rightarrow f'_{(x)} > f'_{(0)} = 0$$

$$\Rightarrow f_{(x)} \text{ là hàm tăng ; } \forall x > 0 \Rightarrow f_{(x)} > f_{(0)}$$

$$\Rightarrow e^{2x} > 2(x^2 + x) \Rightarrow x^{1999} \cdot e^{2x} > 2 \cdot x^{1999} (x^2 + x)$$

$$\Rightarrow \frac{1}{2} \int_0^{\pi} x^{1999} \cdot e^{2x} dx > \int_0^{\pi} x^{1999} (x^2 + x) dx$$

$$\Rightarrow \frac{1}{2} \int_0^{\pi} x^{1999} \cdot e^{2x} \cdot dx > \frac{\pi^{2001}}{2001} + \frac{\pi^{2001}}{2002}$$

2. Trước hết ta chứng minh : $1 + x \ln(x + \sqrt{1+x^2}) \geq \sqrt{1+x^2} ; \forall x \in \mathbb{R}$

Xét hàm số :

$$f_{(x)} = 1 + x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2}$$

$$f'_{(x)} = \ln(x + \sqrt{1+x^2}) \Rightarrow f'_{(x)} = 0 \Leftrightarrow x + \sqrt{1+x^2} = 1$$

$$\Leftrightarrow \begin{cases} 1-x \geq 0 \\ 1+x^2 = (1-x)^2 \end{cases} \Leftrightarrow x = 0$$

$$\text{và } f'_{(x)} < 0 \Leftrightarrow \ln(x + \sqrt{1+x^2}) < 0 \Leftrightarrow x < 0$$

x	$-\infty$	0	$+\infty$
$f'_{(x)}$	$-$	0	$+$
$f_{(x)}$	$\square$	$\square$	
		0	

$$\Rightarrow f_{(x)} \geq f_{(0)} = 0 ; \forall x \in \mathbb{R}$$

$$\Rightarrow 1 + x \ln(x + \sqrt{1+x^2}) \geq \sqrt{1+x^2}$$

$$\Rightarrow x \ln(x + \sqrt{1+x^2}) \geq \sqrt{1+x^2} - 1$$

$$\Rightarrow \int_0^1 x \ln(x + \sqrt{1+x^2}) dx \geq \int_0^1 (\sqrt{1+x^2} - 1) dx = \frac{1}{2} \left[x\sqrt{x^2+1} + \frac{1}{2} \ln(x + \sqrt{1+x^2}) - x \right]_0^1$$

$$\Rightarrow \int_0^1 x \ln(x + \sqrt{1+x^2}) dx \geq \frac{1}{2} \ln(1 + \sqrt{2}) + \frac{\sqrt{2}}{2} - 1$$

$$\text{3. Xét } f_{(x)} = tgx - x ; \forall x \in \left[0, \frac{\pi}{4}\right]$$

$$f'_{(x)} = \frac{1}{\cos^2 x} - 1 = tg^2 x > 0 ; \forall x \in \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow f_{(x)} \text{ đồng biến trên } \left[0, \frac{\pi}{4}\right] \Rightarrow f_{(x)} \geq f_{(0)} = 0$$

$$\Rightarrow tgx \geq x ; \forall x \in \left[0, \frac{\pi}{4}\right] \Rightarrow tg^n x \geq x^n$$

$$\Rightarrow xtg^n x \geq x^{n+1} \Rightarrow \int_0^{\pi/4} xtg^n x dx \geq \int_0^{\pi/4} x^{n+1} dx$$

$$\Rightarrow \int_0^{\pi/4} xtg^n x dx \geq \frac{1}{n+2} \left(\frac{\pi}{4}\right)^{n+2}$$

Gia sử $f(x)$ có đạo hàm liên tục trên $[0,1]$ và $f(1) - f(0) = 1$

Chứng minh rằng : $\int_0^1 (f'_{(x)})^2 dx \geq 1$

Ta có : $\int_0^1 (f'_{(x)} - 1)^2 dx \geq 0 ; \forall x \in [0,1]$

$$\Rightarrow \int_0^1 (f'_{(x)})^2 dx - 2 \int_0^1 f'_{(x)} dx \geq 1 + \int_0^1 dx \geq 0 \Leftrightarrow \int_0^1 (f'_{(x)})^2 dx - 2[f_{(1)} - f_{(0)}] + 1 \geq 0$$

$$\Leftrightarrow \int_0^1 (f'_{(x)})^2 dx - 2 + 1 \geq 0 \Rightarrow \int_0^1 (f'_{(x)})^2 dx \geq 1$$

Cho f là hàm liên tục trên $[0;1]$ thỏa mãn

$$\begin{cases} 1 \leq f(x) \leq 2; \forall x \in [0,1] & (a) \\ \int_0^1 f(x) dx = \frac{3}{2} & (b) \end{cases}$$

Chứng minh $\frac{2}{3} \leq \int_0^1 \frac{1}{f(x)} dx < \frac{3}{4}$

Theo BNT Bunhiacovsky

$$1 \leq \left(\int_0^1 1 \cdot dx \right)^2 = \left(\int_0^1 \sqrt{f(x)} \cdot \frac{1}{\sqrt{f(x)}} dx \right)^2 \leq \int_0^1 f(x) dx \cdot \int_0^1 \frac{dx}{f(x)}$$

$$= \frac{3}{2} \int_0^1 \frac{dx}{f(x)} \Rightarrow \int_0^1 \frac{dx}{f(x)} \geq \frac{2}{3} \quad (1)$$

Đấu “=” không xảy ra :

$$\frac{\sqrt{f(x)}}{1/\sqrt{f(x)}} = k \Leftrightarrow f(x) = k = \frac{3}{2}$$

$$\text{do } \int_0^1 f(x) dx = \frac{3}{2}$$

Tổ (a) : $1 \leq f(x) \leq 2; \forall x \in [0,1]$ thì $\begin{cases} 2 - f(x) \geq 0 \\ f(x) - 1 \geq 0 \end{cases}$

$$\Leftrightarrow (2 - f(x))(f(x) - 1) \geq 0 \Leftrightarrow f^2(x) - 3f(x) + 2 \leq 0$$

$$f(x) - 3 + \frac{2}{f(x)} \leq 0 \quad (2) \text{ Đặt } t = f(x)$$

$$\Rightarrow 1 \leq t \leq 2 \text{ thì } (2) \Leftrightarrow t - 3 - \frac{2}{t} = f(t) \leq 0$$

t	1	$\sqrt{2}$	2
$f'(t)$	–	0	+
$f(t)$	$\square$	$\square$	
	$2\sqrt{2} - 3$		

$$\Rightarrow \int_0^1 f(x) dx - 3 \int_0^1 dx + 2 \int_0^1 \frac{dx}{f(x)} < 0$$

$$\Rightarrow 2 \int_0^1 \frac{dx}{f(x)} < 3 \int_0^1 dx - \int_0^1 f(x) dx = \frac{3}{2} \Rightarrow \int_0^1 \frac{dx}{f(x)} < \frac{3}{4}$$

Từ (1) và (2) suy ra :

$$\frac{2}{3} \leq \int \frac{1}{f(x)} dx < \frac{3}{4}$$

BAI TẬP TỐI LUYỆN

Chứng minh rằng :

$$1. \frac{\pi}{28} \leq \int_0^{\pi/4} \frac{1}{5+2\cos^2 x} dx \leq \frac{\pi}{24}$$

$$2. \frac{\pi}{24} \leq \int_0^{\pi/4} \frac{1}{3+4\sin^2 x} dx \leq \frac{\pi}{18}$$

$$3. \frac{2}{9} \leq \int_{-1}^1 \frac{1}{x^3+8} dx \leq \frac{2}{7}$$

$$4. \int_0^{10} \frac{x}{x^3+16} dx < \frac{5}{6}$$

$$5. \left| \int_0^{18} \frac{\cos x}{\sqrt{1+x^4}} dx \right| < \frac{5}{6}$$

$$6. \frac{\pi}{16} \leq \int_0^{\pi/2} \frac{1}{5+3\cos^2 x} dx \leq \frac{\pi}{10}$$

$$7. \int_1^{\sqrt{3}} \frac{e^{-x} \cdot \sin x}{x^2+1} dx < \frac{\pi}{12e}$$

$$8. \int_0^1 \sqrt{3+e^{-x}} dx \leq 2$$

$$9. \int_1^{\pi} x^{2001} \cdot \ln x \cdot dx < (\sqrt{\pi})^{4003}$$

$$10. \frac{\pi}{6} \leq \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}} dx \leq \frac{\pi\sqrt{2}}{8}$$

$$11. \frac{1}{2} \leq \int_0^{1/2} \frac{1}{\sqrt{1-x^{2n}}} dx < \frac{\pi}{6} \quad (n=2,3,\dots)$$

$$12. \frac{2}{\sqrt[4]{e}} \leq \int_0^2 e^{x^2-x} dx \leq 2e^2$$

$$13. 0 < \int_0^1 \frac{x^7}{\sqrt[3]{1+x^8}} dx < \frac{1}{8}$$

$$14. 1 < \int_0^1 e^{x^2} dx < e$$

$$15. \frac{1}{20\sqrt[3]{2}} < \int_0^1 \frac{x^{19}}{\sqrt{1+x^6}} dx < \frac{1}{20}$$

$$16. \frac{\pi}{10} \leq \int_0^{\pi/2} \frac{1}{5-3\cos^2 x} dx \leq \frac{\pi}{4}$$

$$17. 0 \leq \int_0^1 \frac{x^n}{1+x} dx \leq \frac{1}{n+1}$$

$$18. 1 \leq \int_0^3 \frac{1}{\sqrt{-x^2+2x+8}} dx \leq \frac{3}{\sqrt{5}}$$

$$19. 1 \leq \int_0^1 \sqrt{1+x^2} dx \leq \sqrt{2}$$

$$20. \sqrt{3} \leq \int_0^1 \sqrt{3+x^2} dx \leq 2$$

$$21. \frac{\pi}{8} \leq \int_{\pi/4}^{3\pi/4} \frac{1}{3+\sin^2 x} dx \leq \frac{\pi}{7}$$

$$22. 2 \leq \int_{-1}^1 \sqrt{5-4x} dx \leq 6$$

$$23. 2 \leq \int_0^1 \sqrt{4+x^2} dx \leq \sqrt{5}$$

$$24. \frac{\pi\sqrt{3}}{18} < \int_0^1 \frac{1}{x^2+x+2} dx < \frac{\pi}{8}$$

$$25. \frac{1}{\sqrt{2}} \leq \int_0^{\sqrt[4]{2}} \frac{1}{\sqrt{1-x^{2004}}} dx \leq \frac{\pi}{4}$$

$$26. \frac{\pi}{18} \leq \int_0^1 \frac{\sqrt{x}}{x^7+x^5+x^3+3} dx < \frac{\pi\sqrt{3}}{27}$$

$$27. 0 \leq \int_0^e x \ln x dx \leq e^2$$

$$28. 9 < \int_0^3 \sqrt{81+x^2} dx < 10$$

$$29. \frac{2\pi}{3} < \int_0^{2\pi} \frac{dx}{10+3\cos x} < \frac{2\pi}{7}$$

$$30. \frac{\pi}{2} < \int_0^{\pi/2} \sqrt{1+\frac{1}{2}\sin^2 x} dx < \frac{\pi\sqrt{6}}{4}$$

$$31. 0 < \int_{-1}^1 tx^2 dx < 2\sqrt{3}$$

$$32. \frac{\pi}{4} \leq \int_{\pi/4}^{3\pi/4} \frac{1}{3-2\sin^2 x} dx \leq \frac{\pi}{2}$$

$$33. \frac{1}{e}(e-1) < \int_0^1 e^{-x^2} dx < 1$$

$$34. \left| \int_0^1 \frac{\sin(nx)}{x+1} dx \right| \leq \ln 2$$

$$35. \int_0^1 \frac{\cos(nx)}{x+1} dx \leq \ln 2$$

$$36. \int_1^{\sqrt[4]{2}} \frac{x}{\sqrt{1-x^{2n}}} dx < \frac{\pi}{12}; n=3,4$$

$$37. \int_0^{\sqrt[4]{2}} \sin(x^2) dx > 0$$

Chứng minh rằng :

1. $\int_0^{\pi} (\sqrt{2 + \cos^2 x} + \sqrt{2 - \cos^2 x}) dx \leq 2\sqrt{2} \pi$
2. $0 < \int_1^{\sqrt{3}} \frac{3\cos x + 4\sin x}{x^2 + 1} dx \leq \frac{5\pi}{12}$
3. $\int_{\pi/6}^{\pi/3} (3 - 2\sqrt{\sin x})(\sin x + 6\sqrt{\sin x} + 5) dx \leq \frac{9\pi}{2}$
4. $\int_0^{\pi/4} \sqrt{\tan x} (2 + 3\sqrt{\tan x})(7 - 4\sqrt{\tan x}) dx \leq \frac{27\pi}{4}$
5. $\int_0^{\pi/4} \sin^2 x (2 + 3\cos^2 x) dx < \frac{25\pi}{48}$
6. $\int_0^{\pi/2} \cos^4 x (2\sin^2 x + 3) dx < \frac{125\pi}{54}$
7. $\int_{-\pi/4}^{\pi/4} (\sqrt{5 - 2\cos^2 x} + \sqrt{3 - 2\sin^2 x}) dx \leq \frac{3\pi\sqrt{3}}{2}$
8. $\int_0^{\pi/2} \sqrt{\sin x} (2 + 3\sqrt{\sin x})(7 - 4\sqrt{\sin x}) dx \leq \frac{27\pi}{2}$
9. $\int_{\pi/6}^{\pi/3} (3 - 2\sqrt{\sin x})(5 + \sqrt{\sin x})(1 + \sqrt{\sin x}) dx \leq \frac{9\pi}{2}$
10. $\int_0^{\pi} (2\sin x + \tan x) dx > 0$
11. $0 \leq \int_0^1 (e^{-x} + x - 1) dx \leq e^{-1}$
12. $\frac{1}{2} \leq \int_0^1 \frac{e^{-x^2}}{x^2 + 1} dx \leq \frac{5}{24} + \frac{1}{2}$

Chứng minh rằng :

1. $\frac{2\pi}{13} \leq \int_0^{2\pi} \frac{1}{10 + 3\cos x} dx \leq \frac{2\pi}{7}$
2. $\frac{\pi}{14} \leq \int_0^{\pi/2} \frac{1}{4 + 3\cos^2 x} dx \leq \frac{\pi}{8}$
3. $\left| \int_0^{18} \frac{\cos x}{\sqrt{1 + x^4}} dx \right| < 0,1$
4. $\int_0^1 \sqrt{1 + x^2} dx > \int_0^1 x dx$
5. $\int_0^1 x^2 \sin^2 x dx < \int_0^1 x \sin^2 x dx$
6. $\int_1^2 e^{x^2} dx > \int_1^2 e^x dx$
11. $-1 \leq \int_0^1 \frac{x \sin a + \sqrt{a+1} \cos a}{x+1} dx \leq 1$
($a \in \mathbb{R}$)
12. $\frac{\pi}{4\sqrt{2}} < \int_0^1 \frac{\sqrt{1-x^2}}{1+x^2} dx < \sqrt{\frac{\pi}{6}}$
13. $1 \leq \int_{-1}^1 2^{x^3} dx \leq 4$
14. $1 \leq \int_0^1 e^{x^2} dx \leq e$
15. $\pi \leq \int_0^{\pi} \frac{1}{\sin^4 x + \cos^4 x} dx \leq 2\pi$
16. $\int_0^1 \frac{1}{x^2 + x + 2} dx < \frac{\pi}{8}$

$$7. \left| \int_{-1}^1 \frac{x^2 \cos \alpha - 2x + \cos \alpha}{x^2 - 2x \cos \alpha + 1} dx \right| \leq 2$$

$$\alpha \in (0, \pi)$$

$$8. \int_0^{\pi/2} \sin^{10} x dx \leq \int_0^{\pi/2} \sin^2 x dx$$

$$9. \int_{-\pi/6}^{\pi/6} \left(\sqrt{\cos^2 x + 2 \sin^2 x} + \sqrt{\sin^2 x + 2 \cos^2 x} \right) dx \leq \pi \sqrt{6}$$

$$10. \int_0^{\pi/2} \left(\sqrt{3 \cos^2 x + \sin^2 x} + \sqrt{3 \sin^2 x + \cos^2 x} \right) dx \leq \pi \sqrt{2}$$

$$17. \int_{\pi/4}^{\pi/2} \sin x dx > \int_{\pi/4}^{\pi/2} \cos x dx$$

$$18. 3\sqrt{6} \leq \int_{-2}^1 (\sqrt{2+x} + \sqrt{4-x}) dx \leq 6\sqrt{3}$$

$$19. 25 \leq \int_3^5 \frac{x^3}{x^2 - 4x + 5} dx \leq 27$$

Chứng minh rằng :

$$1. \frac{\sqrt{3}}{4} < \int_{\pi/6}^{\pi/3} \frac{\sin x}{x} dx < \frac{1}{2}$$

$$2. \frac{\sqrt{3}}{8} < \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx < \frac{\sqrt{2}}{6}$$

$$3. 0 < \int_0^3 x(1-x^2) dx < \frac{4}{27}$$

$$4. \frac{\pi\sqrt{3}}{3} < \int_0^{\pi} \frac{1}{\sqrt{\cos^2 x + \cos x + 1}} dx < \frac{2\pi\sqrt{3}}{3}$$

$$5. \frac{2}{5} < \int_1^2 \frac{x}{x^2+1} dx < \frac{1}{2}$$

$$6. 0 < \int_0^1 x(1-x^2) dx < \frac{2\sqrt{3}}{9}$$

$$28. \int_0^{2\pi} \frac{\cos x}{x} dx < \frac{1}{2\pi}$$

$$29. 0 \leq \int_{-2}^0 x^2 e^x dx \leq 8e^{-2}$$

$$30. -24 \leq \int_{-1}^2 (-x^5 - 5x^3 + 20x + 2) dx \leq 32$$

$$31. -2\sqrt[3]{4} \leq \int_{-1}^1 \sqrt{-x^3 + 3x - 2} dx \leq 0$$

$$32. 0 < \int_0^{e^2} x^{\frac{1}{\sqrt{x}}} dx < e^{2(e+1)}$$

$$33. \frac{1}{2} < \int_0^{2\pi/3} (2 \cos^2 x + 2 \cos x + 1) dx < 5$$

$$34. \left| \int_{-\pi}^{\pi} \sin x (1 + \cos x) dx \right| < \frac{\pi\sqrt{3}}{2}$$

$$35. \left| \int_{-\pi}^{\pi} (\sin x + \cos x) dx \right| < 2\pi\sqrt{2}$$

$$36. \frac{7\pi\sqrt{3}}{3} < \int_0^{2\pi} (x\sqrt{3} + 2 \sin x) dx < \frac{5\pi\sqrt{3} + 6}{3}$$

$$37. \int_0^{2\pi} (\cos x - x) dx < 2\pi$$

$$38. \left| \int_0^{\pi} \left(\frac{\cos x}{\sin^3 x} - 2 \cot gx \right) dx \right| < \frac{\pi 2\sqrt{3}}{9}$$

$$39. -2 \leq \int_2^4 \frac{x^2 - 7x + 5}{x^2 - 5x + 7} dx \leq 6$$

$$40. 0 \leq \int_{-1}^1 \frac{x^2 + 3x + 1}{x^2 - x + 1} dx \leq 10$$

$$12. \frac{5}{2} < \int_2^3 \frac{x^2}{\sqrt{x^2-1}} dx < \frac{9\sqrt{2}}{4}$$

$$13. \int_{100\Pi}^{200\Pi} \frac{\cos x}{x} dx < \frac{1}{200\Pi}$$

$$14. \frac{1}{2} < \int_{\frac{\Pi}{4}}^{\frac{\Pi}{2}} \frac{\sin x}{x} dx < \frac{\sqrt{2}}{2}$$

$$15. 2(e^2 - e) < \int_2^{e^2} \left(3\ln x - \frac{1}{\ln x} \right) dx$$

$$16. \frac{2}{5} < \int_1^2 \frac{x}{x^2+1} dx < \frac{1}{2}$$

$$17. \int_0^1 x(1-x) dx < \frac{1}{2}$$

$$18. \Pi \leq \int_{\Pi/2}^{\Pi} \sqrt{\cos 2x - \cos x + 1} dx \leq 2\Pi$$

$$19. 5\sqrt{2} \leq \int_0^{\sqrt{2}} (-x^4 + 4x^2 + 5) dx \leq 9\sqrt{2}$$

$$20. -141 \leq \int_{-1}^2 (-3x^4 - 8x^3 + 30x^2 + 72x - 20) dx \leq 369$$

$$21. \frac{5}{2} \leq \int_{-2}^{1/2} \frac{2x^2 + 4x + 5}{x^2 + 1} dx \leq 15$$

$$22. 0 \leq \int_0^e x \ln x \leq e^2$$

$$23. e^2(e-1) < \int_e^{e^2} \frac{x}{\ln x} dx < e^3(e-1)$$

$$24. \frac{5}{4} < \int_1^2 \frac{2x}{x^2+1} dx < 1$$

$$25. \ln 2 < \int_1^3 \ln \frac{x+1}{x} dx < \ln 3$$

$$26. \frac{2}{\sqrt[4]{e}} < \int_0^2 e^{x^2-x} dx < 2e^2$$

$$27. e < \int_1^2 \frac{e^x}{x} dx < \frac{1}{2}e^2$$