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ROMANIAN MATHEMATICAL COMPETITIONS 2007

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FOREWORD

The 14^{th} volume of the Romanian Mathematical Contests booklet consists, as usual, of two parts. In the first part we present the problems given at the district and final round of the Romanian National Olympiad along with those given at the selection tests for the Romanian Teams, junior and senior. We collected some of the problems considered by the problem selection committee at different stages of the Olympiad.

The second part provides full solutions to the problems, with emphasis on those given at the selection tests for the IMO. We hope that in this way we contribute to the development of the so-called problem solving community in the world.

We thank the Ministry of Education and Research for permanent involvement in supporting the Olympiads and the participation of our teams in international events.

Special thanks are due to SOFTWIN, Volvo Romania, Medicover, and WBS – sponsors of the Romanian IMO team. Thanks are also due to the "Sigma Foundation" for constant support in the mathematical competitions. Many of the solutions are student's contribution. We thank them all.

Luminiţa Stafi from "The Theta Foundation" helped the editor in the process of producing this booklet.

Last, not least, we are grateful to the Board of the Institute of Mathematics "Simion Stoilow" in Bucharest, for constant technical support in the Mathematical Olympiads and involvement in the training seminars for students.

Bucharest, July 12^{th}, 2007 Radu Gologan

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PART ONE

PROBLEMS

DISTRICT ROUND

March 5$^{\text{th}}$, 2007

7$^{\text{th}}$ GRADE

Problem 1. Point O is the intersection of bisector lines of the sides of triangle ABC. Denote by D the intersection of line AO with the segment BC. If $OD = BD = \frac{1}{3} \cdot BC$, find the angles of the triangle ABC.

Virginia and Vasile Tică

Problem 2. A can contains blue and red stones. Someone invented the following game: extracts successively stones untill, for the first time, the extracted blue stones and the extracted red stones are in the same number. At one of the played games one observes that, at the end, 10 stones were extracted and no 3 consecutive extracted stones are of the same color. Prove that at that game the fifth and the sixth stones have different colors.

Dinu Şerbănescu

Problem 3. Let a and b be integers such that $b > a \geqslant 2$. Prove that if the number $a + k$ is prime with the number $b + k$ for all $k = 1, 2, \ldots, b - a$, then a and b are consecutive.

Aurel Bârsan

Problem 4. Let n be a composite natural number. Prove that there are integers $k > 1$ and $a_1, a_2, \ldots, a_k > 1$ such that

$$a_1 + a_2 + \cdots + a_k = n \cdot \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} \right).$$

Petre Bătrâneţu

8^{th} GRADE

Problem 1. Consider the positive real numbers x, y, z, with the property that $xy = \frac{z-x+1}{y} = \frac{z+1}{2}$. Prove that one of them is the average of the orther two.

Gheorghe Molea

Problem 2. A rectangle $ABCD$ has $AB = 2$ and $BC = \sqrt{3}$. Point M is on AD such that $MD = 2 \cdot AM$ and point N is the midpoint of AB. MP is perpendicular to the rectangle's plane and point Q is on the segment MP such that the angle between planes (MPC) and (NPC) is $45°$, and the angle between plane (MPC) and (QNC) is $60°$.

a) Prove that DN and CM are perpendicular.

b) Prove that Q is the midpoint of MP.

Gheorghe Bumbăcea

Problem 3. Eight consecutive natural numbers are partitioned into two classes, each of four numbers. Prove that if the sum of the squares of the numbers in each class is the same, then the sum of elements in each class is the same.

Adrian Stoica

Problem 4. All points on a circle are painted in green or yellow such that each inscribed equilateral triangle has exactly two yellow vertices. Prove that one can find an inscribed square which has at least three yellow vertices.

Vasile Pop

9^{th} GRADE

Problem 1. Let $k \in \mathbb{N}^*$. We will say a function $f : \mathbb{N} \to \mathbb{N}$ holds property $(\mathcal{P})$ if for any $y \in \mathbb{N}$ the equation $f(x) = y$ has exactly k solutions.

a) Show there exist infinitely many functions holding property $(\mathcal{P})$.

b) Determine the monotone functions holding property $(\mathcal{P})$.

c) Determine if for $k > 1$ there exist monotone functions $f : \mathbb{Q} \to \mathbb{Q}$ holding property $(\mathcal{P})$.

Mihai Piticari

Problem 2. Given triangle ABC and points $M \in (AB)$, $N \in (BC)$, $P \in (CA)$, $R \in (MN)$, $S \in (NP)$, $T \in (PM)$, such that

$$\frac{AM}{MB} = \frac{BN}{NC} = \frac{CP}{PA} = \lambda, \quad \frac{MR}{RN} = \frac{NS}{SP} = \frac{PT}{TM} = 1 - \lambda, \quad \lambda \in (0,1).$$

a) Prove that triangles STR and ABC are similar.

b) Determine the value for parameter λ which makes the area of triangle STR minimal.

Marian Teller

Problem 3. Determine the functions $f : \mathbb{N}^* \to \mathbb{N}^*$ for which

$$x^2 + f(y) \text{ divides } f(x)^2 + y$$

for all $x, y \in \mathbb{N}^*$.

Adapted after Lucian Dragomir

Problem 4. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be coplanar vectors, each of module 1.

a) Prove that we can choose signs $+, -$, such that $| \pm \mathbf{u} \pm \mathbf{v} \pm \mathbf{w}| \leqslant 1$.

b) Exhibit a triplet such that, no matter how we choose signs $+, -$, we get $| \pm \mathbf{u} \pm \mathbf{v} \pm \mathbf{w}| \geqslant 1$.

10^{th} GRADE

Problem 1. The real numbers a, b, c are such that $a, b, c \in (1, \infty)$ or $a, b, c \in (0, 1)$. Prove that

$$\log_a bc + \log_b ca + \log_c ab \geqslant 4(\log_{ab} c + \log_{bc} a + \log_{ca} b).$$

Cezar Lupu

Problem 2. The $2n$ squares composing a rectangle of dimension $2 \times n$ are painted with three colors. We say that a color has a *cutting* if on one of the columns we have two squares of the same color. Find:

a) the number of coloring without cuttings;

b) the number of coloring with exactly one cutting.

Inoan Daniela

Problem 3. Let ABC a triangle with $BC = a$, $CA = b$, $AB = c$. For each line Δ we denote by d_A, d_B, d_C the distances from A, B, C to Δ. Consider

$$E(\Delta) = ad_A^2 + bd_B^2 + cd_C^2.$$

Prove that if the value of $E(\Delta)$ is minimal, then Δ contains the incenter of the triangle.

Vasile Pop

Problem 4. Let u, v, w be complex numbers of modulus 1. Prove that one can choose signs $+$ and $-$ such that

$$|\pm u \pm v \pm w| \leqslant 1.$$

Dan Schwarz

11th GRADE

Problem 1. Let $a \in (0,1)$ and $(a_n)_{n\geqslant 1}$ the sequence defined by $x_{n+1} = x_n(1-x_n^2)$, for any $n \geqslant 10$, starting with $x_0 \in (0,1)$.

Calculate $\lim_{n\to\infty} \sqrt{n} \cdot a_n$.

Farkas Csaba

Problem 2. Let $A \in \mathcal{M}_n(\mathbb{R}^*)$. If $A \cdot {}^tA = I_n$, prove that:

a) $|\mathrm{tr}\,(A)| \leqslant n$;

b) for n odd, $\det(A^2 - I_n) = 0$.

Alin Gălățan

Problem 3. Consider the sequence $(x_n)_{n\geqslant 1}$ given by $x_n = \sqrt{n} - [\sqrt{n}]$. Denote by A the set of its limit points, that is, the set of $x \in \mathbb{R}$ such that there is a subsequence of $(x_n)_n$ with limit x.

a) Prove that $\mathbb{Q} \cap [0,1] \subset A$.

b) Determine the set A.

Tiberiu Trif

Problem 4. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ such that $B^2 = I_n$ and $A^2 = AB + I_n$. Prove that $\det(A) \leqslant \left(\frac{1+\sqrt{5}}{2}\right)^n$.

Marius Cavachi

12th GRADE

Problem 1. Given a group $(G, *)$ and A, B nonempty subsets of G, we define $A * B = \{a * b \mid a \in A \text{ and } b \in B\}$.

a) Prove that for $n \in \mathbb{N}$, $n \geqslant 3$, the group $(\mathbb{Z}_n, +)$ can be written in the form $\mathbb{Z}_n = A + B$, where A and B are two nonempty subsets of $\mathbb{Z}_n$ such that $A \neq \mathbb{Z}_n$, $B \neq \mathbb{Z}_n$ and $|A \cap B| = 1$.

b) If $(G, *)$ is finite, A, B are nonempty subsets of G and $a \in G \setminus (A * B)$, prove that the function $f : A \to G \setminus B$ given by $f(x) = x^{-1} * a$ is well-defined and one-to-one. Conclude that if $|A| + |B| > |G|$, then $G = A * B$.

Farkas Csaba

Problem 2. Consider two continuous functions $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to (0,\infty)$ such that f is non-decreasing. Prove that

$$\int_0^t f(x)g(x)\mathrm{d}x \cdot \int_0^1 g(x)\mathrm{d}x \leqslant \int_0^t g(x)\mathrm{d}x \cdot \int_0^1 f(x)g(x)\mathrm{d}x,$$

for any $t \in [0,1]$.

Cezar Lupu

Problem 3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which verify simultaneously the following two conditions:

a) the limit $\lim_{x\to\infty} f(x)$ exists;

b) $f(x) = \int_{x+1}^{x+2} f(t)\mathrm{d}t$, for all $x \in \mathbb{R}$.

Mihai Piticari

Problem 4. Let k be a field 2^n with $n \in \mathbb{N}^*$ elements and consider the polynomial $f = X^4 + X + 1$. Prove that:

a) for even n, f is reducible in $k[X]$;

b) for odd n, f is irreducible in $k[X]$.

Marian Andronache

FINAL ROUND

Piteşti, March 29th, 2007

7th GRADE

Problem 1. If the side lengths a, b, and c of a triangle satisfy the conditions $a + b - c = 2$, and $2ab - c^2 = 4$, show that the triangle is equilateral.

* * *

Problem 2. Consider a triangle ABC with a right angle at A, and $AC = 2AB$. Let P and Q be the midpoints of the sides AC and AB, respectively. Let further M and N be two points on the side BC such that $BM = CN = x$, with $2x < BC$. Express x in terms of AB, if the area of $MNPQ$ is half the area of ABC.

* * *

Problem 3. Consider a triangle ABC with a right angle at A, and $AB < AC$. Let D be the point of the side AC for which $\angle ACB = \angle ABD$. Drop the altitude DE in triangle BCD. If $AC = BD + DE$, what are the angles ABC and ACB?

Mircea Fianu

Problem 4. If m and n are non-negative integer numbers such that $m > 1$ and $2^{2m+1} \geqslant n^2$, show that $2^{2m+1} \geqslant n^2 + 7$.

Radu Gologan

8th GRADE

Problem 1. Prove that the number 10^{10} cannot be written as a product of two positive integers all of whose base-10 digits are different from zero.

Adrian Stoica

Problem 2. A number of 2007 offices are assigned 6018 desks. Each office is assigned at least one desk. All desks from any one office may be removed from that office and reascribed to other offices to get an equal number of desks in each office other than the one they have been removed from. What are the possible desk ascriptions?

Severius Moldoveanu

Problem 3. a) If all sides of a triangle ABC have length less than 2, show that the length of the altitude from A is less than $\sqrt{4 - BC^2/4}$.

b) Show that the volume of a tetrahedron at most one edge of which has a length greater than or equal to 2 is less than 1.

* * *

Problem 4. Let $ABCD$ be a tetrahedron, and M a point in space such that $MA^2 + MB^2 + CD^2 = MB^2 + MC^2 + DA^2 = MC^2 + MD^2 + AB^2 = MD^2 + MA^2 + BC^2$. Show that M lies on the common perpendicular to the lines AC and BD.

Vasile Pop

9th GRADE

Problem 1. Prove that, for $a_i \in \mathbb{N}^*$, $1 \leqslant i \leqslant n+1$, $a_{n+1} = a_1$, $n \in \mathbb{N}^*$, if the polynomial function

$$P(x) = x^2 - \left(\sum_{i=1}^{n} a_i^2 + 1\right)x + \sum_{i=1}^{n} a_i a_{i+1}$$

admits an integer root, then, if n is a perfect square, so are both its roots.

T. Tămîian

Problem 2. Let ABC be an acute-angled triangle, and M a point in its plane, different from its vertices. Then, with the usual notations,

$$\frac{a}{MA}\overrightarrow{MA} + \frac{b}{MB}\overrightarrow{MB} + \frac{c}{MC}\overrightarrow{MC} = \overrightarrow{0}$$

if and only if $M \equiv H$, the orthocenter of ABC.

Viorel Cornea, Dan Marinescu, and Vasile Pop

Problem 3. Color white or black each band of width 1, determined by partitioning the plane with equidistant parallel lines, distanced 1 apart. Show that one can place an equilateral triangle of side 100, such that its vertices share a same color.

Radu Gologan

Problem 4. For $f : X \to X$, denote $f_0(X) = X$, $f_{n+1}(X) = f(f_n(X))$, for all $n \in \mathbb{N}$. Also denote

$$f_\infty(X) = \bigcap_{n\in\mathbb{N}} f_n(X).$$

Prove that, if X is finite, then $f(f_\infty(X)) = f_\infty(X)$. Does the result still hold when X is infinite?

Dan Schwarz

10th GRADE

Problem 1. Let n be a positive integer. Prove that a complex number of absolute value 1 is a solution to $z^n + z + 1 = 0$ if and only if $n = 3m + 2$ for some positive integer m.

Mihai Băluna̋

Problem 2. Solve the equation $2^{x^2+x} + \log_2 x = 2^{x+1}$ in the set of real numbers.

Lucian Dragomir

Problem 3. For what integer numbers $n \geqslant 2$ is $(n-1)^{n^{n+1}} + (n+1)^{n^{n-1}}$ divisible by n^n ?

* * *

Problem 4. a) Let S be a finite set of numbers, and let $S + S = \{x + y : x, y \in S\}$. Show that

$$|S + S| \leqslant \frac{1}{2}|S|(|S| + 1),$$

where $|X|$ is the cardinal number (that is, the number of elements) of the set X.

b) Given a positive integer m, let $C(m)$ be the greatest positive integer k such that, for some set S of m integers, every integer from 1 to k belongs to S or is a sum of two not necessarily distinct elements of S. For instance, $C(3) = 8$ with $S = \{1, 3, 4\}$. Show that $m(m + 6)/4 \leqslant C(m) \leqslant m(m + 3)/2$.

* * *

11th GRADE

Problem 1. If A and B are 2-by-2 matrices with real numbers as entries, and $A^2 + B^2 = AB$, prove that $(AB - BA)^2 = 0_2$.

Marian Ionescu

Problem 2. Given two real numbers a and b, $a < b$, in the image of a continuous, real-valued function f on $\mathbb{R}$, prove that the closed interval $[a, b]$ is the image under f of some interval $I \subset \mathbb{R}$.

* * *

Problem 3. Given an integer number $n \geqslant 2$, let Σ^{n-1} be the set of all $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ with $|x_1| + \cdots + |x_n| = 1$. Determine the n-by-n matrices A with real numbers as entries such that $xA \in \Sigma^{n-1}$ for all $x \in \Sigma^{n-1}$.

Vasile Pop

Problem 4. A *P-function* is a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with a continuous derivative f' on $\mathbb{R}$ such that $f(x + f'(x)) = f(x)$ for all x in $\mathbb{R}$.

a) Prove that the derivative of a P-function has at least one zero.

b) Provide an example of a non-constant P-function.

c) Prove that a P-function whose derivative has at least two distinct zeros is constant.

Dorin Andrica and Mihai Piticari

12th GRADE

Problem 1. Let $\mathcal{C}$ be the class of all differentiable functions $f : [0,1] \to \mathbb{R}$ with a continuous derivative f' on $[0,1]$, and $f(0) = 0$ and $f(1) = 1$. Determine the minimum value the integral

$$\int_0^1 \left(1 + x^2\right)^{1/2} \left(f'(x)\right)^2 \mathrm{d}x$$

may assume as f runs through all of $\mathcal{C}$, and find all functions in $\mathcal{C}$ that achieve this minimum value.

* * *

Problem 2. Let f be a continuous, positive real-valued function on $[0, 1]$.

a) Given a positive integer number n, prove that there exists a unique subdivision, $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$, of $[0,1]$ such that

$$(*) \qquad \int_{a_k}^{a_{k+1}} f(x)\mathrm{d}x = \frac{1}{n}\int_0^1 f(x)\mathrm{d}x, \quad k = 0, \ldots, n-1.$$

b) For each positive integer number n, let

$$\bar{a}_n = \frac{a_1 + \cdots + a_n}{n},$$

where $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$ is the unique subdivision of $[0,1]$ satisfying $(*)$. Prove that the sequence $(\bar{a}_n)_{n\geqslant 1}$ is convergent and evaluate its limit.

École Polytechnique

Problem 3. Given a positive integer n, determine the rings R with the property that $x^{2^n+1} = 1$ for all $x \in R \setminus \{0\}$.

Dorel Miheţ

Problem 4. Given an integer number $n \geqslant 3$, let G be a subgroup of the symmetric group S_n generated by $n-2$ transpositions. Prove that, for each i in $\{1, \ldots, n\}$, the set $\{\sigma(i) : \sigma \in G\}$ has at most $n-1$ elements.

* * *

SELECTION TESTS FOR THE BALKAN AND INTERNATIONAL MATHEMATICAL OLYMPIADS

FIRST SELECTION TEST

Problem 1. At the vertices of a convex polygon with even number of sides sit hunters, while in the interior of the polygon, and not lying on any of its diagonals, sits a fox. Simultaneously, the hunters shoot at the fox, but the fox ducks in good time, and the bullets go on, hitting sides of the polygon. Prove that at least one side is not hit.

Latvian Textbook

Problem 2. Let $\mathcal{C}(O_1)$ and $\mathcal{C}(O_2)$ be two circles, external to each other. Points A, B, C lie on $\mathcal{C}(O_1)$, while points D, E, F lie on $\mathcal{C}(O_2)$, such that AD and BE are external tangents to the two circles, while CF is an internal common tangent. The lines CO_1 and FO_2 meet the lines AB, respectively DE, at M, respectively N. Show the line MN passes through the midpoint of the segment CF.

* * *

Problem 3. Any $f : \mathbb{Q} \to \mathbb{R}$ with the property below is constant

$$|f(x) - f(y)| \leqslant (x-y)^2, \text{ for all } x, y \in \mathbb{Q}.$$

* * *

Problem 4. For $n \in \mathbb{N}$, $n \geqslant 2$, determine

$$\max \prod_{i=1}^{n}(1 - x_i), \text{ for } x_i \in \mathbb{R}_+,\ 1 \leqslant i \leqslant n,\ \sum_{i=1}^{n} x_i^2 = 1.$$

AMM

SECOND SELECTION TEST

Problem 5. Let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ be a degree $n \geqslant 3$ polynomial with integer coefficients, $a_k + a_{n-k}$ even, for all $k = 1, 2, \ldots, n-1$ and a_0 also even. If $f = gh$, where g and h are polynomials with integer coefficients, and the degree of g at most the degree of h, and all coefficients of h are odd, show that f has (at least) an integer root.

M. Andronache

Problem 6. Let ABC be a triangle. Its incircle is tangent to AB at E, while its excircle relative to BC is tangent to AB at F. Let D be the point lying on side BC for which the incircles of triangles ABD and ACD have equal radii. The lines DE and DB meet a second time the circumcircle of triangle ADF at X and Y. Show that $XY \parallel AB$ if and only if $AB = AC$.

O. Ganea

Problem 7. Find all sets A of at least two positive integers, such that for any distinct $x, y \in A$ we also have $(x+y)/(x,y) \in A$.

Adapted from Swiss Olympiad

Problem 8. Let X be the set of the 2^n points $\{0,1\}^n$, $n \geqslant 3$, in the Euclidean n-space (the vertices of the unit hypercube). Denote by $M(n)$ the least integer such that any subset $Y \subseteq X$, with $M(n)$ elements or more, necessarily contains an equilateral triangle (determined by points from Y). Prove that $M(n) \leqslant \lfloor 2^{n+1}/n \rfloor + 1$, and effectively compute $M(3)$ and $M(4)$.

Putnam Competition

THIRD SELECTION TEST

Problem 9. Let $\mathcal{F}$ be the set of all functions $f : \mathcal{P}(S) \to \mathbb{R}$ with the property that, for any $X, Y \subseteq S$, we have $f(X \cap Y) = \min(f(X), f(Y))$, where S is a finite set. Determine

$$\max_{f \in \mathcal{F}} |\mathrm{Im}(f)|.$$

* * *

Problem 10. Show that, for n, p positive integers, $n \geqslant 4$ and $p \geqslant 4$, the proposition $\mathcal{P}(n,p)$ below is false

$$\sum_{i=1}^{n} \frac{1}{x_i{}^p} \geqslant \sum_{i=1}^{n} x_i{}^p \text{ for } x_i \in \mathbb{R},\ x_i > 0,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} x_i = n.$$

(As a matter of fact, the propositions $\mathcal{P}(4,3)$ and $\mathcal{P}(3,4)$ are true, but hard to prove!)

Dan Schwarz

Problem 11. Let a_i, $i = 1, 2, \ldots, n$, $n \geqslant 3$, be positive integers with their greatest common divisor equal to 1, such that a_j divides $\sum_{i=1}^n a_i$ for all $j = 1, 2, \ldots, n$. Prove that $\prod_{i=1}^n a_i$ divides $(\sum_{i=1}^n a_i)^{n-2}$.

(Also, provide an example showing that the exponent $n-2$ cannot be lowered).

AMM

Problem 12. Points M, N, P on the sides BC, CA, AB of triangle ΔABC are such that triangle MNP is acute-angled. Denote by x the length of the shortest altitude of ABC, and by X the length of the longest altitude of ΔMNP. Prove that $x \leqslant 2X$.

BMO 2007 Short List – Bulgaria

FOURTH SELECTION TEST

Problem 13. Prove that the function $f : \mathbb{N} \to \mathbb{Z}$ defined below is injective

$$f(n) = n^{2007} - n!\,.$$

BMO 2007 Short List – Serbia

Problem 14. Let $A_1A_2A_3A_4A_5$ be a convex pentagon, such that

$$[A_1A_2A_3] = [A_2A_3A_4] = [A_3A_4A_5] = [A_4A_5A_1] = [A_5A_1A_2].$$

Prove there exists a point M in the plane of the pentagon, such that

$$[A_1MA_2] = [A_2MA_3] = [A_3MA_4] = [A_4MA_5] = [A_5MA_1].$$

($[XYZ]$ is the area of ΔXYZ).

BMO 2007 Short List – Moldova

Problem 15. Consider the set $E = \{1, 2, \ldots, 2n\}$. Prove that an element $c \in E$ may belong to a subset $A \subset E$, with n elements, such that for any two distinct elements of A, none divides the other, if and only if $c > n(2/3)^{k+1}$, where k is the exponent of 2 in the factorization of c.

Abouabdillah

Problem 16. i) Determine all infinite arithmetical sequences of positive integers, with the property: there exists $N \in \mathbb{N}$, such that for any p prime, $p > N$, the p^{th} term of the sequence is also a prime.

Adapted after M. Burtea

ii) Determine all polynomials $f(X) \in \mathbb{Z}[X]$, with the property: there exists $N \in \mathbb{N}$, such that for any p prime, $p > N$, $|f(p)|$ is also a prime.

D. Schwarz

FIFTH SELECTION TEST – ALL GEOMETRY

Problem 17. The vertices of a convex polygon are lying on a circle of center O. Prove that, for any triangulation of the polygon made by not self-intersecting diagonals, the sum of the squares of distances, from O to the incenters of the triangles in the triangulation, is the same.

AMM

Problem 18. Let $\Gamma_A, \Gamma_B, \Gamma_C$ be three circles situated in the interior of triangle ABC, such that each is tangent to the two other, Γ_A is tangent to the sides AB and AC, Γ_B is tangent to the sides BC and BA, while Γ_C is tangent to the sides CA and CB. Let D be the tangency point of Γ_B and Γ_C, E be the tangency point of Γ_C and Γ_A, and F be the tangency point of Γ_A and Γ_B. Prove that the lines AD, BE, CF are concurrent.

AMM

Problem 19. Consider the convex pentagon $ABCDE$ where $AB = BC$, $CD = DE$, angles $\angle ABC$ and $\angle CDE$ are supplementary, $\angle ABC = 135°$, and the area of the pentagon is $\sqrt{2}$.

a) Determine the length of BD.

b) Letting $\angle ABC$ be variable within the initial conditions, determine the minimum length of BD.

Adapted from Belarus Olympiad

SIXTH SELECTION TEST

Problem 20. Let $ABCD$ be a parallelogram with no angle equal to 60°. Find all pairs of points E, F, in the plane of $ABCD$, such that triangles AEB and BFC are isosceles, of basis AB, respectively BC, and triangle DEF is equilateral.

V. Vornicu

Problem 21. The world-renowned marxist theorist *Joric* is obsessed with both mathematics and social equalitarism. Therefore, for any positive integer n in its decimal representation, he tries to partition its digits into two groups, such that the difference between the sums of the digits in each group be as small as possible. Joric calls this difference the *defect* of the number n. Determine the average value of the defect (over all positive integers), that is, if we denote by $\delta(n)$ the defect of n, compute

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \delta(k)}{n}.$$

I. Boreico

Problem 22. Three travel companies provide transportation between n cities, such that each connection between a pair of cities is covered by one company only. Prove that, for $n \geqslant 11$, there must exist a round-trip through some four cities, using the services of a same company, while for $n < 11$ this is not anymore necessarily true.

D. Schwarz

SEVENTH SELECTION TEST

Problem 23. For $n \in \mathbb{N}$, $n \geqslant 2$, $a_i, b_i \in \mathbb{R}$, $1 \leqslant i \leqslant n$, such that

$$\sum_{i=1}^{n} a_i^2 = 1, \quad \sum_{i=1}^{n} b_i^2 = 1, \quad \text{and} \quad \sum_{i=1}^{n} a_i b_i = 0,$$

prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \leqslant n.$$

C. & T. Lupu

Problem 24. Let ABC be a triangle, let E, F be the tangency points of the incircle $\Gamma(I)$ to the sides AC, respectively AB, and let M be the midpoint of the side BC. Let $N = AM \cap EF$, let $\gamma(M)$ be the circle of diameter BC, and let X, Y be the other (than B, C) intersection points of BI, respectively CI, with γ. Prove that

$$\frac{NX}{NY} = \frac{AC}{AB}.$$

C. Pohoaţă

Problem 25. i) Prove that a real polynomial function f cannot be a sum of (at most) $\deg f$ real periodical functions.

ii) For $\deg f = 1$, show that f can effectively be construed as the sum of two real periodical functions.

Adapted after V. Pop

iii) For $\deg f = 1$, show that if f is the sum of two real periodical functions, they must be unbounded in any interval (thus quite "wild").

iv) Show that a real, not null, polynomial function f can effectively be construed as the sum of $\deg f + 1$ real periodical functions.

v) Exhibit a real function that cannot be construed as a (finite) sum of real periodical functions.

D. Schwarz

SELECTION TESTS FOR THE JUNIOR BALKAN MATHEMATICAL OLYMPIAD

FIRST SELECTION TEST

Problem 1. Let a and b be integer numbers. Show that there exists a unique pair of integers x, y so that

$$(x + 2y - a)^2 + (2x - y - b)^2 \leqslant 1.$$

Adrian Zahariuc

Problem 2. Consider a trapezoid $ABCD$ with the bases AB and CD so that the circles with the diameters AD and BC are secant; denote by M and N their common points. Prove that the intersection point of the diagonals AC and BD belongs to the line MN.

Severius Moldoveanu

Problem 3. A rectangular cardboard is divided successively into smaller pieces by a straight cut; at each step, only one single piece is divided in two. Find the smallest number of cuts required in order to obtain – among others – 251 polygons with 11 sides.

Marian Andronache

SECOND SELECTION TEST

Problem 4. Find all integers n, $n \geqslant 4$ such that $[\sqrt{n}] + 1$ divides $n - 1$ and $[\sqrt{n}] - 1$ divides $n + 1$.

Marian Andronache

Problem 5. Let $ABCD$ be a convex quadrilateral. The incircle ω_1 of triangle ABD touches the sides AB, AD at points M, N respectively, while the incircle ω_2 of triangle CBD touches the sides CD, CB at points P, Q respectively. Given that ω_1 and ω_2 are tangent, show that:

a) the quadrilateral $ABCD$ is circumscriptible;

b) the quadrilateral $MNPQ$ is cyclic;

c) the incircles of triangles ABC and ADC are tangent.

Vasile Pop

Problem 6. Let ABC be an acute-angled triangle with $AB = AC$. For any point P inside the triangle ABC consider the circle centered at A with radius AP and let M and N be the intersection points of the sides AB and AC with the circle. Determine the position of the point P so that $MN + BP + CP$ is minimum.

Francisc Bozgan

THIRD SELECTION TEST

Problem 7. Let ABC be a triangle. Points M, N, P are given on the sides AB, BC, CA respectively so that $CPMN$ is a parallelogram. Lines AN and MP intersect at point R, lines BP and MN intersect at point S, while Q is the intersection point of the lines AN and BP. Show that $S[MRQS] = S[NQP]$.

Mircea Lascu

Problem 8. Solve in positive integers the equation:

$$(x^2+2)(y^2+3)(z^2+4) = 60\,xyz.$$

Flavian Georgescu

Problem 9. Consider a $n \times n$ array divided into unit squares which are randomly colored in black or white. Three of the four corner squares are colored in white and the fourth is colored in black. Prove that there exists a 2×2 square which contains an odd number of white squares.

Livia Ilie

Problem 10. Suppose a, b, c are positive real numbers satisfying:

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \geqslant 1.$$

Show that

$$a+b+c \geqslant ab+bc+ca.$$

Andrei Ciupan

FOURTH SELECTION TEST

Problem 11. Find all non-empty subsets A of the set $\{2, 3, 4, 5, \ldots\}$ so that for any $n \in A$, both $n^2 + 4$ and $[\sqrt{n}] + 1$ also belong to A.

Lucian Ţurea

Problem 12. Circles ω_1 and ω_2 intersect at points A and B. A third circle ω_3, which intersects ω_1 at points D and E, is internally tangent to ω_2 at point C and tangent to the line AB at point F, and lines DE and AB meet at point G. Let H be the mirror image of F across G. Calculate the measure of the angle $\angle HCF$.

Lucian Ţurea

Problem 13. Consider the numbers from 1 to 16. A *solitaire* game is played in the following manner: the numbers are paired and each pair is replaced by the greatest prime divisor of the sum of the numbers in that pair – for example, $(1, 2); (3, 4); (5, 6); \ldots; (15, 16)$ produces the sequence 3,7,11,5,19,23,3,31. The game continues similarly until one single number is left. Find the greatest possible value of the number which ends the game.

Adrian Stoica

Problem 14. Determine all positive integers n which can be represented in the form

$$n = [a, b] + [b, c] + [c, a],$$

where a, b, c are positive integers.

Note: $[p, q]$ is the lowest common multiple of the integers p and q.

Adrian Zahariuc

FIFTH SELECTION TEST

Problem 15. Let ρ be a semicircle of diameter AB. A parallel line to AB intersects the semicircle in C and D so that points B and C lie on opposite sides of the line AD. The parallel line from C to AD meets ρ again at point E. Lines BE and CD meet at point F and the parallel line from F to AD intersects AB at point P. Prove that the line PC is tangent to the semicircle ρ.

Cosmin Pohoaţă

Problem 16. Prove that

$$\frac{x^3+y^3+z^3}{3} \geqslant xyz + \frac{3}{4}|(x-y)(y-z)(z-x)|,$$

for any real numbers $x, y, z \geqslant 0$.

Viorel Vâjâitu

Problem 17. Eight persons attend a party, and each participant has at most three others to whom he/she cannot speak. Show that the persons can be grouped in 4 pairs so that each pair can converse.

Mihai Bălună

Problem 18. A set of points is called *free* if there is no equilateral triangle whose vertices are among the points in the set. Show that any set of n points in the plane contains a free subset of at least $\sqrt{n}$ points.

Călin Popescu

SIXTH SELECTION TEST

Problem 19. A 8×8 square board is divided into 64 unit squares. A "skew-diagonal" of the board is a set of 8 unit squares with the property that each row or column of the board contains only one unit square of the set. Checkers are placed in some of the unit squares so that each "skew-diagonal" has exactly 2 squares occupied by checkers. Prove that there exist two rows or two columns which contain all the checkers.

Dinu Şerbănescu

Problem 20. Let $1 \leqslant m < n$ be positive integers, and consider the set $M = \{(x,y); x, y \in \mathbb{N}^*,\ 1 \leqslant x, y \leqslant n\}$. Determine the least value $v(m,n)$ with the property that for any subset $P \subseteq M$ with $|P| = v(m,n)$ there exist $m+1$ elements $A_i = (x_i, y_i) \in P$, $i = 1, 2, \ldots, m+1$, for which the values x_i are all distinct, and y_i are also all distinct.

Vasile Pop

Problem 21. Let ABC be a triangle right-angled at A and let D be a point on the side AC. Point E is the mirror image of A across BD and point F is the intersection of the line CE with the perpendicular line from D to CB. Show that the lines AF, DE and CB are concurrent.

Dinu Şerbănescu

Problem 22. An irrational number x, $0 < x < 1$ is called *suitable* if its first 4 decimals in the decimal representation are equal. Find the smallest positive integer n such that any real number t, $0 < t < 1$ may be written as a sum of n distinct *suitable* numbers.

Lucian Ţurea

THE 24th BALKAN MATHEMATICAL OLYMPIAD

Rhodos, April 30 – May 5, 2007

Problem 1. Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$, $AC \neq BD$, and let E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120^\circ$.

Albania

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

Bulgaria

Problem 3. Find all positive integers n such that there exists a permutation σ of the set $\{1, 2, \ldots, n\}$ for which

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}} \in \mathbb{Q}.$$

Serbia

Problem 4. For a given positive integer $n > 2$, let C_1, C_2, C_3 be the boundaries of three convex n-gons in the plane, such that all three sets $C_1 \cap C_2, C_2 \cap C_3, C_3 \cap C_1$, are finite. Find the maximum number of points of the set $C_1 \cap C_2 \cap C_3$.

Turkey

SHORTLISTED PROBLEMS FOR THE 2007 OLYMPIAD

JUNIORS

Problem 1. Show that 2007 cannot be represented as the sum of the squares of four primes.

* * *

Problem 2. In a quadrilateral $ABCD$ the bisectors of the angles $\angle A$ and $\angle C$ meet in I, situated on the diagonal BD. Prove that:

a) $AB \cdot CD = AD \cdot BC$;

b) the bisectors of the angles $\angle B$ and $\angle D$ meet in a point situated on the diagonal AC;

c) if the distances from I to the straight lines AB and BC are equal, then the diagonals AC and BD are perpendicular.

Vasile Pop

Problem 3. Let ABC be a triangle, H be its orthocenter, O be its circumcenter and M be the midpoint of the side BC. It is known that $AC = 2AB = 2AH$. Prove that:

a) $AC = 4OM$;

b) it is possible to construct a triangle with the segments AH, OM and BM.

Petre Simion

Problem 4. A square $ABCD$ is folded in such a way that the point D coincides with a point M situated on the side AB. Show that the folded area is at least a quarter of the area of the square.

Dorina Zaharia

Problem 5. Show that the $(2n+2)$-digit number $\sqrt{44\ldots 44355\ldots 556}$, ($n$ 4's and n 5's) is an integer.

Petre Bătrânețu

Problem 6. A triangle ABC has $AB = AC$ and $m(\angle BAC) = 120°$. A point D is taken on $[BC]$ such that $\frac{BD}{DC} = 2$. Prove that $BC \cdot AD = AB^2$.

Adriana and Lucian Dragomir

Problem 7. a) Prove that there does not exist a pair (a, b) of positive integers such that $3a^2 + 223b^2 = 20072007$.

b) Find all triples (a, b, k) of positive integers under the conditions:

i) $3a^2 + 223b^2 = 2007k^2$;

ii) a has exactly four positive divisors.

Adrian Turcanu

Problem 8. Let ABC be an acute angled triangle, ω its circumcircle, and H, O, I, G be the usual notations. Let H', O', I', G' be the reflections of these points into BC. Prove that:

a) $H' \in \omega$;

b) $O' \in \omega$ if and only if $m(\angle A) = 30°$;

c) $I' \in \omega$ if and only if $m(\angle A) = 60°$;

d) $G' \in \omega$ if and only if $3bc = 4m_b \cdot m_c$.

Vasile Pop

Problem 9. Consider a triangle ABC, D, M, N the midpoints of the sides (BC), (AB), (AC) respectively and P the midpoint of (AD). The parallel from the baricenter G to BC meets AB in E and AC in F.

a) Prove that the points $A, GM \cap PF$ and $GN \cap PE$ are collinear.

b) Prove that the straight lines EP, GN and DF are concurrent.

Romanța and Ioan Ghiță

Problem 10. In a tetrahedron $ABCD$, the vertex A is situated at equal distances from the sides of the triangle BCD and $\angle BAC = \angle BDC = 90°$. Let I be the incenter of the triangle BCD and P be the orthogonal projection of I on BC. Prove that:

a) $BD \cdot DC = 2 \cdot BP \cdot PC$;

b) $\frac{S_{ABC}}{S_{BCD}} = \frac{BC}{2 \cdot AP}$.

* * *

Problem 11. Prove that if in a tetrahedron the incircles of three faces are mutually tangent then all the four incircles are mutually tangent.

* * *

Problem 12. Find the maximum value of the expression $E = x + 3y$ if $x, y \in \mathbb{R}$ and $x^2 + y^2 \leqslant 2x + 6y$.

Virginia and Vasile Tică

Problem 13. Let $n > 2$ be a positive integer and let $x_1, x_2, \ldots, x_n \in [1, \infty)$ such that

$$\sum_{i=1}^{n} \sqrt{x_i - 1} \geqslant \frac{1}{n-1}\left(\sum_{i \neq j} \sqrt{x_i x_j}\right).$$

Compute $(x_1 - 1)(x_2 - 1)\ldots(x_n - 1)$.

Gheorghe Molea

Problem 14. Let a, b be positive integers such that $d = (a, b) > 1$.

a) Show that there exists a partition of $\mathbb{N}^*$ into two nonempty sets A, B with the property

$$(P): \quad x \in A \Rightarrow x + a \in A \text{ and } y \in B \Rightarrow y + b \in B.$$

b) Show that if (A, B) is a partition of $\mathbb{N}^*$ with the property (P), then, for every $z \in \mathbb{N}^*$, the numbers z and $z + d$ are both in A or in B.

Vasile Pop

Problem 15. Let $a, b, c > 0$ be real numbers such that $a+b+c = 1$. Prove that

$$\frac{\sqrt{ab}}{1-c} + \frac{\sqrt{bc}}{1-a} + \frac{\sqrt{ca}}{1-b} \leqslant \frac{1}{8}\left(3 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Laura Molea and Gheorghe Molea

Problem 16. Let $VABC$ be a triangular pyramid and M be a variable point on the small arc AB from the circumcircle $C(O, R)$ of triangle ABC. Let VD, VE, VF be the distances from V to the straight lines MC, MA, MB respectively.

Prove that

$$\max(VD^2 + VE^2 + VF^2) = 3\left(VO^2 + \frac{R^2}{2}\right).$$

Virginia and Vasile Tică

Problem 17. Let $ABCDA'B'C'D'$ be a cube and M, N, P, Q be points on the segments $(A'B)$, $(B'C)$, $(C'D)$, respectively $(D'A)$, such that $\frac{A'M}{MB} = \frac{B'N}{NC} = \frac{DP}{PC'} = \frac{AQ}{QD'} = k$.

a) Prove that the points M, N, P, Q are coplanar if and only if $k = 1$.

b) Find k if the angle between the straight line $C'M$ and the plane ADB' has measure $15°$.

Petre Simion

SENIORS

Problem 18. Let $ABCD$ be a rectangle and M, N be two points separated from D by AB, respectively BC, such that $MA = MB$ and $NB = NC$. Prove that the triangle DMN is equilateral if and only if the triangles MAB and NBC are equilateral.

* * *

Problem 19. The points A, B, C are inside the convex hexagon $MNPQRS$, such that the triangles ABC, NAM, PQB and CRS are similar. Let X, Y, Z be the midpoints of the segments $[NP]$, $[QR]$, respectively $[SM]$ and G, K, I the baricenters of the triangles ABC, MPR, respectively NQS. Prove that:

a) if triangle ABC is equilateral then triangle GKI is equilateral;

b) triangles ABC and XYZ are similar if and only if triangle ABC is equilateral.

Dana Heuberger

Problem 20. Let $p \in \mathbb{N}$, $p \geqslant 2$. Find all the increasing functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f_1(n) + f_2(n) + \cdots + f_p(n) = pn + \frac{1}{2}p(p+1), \quad \forall n \in \mathbb{Z},$$

where $f_k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$.

Marin Ionescu

Problem 21. Prove that if a function $f : \mathbb{R} \to \mathbb{R}$ has the property

$$\left|\sum_{k=1}^{n} 2^k \left(f(x+ky) - f(x-ky)\right)\right| \leqslant 1, \quad \forall n \in \mathbb{N}^*, \forall x, y \in \mathbb{R}$$

then f is constant.

Farcas Csaba

Problem 22. Prove that, for every real numbers $a, b, c > 0$,

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{a+c} + \frac{c^2+1}{a+b} \geqslant 3.$$

Petre Bătrânețu

Problem 23. Prove that if $a, b, c \in [0,1]$ then

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} + abc \leqslant \frac{5}{2}.$$

Vasile Pop

Problem 24. Prove that if $a_1, a_2, \ldots, a_n > 0$ and $a_1 + a_2 + \cdots + a_n = 1$ then

$$\frac{(a_2 + a_3 + \cdots + a_n)^2}{1+a_1} + \frac{(a_1 + a_3 + \cdots + a_n)^2}{1+a_2} + \cdots + \frac{(a_1 + a_2 + \cdots + a_{n-1})^2}{1+a_n} \geqslant \frac{(n-1)^2}{n+1}.$$

Traian Tămâian

Problem 25. Prove that, in every triangle,

$$\frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \geqslant \frac{2(a+b+c)}{\sqrt{a^2+b^2+c^2}}.$$

Petre Bătrânețu

Problem 26. Prove that, for every real numbers $a, b, c > -\frac{1}{2}$,

$$\frac{a^2+2}{b+c+1} + \frac{b^2+2}{a+c+1} + \frac{c^2+2}{a+b+1} \geqslant 3.$$

Marin Ionescu

Problem 27. Let $n \in \mathbb{N}^*$. Solve in $\mathbb{N}^*$ the equation

$$\left\lfloor \frac{x^2}{n} \right\rfloor + \left\lfloor \frac{n^2}{x} \right\rfloor = \left\lfloor \frac{x}{n} + \frac{n}{x} + nx \right\rfloor.$$

Marcel Chiriţă

Problem 28. Solve the system

$$\begin{cases} x(3y^2+1) = y(y^2+3) \\ y(3z^2+1) = z(z^2+3) \\ z(3x^2+1) = x(x^2+3). \end{cases}$$

Marcel Chiriţă

Problem 29. Find all the prime numbers $p \geqslant 3$ such that, for every $k \in \overline{1, \frac{p-1}{2}}$, the number $1 + k(p-1)$ is prime.

Adrian Stoica

Problem 30. Let z_1, z_2, z_3 be complex distinct numbers, with $|z_1| = |z_2| = |z_3|$. Prove that the points of complex coordinate z_1, z_2, z_3 are the vertices of an equilateral triangle if and only if there exists $k \in \mathbb{R} \setminus \{1\}$ such that

$$|kz_1 + z_2 + z_3| = |kz_2 + z_3 + z_1| = |kz_3 + z_1 + z_2|.$$

Marin Ionescu

Problem 31. A set M of real numbers fulfils:

i) $0 \in M$;

ii) if $x, y > 0$ and $\log_2(x+y) \in M$ then $3^x \in M$ and $\log_4 y \in M$.

Prove that $\frac{2008}{2007} \in M$.

Lucian Dragomir

Problem 32. Solve the equation $2^{\tan x} + 2^{\cot x} = 2\cot 2x$.

Traian Tămâian

Problem 33. Prove that, in every triangle,

$$\left(\frac{4R+r}{p}\right)^2 + \frac{9r}{4R+r} \geqslant 4.$$

Cosmin Pohoaţă

Problem 34. Let z be a complex number such that $|z|$, $|z^2 - 2z + 2|$ and $|z^2 - 3z + 3|$ are at most 1. Prove that $z = 1$.

Virgil Nicula

Problem 35. Let z_1, z_2, z_3 be distinct complex numbers. Prove that the following properties are equivalent:

i) z_1, z_2, z_3 are the vertices of an equilateral triangle;

ii) there exists $\lambda \in \mathbb{C}$ such that the polynomial $(X - z_1)(X - z_2)(X - z_3) - \lambda$ has a triple root.

Sever Moldoveanu, Constantin Buşe

PUTNAM SENIORS

Problem 36. Prove that the sequence $(a_n)_{n \geqslant 1}$ given by

$$a_n = \{n\sqrt{2}\} + \{n\sqrt{3}\}, \quad \forall n \geqslant 1,$$

is divergent.

Alin Gălăţan, Cezar Lupu

Problem 37. Prove that the sequence defined by $a_0 = a$, $a_1 = b$, $a_{n+1} = \frac{a_n^x + a_{n-1}^x}{a_n^{x-1} + a_{n-1}^{x-1}}$, $n \in \mathbb{N}^*$, where $a > 0$, $b > 0$, $x > 1$, is convergent.

Marian Ionescu

Problem 38. Suppose that a differentiable real function has a periodical non-constant derivative $f : \mathbb{R} \to \mathbb{R}$. Prove that f has at least positive period.

Cristinel Mortici

Problem 39. Prove that, for every continuous function $f : [0,1] \to \mathbb{R}$,

$$\frac{1}{4}\int_0^1 f^2(x)\mathrm{d}x + 2\left(\int_0^1 f(x)\mathrm{d}x\right)^2 \geqslant 3\int_0^1 f(x)\mathrm{d}x \cdot \int_0^1 xf(x)\mathrm{d}x.$$

Cezar Lupu

Problem 40. a) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which has lateral limits in every point and $f(\mathbb{R}) = \mathbb{Q}$.

b) Is there a function $f : [0,1] \to \mathbb{R}$ which has lateral limits in every point and $f([0,1]) = [0,1] \cap \mathbb{Q}$?

c) Is it possible for the function from a) to be nonconstant on every interval?

Gabriel Dospinescu, Adrian Zahariuc

Problem 41. Find card $\{A \in \mathcal{M}(\mathbb{N}))|) \det A \neq 0, A^{-1} \in \mathcal{M}_n(\mathbb{N})\}$.

Cecilia Diaconescu, Piteşti

Problem 42. A function $f : \mathbb{Q} \to \mathbb{R}$ has the property: there exists $a > 0$ such that $|f(x) - f(y)| \leqslant |x - y|^a$ for every $x, y \in \mathbb{Q}$.

a) Prove that there exists an unique continuous function $\bar{f} : \mathbb{R} \to \mathbb{R}$ such that $\bar{f}(x) = f(x)$ for every $x \in \mathbb{Q}$.

b) Find all possible f in the case $a > 1$.

Atila Abdula

Problem 43. Let G be a finite group and $p_1, p_2, \ldots, p_n$ be distinct prime divisors of card (G), such that for each p_i there exists x_i, y_i in G with $\operatorname{ord}(x_i) = \operatorname{ord}(y_i) = p_i$ and y_i is not a power of x_i.

Prove that

$$\operatorname{card} G \geqslant 1 + \prod_{i=1}^{n}(p_i^2 - 1).$$

Octavian Ganea

Problem 44. Let $f : [0,1] \to \mathbb{R}$ be a concave function, with $f(0) = 1$. Prove that

$$\frac{3}{2}\int_0^1 xf(x)\mathrm{d}x \leqslant \int_0^1 f(x)\mathrm{d}x - \frac{1}{4},$$

and find the cases of equality

Dan Marinescu, Viorel Cornea

Problem 45. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that, for every a, b, c, d,

$$\int_a^b f(t)\mathrm{d}t + \int_c^d f(t)\mathrm{d}t = \int_{a+c}^{b+d} f(t)\mathrm{d}t.$$

Romanţa and Ioan Ghiţă

Problem 46. Let p be a prime number and A be an unitary ring with p^2 elements and less than $p - 1$ bilateral zero divisors. Prove that A is a field.

Alin Gălăţan

PART TWO

SOLUTIONS

PROBLEMS AND SOLUTIONS

DISTRICT ROUND

7^{th} GRADE

Problem 1. Point O is the intersection of bisector lines of the sides of triangle ABC. Denote by D the intersection of line AO with the segment BC. If $OD = BD = \frac{1}{3} \cdot BC$, find the angles of the triangle ABC.

Solution. The following two cases are possible:

I. Triangle ABC is acute. Then O is an interior point of ABC. Denoting by E the midpoint of DC, the triangle ODE is equilateral.

As a consequence, $\angle OEC = \angle ODB = 120°$ and $\angle OBC = \angle OCB = 30°$. As triangle AOC is right and isosceles, we get $\angle OAC = \angle OCA = 45°$. In the triangle AOB we have $\angle AOB = 150°$ and $\angle OAB = \angle OBA = 15°$. Thus $\angle ABC = 45°, \angle ACB = 75°$ and $\angle BAC = 60°$.

II. Triangle ABC is not acute. As line AO is incidental to the segment BC, A is the obtuse angle of the triangle ABC. As above, we get $\angle OAC = \angle OCA = 45°$, implying $\angle ACB = \angle OCA - \angle ECO = 15°$. As the triangle AOB is isosceles with $\angle AOB = 30°$, we get $\angle ABO = \angle BAO = 75°$. As a consequence $\angle BAC = 120°$ and $\angle ABC = 45°$.

Problem 2. A can contains blue and red stones. Someone invented the following game: extracts succesively stones until, for the first time, the extracted blue stones and the extracted red stones are in the same number. At one of the played games one observes that, at the end, 10 stones were extracted and no 3 consecutive extracted stones are of the same color. Prove that at that game the fifth and the sixth stones have different colors.

Solution. WLOG, consider that the last stone is red. Then, the ninth is red also, otherwise the extracting procedure had to be stopped before. The eighth stone has to be blue, otherwise the last three will have the same color. The seventh is then red, otherwise the game has to be over after six extractions. Now, if stones 5 and 6 are both red, we get 3 consecutive red, a contradiction. If stones 5 and 6 are both blue, the game has to be over after four extractions, a contradiction again. Thus, stones 5 and 6 have different colors.

Problem 3. Let a and b be integers such that $b > a \geqslant 2$. Prove that if the number $a+k$ is prime with the number $b+k$ for all $k = 1, 2, \ldots, b-a$, then a and b are consecutive.

Solution. Denote $n = b-a$. We have $(a+k, b+k) = (a+k, b+k-a-k) = (a+k, n) = 1$, for any $k = 1, 2, \ldots, n$. The sequence $a+1, a+2, \ldots, a+n$ contains n consecutive numbers, thus one of them is divisible by n.

The only possibility is $n = 1$, otherwise n implying that a and b are consecutive.

Problem 4. Let n be a composite natural number. Prove that there are integers $k > 1$ and $a_1, a_2, \ldots, a_k > 1$ such that

$$a_1 + a_2 + \cdots + a_k = n \cdot \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} \right).$$

Solution. Take k to be the number of the proper divisors of n; as n is composite we have $k > 1$. Let $1 < a_1 < a_2 < \cdots < a_k$ be the proper divisors of n; then $\frac{n}{a_1} > \frac{n}{a_2} > \cdots > \frac{n}{a_k}$ are the same set of divisors, implying the result.

8th GRADE

Problem 1. Consider the positive real numbers x, y, z, with the property that $xy = \frac{z-x+1}{y} = \frac{z+1}{2}$. Prove that one of them is the average of the other two.

Solution. We have $z = xy^2 + x - 1$ and $z = 2xy - 1$. We get $x(y^2 - 2y + 1) = x(y-1)^2 = 0$, so $y = 1$ because $x \neq 0$. From $xy = \frac{z+1}{2}$ we obtain $x = \frac{z+y}{2}$, that is x the average of y and z.

Problem 2. A rectangle $ABCD$ has $AB = 2$ and $BC = \sqrt{3}$. Point M is on AD such that $MD = 2 \cdot AM$ and point N is the midpoint of AB. MP is perpendicular to the rectangle's plane and point Q is on the segment MP such that the angle between planes (MPC) and (NPC) is 45°, and the angle between planes (MPC) and (QNC) is 60°.

a) Prove that DN and CM are perpendicular.

b) Prove that Q is the midpoint of MP.

Solution. a) As $AN = 1$ and $DM = \frac{2\sqrt{3}}{3}$ we get $\frac{AN}{AD} = \frac{1}{\sqrt{3}} = \frac{DM}{DC}$, that is $\angle MCD = \angle ADN$ implying $DN \perp CM$.

b) By the above, DN is perpendicular to the plane (PMC). Let $T \in CM \cap DN$ and let R, S be the projections of T on CQ and on PC, respectively. By the three perpendiculars theorem $\angle TRN = 60°$ and $\angle TSN = 45°$. Easy calculations give $CT = \sqrt{3}, RC = \frac{2\sqrt{6}}{3}$ and $MC = \frac{4\sqrt{3}}{3}$. As a consequence $TN = 1$, giving $TR = \frac{1}{\sqrt{3}}$ and $TS = 1$. Similarities $\Delta CRT \sim \Delta CQM$ and $\Delta CTS \sim \Delta CMP$ imply $QM = \frac{RT \cdot MC}{RC} = \frac{\sqrt{6}}{3}$ and $MP = \frac{ST \cdot MC}{SC} = \frac{2\sqrt{6}}{3}$, that is $MQ = QP$.

Problem 3. Eight consecutive natural numbers are partitioned into two classes, each of four numbers. Prove that if the sum of the squares of the numbers in each class is the same, then the sum of elements in each class is the same.

Solution. Let $a, a+1, a+2, \ldots, a+7$ be the given eight numbers. The sum of their squares is $8a^2 + 56a + 140$, implying that the sum of squares of each class is $4a^2 + 28a + 70$. Denote by X the class that contains $a+7$ and let $a+i, a+j, a+k$ the other three of its elements. Then $3a^2 + 2a(i+j+k) + i^2 + j^2 + k^2 = 4a^2 + 28a + 70 - (a+7)^2 = 3a^2 + 14a + 21$, that is $2a(i+j+k-7) = 21 - (i^2+j^2+k^2)$. Let us remark, at this point, that the problem consists in showing that $i+j+k = 7$.

Suppose that $i+j+k \geqslant 8$. Then $i^2 + j^2 + k^2 \geqslant \frac{(i+j+k)^2}{3} \geqslant \frac{64}{3} > 21$, giving that the right side of the previous inequality is negative, a contradiction.

Suppose $i+j+k \leqslant 6$. As $21-(i^2+j^2+k^2)$ is even, $i^2+j^2+k^2$ is odd and then $i+j+k$ is odd. This implies that i, j, k is the triple 0,1,2 or 0,1,4 or 0,2,3. In all cases $21-(i^2+j^2+k^2)>0>2a(i+j+k-7)$, a contradiction.

Problem 4. All points on a circle are painted in green or yellow such that each inscribed equilateral triangle has exactly two yellow vertices. Prove that one can find an inscribed square which has at least three yellow vertices.

Solution. Let $A_1A_2\ldots A_{12}$ be a regular dodecagon inscribed in the given circle. Triangles $A_1A_5A_9$, $A_2A_6A_{10}$, $A_3A_7A_{11}$ and $A_4A_8A_{12}$ are equilateral, so 8 vertices of the 12 are yellow. Consider the squares $A_1A_4A_7A_{10}$, $A_2A_5A_8A_{11}$ and $A_3A_6A_9A_{12}$. By Dirichlet principle we get that at least one of the squares has at least three vertices colored in yellow.

9th GRADE

Problem 1. Let $k \in \mathbb{N}^*$. We will say a function $f : \mathbb{N} \to \mathbb{N}$ holds property $(\mathcal{P})$ if for any $y \in \mathbb{N}$ the equation $f(x) = y$ has exactly k solutions.

a) Show there exist infinitely many functions holding property $(\mathcal{P})$.

b) Determine the monotone functions holding property $(\mathcal{P})$.

c) Determine if for $k > 1$ there exist monotone functions $f : \mathbb{Q} \to \mathbb{Q}$ holding property $(\mathcal{P})$.

Solution. (D. Schwarz) a) Let $\sigma : \mathbb{N} \to \mathbb{N}$ be one-to-one, and let $f_\sigma : \mathbb{N} \to \mathbb{N}$ be defined through $f_\sigma(x) = \sigma(\lfloor \frac{x}{k} \rfloor)$. Obviously, functions f_σ hold property $(\mathcal{P})$, while the set of these functions is infinite (in fact not even countable).

b) Were f to be decreasing, the sequence $(f(n))_{n\geqslant 0}$ will eventually become stationary, in contradiction with $(\mathcal{P})$. Therefore, we shall look for f monotonically increasing. But then the first k values in $(f(n))_{n\geqslant 0}$ have to be 0, the next k have to be 1, and so on. The only possibility therefore remains $f(x) = f_{\text{id}}(x) = \lfloor \frac{x}{k} \rfloor$.

c) If $f(x_1) = f(x_2) = y$, $x_1 \neq x_2$, as between x_1 and x_2 there exist infinitely many rational numbers x (for example taking arithmetical means), and f is monotone, it follows $f(x) = y$ for all these values x, in contradiction with $(\mathcal{P})$, hence no such functions do exist.

Problem 2. Given triangle ABC and points $M \in (AB)$, $N \in (BC)$, $P \in (CA)$, $R \in (MN)$, $S \in (NP)$, $T \in (PM)$, such that

$$\frac{AM}{MB} = \frac{BN}{NC} = \frac{CP}{PA} = \lambda, \quad \frac{MR}{RN} = \frac{NS}{SP} = \frac{PT}{TM} = 1-\lambda, \quad \lambda \in (0,1),$$

a) Prove that triangles STR and ABC are similar.

b) Determine the value for parameter λ which makes the area of triangle STR minimal.

Solution. a) Using **bold** letters to denote the vectors determined by segments

$$\mathbf{RT} = \mathbf{RM} + \mathbf{MT} = \frac{1-\lambda}{2-\lambda}\mathbf{NM} + \frac{1}{2-\lambda}\mathbf{MP},$$

hence

$$\mathbf{RT} = \frac{1-\lambda}{2-\lambda}(\mathbf{NB} + \mathbf{BM}) + \frac{1}{2-\lambda}(\mathbf{MA} + \mathbf{AP}).$$

After *de rigueur* computations, $\mathbf{RT} = \rho\mathbf{BC}$, $\mathbf{TS} = \rho\mathbf{AB}$, $\mathbf{SR} = \rho\mathbf{CA}$, so $\mathbf{RT}, \mathbf{TS}, \mathbf{SR}$ are parallel respectively to $\mathbf{BC}, \mathbf{AB}, \mathbf{CA}$, the triangles STR and ABC are similar, and their similarity ratio is $\rho = \frac{1-\lambda+\lambda^2}{2+\lambda-\lambda^2} > 0$ for $\lambda \in (0,1)$.

b) Because the ratio of the areas of the two triangles is ρ^2, the area of triangle STR will be minimal when the value of ρ is minimal. From the formula for ρ follows the equation $(1+\rho)\lambda^2 - (1+\rho)\lambda + 1 - 2\rho = 0$. Its discriminant is $\Delta = 3(1+\rho)(3\rho-1)$, so the condition $\Delta \geqslant 0$ (to warrant λ's existence) comes to $\rho \notin (-1, \frac{1}{3})$, therefore the minimal value of ρ is $\frac{1}{3}$, when $\lambda = \frac{1}{2}$.

Problem 3. Determine the functions $f : \mathbb{N}^* \to \mathbb{N}^*$ for which

$$x^2 + f(y) \text{ divides } f(x)^2 + y$$

for all $x, y \in \mathbb{N}^*$.

Solution. (D. Schwarz) For $x = y = 1$ we get

$$\frac{f(x)^2 + y}{x^2 + f(y)} = \frac{f(1)^2 + 1}{1 + f(1)} = f(1) - 1 + \frac{2}{1 + f(1)} \in \mathbb{N}^*,$$

therefore $f(1) = 1$. Now, as $x^2 + f(y)$ divides $f(x)^2 + y$, it follows that

$$x^2 + f(y) \leqslant f(x)^2 + y.$$

For $y = 1$, $f(x)^2 - x^2 \geqslant f(1) - 1 = 0$, therefore, for all $x \in \mathbb{N}^*$,

$$(1) \qquad f(x) \geqslant x.$$

For $x = 1$, we get $1 + y \geqslant 1 + f(y)$, therefore, for all $y \in \mathbb{N}^*$,

$$(2) \qquad f(y) \leqslant y.$$

Relations (1) and (2) together yield as unique solution f the *identity* function $\mathrm{id}(n) \equiv n$, while divisibility is trivially fulfilled by $x^2 + f(y) = f(x)^2 + y = x^2 + y$.

Problem 4. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be coplanar vectors, each of module 1.

a) Prove that we can choose signs $+, -$, such that $| \pm \mathbf{u} \pm \mathbf{v} \pm \mathbf{w}| \leqslant 1$.

b) Exhibit a triplet such that, no matter how we choose signs $+, -$, we get $| \pm \mathbf{u} \pm \mathbf{v} \pm \mathbf{w}| \geqslant 1$.

Solution. We will consider all vectors *anchored* at the origin O of a coordinates system.

a) If for any two of the given vectors, be them $\mathbf{x}, \mathbf{y}$ we have $\mathbf{x} = \pm\mathbf{y}$, the conclusion is clear. We may thus assume, in the sequel, that this does not occur.

Solution 1. (B. Enescu) The endpoints of vectors $\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, -\mathbf{u}-\mathbf{v}$ and $-\mathbf{u}+\mathbf{v}$ make out a rhombus of side 2. The radius 1 discs centered at these endpoints will evidently cover the circumference of the radius 1 circle centered at O, therefore the endpoint of vector $\mathbf{w}$ is situated at a distance not larger than 1 from one of these endpoints.

Solution 2. (M. Andronache) If the endpoints of the given vectors do not make out an acute triangle, they will be lying on a hemicircle of the radius 1 circle centered at O, and then, considering the opposite vector to the one with its endpoint lying between the other two, its endpoint, together with the other two, will make out an acute triangle. Let then $\mathbf{x}, \mathbf{y}, \mathbf{z}$ represent $\mathbf{u}, \mathbf{v}, \mathbf{w}$, respectively $-\mathbf{u}, \mathbf{v}, \mathbf{w}$ or $\mathbf{u}, -\mathbf{v}, \mathbf{w}$ or $\mathbf{u}, \mathbf{v}, -\mathbf{w}$, such that their endpoints make out an acute triangle XYZ. Consider the vector $\mathbf{h} = \mathbf{x} + \mathbf{y} + \mathbf{z}$. We have

$$\langle \mathbf{h} - \mathbf{x}, \mathbf{y} - \mathbf{z} \rangle = \langle \mathbf{y} + \mathbf{z}, \mathbf{y} - \mathbf{z} \rangle = |\mathbf{y}|^2 - |\mathbf{z}|^2 = 0$$

and the related other relations, therefore $\mathbf{h}$ corresponds to the orthocenter H of triangle XYZ. But then $H \in \Delta XYZ$, hence $OH = |\mathbf{h}| \leqslant 1$.

In effect, a purely vectorial solution (with no geometrical justifications) can be produced, but it will have to contain precise and detailed considerations, in order to remedy the lack of the geometrical argument.

b) A possible example is given by $\mathbf{u} = (1,0)$, $\mathbf{v} = (0,1)$, $\mathbf{w} = (0,-1)$.

10th GRADE

Problem 1. The real numbers a, b, c are such that $a, b, c \in (1, \infty)$ or $a, b, c \in (0, 1)$. Prove that

$$\log_a bc + \log_b ca + \log_c ab \geqslant 4(\log_{ab} c + \log_{bc} a + \log_{ca} b).$$

Solution. Take a logarithmic basis d belonging to the same interval as a, b, c. Taking logarithms in that base, the inequality becomes

$$\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} \geqslant \frac{4x}{y+z} + \frac{4y}{z+x} + \frac{4z}{x+y}, \quad \forall x, y, z > 0$$

It will suffice to show that $\frac{z}{x} + \frac{z}{y} \geqslant 4\frac{z}{x+y}$.

The last is equivalent to the arithmetic-harmonic inequality for $1/x, 1/y$.

Problem 2. The $2n$ squares composing a rectangle of dimension $2 \times n$ are painted with three colors. We say that a color has a *cutting* if on one of the columns we have two squares of the same color. Find:

a) the number of coloring without cuttings;

b) the number of coloring with exactly one cutting.

Solution. a) Colors on the first row can be chosen in 3^n ways. A coloring of the rectangle corresponds to a coloring of the first row associated with the admitted coloring of the second one. For each coloring of the first row we have 2^n colorings that satisfy the condition for the second one; thus the number is $3^n 2^n = 6^n$.

b) A coloring with a unique cutting corresponds to an arbitrary coloring of the first row combined with choosing the columns containing the cutting and to the coloring of the rest of the second row; thus in this case we have $3^n \cdot n \cdot 2^{n-1} = 3n \cdot 6^{n-1}$ colorings.

Problem 3. Let ABC a triangle with $BC = a$, $CA = b$, $AB = c$. For each line Δ we denote by d_A, d_B, d_C the distances from A, B, C to Δ. Consider

$$E(\Delta) = ad_A^2 + bd_B^2 + cd_C^2.$$

Prove that if the value of $E(\Delta)$ is minimal, then Δ contains the incenter of the triangle.

Solution. We show that if Δ does not contain I, the incenter, and Δ' is the line parallel to Δ that contains I, then $E(\Delta') < E(\Delta)$. Denote by A_1, B_1, C_1, I_1 the projections of A, B, C, I onto Δ and by A', B', C' the projections of A, B, C onto Δ'. We have

$$(d_A)^2 - (d'_A)^2 = AA_1^2 - AA'^2 = (AI_1^2 - A_1I_1^2) - (AI^2 - A'I^2) = AI_1^2 - AI^2$$

and the other similar ones. Define $f(M) = aMA^2 + bMB^2 + cMC^2$, where M is a point in the plane. We get $E(\Delta) - E(\Delta') = f(I_1) - f(I)$. For an arbitrary point O we have

$$\begin{aligned} f(M) &= a(\overrightarrow{OA} - \overrightarrow{OM})^2 + b(\overrightarrow{OB} - \overrightarrow{OM})^2 + c(\overrightarrow{OC} - \overrightarrow{OM})^2 \\ &= a\overrightarrow{OA}^2 + b\overrightarrow{OB}^2 + c\overrightarrow{OC}^2 - 2\overrightarrow{OM}(a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}) + (a+b+c)\overrightarrow{OM}^2 \\ &= a\overrightarrow{OA}^2 + b\overrightarrow{OB}^2 + c\overrightarrow{OC}^2 - 2\overrightarrow{OM}(a+b+c)\overrightarrow{OI} + (a+b+c)\overrightarrow{OM}^2 \\ &= a\overrightarrow{OA}^2 + b\overrightarrow{OB}^2 + c\overrightarrow{OC}^2 - (a+b+c)\overrightarrow{OI}^2 + (a+b+c)\overrightarrow{IM}^2, \end{aligned}$$

where we used $\overrightarrow{OI} = \frac{a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC}}{a+b+c}$.

For $O = I$ we obtain $f(M) = (a+b+c)IM^2 + aIA^2 + bIB^2 + cIC^2$, thus $E(\Delta) - E(\Delta') = f(I_1) - f(I) = (a+b+c)II_1^2 > 0$.

Problem 4. Let u, v, w be complex numbers of modulus 1. Prove that one can choose signs $+$ and $-$ such that

$$|\pm u \pm v \pm w| \leqslant 1.$$

Solution. Denote by capitals the points having complex coordinates the corresponding small letter. Using the Sylvester formula, we get that $u + v + w$ is the complex coordinate of the orthocenter H of the triangle UVW.

In case UVW is acute or right we take all signs to be $+$ and this gives the solution because H is interior to UVW, so interior to the circumcircle.

In the other case, one angle is obtuse, say W. Then for $w' = -w$, we get the acute triangle UVW', reducing the problem to the first case.

11th GRADE

Problem 1. Let $a \in (0,1)$ and $(a_n)_{n\geqslant 1}$ the sequence defined by $x_{n+1} = x_n(1 - x_n^2)$, for any $n \geqslant 10$, starting with $x_0 \in (0,1)$.

Calculate $\lim\limits_{n\to\infty} \sqrt{n} \cdot a_n$.

Solution. (D. Schwarz) It is immediately established, through simple induction, that $x_n > 0$. Now, $x_{n+1} - x_n = -x_n^3 < 0$, so $(x_n)_{n\geqslant 0}$ is strictly decreasing, therefore convergent to $\lim\limits_{n\to\infty} x_n = \inf\limits_{n\geqslant 0} x_n = l$, so $l = l(1 - l^2)$, yielding $l = 0$.

Define $y_n := \frac{1}{x_n}$, and $z_n := y_n^2$, so $(z_n)_{n\geqslant 0}$ is strictly increasing, $z_1 > 1$, and $\lim\limits_{n\to\infty} z_n = \infty$. Since $y_{n+1} = y_n(1 + \frac{1}{y_n^2 - 1})$, it follows

$$z_{n+1} = z_n + 2 + \frac{3}{z_n - 1} + \frac{1}{(z_n - 1)^2}.$$

Finally, we get

$$\lim_{n\to\infty} \frac{z_n - 2n}{\frac{3}{2}\ln n} = 1, \quad \text{so } \lim_{n\to\infty} \frac{z_n}{n} = 2.$$

This yields more than the asked result.

Remark. The limit can be also easily found using the Stolz-Cesaro lemma in an obviuos way.

Problem 2. Let $A \in \mathcal{M}_n(\mathbb{R}^*)$. If $A \cdot {}^tA = I_n$, prove that:

a) $|\mathrm{tr}\,(A)| \leqslant n$;

b) for n odd, $\det(A^2 - I_n) = 0$.

Solution. a) Consider $\lambda \in \mathbb{C}$ a root of the polynomial $\det(A - XI_n) = 0$ (a proper value of A). The system of equations $AX = \lambda X$, where X is a column complex matrix, has a non-zero solution. By complex conjugation and transposition, we get $\overline{{}^tX \cdot {}^tA} = \overline{\lambda {}^tX}$. Multiplying both relations we obtain $\overline{{}^tX}{}^tA \cdot A \cdot X = |\lambda|^2 \overline{{}^tX} \cdot X$. Because $A \cdot {}^tA = I_n$ and $\overline{{}^tX} \cdot X$ is a positive number, we get $|\lambda|^2 = 1$ so $|\lambda| = 1$.

As $\operatorname{tr}(A) = \sum \lambda$, the sum being taken over all the n proper values, including multiplicities, the triangle inequality finishes the proof.

b) If n is odd, the characteristic polynomial has at least a real root. It has to be 1 or -1, so $\det(A - I_n) = 0$ or $\det(A + I_n) = 0$, implying $\det(A^2 - I_n) = 0$.

Remark. An elementary solution can be also given. By matrix calculations, the main diagonal of ${}^tA \cdot A$ contains sums of the form $\sum a_{ij}^2$, so obviously $a_{ii}^2 \leqslant 1$.

For the second part, the given property makes the equality $\det(A^2 - I_n) = 0$ equivalent to $\det({}^tA - A) = 0$. For odd n, the system of linear equations $({}^tA - A)X = 0$, with X column matrix, has the nontrivial solution $X = {}^t(1, -1, 1, \dots, 1)$, implying that the determinant is zero.

Problem 3. Consider the sequence $(x_n)_{n\geqslant 1}$ given by $x_n = \sqrt{n} - [\sqrt{n}]$. Denote by A the set of its limit points, that is, the set of $x \in \mathbb{R}$ such that there is a subsequence of $(x_n)_n$ with limit x.

a) Prove that $\mathbb{Q} \cap [0,1] \subset A$.

b) Determine the set A.

Solution. We shall present a completely elementary solution, even if by taking the subsequence $x_{2n^2} = n\sqrt{2}$ the Kroneker theorem gives the result.

a) Let $r = \frac{p}{q}$ be an arbitrary rational number in $[0,1]$, $p, q \in \mathbb{N}$, $q \neq 0$. The sequence $(n_k)_k$ given by $n_k = q^2k^2 + 2pk$ is increasing and $[\sqrt{n_k}] = qk$. Thus

$$x_{n_k} = \sqrt{q^2k^2 + 2pk} - qk = \frac{2p}{q\left(1 + \sqrt{1 + \frac{2p}{q^2k}}\right)} \to \frac{p}{q}.$$

b) We show that $A = [0,1]$. Obviously $A \subset [0,1]$, and by a) we get that for any $r \in [0,1] \cap \mathbb{Q}$, any $\varepsilon > 0$ and any $n \in \mathbb{N}^*$, there is $n > n_0$ such that $|x_n - r| < \varepsilon$.

We shall prove that any irrational $\alpha \in (0,1)$ is in A. Inductively we find $(n_k)_k$ such that $|x_{n_k} - \alpha| < \frac{1}{k}$: for chosen $n_1 < n_2 < \cdots < n_k$, find $r \in \mathbb{Q} \cap [0,1]$ such that $|r - \alpha| < \frac{1}{2(k+1)}$ and using the above remark with $n_0 = n_k$ and $\varepsilon = \frac{1}{2(k+1)}$, there is $n_{k+1} > n_k$ with $|x_{n_{k+1}} - r| < \frac{1}{2(k+1)}$. Then

$$|x_{n_{k+1}} - \alpha| \leqslant |x_{n_{k+1}} - r| + |r - \alpha| < \frac{1}{k+1}.$$

Obviously, $\lim\limits_{k} x_{n_k} = \alpha$.

Problem 4. Let $A, B \in \mathcal{M}_n(\mathbb{R})$ such that $B^2 = I_n$ and $A^2 = AB + I_n$. Prove that $\det(A) \leqslant \left(\frac{1+\sqrt{5}}{2}\right)^n$.

Solution. Denote by $(f_k)_k$ the Fibonacci sequence:

$$f_k = \frac{1}{\sqrt{5}}\left[\varphi^k - \bar{\varphi}^k\right],$$

where $\varphi = \frac{1+\sqrt{5}}{2}$. By easy induction we obtain $A^k = f_kA + f_{k-1}B$ for any odd k. Then $\frac{1}{f_k}A^k = A + \frac{f_{k-1}}{f_k}B$ and making $k \to \infty$ we get

$$\det\left(\frac{1}{f_k}A^k\right) \to \det(A - \bar{\varphi}B) \in \mathbb{R}.$$

On the other side, if we suppose that $\det A > \varphi^n$, we obtain

$$\det\left(\frac{1}{f_k}A^k\right) = \frac{1}{f_k^n}(\det A)^k = \left(\frac{\varphi^k}{f_k}\right)^n \left(\frac{\det A}{\varphi^n}\right)^k,$$

which is infinity when k tends to infinity.

Remarks. This is the best possible result: take $B = I_n$ and $A = \varphi I_n$.

One easily obtains from the hypothesis $AB = BA$. Squaring the second relation we get $A^4 - 3A^2 + I = 0$. The equation $t^4 - 3t^2 + 1 = 0$ has the golden number as the root of maximal modulus. This concludes the problem.

12th GRADE

Problem 1. Given a group $(G, *)$ and A, B nonempty subsets of G, we define $A * B = \{a * b)|\rangle a \in A\rangle\rangle\text{and}\rangle b \in B\}$.

a) Prove that for $n \in \mathbb{N}$, $n \geqslant 3$, the group $(\mathbb{Z}_n, +)$ can be written in the form $\mathbb{Z}_n = A + B$, where A and B are two nonempty subsets of $\mathbb{Z}_n$ such that $A \neq \mathbb{Z}_n$, $B \neq \mathbb{Z}_n$ and $|A \cap B| = 1$.

b) If $(G, *)$ is finite, A, B are nonempty subsets of G and $a \in G \setminus (A * B)$, prove that the function $f : A \to G \setminus B$ given by $f(x) = x^{-1} * a$ is well-defined and one-to-one. Conclude that if $|A| + |B| > |G|$, then $G = A * B$.

Solution. a) We choose $A = \{\hat{0}, \hat{1}\}$ and $B = \{\hat{1}, \hat{2}, \ldots, \widehat{n-1}\}$. As $\hat{0} = \hat{1} + \widehat{n-1}$ and $B \subset A + B$, we get $\mathbb{Z}_n = A + B$.

b) Suppose $G \setminus (A * B) \neq \emptyset$ and let $a \in G \setminus (A * B)$. Define $f : A \to G \setminus B$, by $f(x) = x^{-1} * a$.

If $x^{-1} * a = b \in B$, then $a = x * b \in A * B$, in contradiction with the way we picked a. So f is well defined. As $f(x) = f(y)$ implies $x^{-1} * a = y^{-1} * a$, that is $x = y$, so f is injective and thus $|A| \leqslant |G \setminus B|$.

As a consequence $|G| = |G \setminus B| + |B| \geqslant |A| + |B|$, a contradiction. Thus $G = A * B$.

Problem 2. Consider two continuous functions $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to (0, \infty)$ such that f is non-decreasing. Prove that

$$\int_0^t f(x)g(x)\mathrm{d}x \cdot \int_0^1 g(x)\mathrm{d}x \leqslant \int_0^t g(x)\mathrm{d}x \cdot \int_0^1 f(x)g(x)\mathrm{d}x,$$

for any $t \in [0, 1]$.

Solution. Let $k = \int_0^1 g(x)\mathrm{d}x > 0$. Replacing, if necessary, g by $g_1 = \frac{1}{k}g$, we may suppose that $\int_0^1 g(x)\mathrm{d}x = 1$.

Define $F : (0, 1] \to \mathbb{R}$ by

$$F(t) = \frac{\int_0^t f(x)g(x)\mathrm{d}x}{\int_0^t g(x)\mathrm{d}x}.$$

F is obviously differentiable on $(0, 1]$ and

$$\begin{aligned}
F'(t) &= \frac{f(t)g(t)\int_0^t g(x)\mathrm{d}x - g(t)\int_0^t f(x)g(x)\mathrm{d}x}{\left(\int_0^t g(x)\mathrm{d}x\right)^2} \\
&= g(t)\frac{f(t)\int_0^t g(x)\mathrm{d}x - \int_0^t f(x)g(x)\mathrm{d}x}{\left(\int_0^t g(x)\mathrm{d}x\right)^2} \\
&\geqslant g(t)\frac{\int_0^t (f(t) - f(x))g(x)\mathrm{d}x}{\left(\int_0^t g(x)\mathrm{d}x\right)^2} \geqslant 0.
\end{aligned}$$

Thus F is non-decreasing. In particular, $F(t) \leqslant F(1)$, as asked.

Problem 3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which verify simultaneously the following two conditions:

a) the limit $\lim\limits_{x \to \infty} f(x)$ exists;

b) $f(x) = \int_{x+1}^{x+2} f(t)\mathrm{d}t$, for all $x \in \mathbb{R}$.

Solution. For $a \in \mathbb{R}$ define inductively the sequence $(a_n)_n$ by $a_0 = a$, and, for given a_n, choose $a_{n+1} \in [a_n + 1, a_n + 2]$ such that

$$\int_{a_n+1}^{a_n+2} f(x)\mathrm{d}x = f(a_{n+1}).$$

This can be done by the mean value theorem. Thus $a_n \to \infty$.

As $f(a_n) \to \lim\limits_{x \to \infty} f(x)$ and $f(a_n) = f(a)$ for any $n \in \mathbb{N}$, we have $f(a) = \lim\limits_{x \to \infty} f(x)$. As a consequence f is constant, and reciprocally, any constant function verifies the given condition.

Problem 4. Let k be a field with 2^n elements where $n \in \mathbb{N}^*$ and consider the polynomial $f = X^4 + X + 1$. Prove that:

a) for even n, f is reducible in $k[X]$;

b) for odd n, f is irreducible in $k[X]$.

Solution. a) As $3 | 2^n - 1$, we can find $\alpha \in k$ with $\mathrm{ord}(\alpha) = 3$ in the group $(k^*, \cdot)$. Then $(\alpha - 1)(\alpha^2 + \alpha + 1) = 0$, thus $\alpha^2 + \alpha + 1 = 0$. We conclude $f = (X^2 + X + \alpha)(X^2 + X + \alpha + 1)$.

b) Suppose there is a $\beta \in k$ with $f(\beta) = 0$. We have $(\beta^2 + \beta)^2 + (\beta^2 + \beta) + 1 = 0$, so $(\beta^2 + \beta)^3 = 1$ and, because $2^n - 1$ is not a multiple of 3, we have $\beta^2 + \beta + 1 = 0$. Thus $\beta^3 = 1$, that is $\beta = 1$. In turn $3 = 0$, that is $1 = 0$, a contradiction. As a consequence f has no roots in k.

If f is reducible, $f = (X^2 + mX + n)(X^2 + pX + q)$ with $m, n, p, q \in k$. Identifying coefficients gives $m + p = 0$, $n + q + mp = 0$, $mq + np = 1$ and $nq = 1$. We easily obtain $m = p$, $n + q = p^2$, $n + q = p^{-1}$, from where $p^3 = 1$. Thus $p = 1$, $n = 1 + q$, so $q^2 + q + 1 = 0$, an obvious contradiction.

PROBLEMS AND SOLUTIONS

FINAL ROUND

7th GRADE

Problem 1. If the side lengths a, b, and c of a triangle satisfy the conditions $a + b - c = 2$, and $2ab - c^2 = 4$, show that the triangle is equilateral.

Solution. Take the square of $c = a + b - 2$ and use $c^2 = 2ab - 4$ to get $a^2 + b^2 - 4a - 4b + 8 = 0$; that is, $(a - 2)^2 + (b - 2)^2 = 0$. This forces $a = 2$ and $b = 2$, so $c = a + b - 2 = 2$. The conclusion follows.

Problem 2. Consider a triangle ABC with a right angle at A, and $AC = 2AB$. Let P and Q be the midpoints of the sides AC and AB, respectively. Let further M and N be two points on the side BC such that $BM = CN = x$, with $2x < BC$. Express x in terms of AB, if the area of $MNPQ$ is half the area of ABC.

Solution. Set $AB = c$. Then $AC = 2c$ and $BC = c\sqrt{5}$. Let y be the length of the altitude from M in triangle BMQ, and let z be the length of the altitude from N in triangle CNP. Obvious, similarities yield $y = 2x/\sqrt{5}$, and $z = x/\sqrt{5}$. Rewrite the condition on the area of $MNPQ$ as

$$\text{area}(APQ) + \text{area}(BMQ) + \text{area}(CNP) = \frac{1}{2}\text{area}(ABC)$$

to get $c^2/4 + cy/4 + cz/2 = c^2/2$. Finally, substitution of $y = 2x/\sqrt{5}$ and $z = x/\sqrt{5}$ yields $x = c\sqrt{5}/4$.

Problem 3. Consider a triangle ABC with a right angle at A, and $AB < AC$. Let D be the point of the side AC for which $\angle ACB = \angle ABD$. Drop the altitude DE in triangle BCD. If $AC = BD + DE$, what are the angles ABC and ACB ?

Solution. Triangles ABD and ECD are similar, so $AD \cdot CD = BD \cdot DE$. On the other hand, $AD + CD = AC = BD + DE$. Since $AD < BD$ (from the right triangle ABD), and $CD > DE$ (from the right triangle CDE), the two relations above yield $AD = DE$ (equivalently, $BD = CD$). Consequently, the line BD bisects the angle ABC, so the latter is $60°$, and the angle at C is $30°$. (Equivalently, $BD = CD$ shows that the angle at C is equal to each of the angles formed at B, and since the three add up to $90°$ the conclusion follows.)

Alternative solution. Reflect E through AC to get F. Notice that B, D, and F are collinear, so $\angle BFC = 90°$. Therefore, A, B, C, F are cocyclic. Observe further that $AC = BD + DE = BD + DF = BF$, to deduce that $ABCF$ is an isosceles trapezium, so $\angle ACB = \angle AFB = \angle CBF$. Since $\angle ACB = \angle ABF$, by hypothesis, we are back in the situation at the end of the first solution, and we can repeat that argument *verbatim*.

Problem 4. If m and n are non-negative integer numbers such that $m > 1$ and $2^{2m+1} \geqslant n^2$, show that $2^{2m+1} \geqslant n^2 + 7$.

Solution. Since $m > 1$, the conclusion holds for $n = 0$: $2^{2m+1} - n^2 = 2^{2m+1} \geqslant 8 > 7$. For $n \geqslant 1$, write $n = 2^p q$ with p and q non-negative integers, and q odd (so $q \geqslant 1$). Thus, $2^{2m+1} - n^2 = 2^{2p}\left(2^{2(m-p)+1} - q^2\right)$. Since $q \geqslant 1$, the condition $2^{2m+1} \geqslant n^2$ implies that $p \leqslant m$. The case $p = m$ forces $q = 1$, so $2^{2m+1} - n^2 = 2^{2m} \geqslant 16 > 7$. For $p < m$ (that is, $p + 1 \leqslant m$), recall that q is odd to write $q^2 = 8r + 1$ for some non-negative integer r, so $2^{2m+1} - n^2 = 2^{2p}\left(8\left(2^{2(m-p-1)} - r\right) - 1\right)$. Since $m \geqslant p + 1$, it follows that $2^{2(m-p-1)} - r$ is integer. In conjunction with $2^{2m+1} \geqslant n^2$, this implies that $2^{2(m-p-1)} - r \geqslant 1$, so $2^{2m+1} - n^2 \geqslant 7 \cdot 2^{2p} \geqslant 7$.

Alternative solution. We show that there are no integer solutions $m > 1$ and $n \geqslant 0$ to $2^{2m+1} = n^2 + k$ for $k = 0, 1, \ldots, 6$. Notice that 2^{2m+1} is not a perfect square to rule out the case $k = 0$. For the remaining cases, notice that 2^{2m+1} is divisible by 8 for $m > 1$ and argue by contradiction. For $k = 1, 3, 5$, the number n should be odd, so $n^2 = 8\mathcal{M} + 1$ and $k = 8\mathcal{M} - 1$ which is not the case. Finally, for $k = 2$ or 6, the number n should be even, so $k = 2^{2m+1} - n^2$ should be divisible by 4 which is not the case, either.

Remark. Notice that the equation $2^{2m+1} = n^2 + 7$ has integer solutions $m > 1$ and $n \geqslant 0$: for instance, $m = 2$ and $n = 5$ or $m = 3$ and $n = 11$ or $m = 7$ and $n = 181$.

8th GRADE

Problem 1. Prove that the number 10^{10} cannot be written as a product of two positive integers all of whose base-10 digits are different from zero.

Solution. Suppose, if possible, that 10^{10} can be written as a product of two positive integers all of whose base-10 digits are different from zero. In particular, neither factor is divisible by 10. Since $10^{10} = 2^{10} \cdot 5^{10}$, this forces one of the factors to be $2^{10} = 1024$ which contradicts the assumption.

Problem 2. A number of 2007 offices are assigned 6018 desks. Each office is assigned at least one desk. All desks from any one office may be removed from that office and reascribed to other offices to get an equal number of desks in each office other than the one they have been removed from. What are the possible desk ascriptions?

Solution. Since the average number of desks per office is $6018/2007 > 2$, some offices must be assigned at least 3 desks each.

We claim that no office may be assigned more than 3 desks. Indeed, were some office assigned more than 3 desks, removal of all desks from another office and their reassignment, whatsoever it be, would yield 2006 offices with at least 4 desks each — a total of at least $4 \cdot 2006 = 8024$ desks which is impossible.

In what follows, a *k-office* is an office which is assigned k desks — by hypothesis, $k \geqslant 1$. Let n_k denote the number of k-offices. By the preceding, $n_k = 0$ for $k \geqslant 4$, so $n_1 + n_2 + n_3 = 2007$, and $n_1 + 2n_2 + 3n_3 = 6018$. Hence $2n_1 + n_2 = 3$, so either $n_1 = 0$ and $n_2 = 3$, in which case $n_3 = 2004$, or $n_1 = 1$ and $n_2 = 1$, in which case $n_3 = 2005$.

We now show that both fill in the bill. In the first case, remove the two desks from any 2-office and assign the other two 2-offices one each to get a balanced

arrangement; similarly, remove the three desks in any 3-office and assign the three 2-offices one each to get a balanced configuration.

In the second case, the one desk in the 1-office may be transferred to the 2-office, and the two desks in the 2-office may be transferred to the 1-office — the resulting arrangements are both balanced; finally, remove the three desks from any 3-office and ascribe two to the 1-office and one to the 2-office to get a balanced configuration.

Problem 3. a) If all sides of a triangle ABC have length less than 2, show that the length of the altitude from A is less than $\sqrt{4 - BC^2/4}$.

b) Show that the volume of a tetrahedron at most one edge of which has a length greater than or equal to 2 is less than 1.

Solution. a) The length of the altitude from A does not exceed that of the median from A whose square is $(AB^2 + AC^2)/2 - BC^2/4 < 4 - BC^2/4$. The conclusion follows.

b) Let AB denote the longest edge of the tetrahedron, and let a denote the length of the edge CD opposite AB. By hypothesis, triangles ACD and BCD both have all sides of length less than 2; in particular, $a < 2$. The length of the altitude from A in the tetrahedron does not exceed the length of the altitude from A in triangle ACD which by a) is less than $\sqrt{4 - a^2/4}$. With reference again to a), the length of the altitude from B in triangle BCD is less than $\sqrt{4 - a^2/4}$, so the triangle has an area less than $(a/8)\sqrt{4 - a^2/4}$. Consequently, the volume of the tetrahedron is less than $a(16 - a^2)/24 = 1 - (2 - a)(4 + (2 - a)(4 + a))/24 < 1$.

Problem 4. Let $ABCD$ be a tetrahedron, and M a point in space such that $MA^2 + MB^2 + CD^2 = MB^2 + MC^2 + DA^2 = MC^2 + MD^2 + AB^2 = MD^2 + MA^2 + BC^2$. Show that M lies on the common perpendicular to the lines AC and BD.

Solution. Note that $AB^2 - AD^2 = MB^2 - MD^2 = CB^2 - CD^2$ to deduce that the points A, C and M lie in a plane α perpendicular to the line BD. Similarly, the points B, D and M lie in a plane β perpendicular to the line AC. The planes α and β both contain the point M, and do not coincide, for the lines AC and BD

are not parallel. Consequently, they meet along a line through M which is the common perpendicular to the lines AC and BD.

9th GRADE

Problem 1. Prove that, for $a_i \in \mathbb{N}^*$, $1 \leqslant i \leqslant n+1$, $a_{n+1} = a_1$, $n \in \mathbb{N}^*$, if the polynomial function

$$P(x) = x^2 - \Big(\sum_{i=1}^{n} a_i^2 + 1\Big)x + \sum_{i=1}^{n} a_i a_{i+1}$$

admits an integer root, then, if n is a perfect square, so are both its roots.

Solution. (Dan Schwarz) Since $P(x)$ is monic, and has integer coefficients, it follows that its second root is also an integer. If we compute

$$P(0) = \sum_{i=1}^{n} a_i a_{i+1} > 0 \quad \text{and} \quad P(1) = -\frac{1}{2}\sum_{i=1}^{n}(a_i - a_{i+1})^2 \leqslant 0,$$

then it follows that $P(x)$ has a root in $(0,1]$, therefore $P(1) = 0$, which only occurs when all a_i are equal, $a_i = a$, $1 \leqslant i \leqslant n$, when $P(x) = (x-1)(x-na^2)$, and the second root is na^2, a perfect square when n is a perfect square.

Alternatively, without even having to show the second root is also an integer, it is needed that

$$\Delta = \Big(\sum_{i=1}^{n} a_i^2 + 1\Big)^2 - 4\Big(\sum_{i=1}^{n} a_i a_{i+1}\Big)$$

be a perfect square, and having same parity as $(\sum_{i=1}^{n} a_i^2 + 1)^2$, from having $(\sum_{i=1}^{n} a_i^2 - 1)^2 \leqslant \Delta < (\sum_{i=1}^{n} a_i^2 + 3)^2$ follows $a_i = a$, $1 \leqslant i \leqslant n$, etc.

Remarks. Notice that, for $a_i \in \mathbb{Z}$, having $\sum_{i=1}^{n} a_i a_{i+1} \neq 0$ is enough for the first solution, as then $P(-1) = 2 + \frac{1}{2}\sum_{i=1}^{n}(a_i + a_{i+1})^2 \geqslant 2$, while $P(0) \neq 0$, so again $P(1) = 0$, etc. In fact, knowing $P(x)$ has an integer root allows solving it, as the case $\sum_{i=1}^{n} a_i a_{i+1} = 0$ leads to one root 0, and the other being $\sum_{i=1}^{n} a_i^2 + 1$. The second solution also works, as may be seen.

Problem 2. Let ABC be an acute-angled triangle, and M a point in its plane, different from its vertices. Then, with the usual notations,

$$\frac{a}{MA}\overrightarrow{MA} + \frac{b}{MB}\overrightarrow{MB} + \frac{c}{MC}\overrightarrow{MC} = \overrightarrow{0}$$

if and only if $M \equiv H$, the orthocenter of ABC.

Solution. (Dan Schwarz) It is clear that if M is not interior to ABC, all three vectors point to a same semiplane, therefore their sum cannot be $\overrightarrow{0}$ (this trivial observation also accounts for the requirement that ABC be acute-angled). Rewrite

$$a\frac{\overrightarrow{MA}}{MA} + b\frac{\overrightarrow{MB}}{MB} + c\frac{\overrightarrow{MC}}{MC} = \overrightarrow{0}$$

and notice that this is equivalent to the three vectors forming a closed triangular contour of sides a, b, c, therefore congruent to ABC. Denote A', B', C', the feet of cevians AM, BM, CM, respectively. Then $\angle AMB + \angle ACB = \pi$, so quadrilateral $CA'MB'$ is cyclic, and similar. Therefore, $\angle CA'M = \angle AB'M = \angle BC'M$ and $\angle CB'M = \angle BA'M = \angle AC'M$. But this can only happen when M is lying on the same (oriented) side of the three altitudes through H, which in turn only happens when $M \equiv H$.

Conversely, when $M \equiv H$, the reverse reasoning of the above leads to the fact that the three vectors close their contour, therefore have sum $\overrightarrow{0}$.

Alternative, more computational solutions are available.

Problem 3. Color white or black each band of width 1, determined by partitioning the plane with equidistant parallel lines, distanced 1 apart. Show that one can place an equilateral triangle of side 100, such that its vertices share a same color.

Solution. (Dan Schwarz) All that is needed is for the triangle to have a non-integer altitude! If the coloring is monochromatic, the result is trivial. If not, place the triangle with a side on a boundary line between white and black bands, pointing towards the white. Then the apex will fall into a band which, if colored white, allows sliding the triangle in the direction of its apex, with its base kept parallel to the boundaries, to find a triangle with all white vertices, while if colored black, allows sliding the triangle in the opposite direction, to find a triangle with all black vertices.

As the altitude is $50\sqrt{3} \notin \mathbb{Z}$, the result follows.

Remarks. As one can see, this method works for any triangle having at least a non-integer altitude. On the other hand, for an equilateral triangle with integer altitude h, consider a coloring made of alternating monochromatic black and white groups of h bands. Then any equilateral triangle in the interior of a monochromatic group will have its altitude lower than h, while an equilateral triangle spanning a monochromatic group will have its altitude larger than h. Thus, the integer altitude case is a relevant counter-example.

Problem 4. For $f : X \to X$, denote $f_0(X) = X$, $f_{n+1}(X) = f(f_n(X))$, for all $n \in \mathbb{N}$. Also denote

$$f_\infty(X) = \bigcap_{n\in\mathbb{N}} f_n(X).$$

Prove that, if X is finite, then $f(f_\infty(X)) = f_\infty(X)$. Does the result still hold when X is infinite?

Solution. It is clear that $f_{n+1}(X) \subseteq f_n(X)$, for all $n \in \mathbb{N}$. When X is finite, the sequence $(f_n(X))_{n\in\mathbb{N}}$ will become stationary, and so $f_\infty(X) = f_n(X)$ for some $n \in \mathbb{N}$ for which $f_{n+1}(X) = f_n(X)$, therefore $f(f_\infty(X)) = f_\infty(X)$.

When X is infinite, if $x \in X$ belongs to a finite cycle $\mathcal{C}_x = \{x, f(x), \dots\}$, then $x \in \mathcal{C}_x \subseteq f_\infty(X)$ and $f(\mathcal{C}_x) = \mathcal{C}_x$. If not, when we denote

$$f^0(x) = \{x\}, \quad f^{-1}(x) = \{y \in X; f(y) = x\},$$

$$f^{-(n+1)}(x) = \bigcup_{y\in f^{-n}(x)} f^{-1}(y),$$

the sets $f^{-n}(x)$, $n \in \mathbb{N}$, are disjoint, and König's Infinity Lemma [1] implies that, if all sets $f^{-n}(x)$ are finite, but not empty, there exists an infinite path $\mathcal{P}_x = \{x, x_1 \in f^{-1}(x), \dots, x_n \in f^{-n}(x), \dots\}$, and then again $x \in \mathcal{P}_x \subseteq f_\infty(X)$ and $f(\mathcal{P}_x) = \mathcal{P}_x$, while if some $f^{-n}(x) = \emptyset$, then $x \notin f_\infty(X)$.

[1] See, for example, [Reinhard Diestel – *Graph Theory*].

It follows that, in order to have $f(f_\infty(X)) \neq f_\infty(X)$, we need some $x \in X$, such that x does not belong to a finite cycle $\mathcal{C}_x$, and infinitely often $f^{-n}(x)$ is infinite, while no infinite path $\mathcal{P}_x$ exists. Also, f can be neither into, nor onto. Clearly, $f(f_\infty(X)) \subseteq f_\infty(X)$ always.

For $n \in \mathbb{N}$, denote $\mathbb{P}_n = \{\frac{p}{q};\ p, q \in \mathbb{N}^*,\ n < p < q,\ q \text{ prime}\}$, look at $\mathbb{N}^*$ as $\mathbb{N}^* = \{\frac{p}{q};\ p \in \mathbb{N}^*,\ q = 1\}$, and take $X = \mathbb{P}_0 \cup \mathbb{N}^*$ and $f(\frac{p}{q}) = \frac{p+1}{q}$. It is readily seen that $f_n(X) = \mathbb{P}_n \cup \mathbb{N}^*$, hence $f_\infty(X) = \mathbb{N}^*$, and clearly $f(\mathbb{N}^*) = \mathbb{N}^* \setminus \{1\} \neq \mathbb{N}^*$, therefore the result for finite sets does not anymore necessarily hold for infinite sets.

Remarks. The infinite sets result is counter-intuitive, as one would expect f to invariate $f_\infty(X)$. Another, trivial, remark is that for infinite sets $f_\infty(X)$ may be empty (e.g., $f(n) = n + 1$ on $\mathbb{N}$), while for finite sets $f_\infty(X) \neq \emptyset$.

10th GRADE

Problem 1. Let n be a positive integer. Prove that a complex number of absolute value 1 is a solution to $z^n + z + 1 = 0$ if and only if $n = 3m + 2$ for some positive integer m.

Solution. If $n = 3m+2$ for some positive integer m, then the complex number $\cos(2\pi/3) + \mathrm{i}\sin(2\pi/3)$ is clearly a solution of absolute value 1. Conversely, if z is a solution of absolute value 1, then so is $\bar{z} = 1/z$. Hence $z^n + z + 1 = 0 = z^n + z^{n-1} + 1$ which yields successively $z^{n-2} = 1$, $z^2 + z + 1 = 0$, $z^3 = 1$ with $z \neq 1$, so $n = 3m + 2$ for some positive integer m.

Alternative solution. Let $P(z) = z^n + z + 1 = 0$. If $P(\omega) = 0$, $|\omega| = 1$, then $\omega = \cos\theta + \mathrm{i}\sin\theta$, and so, using de Moivre's formula, $\omega^n = \cos n\theta + \mathrm{i}\sin n\theta$. Then $0 = (\cos n\theta + \cos\theta + 1) + \mathrm{i}(\sin n\theta + \sin\theta)$, hence $\sin^2 n\theta = \sin^2\theta$, and $\cos^2 n\theta = \cos^2\theta + 2\cos\theta + 1$, so $\cos\theta = -\frac{1}{2}$. It follows $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$, therefore $\omega^n = \omega^2$, so $n \equiv 2 \pmod 3$.

Conversely, if $n \equiv 2 \pmod 3$, for $\omega \neq 1$, ω root of unity of order 3, $P(\omega) = 0$. In fact then $P(z) = z^n + z + 1 = (z^2 + z + 1)Q(z)$ for some Q (with integer coefficients).

Problem 2. Solve the equation $2^{x^2+x} + \log_2 x = 2^{x+1}$ in the set of real numbers.

Solution. Clearly, x must be positive. Rewrite the equation as $\log_2 x = 2^{x+1} - 2^{x^2+x}$ and notice that for $x > 0$ the two members have opposite signs, unless $x = 1$ in which case both vanish. Consequently, $x = 1$ is the unique solution of the equation.

Alternative solution. As before, x must be positive. Add $\log_2(x+1)$ both sides to get $2^{x^2+x} + \log_2(x^2+x) = 2^{x+1} + \log_2(x+1)$. Since $t \mapsto 2^t + \log_2 t$, $t > 0$, is a strictly increasing function (it is the sum of two strictly increasing functions), it follows that $x^2 + x = x + 1$ with $x > 0$; that is, $x = 1$ which clearly is the solution of the equation.

Problem 3. For what integer numbers $n \geqslant 2$ is $(n-1)^{n^{n+1}} + (n+1)^{n^{n-1}}$ divisible by n^n ?

Solution. We show that $a_n = (n-1)^{n^{n+1}} + (n+1)^{n^{n-1}}$ is divisible by n^n if and only if n is odd. Necessity follows from the fact that $a_2 = 10$ which is not divisible by $2^2 = 4$, and $a_n = \mathcal{M}n + 2$ for n even. Sufficiency is a consequence of the following slight generalization: For any odd integer $n > 1$, n^n is the highest power of n dividing a_n. To show this, write

$$a_n = \sum_{k=1}^{n} (-1)^{k-1} n^k \binom{n^{n+1}}{k} + \sum_{k=1}^{n} n^k \binom{n^{n-1}}{k} + \mathcal{M}n^{n+1};$$

since n is odd, the two terms corresponding to $k = 0$ cancel out, and the generic exponent of -1 in the first expansion is $k - 1$. Noting that the first term in the second sum is exactly n^n, we show that the other summands in both sums are all divisible by n^{n+1}. To this end, we merely have to show that every prime factor of n^{n+1} divides those summands to at least as high a power as it divides n^{n+1}. So let $p \geqslant 3$ be a prime factor of n (recall that n is odd, so $p \geqslant 3$), and let p^α be the highest power of p dividing n (so $\alpha \geqslant 1$). Clearly, $p^{\alpha(n+1)}$ is the highest power of p that divides n^{n+1}.

The generic term of the first sum is divided by p to at least as high a power as

$$\begin{aligned}
\beta &= \alpha k + \alpha(n+1) - \left(\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{p^2} \right\rfloor + \left\lfloor \frac{k}{p^3} \right\rfloor + \cdots \right) \\
&> \alpha k + \alpha(n+1) - \frac{k}{p}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) \\
&\geqslant \alpha k + \alpha(n+1) - \frac{k}{p-1} \\
&= \alpha(n+1) + k\left(\alpha - \frac{1}{p-1}\right) \\
&> \alpha(n+1) \quad \text{for } k \geqslant 1,\ \alpha \geqslant 1, \text{ and } p \geqslant 3.
\end{aligned}$$

Consequently, each term in the first sum is divisible by n^{n+1}.

For the generic term of the second sum, merely replace $n+1$ by $n-1$ in the first three estimates above to see that p divides it to a (strictly) higher power than

$$\begin{aligned}
\alpha k + \alpha(n-1) - \frac{k}{p-1} &= \alpha(n+1) + k\left(\alpha - \frac{1}{p-1}\right) - 2\alpha \\
&\geqslant \alpha(n+1) + k\left(\alpha - \frac{1}{2}\right) - 2\alpha \quad \text{for } p \geqslant 3.
\end{aligned}$$

For $k \geqslant 4$, note that $k(\alpha - 1/2) - 2\alpha \geqslant 2(\alpha - 1) \geqslant 0$ to deduce that the corresponding summands are all divisible by n^{n+1}. Finally, for $k = 1, 2, 3$, the first term in the second sum is exactly n^n, and the second and third are easily seen to be divisible by n^{n+1}. The conclusion follows.

Remark. Along the same lines, it can be shown that the number $b_n = (n+1)^{n^{n-1}} - (n-1)^{n^{n+1}}$ is divisible by n^n if and only if n is even. Moreover, for even $n > 2$, n^n is the highest power of n that divides b_n. For $n = 2$, $b_2 = 9 - 1 = 8 = 2^3$. The latter fact is not at all accidental. It can be shown that b_n is divisible by $2n^n$ for $n \equiv 2 \pmod 4$.

Problem 4. a) Let S be a finite set of numbers, and let $S + S = \{x + y : x, y \in S\}$. Show that

$$|S + S| \leqslant \frac{1}{2}|S|(|S| + 1),$$

where $|X|$ is the cardinal number (that is, the number of elements) of the set X.

b) Given a positive integer m, let $C(m)$ be the greatest positive integer k such that, for some set S of m integers, every integer from 1 to k belongs to S or is a sum of two not necessarily distinct elements of S. For instance, $C(3) = 8$ with $S = \{1, 3, 4\}$. Show that $m(m+6)/4 \leqslant C(m) \leqslant m(m+3)/2$.

Solution. a) Clearly,

$$|S+S| \leqslant |S| + \binom{|S|}{2} = \frac{1}{2}|S|\left(|S|+1\right).$$

b) The number $C(m)$ is the greatest positive integer k such that $\{1, 2, \ldots, k\} \subseteq S \cup (S+S)$ for some set S of m integers. Since

$$\begin{aligned} |S \cup (S+S)| &\leqslant |S| + |S+S| \\ &\leqslant |S| + \frac{1}{2}|S|\left(|S|+1\right) \text{ by a)} \\ &= \frac{1}{2}|S|\left(|S|+3\right), \end{aligned}$$

it follows that $C(m) \leqslant m(m+3)/2$. To get a lower bound for $C(m)$, take a positive integer $t < m$, and

$$S = \{1, 2, \ldots, t\} \cup \{k + (k+1)t : k = 1, 2, \ldots, m-t\}.$$

Clearly, $|S| = m$, and

$$\{1, 2, \ldots, (m-t+1)t + (m-t) + t\} = \{1, 2, \ldots, m + (m+1)t - t^2\} \subseteq S \cup (S+S).$$

Maximizing the cardinality of the set on the left as t runs through the positive integers less than m, yields the best lower bound we can get this way. It is readily checked that the maximum is $\lceil m(m+6)/4 \rceil$ and is achieved for $t = \lfloor (m+1)/2 \rfloor$. Consequently, $C(m) \geqslant m(m+6)/4$.

Remarks. 1) For $2 < m < 8$, a different construction yields a better lower bound: Take $S = \{1, 3, \ldots, 2m-3, 2m-2\}$ and note that $\{1, \ldots, 4m-4\} \subseteq S \cup (S+S)$, to get $m(m+6)/4 < 4m-4 \leqslant C(m)$.

2) Given $\varepsilon > 0$, our estimates yield $1/4 < C(m)/m^2 < 1/2 + \varepsilon$ for all sufficiently large m. A more careful choice of the set S yields better asymptotic bounds: $9/32 < C(m)/m^2 < 4/9 + \varepsilon$ for all sufficiently large m.

11th GRADE

Problem 1. If A and B are 2-by-2 matrices with real numbers as entries, and $A^2 + B^2 = AB$, prove that $(AB - BA)^2 = 0_2$.

Solution. If ε is a primitive third root of unity, then

$$\begin{aligned} |\det(\varepsilon A + B)|^2 &= \det(\varepsilon A + B)\,\overline{\det(\varepsilon A + B)} \\ &= \det(\varepsilon A + B)\det(\bar{\varepsilon} A + B) \\ &= \det\left((\varepsilon A + B)(\bar{\varepsilon} A + B)\right) \\ &= \det\left(\varepsilon(AB - BA)\right) \\ &= \varepsilon^2 \det(AB - BA), \end{aligned}$$

so $\det(AB - BA) = 0$. Since $X^2 - (\operatorname{tr} X)\cdot X + (\det X)\cdot I_2 = 0_2$ for any 2-by-2 matrix, and $\operatorname{tr}(AB - BA) = 0$, the conclusion follows.

Problem 2. Given two real numbers a and b, $a < b$, in the image of a continuous, real-valued function f on $\mathbb{R}$, prove that the closed interval $[a, b]$ is the image under f of some interval $I \subset \mathbb{R}$.

Solution. Since a and b lie in the image of f, $a = f(a')$ and $b = f(b')$ for some $a', b' \in \mathbb{R}$. Without loss of generality, we may and will assume that $a' < b'$. The set $A = \{x : a' \leqslant x \leqslant b' \text{ and } f(x) = a\}$ is non-empty ($a' \in A$) and bounded from above by b'. Let $\alpha = \sup A$. By continuity, $f(\alpha) = a$. The set $B = \{x : \alpha \leqslant x \leqslant b' \text{ and } f(x) = b\}$ is non-empty ($b' \in B$) and bounded from below by α. Let $\beta = \inf B$. By continuity, $f(\beta) = b$. We now show that the closed interval $[a, b]$ is the image of $I = [\alpha, \beta]$ under f. By the intermediate value theorem, $[a, b] \subseteq f(I)$. To prove the reverse inclusion, suppose first that $f(x) < a$ for some $x \in (\alpha, \beta)$. Then $f(x') = a$ for some $x' \in (x, \beta)$, and we would get a contradiction with the choice of α. Similarly, were $f(x) > b$ for some $x \in (\alpha, \beta)$, we would reach a contradiction with the choice of β. The conclusion follows.

Problem 3. Given an integer number $n \geqslant 2$, let Σ^{n-1} be the set of all $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ with $|x_1| + \cdots + |x_n| = 1$. Determine the n-by-n matrices A with real numbers as entries such that $xA \in \Sigma^{n-1}$ for all $x \in \Sigma^{n-1}$.

Solution. Let $A = (a_{ij})$ be an n-by-n matrix with real numbers as entries such that $xA \in \Sigma^{n-1}$ for all $x \in \Sigma^{n-1}$. For each i in $\{1, \ldots, n\}$, let e_i be the row all of whose entries are zero except the i-th which is 1. Since $\frac{1}{2}(e_i + e_j) \in \Sigma^{n-1}$ for all i and all j, the condition $\frac{1}{2}(e_i + e_j)A \in \Sigma^{n-1}$ yields $\sum_{k=1}^{n} |a_{ik} + a_{jk}| = 2$ for all i and all j. In particular, $|a_{i1}| + \cdots + |a_{in}| = 1$ for all i, so each row has some non-zero entry. On the other hand, for all i and all j with $i \neq j$, $\frac{1}{2}(e_i - e_j) \in \Sigma^{n-1}$, and the condition $\frac{1}{2}(e_i - e_j)A \in \Sigma^{n-1}$ yields $\sum_{k=1}^{n} |a_{ik} - a_{jk}| = 2$ for all i and all j with $i \neq j$. Hence, $\sum_{k=1}^{n} (|a_{ik}| + |a_{jk}| - |a_{ik} \pm a_{jk}|) = 0$ for all i and all j with $i \neq j$. Since each summand is non-negative, it follows that $a_{ik}a_{jk} = 0$, $k = 1, \ldots, n$, for every pair of distinct indices i and j. Recall now that each row has some non-zero entry to infer that on each row and each column there must be exactly one non-zero entry; by the preceding, this non-zero entry must be ± 1. Consequently, the rows of A are a permutation of the $\pm e_i$, $i = 1, \ldots, n$, with signs chosen arbitrarily — a total of $2^n n!$ such matrices. It is readily checked that any such matrix satisfies the required condition.

Remarks. Let S^{n-1} be the standard $(n-1)$-sphere in $\mathbb{R}^n$ with the Euclidean norm $\|x\|_2 = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}$: $S^{n-1} = \{x : x \in \mathbb{R}^n, \|x\|_2 = 1\}$. To avoid confusion, let $\mathbb{E}^n$ denote $\mathbb{R}^n$ with the Euclidean norm. The set Σ^{n-1} considered in the problem is the $\| \, \|_1$-counterpart of S^{n-1}: it is the standard $(n-1)$-sphere in $\mathbb{R}^n$ with the norm $\|x\|_1 = |x_1| + \cdots + |x_n|$. For $n = 1$ the two spheres are the same, but for $n \geqslant 2$ this is no longer the case: not only are they different as sets, but S^{n-1} is smooth, strictly convex, with no particular features, while Σ^{n-1} is not: literally, it has vertices, edges, faces etc. (it is just piecewise smooth) — for instance, Σ^1 is the boundary of the square with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$, Σ^2 is the boundary of the octahedron with vertices at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$, and so on. Note that the extremal points of the convex hull of S^{n-1} (the Euclidean n-ball) are all of S^{n-1}, whereas the extremal points of the convex hull of Σ^{n-1} are located at the $\pm e_i$ — and this is precisely what forced the result. The problem essentially asks for a complete description of the linear transformations of $\mathbb{R}^n$ preserving the $\| \, \|_1$-norm: $\|xA\|_1 = \|x\|_1$ for all x in $\mathbb{R}^n$. The answer shows that these transformations are but a tiny fraction of their Euclidean counterparts: they form a *finite* group embedded in the *uncountable* continuous group of orthogonal transformations (rotations and rotoinversions) of $\mathbb{E}^n$. Roughly speaking, the norm $\| \, \|_1$ deprives $\mathbb{R}^n$ of most of its Euclidean rotational symmetry. It brings in a rigidity the Euclidean structure is unaware of. Not surprisingly, this rigidity reflects on the stiff behaviour of $\| \, \|_1$-preserving linear transformations relative to subspaces spanned by a finite number of the e_i. For example, the subspace spanned by an e_i is sent onto that spanned by some e_j — in contrast with the Euclidean case, where any given 1-dimensional subspace may be mapped onto any prescribed 1-dimensional subspace by some $\| \, \|_2$-preserving linear transformation (and there are uncountably many ways for a 1-dimensional subspace to be prescribed). Finally, note that the stiffness of $\| \, \|_1$-preserving linear transformations forces them to be *simplicial* on Σ^{n-1}: vertices are mapped to vertices, edges to edges, faces to faces etc. At the other extreme, we may consider the norm $\| \, \|_\infty$: for $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, $\|x\|_\infty = \max\{|x_i| : i = 1, \ldots, n\}$. This time, the standard $(n-1)$-sphere is the set Γ^{n-1} consisting of those x with at least one coordinate ± 1; it is the boundary of the n-cube with vertices at $(\pm 1, \ldots, \pm 1)$. The corresponding problem can be dealt with similarly.

Problem 4. A *P-function* is a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with a continuous derivative f' on $\mathbb{R}$ such that $f(x + f'(x)) = f(x)$ for all x in $\mathbb{R}$.

a) Prove that the derivative of a P-function has at least one zero.

b) Provide an example of a non-constant P-function.

c) Prove that a P-function whose derivative has at least two distinct zeros is constant.

Solution. a) If f is a P-function, and $f'(x) \neq 0$ for some $x \in \mathbb{R}$, the mean value theorem shows that f' vanishes at some point ξ between x and $x + f'(x)$: $0 = f(x + f'(x)) - f(x) = f'(x)f'(\xi)$.

b) Try a non-constant polynomial function f. Identification of coefficients forces $f(x) = -x^2 + px + q$, where p and q are two arbitrarily fixed real numbers. This is not at all accidental. As shown in the comment that follows the solution, every non-constant P-function whose derivative vanishes at a single point is of this form.

c) Let f be a P-function. By a), the set $Z = \{x : x \in \mathbb{R} \text{ and } f'(x) = 0\}$ has at least one element. We now show that if it has more than one element, then it must be all of $\mathbb{R}$. The conclusion will follow. The proof is broken into three steps.

Step 1. If f' vanishes at some point a, then $f'(x) \geqslant 0$ for $x \leqslant a$, and $f'(x) \leqslant 0$ for $x \geqslant a$. The argument is essentially the same in both cases, so we only deal with the first one. We argue by *reductio ad absurdum*. Suppose $f'(x_0) < 0$ for some $x_0 < a$ and let $\alpha = \inf\{x : x > x_0 \text{ and } f'(x) = 0\}$ — clearly, this infimum exists. By continuity of f', $f'(\alpha) = 0$, and $f'(x) < 0$ for $x_0 < x < \alpha$; in particular, f is strictly monotonic (decreasing) on (x_0, α). Consider further the continuous, real-valued function $g : x \mapsto x + f'(x)$, $x \in \mathbb{R}$, and note that $g(x) < x$ for $x_0 < x < \alpha$, and $g(\alpha) = \alpha$. Since $g(\alpha) = \alpha > x_0$ and g is continuous, $g(x) > x_0$ for x in (x_0, α), sufficiently close to α. Consequently, for any such x, $x_0 < g(x) < x < \alpha$, and $f(g(x)) = f(x)$, which contradicts the strict monotonicity of f on (x_0, α).

Step 2. If f' vanishes at two points a and b, $a < b$, then f is constant on $[a, b]$. By Step 1, $f'(x) \geqslant 0$ for $x \leqslant b$, and $f'(x) \leqslant 0$ for $x \geqslant a$, so f' vanishes identically on $[a, b]$. Consequently, f is constant on $[a, b]$.

We are now in a position to conclude the proof.

Step 3. If the set $Z = \{x : x \in \mathbb{R} \text{ and } f'(x) = 0\}$ has more than one element, then Z is all of $\mathbb{R}$. By Step 2, Z is a non-degenerate interval, and f is constant on Z: $f(x) = c$ for all x in Z. We show that $\alpha = \inf Z = -\infty$ and $\beta = \sup Z = +\infty$. Suppose, if possible, that $\alpha > -\infty$. Then α is a member of Z, by continuity of f'. Recall the function g from Step 1. By Step 1, $f'(x) > 0$ for $x < \alpha$, so $g(x) > x$, $f(x)$ is strictly monotonic (increasing), and $f(x) < c$ for $x < \alpha$. Since $f(x)$ is strictly monotonic for $x < \alpha$, the conditions $f(g(x)) = f(x)$ and $g(x) > x$ force $x < \alpha < g(x)$. Since $g(\alpha) = \alpha < \beta$, and g is continuous, it follows that $g(x) < \beta$ for $x < \alpha$, sufficiently close to α. Finally, take any such x and recall that Z is an interval to conclude that $g(x) \in Z$, so $f(x) = f(g(x)) = c$ — in contradiction with $f(x) < c$ established above. Consequently, $\alpha = -\infty$. A similar argument shows that $\beta = +\infty$.

Alternative solution. (Dan Schwarz) a) Let us define $g(x) := x + f'(x)$. Supposing $f'(x) \neq 0$ for some $x \in \mathbb{R}$, it follows $g(x) \neq x$, but $f(x) = f(g(x))$, so Rolle's Theorem implies the existence of $\zeta \in (x, g(x))$ with $f'(\zeta) = 0$.

Therefore, $\mathcal{F} := \{x \in \mathbb{R}; f'(x) = 0\} \neq \emptyset$. It is evident that $f(x) \equiv C$, C real constant, is a solution (and then $\mathcal{F} = \mathbb{R}$).

b) If we look for polynomial solutions of degree > 0, we readily find (identifying coefficients) that the degree must be $\leqslant 2$, and *all* such solutions are of the form $f(x) = -x^2 + px + q$, with $p, q \in \mathbb{R}$ arbitrarily fixed.

c) Let us first show that if $|\mathcal{F}| > 1$, then $\mathcal{F}$ is a convex set, therefore a (closed, as f' is continuous) interval. Let $x_1 < x_2$, with $f'(x_1) = f'(x_2) = 0$, and let $x \in (x_1, x_2)$. Denote $a = \sup\{z; z \in [x_1, x] \cap \mathcal{F}\}$ and $b = \inf\{z; z \in [x, x_2] \cap \mathcal{F}\}$ (clearly $a, b \in \mathcal{F}$, as f' is continuous). If $a = b$, then also $x = a \in \mathcal{F}$. Supposing $a < b$, it follows that f is strictly monotonical on $[a, b]$. Be f increasing, there exist $0 < \varepsilon < \frac{b-a}{2}$ and $0 < \delta < \frac{b-a}{2}$ such that $a < a + \varepsilon < b$ and $0 < f'(a + \varepsilon) < \delta$, so $a < g(a + \varepsilon) = a + \varepsilon + f'(a + \varepsilon) < a + \varepsilon + \delta < b$. If $a + \varepsilon = g(a + \varepsilon)$, then $f'(a + \varepsilon) = 0$, with $a + \varepsilon \in (a, b)$ while otherwise, as $f(a + \varepsilon) = f(g(a + \varepsilon))$, there exists $\zeta \in (a + \varepsilon, g(a + \varepsilon)) \subset (a, b)$ with $f'(\zeta) = 0$, absurd, as (a, b) cannot contain any more zeros of f'. Similarly, be f decreasing, one will work with $b - \varepsilon$ and reach the same contradiction.

If now $\mathcal{F} = [\alpha, \beta]$, for $x < \alpha$ it follows $g(x) > \beta$, because f is strictly monotonical on $(-\infty, \alpha]$ and constant on $[\alpha, \beta]$, so we cannot have $g(x) \in (-\infty, \beta]$ and $f(x) = f(g(x))$ unless $g(x) = x$, that is $f'(x) = 0$, absurd. So $g(x) > \beta$, but then $f'(x) > \beta - \alpha$. Similarly, for $x > \beta$ it follows $f'(x) < \alpha - \beta$. Therefore $|f'(x)| > \beta - \alpha$ on $(-\infty, \alpha) \cup (\beta, +\infty)$, while $f'(x) = 0$ on $[\alpha, \beta]$, absurd, as f' is continuous, and thus holds the Intermediate Value Property (Darboux).

If $\mathcal{F} = [\alpha, +\infty)$, for $x < \alpha$ it follows as above $g(x) = x$, so $f'(x) = 0$, absurd, and similarly if $\mathcal{F} = (-\infty, \beta]$, with $x > \beta$. It finally follows, as only possibility, $|\mathcal{F}| > 1$ is $\mathcal{F} = \mathbb{R}$, therefore $f'(x) \equiv 0$ and so $f(x) \equiv C$, C real constant.

Remark. (Barbu Berceanu) As stated, the problem leaves open the case of a non-constant P-function f whose derivative f' vanishes at a single point a. Our purpose here is to settle the case. Recall the continuous, real-valued function $g : x \mapsto x + f'(x)$, $x \in \mathbb{R}$, considered above. We shall prove that $g(x) = -x + 2a$ for all x in $\mathbb{R}$, so $f(x) = -x^2 + 2ax + b$, $b \in \mathbb{R}$, for all x in $\mathbb{R}$.

To begin with, note that f restricts injectively to either side of a. We now show that g is an involution on $\mathbb{R}$: $g(g(x)) = x$ for all x in $\mathbb{R}$. Clearly, this holds at $x = a$. For $x \neq a$, injectivity of f on either side of a shows that x and $g(x)$ always fall on opposite rays: either $x < a < g(x)$ or $g(x) < a < x$. Consequently, x and $g(g(x))$ always fall on the same ray. Since $f(x) = f(g(x)) = f(g(g(x)))$, the conclusion follows by injectivity of f on the ray containing both x and $g(g(x))$.

Next, we show that g is differentiable at any $x \neq a$, and $g'(x) = -1$ for all $x \neq a$. To this end, fix any such x and let $0 < h < |x - a|$. Then

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{\dfrac{f(x+h)-f(x)}{h}}{\dfrac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}} \xrightarrow{h\to 0} \frac{f'(x)}{f'(g(x))}\\ &= \frac{g(x)-x}{g(g(x))-g(x)} = \frac{g(x)-x}{x-g(x)} = -1,\end{aligned}$$

and we are done; note that all quotients above make sense.

Finally, continuity at $x = a$ yields $g(x) = -x + 2a$.

12th GRADE

Problem 1. Let $\mathcal{C}$ be the class of all differentiable functions $f : [0,1] \to \mathbb{R}$ with a continuous derivative f' on $[0,1]$, and $f(0) = 0$ and $f(1) = 1$. Determine the minimum value the integral $\int_0^1 \left(1+x^2\right)^{1/2} \left(f'(x)\right)^2 \mathrm{d}x$ may assume as f runs through all of $\mathcal{C}$, and find all functions in $\mathcal{C}$ that achieve this minimum value.

Solution. Apply the Cauchy-Schwarz inequality to get

$$\begin{aligned}1 =& f(1) - f(0) = \int_0^1 f'(x)\mathrm{d}x\\ &= \int_0^1 \left(1+x^2\right)^{-1/4} \left(\left(1+x^2\right)^{1/4} f'(x)\right) \mathrm{d}x\\ &\leqslant \left(\int_0^1 \left(1+x^2\right)^{-1/2} \mathrm{d}x\right)^{1/2} \left(\int_0^1 \left(1+x^2\right)^{1/2} \left(f'(x)\right)^2 \mathrm{d}x\right)^{1/2}\\ &= \left(\ln\left(1+\sqrt{2}\right)\right)^{1/2} \left(\int_0^1 \left(1+x^2\right)^{1/2} \left(f'(x)\right)^2 \mathrm{d}x\right)^{1/2}.\end{aligned}$$

Consequently,

$$\int_0^1 \left(1+x^2\right)^{1/2} \left(f'(x)\right)^2 \mathrm{d}x \geqslant \frac{1}{\ln\left(1+\sqrt{2}\right)},$$

for all $f \in \mathcal{C}$. Equality holds if and only if $f'(x) = k\left(1+x^2\right)^{-1/2}$; that is,

$$f(x) = k \ln\left(x + \sqrt{1+x^2}\right) + c.$$

The conditions $f(0) = 0$ and $f(1) = 1$ yield

$$f(x) = \frac{1}{\ln(1+\sqrt{2})} \ln\left(x + \sqrt{1+x^2}\right), \quad 0 \leqslant x \leqslant 1,$$

a function which clearly belongs to $\mathcal{C}$.

Problem 2. Let f be a continuous, positive real-valued function on $[0,1]$.

a) Given a positive integer number n, prove that there exists a unique subdivision, $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$, of $[0,1]$ such that

$$(*) \qquad \int_{a_k}^{a_{k+1}} f(x)\mathrm{d}x = \frac{1}{n}\int_0^1 f(x)\mathrm{d}x, \quad k = 0, \ldots, n-1.$$

b) For each positive integer number n, let

$$\bar{a}_n = \frac{a_1 + \cdots + a_n}{n},$$

where $0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1$ is the unique subdivision of $[0,1]$ satisfying $(*)$. Prove that the sequence $(\bar{a}_n)_{n\geqslant 1}$ is convergent and evaluate its limit.

Solution. a) Let $F : [0,1] \to [0,\infty)$,

$$F(x) = \int_0^x f(t)\mathrm{d}t.$$

Rewrite $(*)$ as

$$F(a_{k+1}) - F(a_k) = \frac{1}{n}F(1), \quad k = 0, \ldots, n-1,$$

to get

$$(**) \qquad F(a_k) = \frac{k}{n}F(1), \quad k = 0, \ldots, n.$$

Since f takes on positive values, F is strictly increasing, so its restriction $F : [0,1] \to [0, F(1)]$ is one-to-one and onto. Consequently,

$$a_k = F^{-1}\left(\frac{k}{n}F(1)\right), \quad k = 0, \dots, n,$$

is the unique solution to $(**)$.

b) Given an integer $n \geqslant 1$, by part a),

$$a_k = F^{-1}\left(\frac{k}{n}F(1)\right), \quad k = 0, \dots, n,$$

so

$$\bar{a}_n = \frac{1}{n}\sum_{k=1}^{n} F^{-1}\left(\frac{k}{n}F(1)\right) = \frac{1}{F(1)} \cdot \frac{F(1)}{n}\sum_{k=1}^{n} F^{-1}\left(\frac{k}{n}F(1)\right).$$

Since

$$\frac{F(1)}{n}\sum_{k=1}^{n} F^{-1}\left(\frac{k}{n}F(1)\right)$$

is a Riemann sum for $F^{-1} : [0, F(1)] \to [0,1]$, it converges to

$$\int_0^{F(1)} F^{-1}(t)\mathrm{d}t = \int_0^1 xf(x)\mathrm{d}x.$$

Consequently, $(\bar{a}_n)_{n\geqslant 1}$ converges to $\dfrac{\int_0^1 xf(x)\mathrm{d}x}{\int_0^1 f(x)\mathrm{d}x}$.

Remark. As expected, $(\bar{a}_n)_{n\geqslant 1}$ converges to the abscissa of the centroid of the plane domain bounded by the lines $x = 0$, $x = 1$, $y = 0$ and the graph of the function $y = f(x)$.

Problem 3. Given a positive integer n, determine the rings R with the property that $x^{2^n+1} = 1$ for all $x \in R \setminus \{0\}$.

Solution. Clearly, R is of characteristic 2. If $R = \{0,1\}$, then $R = \mathbb{F}_2$, which obviously fulfills the required condition for any positive integer n. Otherwise, for any $x \in R \setminus \{0,1\}$, $(x+1)^{-1} = (x+1)^{2^n} = x^{2^n} + 1 = x^{-1} + 1$, whence $1 = x^{-1}(x+1) + x + 1$, or $x^{-1} + x + 1 = 0$; that is, $x^2 + x + 1 = 0$. Now, let $x \in R\setminus\{0,1\}$ and $y \in R\setminus\{0,1,x\}$. Were $x+y \neq 1$, then $(x+y)^2+(x+y)+1 = 0$, whence $xy + yx = 1$. It would then follow that $x^2y + xyx = x = xyx + yx^2$, so $x^2y = yx^2$, whence $(x+1)y = y(x+1)$; that is, $xy = yx$ — a contradiction. Consequently, $y = 1 + x$, and $R = \{0, 1, x, 1+x\}$, i.e., $R = \mathbb{F}_4$. This R satisfies the condition in the statement if and only if $2^n + 1$ is divisible by 3, which is the case if and only if n is odd. We conclude that $R = \mathbb{F}_2$ for n even, and $R = \mathbb{F}_2$ or $R = \mathbb{F}_4$ for n odd.

Remark. Several entrants derived commutativity from Jacobson's celebrated criterion: If for every element x of a ring, $x^n = x$ for some integer $n \geqslant 2$, then the ring is commutative. For $R \neq \mathbb{F}_2$, this forces exactly two more elements, x and $1+x$, so $R = \mathbb{F}_4$.

Problem 4. Given an integer number $n \geqslant 3$, let G be a subgroup of the symmetric group S_n generated by $n-2$ transpositions. Prove that, for each i in $\{1, \dots, n\}$, the set $\{\sigma(i) : \sigma \in G\}$ has at most $n-1$ elements.

Solution. Associate with the group G a graph Γ on n vertices labeled $1, \dots, n$, by letting ij be an edge of Γ whenever (i,j) is one of the $n-2$ generating transpositions. We thus get a graph with n vertices and $n-2$ edges. Combinatorially, the problem amounts to showing that such a graph is not connected — roughly speaking, there are not enough edges to get from any one vertex to any other vertex. This is a straightforward consequence of the well-known fact that any spanning tree of a connected graph on n vertices has $n-1$ edges. However, in what follows, we provide a proof which does not resort to that argument. Proceed by induction on $n \geqslant 3$. Clearly, the conclusion holds for $n = 3$. For $n > 3$, let i_0 be a vertex of minimal degree. Then

$$n \deg i_0 \leqslant \sum_{i=1}^{n} \deg i = 2(n-2),$$

so either $\deg i_0 = 0$ — in which case we are done — or $\deg i_0 = 1$. In the latter case, the induced subgraph $\Gamma - i_0$ has $n-1$ vertices and $n-3$ edges. By the induction hypothesis, $\Gamma - i_0$ has at least two components. Since $\deg i_0 = 1$, the vertex i_0 is adjoined to exactly one of those components. The conclusion follows.

Remarks. 1) If G is generated by the $n-2$ transpositions $(1,i)$, $i = 2, \dots, n-1$ — that is, G is the standard embedding of S_{n-1} in S_n —, then each set of the form $\{\sigma(i) : \sigma \in G\}$, $i = 1, \ldots, n-1$, has exactly $n-1$ elements — the largest possible cardinality; of course, $\{\sigma(n) : \sigma \in G\} = \{n\}$.

2) The sets $\{\sigma(i) : \sigma \in G\}$ are the orbits of the action of G on $\{1, \ldots, n\}$. The conclusion is that the length of any orbit does not exceed $n-1$. In particular, G cannot act transitively on $\{1, \ldots, n\}$: there exist i and j in $\{1, \ldots, n\}$ such that $j \neq \sigma(i)$, whatever σ in G; that is, j cannot be reached from i via a permutation in G — combinatorially, the graph Γ associated with G has not enough edges to get from i to j.

PROBLEMS AND SOLUTIONS

BMO AND IMO SELECTION TESTS

Problem 1. At the vertices of a convex polygon with even number of sides sit hunters, while in the interior of the polygon, and not lying on any of its diagonals, sits a fox. Simultaneously, the hunters shoot at the fox, but the fox ducks in good time, and the bullets go on, hitting sides of the polygon. Prove that at least one side is not hit.

Solution. (D. Schwarz) Let us use the *Bow-tie Lemma* from the 2006 IMO Problem 6.

Given a convex polygon with even number of sides, call bow-tie the figure formed by the two vertex-opposed triangles obtained by tracing two consecutive "big" diagonals. The reunion of all bow-ties entirely covers the polygon.

Fix an oriented "big" diagonal. All interior points lying on "big" diagonals are obviously covered by the bow-ties; consider then a point not lying on any "big" diagonal, wlog situated to the left of the oriented diagonal. By anti-clockwise "rotating" the diagonal through consecutive vertices, we will reach its initial position, only that its orientation will be changed, so the point will now be situated to its right. It means that somewhere in the process there has been a moment when the point passed from the left to the right of the "rotating" diagonal — so at that time it would have been contained within the bow-tie corresponding to those positions.

It means that the fox is contained in some bow-tie. Then the side of the bow-tie's triangle *not* containing the fox is not hit by any bullet.

Alternative solution. (T. Dumitrescu) Prolong the rays connecting the fox with the hunters to a length greater than any of the distances between fox and hunters, and consider that circle having the fox as center. Also, consider the diametrically opposite points of those occupied by the hunters, and flag them with the corresponding hunter's id. (No conflict arises, as the fox is not lying on any diagonal of the polygon.) Assuming every side would be hit, it follows that each will be hit by exactly one bullet, so hunters and flags must alternate on the circle. Consider a pair hunter-flag, splitting the other hunters in k on one semicircle, and l on the other. But then we will have l flags on the semicircle of the k hunters (and conversely, k flags on the semicircle of the l hunters). The alternating condition then requires that $k = l$, so in all we must have $k + l + 1 = 2k + 1$ hunters, but this is an odd number, absurd.

Alternative solution. (Given by several contestants) Similar with the one above, counting hunters and bullets on both sides of a diagonal (or a main diagonal), and applying a similar parity argument. We leave details to the reader.

Remarks. Last minute info traces this problem (and more) to some issue of [AMM]. Also, it has been used in the [Moldova 2007 Selection Tests], unbeknownst to the Romanian selectioners, but no harm was done.

Problem 2. Let $\mathcal{C}(O_1)$ and $\mathcal{C}(O_2)$ be two circles, external to each other. Points A, B, C lie on $\mathcal{C}(O_1)$, while points D, E, F lie on $\mathcal{C}(O_2)$, such that AD and BE are external tangents to the two circles, while CF is an internal common tangent. The lines CO_1 and FO_2 meet the lines AB, respectively DE, at M, respectively N. Show the line MN passes through the midpoint of the segment CF.

Solution. (C. Pohoaţă) The conclusion is obvious if the radii of the two circles are equal. In order for the solution to match a possible drawing, we will assume the radius of $\mathcal{C}(O_1)$ to be less than that of $\mathcal{C}(O_2)$.

Let $P = AD \cap BE$, let X, Y be the points where the internal tangent CF meets AD, respectively BE, and let X', Y' be the points where the parallel line through N to XY meets AD, respectively BE. Notice the quadrilaterals $O_2NX'D$ and O_2NEY' are cyclic, since each has a pair of right homological angles.

Therefore, $\angle O_2X'N = \angle O_2DN = \angle O_2EN = \angle O_2Y'N$, hence triangle $O_2X'Y'$ is isosceles, therefore O_2N, being altitude, is also median, so $NX' = NY'$. Similarly, $MX'' = MY''$, for the corresponding construction. Taking $K = PN \cap XY$, since $XY \parallel X'Y' \parallel X''Y''$, it follows that $M \in PN$, and K is the midpoint of XY. Since $\mathcal{C}(O_1)$ is the incircle of triangle PXY, while $\mathcal{C}(O_2)$ is its excircle (relative to P), and their points C, respectively F, of contact with XY are isotonic with respect to XY, it follows that K is the midpoint of CF, which solves the problem, since $K \in MN$.

Alternatively, using Lemma 2 presented in the solution of Problem 24, one gets that M lies on the median from P in triangle PXY, and similarly N, therefore P, M, N are collinear with the midpoint K of XY, which is in the same time the midpoint of CF.

Problem 3. Any $f : \mathbb{Q} \to \mathbb{R}$ with the property below is constant

$$|f(x) - f(y)| \leqslant (x-y)^2, \text{ for all } x, y \in \mathbb{Q}.$$

Solution. Let $a < b$, $a, b \in \mathbb{Q}$, and let $a = x_0 < x_1 < \cdots < x_n = b$ be an equidistant division of the interval $[a, b]$. Then

$$|f(a) - f(b)| = \left|\sum_{k=0}^{n-1}(f(x_{k+1}) - f(x_k))\right| \leqslant \sum_{k=0}^{n-1}|f(x_{k+1}) - f(x_k)|.$$

Since $x_{k+1} - x_k = \frac{b-a}{n}$ for all $0 \leqslant k < n$, it follows that

$$|f(a) - f(b)| \leqslant \frac{(b-a)^2}{n}, \text{ for all } n \in \mathbb{N}^*.$$

If $f(a) \neq f(b)$, for $n > \frac{(b-a)^2}{|f(a)-f(b)|}$, we reach a contradiction.

Alternative solution. (D. Schwarz) Since clearly f is continuous on $\mathbb{Q}$, one can use the fact that a function, continuous on a dense subset of $\mathbb{R}$ (in the induced topology), can be uniquely prolonged to a continuous function on $\mathbb{R}$. Now the given property also extends, since the function modulus is also continuous. Therefore

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leqslant |x - y|, \text{ for all } x \neq y \in \mathbb{R},$$

so f is differentiable on $\mathbb{R}$, and its derivative is null, hence f is constant.

Remarks. It is enough to have

$$|f(x) - f(y)| \leqslant (x-y)^p, \text{ for all } x, y \in \mathbb{Q}, \text{ with } p \in \mathbb{R},\ p > 1.$$

Problem 4. For $n \in \mathbb{N}$, $n \geqslant 2$, determine

$$\max \prod_{i=1}^{n} (1 - x_i), \text{ for } x_i \in \mathbb{R}_+,\ 1 \leqslant i \leqslant n,\ \sum_{i=1}^{n} x_i^2 = 1.$$

Solution. (D. Schwarz) Let us analyze $E(x,y) = (1-x)(1-y)$, $x, y \geqslant 0$, $x^2 + y^2 = k^2$, $0 < k \leqslant 1$. Take $x = k\sin\theta$, $y = k\cos\theta$, $\theta \in [0, \frac{\pi}{2}]$. Now, $x + y = k(\sin\theta + \cos\theta) = k\sqrt{2}\sin(\theta + \frac{\pi}{4}) = k\sqrt{2}\cos(\theta - \frac{\pi}{4})$, and $xy = k^2 \sin\theta\cos\theta = \frac{k^2}{2}\sin 2\theta = \frac{k^2}{2}\cos(2\theta - \frac{\pi}{2}) = \frac{k^2}{2}\cos 2(\theta - \frac{\pi}{4})$, hence $xy = k^2\cos^2(\theta - \frac{\pi}{4}) - \frac{k^2}{2}$.

Take $u = \cos(\theta - \frac{\pi}{4})$, so $u \in [\frac{1}{\sqrt{2}}, 1]$, and then $E(x,y) = E(u) = k^2z^2 - k\sqrt{2}z + 1 - \frac{k^2}{2}$. Its minimum value is reached for $u_0 = \frac{1}{k\sqrt{2}} \geqslant \frac{1}{\sqrt{2}}$, therefore

• for $1 < \frac{1}{k\sqrt{2}} + (\frac{1}{k\sqrt{2}} - \frac{1}{\sqrt{2}}) = \frac{2-k}{k\sqrt{2}}$, i.e. $k < 2(\sqrt{2}-1)$, the maximum value for $E(u)$ is reached for $u = \frac{1}{\sqrt{2}}$, i.e. when $\theta \in \{0, \pi\}$, which means x or y being zero;

• for $k \geqslant 2(\sqrt{2}-1)$, the maximum value for $E(u)$ is reached for $u = 1$, i.e. when $\theta = \frac{\pi}{4}$, which means $x = y$.

Now, for $x^2 + y^2 + z^2 = 1$, there will be two, wlog be them $x^2 + y^2 \leqslant \frac{2}{3} < (2(\sqrt{2}-1))^2$ (it comes to $288 < 289$), so first case applies for $x^2 + y^2 = k^2$, $k < 2(\sqrt{2}-1)$ (case $k = 0$ is trivial), therefore the maximum value is reached when one of the three variables is zero.

So, when

$$\sum_{i=1}^{n} x_i^2 = 1, \quad E = \prod_{i=1}^{n}(1 - x_i)$$

takes maximum value when *all* but two variables (be them x, y) are zero; then $E = E(x,y) = (1-x)(1-y)$, with $x^2 + y^2 = 1$, and second case applies, yielding $\max E = (1 - \frac{1}{\sqrt{2}})^2$, for $x = y = \frac{1}{\sqrt{2}}$.

Remarks. The result for $n = 2$, $x^2 + y^2 = k^2$, may be obtained through other methods, e.g. by partial differentiation of $E(x,y) = (1-x)(1-y)$. Also, an alternative solution using Lagrange multipliers is available, but must be used with utmost care, as the maximum is to be obtained on the boundary.

Problem 5. Let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ be a degree $n \geqslant 3$ polynomial with integer coefficients, $a_k + a_{n-k}$ even, for all $k = 1, 2, \ldots, n-1$ and a_0 also even. If $f = gh$, where g and h are polynomials with integer coefficients, and the degree of g at most the degree of h, and all coefficients of h are odd, show that f has (at least) an integer root.

Solution. Polynomials g and h are obviously monic. The condition on the coefficients of f and h implies $\deg g > 0$, hence $m = \deg h < n$. Considering the relation $f = gh$ in $\mathbb{Z}_2$, one gets

$$(1) \qquad f_1 + \hat{1} = \hat{g} \cdot \left(X^m + X^{m-1} + \cdots + X + \hat{1}\right),$$

where $f_1 = \hat{f} + \hat{1}$. Since the polynomial f_1 is reciprocal, applying the mapping $X \mapsto \frac{1}{X}$ into (1) yields

$$(2) \qquad f_1 + X^n = g_1 \cdot \left(X^m + X^{m-1} + \cdots + X + \hat{1}\right).$$

Summing relations (1) and (2) yields

$$X^n + \hat{1} = g_2 \cdot \left(X^m + X^{m-1} + \cdots + X + \hat{1}\right)$$

or

$$\left(X^n - X^{n-m-1}\right) + \left(X^{n-m-1} + \hat{1}\right) = g_2 \cdot \left(X^m + X^{m-1} + \cdots + X + \hat{1}\right).$$

Therefore, the polynomial $X^m + X^{m-1} + \cdots + X + \hat{1}$ is a divisor of the polynomial $X^{n-m-1} + \hat{1}$, hence $m = n-1$, because $m = \deg h \geqslant \deg g = n - m$. It follows that $\deg g = 1$, i.e. f has (at least) an integer root.

Problem 6. Let ABC be a triangle. Its incircle is tangent to AB at E, while its excircle relative to BC is tangent to AB at F. Let D be the point lying on side BC for which the incircles of triangles ABD and ACD have equal radii. The lines DE and DB meet a second time the circumcircle of triangle ADF at X and Y. Show that $XY \parallel AB$ if and only if $AB = AC$.

Solution. Let us start with the following

LEMMA. *For a triangle ABC, where D is the point lying on side BC, such that triangles ABD and ACD have incircles of equal radii, the relation $AD^2 = p(p-a)$ holds, where the usual notations in a triangle are used.*

Proof. Let I, I_1 and I_2 be the centers of the incircles of triangles ABC, ABD and ACD, let r, r_1 and r_2 be their radii, and let D' be the foot of the perpendicular from I on BC. Since it is given that $r_1 = r_2 = \rho$, we get $I_1I_2 \parallel BC$. Let now D_1, D_2 be the feet of the perpendiculars from I_1, I_2 on BC. Then $I_1I_2 = D_1D_2 = (BD' - BD_1) + (CD' - CD_2) = a - (BD_1 + CD_2) = a - (p_1 - AD + p_2 - AD) = a - p + AD$. On the other hand, $rp = [ABC] = [ABD] + [ACD] = \rho(p_1 + p_2) = \rho(p + AD)$, therefore $\frac{\rho}{r} = \frac{p}{p+AD}$. Finally, since $I_1I_2 \parallel BC$, one gets $\frac{I_1I_2}{BC} = \frac{a-p+AD}{a} = \frac{r-\rho}{r} = \frac{AD}{p+AD}$, so $(a-p+AD)(p+AD) = a \cdot AD$, or $AD^2 = p(p-a)$.

Thus $AD^2 = p(p-a) = AF \cdot AE$, so AD is tangent to the circumcircle of triangle DEF, hence $\angle ADE = \angle AFD$.

Now, from $XY \parallel AB$, since the quadrilateral $AYFD$ is cyclic, and so $\angle AYD = \angle ADY = \alpha$, whence $\angle YXD = 2\alpha$. Then $ABC = 2\alpha$; but $\angle AFD = \alpha$, so $\angle FDB = \angle DFB = \alpha$, hence triangle BFD is isosceles, so $BF = BD = p - c$. It follows that D is the point of tangency of the ex-circle with BC, hence AD is the Nagel line relative to BC. Since we have $AD^2 = p(p-a) \leqslant m_a^2$, and it is readily seen that $AD \geqslant m_a$ (where m_a is the median from A), it follows that equality occurs, i.e. $AB = AC$.

Conversely, from $AB = AC$, since, as proven, AD is tangent to the circumcircle of triangle DEF, and $AD \perp DB$, it follows the circumcenter of triangle DEF lies on DB, hence B is the midpoint of EF. Therefore $\angle BFD = \angle BDF = \angle ADE = \angle BAY = \angle XYA$, so $XY \parallel AB$, since the quadrilaterals $ADFY$ and $ADYX$ are cyclic.

Problem 7. Find all sets A of at least two positive integers, such that for any distinct $x, y \in A$ we also have $(x+y)/(x,y) \in A$.

Solution. Let us denote $f(x,y) = (x+y)/(x,y)$ and take a, b two distinct elements in A.

If $(a,b) = 1$, then $f(a,b) = a+b \in A$, and so $a+2b \in A$ and $2a+b \in A$. Now, $f(a, a+2b) = 2a+2b \in A$, unless $2 \mid a$ and, similarly, $f(b, 2a+b) = 2a+2b \in A$, unless $2 \mid b$; but both cannot be true, so $2(a+b) \in A$. Then $f(a+b, 2(a+b)) = 3$, hence $3 \in A$, and let us take $c \in A$, $(3,c) = 1$ (such c surely exists, as from $(a,b) = 1$ it follows $(3,a) = 1$ or $(3,b) = 1$). Inductively, we get $c + 3k \in A$ for $k \geqslant 1$, and for $k = cm$, $m \geqslant 1$, $f(c, c+3cm) = 3m+2 \in A$. Similarly, $kc + 3 \in A$ for $k \geqslant 1$, and for $k = 3$, $f(3, 3c+3) = c+2 \in A$. Then, for $c = 3m+2$, $c+2 = 3m+4 \in A$ and, for $c = 3m+4$, $c+2 = 3m+6 \in A$. But then $f(3, 3m+6) = m+3 \in A$ for $m \geqslant 1$.

It follows then that the possible sets are $\mathbb{N}^*$, $\mathbb{N}^* \setminus \{1\}$, $\mathbb{N}^* \setminus \{2\}$, $\mathbb{N}^* \setminus \{1,2\}$.

If $(x,y) > 1$ for all pairs of distinct elements $x, y \in A$, then take $1 < a < b$ to be the least two elements in A. Then $f(a,b) = (a+b)/(a,b) < 2b/(a,b) \leqslant 2b/2 = b$, hence $(a+b)/(a,b) = a$, therefore $a[(a,b)-1] = b$, so $a \mid b$, $(a,b) = a$, and so $b = a(a-1)$ with $a > 2$, providing the extra family of solutions $A = \{n, n(n-1)\}$ for $n > 2$.

Assume now $A \neq \{a,b\}$; take $c > b$ to be the least such element. Then, as above, $f(a,c) < c$, and $f(a,c) = a$ leads to $c = a(a-1) = b$, absurd, so we must have $f(a,c) = b = a(a-1)$, therefore $a \mid c$, $(a,c) = a$ and so $c = a(a^2-a-1)$. But then $f(b,c) = (a-1) + (a^2-a-1) = a^2-2$ (as $(a-1, a^2-a-1) = 1$), so $a^2 - 2 \in A$. As $f(b,c) < c$, this means either $a^2-2 = a$ or $a^2-2 = b = a(a-1)$, both leading to $a = 2$, absurd.

Remarks. This problem is based on a Swiss problem, where the condition x, y distinct was not present (in fact, it was explicitly not enforced). In turn, the Swiss have borrowed it, in precisely that format, from an APMO examination paper of the 2000's. Then, for $a \in A$, one has $f(a,a) = 2 \in A$ (which offers the solution with one element $A = \{2\}$) and the proof is greatly simplified:

If $1 \in A$, then $f(1,2) = 3 \in A$, so for $A \neq \{2\}$ there is always $a \in A$, $a > 2$. Let's take a to be the least with this property. If a is even, then $f(2,a) = 1 + a/2 < a$ and $1 + a/2 > 2$, contradiction. Therefore a must be odd, and then inductively $a + 2k \in A$, for $k \geqslant 1$. For $k = ma$, $m \geqslant 1$, it follows $f(a, (2m+1)a) = 2(m+1) \in A$, and $f(2, 2(m+1)) = m+2 \in A$, therefore the other possible sets are $\mathbb{N}^*$ and $\mathbb{N}^* \setminus \{1\}$.

Problem 8. Let X be the set of the 2^n points $\{0,1\}^n$, $n \geqslant 3$, in the Euclidean n-space (the vertices of the unit hypercube). Denote by $M(n)$ the least integer such that any subset $Y \subseteq X$, with $M(n)$ elements or more, necessarily contains an equilateral triangle (determined by points from Y). Prove that $M(n) \leqslant \lfloor 2^{n+1}/n \rfloor + 1$, and effectively compute $M(3)$ and $M(4)$.

Solution. (D. Schwarz) For $p \in X$, consider S_p to be the set of points q at distance 1 from p, i.e. differing from p in exactly one coordinate. Then $|S_p| = n$, and all points in S_p are equidistant (so it is a simplex of dimension $n-1$). Each point in X belongs to exactly n such sets S_p, so for a subset $Y \subseteq X$ with $|Y| > 2^{n+1}/n$ we have

$$\sum_{p \in X} |S_p \cap Y| = n|Y| > 2^{n+1},$$

only that there are 2^n sets $S_p \cap Y$, so, by the pigeonhole principle, (at least) one must contain at least 3 points, which will form an equilateral triangle contained in Y.

For $n = 3$, X is the unit cube, S_p are triangles, and it is easily seen that $M(3) = 5$ (while $\lfloor 2^{3+1}/3 \rfloor + 1 = 6$). For $n = 4$, X is the unit hypercube, S_p are tetrahedra, and considering the set with 8 elements

$$\{(0,0,0,0),(0,0,1,1),(0,1,0,0),(0,1,1,1)\} \cup$$
$$\cup \{(1,1,1,1),(1,1,0,0),(1,0,1,1),(1,0,0,0)\},$$

one gets that $M(4) = \lfloor 2^{4+1}/4 \rfloor + 1 = 9$, as prescribed by the formula.

Remarks. The case $|Y| = 2^{n+1}/n$, requiring $n = 2^k$, to yield $|Y| = 2^{2^k-k+1}$ and allow for Y to intersect each S_p in exactly 2 points, therefore containing no equilateral triangle, could maybe be extrapolated from the case $n = 4$?!?

Problem 9. Let $\mathcal{F}$ be the set of all functions $f : \mathcal{P}(S) \to \mathbb{R}$ with the property that, for any $X, Y \subseteq S$, we have $f(X \cap Y) = \min(f(X), f(Y))$, where S is a finite set. Determine

$$\max_{f \in \mathcal{F}} |\mathrm{Im}(f)|.$$

Solution. We will determine quite precisely the structure of such functions, and prove in the process that $\max_{f \in \mathcal{F}} |\mathrm{Im}(f)| = 1 + |S|$.

Let the values in $\mathrm{Im}(f)$ be $a_0 < a_1 < \cdots < a_k$ for some $0 \leqslant k \leqslant 2^{|S|} - 1$, and take subsets $S_i \subseteq S$ of lowest cardinality such that $f(S_i) = a_i$ (an application of the *extremal element principle*). Clearly, the subsets S_i are distinct, as f takes distinct values on them.

For $X \subseteq S$ there will thus exist $0 \leqslant x \leqslant k$ such that $f(X) = a_x$, and so for all $0 \leqslant j \leqslant x$ we have $f(X \cap S_j) = \min(f(X), f(S_j)) = \min(a_x, a_j) = a_j = f(S_j)$. But $X \cap S_j \subseteq S_j$, so the extremality of S_j implies $X \cap S_j = S_j$, i.e. $S_j \subseteq X$, in particular $S_x \subseteq X$. As $0 \leqslant x$ for any x, it follows that $S_0 \subseteq X$ for any X, therefore $S_0 = \emptyset$. Also, for $X = S_i$, therefore $x = i$, it follows $S_j \subsetneq S_i$ for all $0 \leqslant j < i$. Finally, for $|X| = |S_x|$, as $S_x \subseteq X$, it follows $X = S_x$, by having the same cardinality, so the subsets S_i are uniquely determined.

On the other hand, for $k \geqslant j > x$ we cannot have $S_j \subseteq X$, as then $a_j = f(S_j) = f(X \cap S_j) = \min(f(X), f(S_j)) = \min(a_x, a_j) = a_x$, absurd.

The results above lead to $\emptyset = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subseteq S$, a chain of $1 + k$ subsets from $\mathcal{P}(S)$, starting with $\emptyset$. These subsets are uniquely determined, and in turn they uniquely determine f, as $f(X) = \max\{f(S_i); S_i \subseteq X\}$. As a chain in $\mathcal{P}(S)$ can have at most $1 + |S|$ elements, it follows $0 \leqslant k \leqslant |S|$ and so $|\mathrm{Im}(f)| \leqslant 1 + |S|$.

Conversely, for any chain in $\mathcal{P}(S)$, starting with $\emptyset$, we can uniquely define a function with the structure of the above (therefore establishing a one-to-one correspondence between such functions and such chains – up to the actual values taken by f). This shows that the value $1 + |S|$ (as well as any lower values) can be reached for $|\mathrm{Im}(f)|$.

Problem 10. Show that, for n, p positive integers, $n \geqslant 4$ and $p \geqslant 4$, the proposition $\mathcal{P}(n,p)$ below is false

$$\sum_{i=1}^{n} \frac{1}{x_i{}^p} \geqslant \sum_{i=1}^{n} x_i{}^p \text{ for } x_i \in \mathbb{R},\ x_i > 0,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} x_i = n.$$

(As a matter of fact, the propositions $\mathcal{P}(4,3)$ and $\mathcal{P}(3,4)$ are true, but hard to prove!)

Solution. Notice first that it is enough to find a set of values x_i for $n = 4$ such that

$$E = \sum_{i=1}^{n} \frac{1}{{x_i}^p} - \sum_{i=1}^{n} {x_i}^p < 0$$

as then for any $n > 4$ we can extend this set of values by taking the extra $n - 4$ ones to be equal to 1.

Now that we have reduced it to the case $n = 4$, it makes sense to look for "simple" cases:

- some x_i very small – it yields $E > 0$, no good;
- all x_i equal – it yields common value 1, for which $E = 0$, no good;
- let's then try taking the smallest three x_i equal to some value $0 < x < 1$; the fourth one, denoted by y, will be $1 < y < 4$, $y = 4 - 3x$.

Then $E = 3/x^p + 1/y^p - 3x^p - y^p = (1/x^p)[3 + (x/y)^p - 3x^{2p} - (xy)^p] = (1/x^p)[3 - (xy)^p] + (1/y^p)[1 - 3(xy)^p]$.

It seems natural now to look for the maximum possible value for xy; it is not difficult to see that $xy = (1/3)(3x)(4 - 3x) \leqslant (1/3)(2)^2 = 4/3$ (by AM-GM), with equality for $3x = 4 - 3x$, i.e. $x = 2/3$ and $y = 2$.

Then, as $4/3 > 1$ and $p \geqslant 4$, we have $(4/3)^p \geqslant (4/3)^4 = 256/81 > 3$, hence $E < 0$ for the set of values ($2/3, 2/3, 2/3, 2$ and 1 repeated $n - 4$ times).

Problem 11. Let a_i, $i = 1, 2, \dots, n$, $n \geqslant 3$, be positive integers with their greatest common divisor equal to 1, such that a_j divides $\sum_{i=1}^n a_i$ for all $j = 1, 2, \dots, n$. Prove that $\prod_{i=1}^n a_i$ divides $(\sum_{i=1}^n a_i)^{n-2}$.

(Also, provide an example showing that the exponent $n-2$ cannot be lowered.)

Solution. It is enough to prove that the exponent of a prime factor p in the canonical factorization of $\prod_{i=1}^n a_i$ is at most the exponent of p in the canonical factorization of $(\sum_{i=1}^n a_i)^{n-2}$. For each i, let m_i be the exponent of p in the factorization of a_i; finally, let $m = \max\{m_i; i = 1, 2, \dots, n\}$, and j an index for which $m = m_j$. As a_j divides $\sum_{i=1}^n a_i$, it follows that p^m also divides it, therefore $p^{m(n-2)}$ divides $(\sum_{i=1}^n a_i)^{n-2}$. On the other hand, at least one of the numbers a_i, be it a_k, is not divisible by p. As p divides $\sum_{i=1}^n a_i$, it follows that at least two of its terms are not divisible by p, therefore the exponent of p in the factorization of $\prod_{i=1}^n a_i$ is at most $m(n-2)$. (Let us now take $a_1 = 1$, $a_2 = n - 1$, and $a_j = n$ for $j = 3, \dots, n$, satisfying problem's requirements. Then $\sum_{i=1}^n a_i = n(n-1)$, while $\prod_{i=1}^n a_i = n^{n-2}(n-1)$. As n and $n-1$ are co-prime, it follows that the exponent $n-2$ cannot be lowered. Other examples may be found.)

Problem 12. Points M, N, P on the sides BC, CA, AB of triangle ABC are such that triangle MNP is acute-angled. Denote by x the length of the shortest altitude of ΔABC, and by X the length of the longest altitude of ΔMNP. Prove that $x \leqslant 2X$.

Solution. (Greek Problem Selection Committee) Denote by H the orthocenter of ΔMNP, and by A', B', C' its projections onto BC, CA, AB respectively. Since ΔMNP is acute-angled, H lies in its interior, thus also in the interior of ΔABC. Then

$$x \leqslant HA' + HB' + HC' \leqslant HM + HN + HP \leqslant 2X,$$

as needed, where the first inequality holds by the (well-known) Lemma 1 below, while the last one holds by (not so well-known) Lemma 2.

Clearly, equality may occur, for ΔABC equilateral, and ΔMNP its median triangle. Other solutions, trigonometrical or otherwise, are available, but cannot match the elegance of the above.

LEMMA 1. *If H is any point in the interior or on the sides of a triangle ΔABC, and A', B', C' are its projections onto BC, CA, AB respectively, then $x \leqslant HA' + HB' + HC'$, where x is the shortest altitude of ΔABC.*

Let x_a, x_b, x_c be the altitudes from A, B, C respectively. Then

$$\frac{HA' + HB' + HC'}{x} \geqslant \frac{HA'}{x_a} + \frac{HB'}{x_b} + \frac{HC'}{x_c} = \frac{[BHC]}{[ABC]} + \frac{[CHA]}{[ABC]} + \frac{[AHB]}{[ABC]} = 1,$$

hence the claimed result.

LEMMA 2. *If MNP is an acute-angled triangle, and H is its orthocenter, then $HM + HN + HP \leqslant 2X$, where X is the longest altitude of ΔMNP.*

Let us assume $\angle M \leqslant \angle N \leqslant \angle P$; then $NP \leqslant PM \leqslant MN$, and X equals the altitude MM'. Thus, we need prove $HM + HN + HP \leqslant 2MM'$, or equivalently

$HN + HP \leqslant HM + 2HM'$. But the symmetrical point H' of H, with respect to NP, lies on the circumcircle of ΔMNP, hence the relation is also equivalent to $H'N + H'P \leqslant H'M$. Applying Ptolemy's Theorem to the cyclic quadrilateral $H'NMP$, we get $H'M \cdot NP = H'N \cdot MP + H'P \cdot MN \geqslant H'N \cdot NP + H'P \cdot NP$, therefore $H'M \geqslant H'N + H'P$, as wanted.

Problem 13. Prove that the function $f : \mathbb{N} \to \mathbb{Z}$ defined below is injective

$$f(n) = n^{2007} - n!\,.$$

Solution. Suppose $f(x) = f(y)$, with $x > y$. First we discuss the case $y \leqslant 2007$. Clearly, there are no solutions for $y = 0$ or $y = 1$, so assume $y > 1$. Denote by $\nu_p(n)$ the exponent of the prime p in the factorization of n. For $x! - y! = x^{2007} - y^{2007}$, and q a prime divisor of y, it must be that q divides x. Thus $\nu_q(x!) > \nu_q(y!)$, hence $\nu_q(x! - y!) = \nu_q(y!)$. On the other hand, we have $q^{2007} \mid x^{2007} - y^{2007}$, so we must have $2007 \leqslant \nu_q(y!)$. However, this is impossible, since

$$\nu_q(y!) = \left\lfloor \frac{y}{q} \right\rfloor + \left\lfloor \frac{y}{q^2} \right\rfloor + \cdots < \frac{y}{q-1} \leqslant 2007.$$

For the case $y > 2007$, every prime $p < 2007$ divides $x! - y! = x^{2007} - y^{2007}$. If $\gcd(p-1, 2007) = 1$, there exists a positive integer r such that $2007r \equiv 1 \pmod{p-1}$. Therefore, $x \equiv x^{2007r} \equiv y^{2007r} \equiv y \pmod{p}$ (from Fermat's Little Theorem), i.e. $p \mid x - y$.

In particular, this holds for $p = 23$ and 101, hence $x - y \geqslant 23 \cdot 101 = 2323 > 2008$. But then we have $x! - y! > (x(x-1)\cdots(x-2007) - 1)y! > x(x-1)\cdots(x-2006) \cdot 2007! > x^{2007}$, where the last inequality follows from $(x-k)(k+1) > x$, for $k = 1, 2, \ldots, 2006$, a contradiction.

GENERALIZATION. (Idea of M. Dumitrescu) We will prove that f given by $f(n) = n^k - n!$, for integer $k > 1$, is injective (for $k = 1$ it is clear that $1^1 - 1! = 2^1 - 2! = 0$, but this is the only possible case).

First prove that $f(2k) < 0$. We have $(2k)! > 2 \cdot k^k \cdot k! \geqslant 2^k \cdot k^k = (2k)^k$. Now, if $n \geqslant k$, such that $f(n) < 0$, we have then $f(n) - f(n+1) = n \cdot n! - n^k \cdot [(1 + \frac{1}{n})^k - 1] > 2(n! - n^k) = -2f(n)$, since $(1 + \frac{1}{n})^k \leqslant (1 + \frac{1}{k})^k < e < 3$. Therefore, $f(n+1) < 3f(n) < 0$, and $f(n) - f(n+1) > 0$. By simple induction we have then that, for $n > n_0 \geqslant k$, such that $f(n_0) < 0$, it follows that $f(n) < f(n_0)$, hence $f(n)$ is negative, strictly decreasing.

Now, similarly with the first solution, assuming $f(x) = f(y)$, for $x > y > 1$ and q prime divisor of y, we get $k \leqslant \nu_q(y!) < \frac{y}{q-1}$, so for $q \geqslant 3$ it follows $y > 2k$, and then, according to the above, we have $f(x) < f(y)$.

The only possible case remains $k < y$, $y = 2^m$, $m > 1$, and, moreover, x even. Then, for z odd factor of x, if $z < y$, since z then divides x^k, $x!$, and $y!$, it follows it must divide y^k, but as z is odd, and $y = 2^m$, it follows $z = 1$. Then x is also a power of 2, and $x > y$ implies then $x \geqslant 2y > 2k$. On the other hand, if $z > y$, we again have $x > 2y > 2k$. In both cases it follows $f(x) < 0$. But either $f(y) \geqslant 0 > f(x)$, or else $f(y) < 0$, and then again, according to the above, $f(x) < f(y)$.

Further results. (D. Schwarz) For $p \in \mathbb{N}^*$, analyze the function $f_p : \mathbb{N} \to \mathbb{Z}$ defined by

$$f_p(n) = n^p - n!\,.$$

Analysis. One has $f_p(0) = -1$, $f_p(1) = 0$, for all p.

For $p = 1$, one has $f_1(2) = 0$, $f_1(3) = -3 < 0$, $f_1(n) = n(1 - (n-1)!)$ for $n \geqslant 3$, strictly decreasing thereafter, and $f_1(1) = f_1(2)$; as we will see — the only case of non-injectivity. We will therefore consider in the sequel $p \geqslant 2$.

For $p = 2$, one has $f_2(2) = 2$, $f_2(3) = 3$, $f_2(4) = -8 < 0$.

LEMMA. For $k \in \mathbb{N}^*$

$$\left(1 + \frac{1}{k}\right)^k < e < \left(1 + \frac{1}{k}\right)^{k+1}.$$

For $n \leqslant p$, $p \geqslant 3$

$$\left(1 + \frac{1}{n}\right)^p \geqslant \left(1 + \frac{1}{p}\right)^p = \frac{p}{p+1}\left(1 + \frac{1}{p}\right)^{p+1} > \frac{pe}{p+1} > 2.$$

Then one has $f_p(n) - f_p(n+1) = n \cdot n! - n^p[(1 + \frac{1}{n})^p - 1] < n \cdot n! - n^p = n^2[(n-1)! - n^{p-2}] < n^2[(n-1)! - n^{n-2}] < 0$, hence $f_p(n) < f_p(n+1)$, for $p \geqslant 2$ (at limit, $f_p(p) < f_p(p+1)$).

For $n \geqslant p+1$

$$\left(1+\frac{1}{n}\right)^p < \left(1+\frac{1}{n-1}\right)^p \leqslant \left(1+\frac{1}{p}\right)^p < e < 3.$$

Then $f_p(n)-f_p(n+1) = n\cdot n! - n^p[(1+\frac{1}{n})^p - 1] = [(1+\frac{1}{n-1})n](n-1)\cdot(n-1)! - (1+\frac{1}{n-1})^p(n-1)^p[(1+\frac{1}{n})^p-1] > 3(n-1)\cdot(n-1)! - 3(n-1)^p[(1+\frac{1}{n-1})^p-1] = 3(f_p(n-1)-f_p(n))$ (at limit $f_p(p+1)-f_p(p+2) > 3(f_p(p)-f_p(p+1))$).

Finally, $f_p(2k) < 0$, since $(2p)! > 2\cdot p^p \cdot p! \geqslant 2^p \cdot p^p = (2p)^p$.

This shows there exist $p < n_m < n_0 \leqslant 2p$ such that f_p strictly increasing to $f_p(n_m)$, and decreasing from $f_p(n_m)$ (with possibly $f_p(n_m+1) = f_p(n_m)$, then strictly decreasing thereafter), and $f_p(n_0) < 0$, but $f_p(n_0-1) > 0$ (for $p=2$, $n_m = p+1 = 3$, and $n_0 = 2p = 4$). It would be worth estimating (asymptotically) the values for n_m and n_0, as functions of p, probably needing Stirling's formula.

As a corollary, for $x > y > 1$, $x \geqslant n_0$, one has $f_p(x) < f_p(y)$.

We will prove injectivity; assume now $f_p(x) = f_p(y)$, $x > y > 1$, and consider q a prime divisor of y. (It is trivial that for $y = 0$ or $y = 1$ one cannot have $f_p(x) = f_p(y)$.) Since then q divides y^p, $y!$ and $x!$, it follows that it divides x^p, hence x. If we denote by $\nu_q(z)$ the exponent of q in the canonical factorisation of z, then $p \leqslant \nu_q(x^p - y^p) = \nu_q(x! - y!) = \nu_q(y!) = \lfloor \frac{y}{q} \rfloor + \lfloor \frac{y}{q^2} \rfloor + \cdots < \frac{y}{q-1}$ (by Legendre's formula), therefore $y > (q-1)p$. For $q \geqslant 3$, this implies $x > y > 2p$, hence, according to the above, $f_p(x) < f_p(y)$. It remains the case $q = 2$, when y is a power of 2, $y > p$, and then x must be even. Then an odd divisor of x is either larger than y, or else it must divide y^p, hence it equals 1, so $x > 2p$, hence again, according to the above, $f_p(x) < f_p(y)$.

Alternatively, for $n \geqslant p$, such that $f_p(n) < 0$, we have then $f_p(n) - f_p(n+1) = n\cdot n! - n^p[(1+\frac{1}{n})^p - 1] > 2(n! - n^p) = -2f_p(n)$. Therefore $f_p(n+1) < 3f_p(n) < 0$, and $f_p(n) - f_p(n+1) > 0$. By simple induction we have then that, for $n > n_0 \geqslant k$, such that $f_p(n_0) < 0$, it follows that $f_p(n) < f_p(n_0)$, hence $f_p(n)$ is negative, strictly decreasing, enough to prove injectivity.

Problem 14. Let $A_1A_2A_3A_4A_5$ be a convex pentagon, such that

$$[A_1A_2A_3] = [A_2A_3A_4] = [A_3A_4A_5] = [A_4A_5A_1] = [A_5A_1A_2].$$

Prove there exists a point M in the plane of the pentagon, such that

$$[A_1MA_2] = [A_2MA_3] = [A_3MA_4] = [A_4MA_5] = [A_5MA_1].$$

($[XYZ]$ is the area of ΔXYZ).

Solution. Since $[A_1A_2A_3] = [A_2A_3A_4]$, we have $A_1A_4 \parallel A_2A_3$. Similarly are proven the relations $A_{i-1}A_{i+2} \parallel A_iA_{i+1}$, where all indices are taken modulo 5. Denote by U the intersection of A_1A_3 and A_2A_4. Notice that $UA_4 \parallel A_1A_5$ and $UA_1 \parallel A_4A_5$. Thus $A_1A_5A_4U$ is a parallelogram, and A_5U passes through the midpoint H of A_1A_4. Denote by V the midpoint of A_2A_3. Since $A_1A_2A_3A_4$ is a trapezoid, it is well-known that HU (i.e. A_5U) passes through V. Let G be the centroid of the pentagon. Notice that G lies on A_5V, since the centroids of A_2A_3, A_1A_4, and A_5, all lie on A_5V. Similarly, the segments joining the vertices A_i with the midpoints of $A_{i+2}A_{i+3}$ will all pass through G.

We shall prove that G satisfies the requirement. Notice that $[A_5GA_2] = [A_5GA_3]$ (from $A_2V = A_3V$). Similarly, it is proven that $[A_2GA_5] = [A_2GA_4] = [A_1GA_4] = [A_1GA_3] = [A_3GA_5]$. Denote by S_2 this common value, by S_1 the common value of $[A_1A_2A_3] = \cdots$, and by S_T the area of the pentagon. It follows that $[A_2GA_3] = S_T - [A_5A_1A_2] - [A_5A_4A_3] - [A_5GA_2] - [A_5GA_3] = S_T - 2S_1 - 2S_2$, therefore all $[A_iGA_{i+1}]$ are equal, and thus G satisfies the desired requirement.

It can further be proven that the point G is unique with the required property.

Alternative solution. (L. Ţurea) One knows that affine transformations preserve collinearity of points, parallelness of lines, concurrence of lines, the property of a point to be a centroid of others, ratios of areas (of triangles), and convexity. On the other hand, one can find an affine transformation that maps the pentagon into one with two equal diagonals, emanating from a same vertex, and making the angle found in a regular pentagon.[2] Similar considerations with those in the solution above lead to the fact that this pentagon must be regular,[3] whence a point like the

[2] Any three non collinear points can be mapped to any three other non collinear points, by a unique affine transformation. An affine transformation is a map from the plane to itself, of the form $(x, y) \mapsto (ax + by + c, dx + ey + f)$, with $ae - bd \neq 0$. See e.g. K. Kedlaya.

[3] See Theorem 3.5.3 from same K. Kedlaya.

one we look for is obviously (and uniquely) the center of the pentagon. Then its pre-image is the centroid of the original pentagon (thus also uniquely determined), due to areas' ratio preservation!

A yet another interesting alternative solution may exploit the fact that areas of triangles are given by determinants on vertices' coordinates!

Problem 15. Consider the set $E = \{1, 2, \ldots, 2n\}$. Prove that an element $c \in E$ may belong to a subset $A \subset E$, with n elements, such that for any two distinct elements of A, none divides the other, if and only if $c > n(2/3)^{k+1}$, where k is the exponent of 2 in the factorization of c.

Solution. (D. Schwarz) Such subsets A are *antichains* (relative to the partial order given by division). A famous and well-known Erdös result states that size n antichains are maximal. Recall that a maximal antichain must contain an element each, having each odd from E as (maximal) odd factor. Therefore, for $c = 2^k d$, with d odd, if $3^{k+1}d \leqslant 2n$, we must have elements $c_t = 2^{k_t}3^t d$ for $0 \leqslant t \leqslant k+1$ (where $c_0 = c, k_0 = k$). But for $s < t$ we need $k_s > k_t$, otherwise $c_s \mid c_t$, hence $k = k_0 > k_1 > \cdots > k_{k+1}$, which is clearly impossible in non-negative integers.

On the other hand, for $3^{k+1}d > 2n$, let us consider those odds from E which are divisible by d, hence of the form $qd \leqslant 2n$, with $q \neq 1$ odd; let us take $k_q = \max\{t; 3^t qd \leqslant 2n\}$ (obviously $k_q \geqslant 0$, and $3^k qd \geqslant 3^{k+1}d > 2n$, hence $k_q < k$). Take $k_1 = k, c_1 = c = 2^k d, c_q = 2^{k_q}qd$ and the other elements of the form $d_r = 2^{t_r}r$, where r from E, odd, not divisible by d, and $t_r = \max\{t; 2^t r \leqslant 2n\}$. Clearly, the only thing to verify is that, for $1 < q < q'$, $q \mid q'$ implies that $k_q > k_{q'}$, otherwise $c_q \mid c_{q'}$; but then $q' \geqslant 3q$ and $3^{k_{q'}+1}qd \leqslant 3^{k_{q'}}q'd \leqslant 2n$, hence $k_q \geqslant k_{q'} + 1$.

Problem 16. i) Determine all infinite arithmetical sequences of positive integers, with the property: there exists $N \in \mathbb{N}$, such that for any p prime, $p > N$, the p^{th} term of the sequence is also a prime.

ii) Determine all polynomials $f(X) \in \mathbb{Z}[X]$, with the property there exists $N \in \mathbb{N}$, such that for any p prime, $p > N$, $|f(p)|$ is also a prime.

Solution. i) For a its first term and r its ratio, the sequence is determined by the formula $a_n = f(n) = rn + (a - r)$. When $r = 0$, the sequence is the constant a, $a_n = a$, for all $n \geqslant 1$, so any prime a will do.

When $r > 0$, denote $p_1 = a_p = a + (p-1)r \geqslant 1 + (p-1) = p$, therefore $a_p = p_1$ being prime, we are allowed to iterate and take $p_{n+1} = a_{p_n}$, for $n \geqslant 1$, $p > N$, p prime. Simple induction yields

$$p_n = (a-r)(1 + r + \cdots + r^{n-1}) + pr^n.$$

Now, $r = 1$ gives $p_n = n(a-1) + p$, so when $a = 1$ it follows $p_n = p$ (so the iteration stops after the first step), $a_n = n$ for all $n \geqslant 1$, and it is a suitable sequence; while when $a > 1$, taking $n = p$ yields $p_p = p(a-1) + p = pa$, which cannot be a prime.

Finally, for $r > 1$, we get

$$p_n = (a-r)\frac{r^n - 1}{r - 1} + pr^n = \frac{[(a-1) + (p-1)(r-1)](r^n - 1)}{r - 1} + p$$

and also $p < p_1 < \cdots < p_n < \cdots$. Take $p > N$, $p > r$, so $(p, r) = 1$, $(p, r-1) = 1$, and so for $n = p-1$ we have $p_{p-1} > p$ and $p \mid p_{p-1}$, as $p \mid r^{p-1} - 1$, from Fermat's little Theorem. This is absurd, as all p_n are supposed to be prime.

Alternative solution. i) The case $r = 0$ is treated as above. For $r > 0$, if $a - r = 0$, the need for $f(p) = rp$ to be prime requires $r = 1$. Finally, for $a - r \neq 0$, take q prime with $(q, r) = 1$; then the equation $rx + (a - r) \equiv 0(\bmod q)$ has a solution $x_0 = r^{-1}(r - a)$, so $(q, x_0) = 1$. But $(x_0 + mq)_{m \geqslant 1}$ contains infinitely many primes (by Dirichlet's Theorem), so by taking one such prime p, with $p > N$ and $f(p) > q$, we have $q \mid f(p)$, therefore $f(p)$ cannot be prime.

One could therefore make the point of forbidding the use of Dirichlet's Theorem, as it is both not actually needed, and it offers a much too easy and rapid way out.

ii) We will start by proving the following

LEMMA.[4] *For $f(X) \in \mathbb{Z}[X]$, $\deg(f) \geqslant 1$, the set of prime divisors of $(|f(n)|)_{n \in \mathbb{N}}$ is infinite.*

[4] Năstăsescu, Niţă, Brandiburu, Joiţa – Culegere de probleme de Algebră.

One can write $f(X) = Xg(X) + f(0)$, $\deg(g) \geqslant 0$, so $g(X) = 0$ has finitely many solutions. For $f(0) = 0$, the result is trivial, as $n \mid |f(n)|$. For $f(0) \neq 0$, then $f(f(0)X) = f(0)[Xg(f(0)X) + 1] = f(0)h(X)$, with $h(0) = 1$, and then $(n, |h(n)|) = 1$, therefore $(|h(n)|)_{n\in\mathbb{N}}$ cannot have a finite set of prime divisors, hence neither $(|f(n)|)_{n\in\mathbb{N}}$.

Now, for $\deg(f) = 0$, $f(X) \equiv \pm p$, p prime, is a possibility. For $\deg(f) \geqslant 1$, if $f(0) = 0$ then $f(X) = Xg(X)$, so $|f(p)| = p|g(p)|$ needs be prime for all p prime, $p > N$. But $g(p) = \pm 1$ can only happen for finitely many p, unless $g(X) \equiv \pm 1$, when $f(X) \equiv \pm X$, yet another possibility.

Finally, if $f(0) \neq 0$, take a prime q, $q > |f(0)|$, such that $q \mid |f(n)|$ for some n, as warranted by the Lemma. Then clearly $(q, n) = 1$, as $f(n) = ng(n) + f(0)$, and $(q, |f(0)|) = 1$. Therefore, $f(n + mq) = Mq + f(n)$ is divisible by q, but in the arithmetical sequence $(n + mq)_{m\in\mathbb{N}}$ there are infinitely many primes (Dirichlet's Theorem). On the other hand, $f(X) = \pm q$ has finitely many solutions. Taking p prime from the arithmetical sequence above, $p > N$, such that $f(p) \neq \pm q$, we need $|f(p)|$ to be prime, but $q \mid |f(p)|$, absurd.

Alternative solution. (Given by several contestants) ii) For p prime, $p > N$, when $|f(p)| = q$ prime, if $q \neq p$, then for $m \in \mathbb{N}$, one has $p + mq > N$, so $|f(p + mq)|$ must be a prime whenever $p + mq$ is itself a prime, which occurs infinitely often, according to Dirichlet's Theorem. On the other hand, $|f(p + mq)| = |Mq + f(p)| = Mq$, so one needs have $|f(p + mq)| = q$ in all those cases, therefore f must be constant, $f(X) \equiv \pm q$. If $|f(p)| = q = p$, for all p prime, $p > N$, then one needs have $f(X) \equiv \pm X$.

It would be worth searching for a solution that makes no use of Dirichlet's Theorem.

Problem 17. The vertices of a convex polygon are lying on a circle of center O. Prove that, for any triangulation of the polygon made by not self-intersecting diagonals, the sum of the squares of distances, from O to the incenters of the triangles in the triangulation, is the same.

Solution. Let Δ be a triangulation made by not self-intersecting diagonals, and let T be a triangular cell in Δ. According to Euler's Theorem, the distance d from O to the incenter of T is given by $d^2 = R^2 - 2Rr$, where R is the circumradius of T (equal to the radius of the given circle), while r is the inradius of T. Let x, y, z be the signed distances from O to the sides of T, where the sign of a distance to a side is positive iff O lies in the same half-plane (determined by that side) as the triangle T. Carnot's Theorem states that $R + r = x + y + z$. It follows that

$$d^2 = R^2 - 2R(x + y + z - R) = 3R^2 - 2R(x + y + z).$$

We now sum over all T in Δ. Let s be any side of a triangle in Δ. If s is a diagonal of the given polygon, then the distance from O to s occurs twice in the summation, but with opposite signs, so the two terms cancel each other. If s is a side of the given polygon, then the distance from O to s occurs once only in the summation (its sign being positive or negative, depending on whether O and the polygon lie on the same side of s or not). Denote by S the sum of the signed distances from O to the sides of the polygon. Since the triangulation Δ always consists of the same number of triangles (2 less than the number of sides), we conclude that the sum of the squares of distances sought after only depends on R and S, therefore is the same for any allowed triangulation.

Alternative solution. (A. Zahariuc) The following problem is known. [5]

Given a cyclic quadrilateral, the sum of the inradii of the two triangles determined by one diagonal is equal to the sum of the inradii of the two triangles determined by the other diagonal.

Simple induction, in conjunction to Euler's Theorem (stated above), extends this result to proving the problem at hand.

Problem 18. Let $\Gamma_A, \Gamma_B, \Gamma_C$ be three circles situated in the interior of triangle ABC, such that each is tangent to the two other, Γ_A is tangent to the sides AB and AC, Γ_B is tangent to the sides BC and BA, while Γ_C is tangent to the sides CA and CB. Let D be the tangency point of Γ_B and Γ_C, E be the tangency point

[5] The solution uses (and proves) the fact that the incenters of the four triangles are the vertices of a rectangle. A trigonometrical solution is also readily available.

of Γ_C and Γ_A, and F be the tangency point of Γ_A and Γ_B. Prove that the lines AD, BE, CF are concurrent.

Solution. Let U, V and W be the centers of the circles Γ_A, Γ_B and Γ_C respectively. The pairs of lines EF and VW, FD and WU, and DE and UV meet at X, Y and Z respectively (one or all three points may be at infinity, but the argument below works in those cases, too). Menelaus' Theorem applied to triangle UVW and transversals EFX, FDY and DEZ yields that X, Y and Z are the intersection points of the common external tangents to Γ_B and Γ_C, Γ_C and Γ_A, and Γ_A and Γ_B respectively. By a Theorem of d'Alembert and Monge, the points X, Y and Z are collinear. Consequently, the pair of triangles ABC and DEF is perspective, therefore the lines AD, BE and CF are concurrent (by the converse to Desargues' Theorem).

Problem 19. Consider the convex pentagon $ABCDE$ where $AB = BC$, $CD = DE$, angles $\angle ABC$ and $\angle CDE$ are supplementary, $\angle ABC = 135°$, and the area of the pentagon is $\sqrt{2}$.

a) Determine the length of BD.

b) Letting $\angle ABC$ be variable within the initial conditions, determine the minimum length of BD.

Solution. It is natural to consider the symmetrical point F to C, with respect to BD. Triangles CBF, ABF, CDF, EDF are all isosceles. The angle bisectors of $\angle ABF$ and $\angle EDF$ meet at K. Then $\angle KBD + \angle KDB = \frac{1}{2}\angle ABC + \frac{1}{2}\angle CDE = \frac{1}{2}\pi$, so $\angle BKD = \frac{1}{2}\pi$. Since BK, DK are also the perpendicular bisectors of AF, respectively EF, it follows that also $\angle AFE = \frac{1}{2}\pi$. Therefore K is the midpoint of AE, since K is the circumcenter of triangle AFE. A simple reading of the configuration yields the fact that the area of the pentagon is exactly twice that of triangle BKD, therefore

$$\begin{aligned}\sqrt{2} &= KB \cdot KD = (BD \sin \angle KDB)(BD \sin \angle KBD) \\ &= BD^2 \sin \frac{1}{2}\angle ABC \cos \frac{1}{2}\angle ABC = \frac{1}{2} BD^2 \sin \angle ABC,\end{aligned}$$

or

$$BD = \sqrt{\frac{2\sqrt{2}}{\sin \angle ABC}} \geqslant \sqrt{2\sqrt{2}},$$

with equality when $\angle ABC$ (and $\angle CDE$) are right angles. For the first part, since $\sin 135° = \frac{\sqrt{2}}{2}$, it follows $BD = 2$.

Problem 20. Let $ABCD$ be a parallelogram with no angle equal to $60°$. Find all pairs of points E, F, in the plane of $ABCD$, such that triangles AEB and BFC are isosceles, of basis AB, respectively BC, and triangle DEF is equilateral.

Solution. Take E and F such that triangles AEB and BFC are equilateral, with interiors disjunct with $ABCD$. Then $AD = BC = CF = BF$, $AE = BE = AB = CD$, and, denoting $\alpha = \angle BAD$, $\angle DAE = \angle FBE = \angle FCD = \alpha + 60°$, therefore triangles DAB, FBE and FCD are congruent (SAS), whence $DE = EF = FD$, i.e. triangle DEF is equilateral. Similarly, for E and F such that triangles AEB and BFC are equilateral, with interiors intersecting $ABCD$.

On the other hand, when triangle DEF is equilateral, a rotation of angle $\pm 60°$ will rotate the perpendicular bisector of AB, meeting the perpendicular bisector of BC in exactly two points, be them F_+, F_-, originating from points E_+, E_-. But the triangles DE_+F_+ and DE_-F_- are clearly equilateral, and this accounts for the two cases proven in the paragraph above, therefore no other cases, with isosceles triangles AEB and BFC may exist anymore.

Problem 21. The world-renowned marxist theorist *Joric* is obsessed with both mathematics and social equalitarism. Therefore, for any positive integer n, written in decimal representation, he tries to partition its digits into two groups, such that the absolute value of the difference between the sums of the digits in each group is as small as possible. Joric calls this value the *defect* of the number n. Determine the average value of the defect (over all positive integers), that is, if we denote by $\delta(n)$ the defect of n, compute

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \delta(k)}{n}.$$

Solution. We claim the answer is $\frac{1}{2}$. When the sum of the digits of n is even, one has $\delta(n) \geqslant 0$, while when it is odd, one has $\delta(n) \geqslant 1$, since the defect has the same parity with the sum of the digits. Since exactly "half" of the positive integers have the sum of their digits even, respectively odd, the average of the defect is at least $\frac{1}{2}$.

Let $\sigma(n) = 0$ if the sum of the digits of n is even, and $\sigma(n) = 1$ if it is odd. We will show that the set of those positive integers n for which $\delta(n) \neq \sigma(n)$ has zero density. There are exactly

$$9^{N-k}\binom{N}{k}$$

integers of at most N digits, containing exactly k digits equal to 1. Then the set of the integers of at most N digits, containing less than 9 digits equal to 1, contains

$$\sum_{k=0}^{8} 9^{N-k}\binom{N}{k} < \sum_{k=0}^{8} 9^{N}\cdot N^{k} < 9^{N}(9N)^{8}$$

elements, hence its density is

$$\lim_{N\to\infty}\left(\frac{9}{10}\right)^{N}\cdot(9N)^{8} = 0.$$

Now, for a number n of N digits, be them ordered by magnitude $9 \geqslant a_1 \geqslant a_2 \geqslant \cdots \geqslant a_N \geqslant 0$, we have $(a_1 + a_3 + \cdots) - (a_2 + a_4 + \cdots) = (a_1 - a_2) + (a_3 - a_4) + \cdots = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots$, hence $0 \leqslant \delta(n) \leqslant 9$.

When n has $k \geqslant 9$ digits 1, the same computation for the other $N - k$ digits yields a value v between 0 and 9, and we have enough digits 1 to compensate and achieve $\delta(n) = \sigma(n)$. Putting together all these results provides the claimed answer.

(As it can easily be seen, the result holds for any numeration basis.)

Problem 22. Three travel companies provide transportation between n cities, such that each connection between a pair of cities is covered by one company only. Prove that, for $n \geqslant 11$, there must exist a round-trip through some four cities, using the services of a same company, while for $n < 11$ this is not anymore necessarily true.

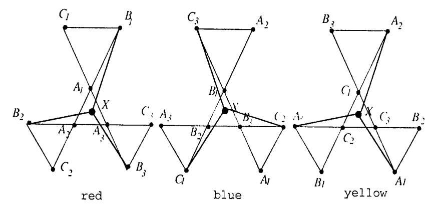
Solution. In graph-theoretical language the problem amounts to considering the following proposition $(\mathcal{P})$, and proving that proposition $(\mathcal{P})$ is true for $n \geqslant 11$, while being false for $n < 11$:

Any 3-coloring of the edges of the complete graph K_n on n vertices contains a monochromatic cycle of length 4 (4-cycle).

It is clear that $(\mathcal{P})$ is a hereditary property: if false for $n = k$, then is also false for any $n < k$. Therefore, it is enough to exhibit a 3-coloring for K_{10}, with no monochromatic cycle of length 4, in order to prove the second part of the assertion. We will use a 3-coloring configuration for K_9: [6] denote the 9 vertices A_i, B_i, C_i for $i = 1, 2, 3$ and

- color in red A_1A_2, A_2A_3, A_3A_1 and A_iB_i, A_iC_i, B_iC_i for $i = 1, 2, 3$;
- color in blue B_1B_2, B_2B_3, B_3B_1 and B_iA_j, A_jC_k, C_kB_i for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$;
- color in yellow C_1C_2, C_2C_3, C_3C_1 and C_iA_j, A_jB_k, B_kC_i for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ and $(3, 1, 2)$.

Take a 10^{th} vertex X and color in red XB_i, in blue XC_i, and in yellow XA_i for $i = 1, 2, 3$. This 3-coloring contains no monochromatic 4-cycle.



On the other hand K_{11} has 55 edges, hence, by pigeonhole principle, any 3-coloring will yield (at least) one color used for (at least) 19 edges. It will suffice to prove that any graph with 19 edges on 11 vertices exhibits a 4-cycle.

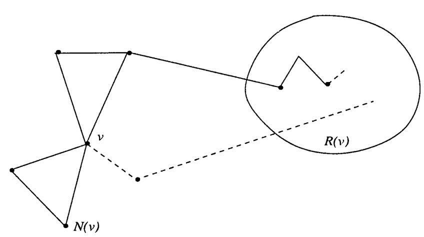
[6] This is a very interesting configuration in its own: it is the largest such as K_n contains no monochromatic cycles of any length larger than 3.

Let us denote by $\mathrm{ex}(n)$ the maximum number of edges a graph on n vertices may have, without containing any 4-cycle (such a graph will be called *extremal*). For example, trivially so, $\mathrm{ex}(0) = \mathrm{ex}(1) = 0$, $\mathrm{ex}(2) = 1$, $\mathrm{ex}(3) = 3$, $\mathrm{ex}(4) = 4$. Let us also denote by $E(S,T)$ the number of *crossing* edges with ends in subsets of vertices S and T.

If we select a vertex v of degree d from an extremal graph on n vertices, let us denote by $N(v)$ the set of its d neighbors, and by $R(v)$ the set of the remaining $n-d-1$ vertices. Then $E(v,v) = 0$, $E(v, N(v)) = d$, $E(v, R(v)) = 0$. Counting edges such as no 4-cycle could possibly occur forces $E(N(v), N(v)) \leqslant \lfloor \frac{d}{2} \rfloor$, $E(N(v), R(v)) \leqslant n-d-1$, $E(R(v), R(v)) \leqslant \mathrm{ex}(n-d-1)$, whence the majoration

$$\mathrm{ex}(n) \leqslant d + \left\lfloor \frac{d}{2} \right\rfloor + (n-d-1) + \mathrm{ex}(n-d-1) = n-1+\left\lfloor \frac{d}{2} \right\rfloor + \mathrm{ex}(n-d-1) = f_n(d).$$

The function $f_n(d)$ is clearly decreasing in d while $d < n-2$, therefore an upper bound for $f_n(d)$ is given by the least guaranteed value for the highest degree d of any vertex v of an extremal graph. [7]



[7] Use the well-known fact that for a graph $G = (V, E)$ the sum of vertices' degrees equals twice the number of edges

$$\sum_{v \in V} \deg(v) = 2\mathrm{card}(E)$$

and an *averaging* argument

$$\max_{v \in V}(\deg(v)) \geqslant \frac{2\mathrm{card}(E)}{\mathrm{card}(V)}.$$

Let us prove that $\mathrm{ex}(6) = 7$. Assuming $\mathrm{ex}(6) \geqslant 8$, there will exist (at least) a vertex v of degree $d \geqslant \lceil \frac{2\cdot 8}{6} \rceil = 3$ (see above footnote). Then $f_6(3) = 5 + \lfloor \frac{3}{2} \rfloor + \mathrm{ex}(2) = 7$, absurd. On the other hand, any configuration with 7 edges for an extremal graph on 6 vertices will have to contain (at least) one triangle — the one counted by $\lfloor \frac{d}{2} \rfloor$, since the degree d can be chosen to be no less than 3.

(There are, in fact, only four such possible configurations: one with a vertex of (maximal) degree 5 (and two triangles), one with a vertex of (maximal) degree 4 (and two triangles), and two with two vertices of (maximal) degree 3 (one with two triangles, one with only one).)

We claim that $\mathrm{ex}(11) \leqslant 18$ (there exist in fact graphs on 11 vertices, with just 18 edges and with no 4-cycles, so $\mathrm{ex}(11) = 18$, but this information is not actually needed; also our method yields $\mathrm{ex}(10) = 16$). Assume $\mathrm{ex}(11) \geqslant 19$; then there will exist (at least) a vertex v of degree $d \geqslant \lceil \frac{2\cdot 19}{11} \rceil = 4$ (see above footnote). Then the total number of edges is bound by $f_{11}(4) = 10 + \lfloor \frac{4}{2} \rfloor + \mathrm{ex}(6) = 19$, but the only configurations for $\mathrm{ex}(6) = 7$ do not allow for the realization of all 19 edges, as a triangle from $R(v)$ cannot be connected with $N(v)$ using full 3 edges, as this would create a 4-cycle. Therefore, the claim is proven, and $\mathrm{ex}(11) < 19$ implies that a graph with 19 edges on 11 vertices will necessarily contain a 4-cycle.

Remarks. The difficulty of the problem would have been further increased if one asked for the value of the threshold value 11, rather than offering it in its statement.

The author acknowledges the literature, as one would expect, is full of similar problems, this being a natural question to ask in extremal graph theory. However, in the eye of the author, the present result, offering an exact answer to the presence of 4-cycles in 3-colorings of complete graphs, although difficult to reach, is suitable as the difficult question for an IMO selection test.

A problem amounting to the fact that $\mathrm{ex}(16) < 36$ was used in the Bulgarian Mathematical Olympiad sometimes in the years 2000' (the true value, using our method, can be computed to be $\mathrm{ex}(16) = 33$).

An upper bound of

$$\mathrm{ex}(n) \leqslant \left\lfloor \frac{n}{4}(1 + \sqrt{4n-3}) \right\rfloor$$

is proven in [Aigner & Ziegler – Proofs from the Book], but is not tight enough to yield $\text{ex}(11) = 18$ (it actually offers $\text{ex}(11) \leqslant 20$, while we need less than 19). Computed for the problem mentioned above, it yields $\text{ex}(16) \leqslant 35$ (which is just enough, suggesting that it might have been the recommended solution for it).

The fact that a 2-coloring of K_5 does exist (K_5 is the union of two disjunct 5-cycles), together with the result of $\text{ex}(6) = 7$, and the fact that K_6 has 15 edges, hence, by pigeonhole principle, any 2-coloring will yield (at least) one color used for (at least) 8 edges, gives a novel proof to our problem posed in a 2005 JBMO-selection test in Romania, further used the year after in Croatia.

Both the 2-coloring and the 3-coloring problems are stated, with very gross estimations for the threshold values (14 instead of the true 6 value for the 2-coloring; 80 instead of the true 11 value for the 3-coloring), in [Engel – Problem Solving Strategies].

Finally, a calculation of $\text{ex}(10) = 16$, together with some extensive bibliography (and a mention of $\text{ex}(16) = 33$), is to be found in the [1991 August-September, AMM].

All these references are of no avail in solving the present problem, as the methods they use are inadequate for the question at hand.

Further Remarks. (Extremal Graph Theory) Denote by $\text{ex}(n, \mathcal{H})$ the maximum number of edges of a graph G (called *extremal*) on n vertices, not having as a subgraph any of the graphs $H \in \mathcal{H}$. An *edge-maximal* graph G would be one such that any added edge will make it contain some $H \in \mathcal{H}$. Not all edge-maximal graphs are extremal (e.g., a graph on 4 vertices, with 2 edges, not containing a P_3, while $\text{ex}(4, P_3) = 3$).

Some notable results:

• $\text{ex}(n, (P_k)_{k\geqslant r}) = \text{ex}(n, P_r) \leqslant \frac{r-1}{2}n$, with equality iff the connected components of the extremal graph are K_r's (Erdös, with difficult proof). If the extremal graph is desired to be connected, results are published by various authors.

• $\text{ex}(n, (C_k)_{k\geqslant r}) \leqslant \frac{r-1}{2}(n-1)$, with equality iff the extremal graph is made of "leaves" which are K_{r-1}'s, connected in a "cactus"-like structure (Erdös and Gallai). In particular, $\text{ex}(n, (C_k)_{k\geqslant 3}) = n - 1$, where the extremal graphs are

maximal acyclic graphs, viz. trees; $\text{ex}(5, (C_k)_{k\geqslant 4}) = 6$, $\text{ex}(6, (C_k)_{k\geqslant 4}) = 7$, $\text{ex}(9, (C_k)_{k\geqslant 4}) = 12$, $\text{ex}(10, (C_k)_{k\geqslant 4}) = 13$.

• $\text{ex}(n, (K_k)_{k\geqslant r}) = \text{ex}(n, K_r) = t_{r-1}(n) \leqslant \frac{1}{2}n^2\frac{r-2}{r-1}$, with equality iff $r-1$ divides n (Turán). In particular, $\text{ex}(n, C_3) = \text{ex}(n, K_3) \leqslant \frac{1}{4}n^2$.

• $\text{ex}(n, C_4) \leqslant \lfloor \frac{n}{4}(1 + \sqrt{4n-3}) \rfloor$ (Reiman), only known asymptotically. (For $n = q^2 + q + 1$, with q a prime power exceeding 13, $\text{ex}(n, C_4) = \frac{1}{2}q(q+1)^2$ (Füredi).) In particular, $\text{ex}(5, C_4) = 6$, $\text{ex}(6, C_4) = 7$, $\text{ex}(10, C_4) = 16$, $\text{ex}(11, C_4) = 18$, $\text{ex}(16, C_4) = 33$ (references in the [AMM] problem).

• Finally, the meta-theorem of extremal graph theory

$$\lim_{n\to\infty} \text{ex}(n, H)\binom{n}{2}^{-1} = \frac{\chi(H) - 2}{\chi(H) - 1}$$

(Erdös-Stone corollary). Therefore, for an odd cycle H, since $\chi(H) = 3$, the limit is $\frac{1}{2}$, while for an even cycle H, since $\chi(H) = 2$, the limit is 0, whence the "strange" difference between avoiding odd/even cycles.

Now, for a K_n whose edges are 2-colored, K_5 is the largest such that it may contain no monochromatic cycle longer than 3 (example), while K_6 must contain a monochromatic C_4.

For a K_n whose edges are 3-colored, K_9 is the largest such that it may contain no monochromatic cycle longer than 3 (example), while K_{10} must contain a monochromatic cycle longer than 3, but not necessarily a C_4 (example), and K_{11} must contain a monochromatic C_4 (the problem at hand). (The color spectrum for the K_{10} needs be 15-15-15, 14-15-16 or 13-16-16.)

Problem 23. For $n \in \mathbb{N}$, $n \geqslant 2$, $a_i, b_i \in \mathbb{R}$, $1 \leqslant i \leqslant n$, such that

$$\sum_{i=1}^{n} a_i^2 = 1, \ \sum_{i=1}^{n} b_i^2 = 1, \text{ and } \sum_{i=1}^{n} a_i b_i = 0,$$

prove that

$$\left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 \leqslant n.$$

Solution. (M. Andronache) The simplest, by far, solution avails itself of methods used to compute Fourier coefficients. Denote $A := \sum_{i=1}^{n} a_i$ and $B :=$

$\sum_{i=1}^n b_i$, then

$$\begin{aligned}0 &\leqslant \sum_{i=1}^n (1 - Aa_i - Bb_i)^2 = \sum_{i=1}^n (1 + A^2a_i^2 + B^2b_i^2 - 2Aa_i - 2Bb_i + 2ABa_ib_i) \\ &= \sum_{i=1}^n 1 + A^2\sum_{i=1}^n a_i^2 + B^2\sum_{i=1}^n b_i^2 - 2A\sum_{i=1}^n a_i - 2B\sum_{i=1}^n b_i + 2AB\sum_{i=1}^n a_ib_i \\ &= n + A^2 + B^2 - 2A^2 - 2B^2 + 0 = n - (A^2 + B^2).\end{aligned}$$

In fact, the inequality follows from an identity.

Alternative solution. (A. Zahariuc) With the notations from the solution above, consider the vector $\mathbf{1} = (1, 1, \dots, 1)$. Denote by O the origin of a coordinates system in $\mathbb{R}^n$, and by P, Q, I the other ends (than O) of vectors $\mathbf{a}, \mathbf{b}, \mathbf{1}$. We now are within a 3-dimensional subspace embedded in $\mathbb{R}^n$. Let J be the foot of the perpendicular from I on the plane (OPQ), and P', Q' be the feet of the perpendiculars from J on the lines $(OP), (OQ)$. From the 3-perpendiculars-theorem it follows that $IP' \perp OP'$ and $IQ' \perp OQ'$ (also, of course, $OP' \perp OQ'$).

But then $A = \langle \mathbf{a}, \mathbf{1}\rangle = OP'$, $B = \langle \mathbf{b}, \mathbf{1}\rangle = OQ'$, $OP'^2 + OQ'^2 = P'Q'^2 = OJ^2 = OI^2 - IJ^2 \leqslant OI^2 = \|\mathbf{1}\|^2 = n$, therefore $A^2 + B^2 \leqslant n$.

Alternative solution. (D. Schwarz) Let us present a solution in terms of linear algebra. For a vector $\mathbf{a} = (a_1, a_2, \dots, a_n)$ given, with $a_i \in \mathbb{R}$ and $\|\mathbf{a}\|^2 := \sum_{i=1}^n a_i^2 = 1$, and vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ with $x_i \in \mathbb{R}$ and $\|\mathbf{x}\|^2 := \sum_{i=1}^n x_i^2 = 1$, together with $\langle \mathbf{a}, \mathbf{x}\rangle := \sum_{i=1}^n a_ix_i = 0$, find $\sup(X^2 + A^2)$, where we denoted $A := \sum_{i=1}^n a_i$ and $X := \sum_{i=1}^n x_i$. We claim $\sup(X^2 + A^2) = n$.

Cauchy-Schwartz inequality (C-S) yields $A^2 \leqslant \|\mathbf{a}\|^2 n = n$ and $X^2 \leqslant \|\mathbf{x}\|^2 n = n$, with equality iff all coordinates are equal, to either $\frac{1}{\sqrt{n}}$ or $-\frac{1}{\sqrt{n}}$.

Applying variational methods, for $\lambda \in \mathbb{R}$, $\|\mathbf{x} - \lambda\mathbf{a}\|^2 = 1 + \lambda^2$. But (C-S) yields $\|\mathbf{x} - \lambda\mathbf{a}\|^2 \geqslant \frac{1}{n}(\sum_{i=1}^n |x_i - \lambda a_i|)^2 \geqslant \frac{1}{n}(\sum_{i=1}^n x_i - \lambda\sum_{i=1}^n a_i)^2 = \frac{1}{n}(X - \lambda A)^2$. Therefore, $(n - A^2)\lambda^2 - 2AX\lambda + (n - X^2) \geqslant 0$ for all $\lambda \in \mathbb{R}$.

When $n = A^2$, occurring iff all a_i are equal, to $\frac{1}{\sqrt{n}}$ or $-\frac{1}{\sqrt{n}}$, it follows $X = 0$, so $X^2 + A^2 = n$, with two solutions $\mathbf{x}$ for $n = 2$, and infinitely many solutions $\mathbf{x}$ for $n > 2$.

When $n > A^2$, we need the discriminant $\Delta = (AX)^2 - (n - A^2)(n - X^2) = n((X^2 + A^2) - n) \leqslant 0$, whence $X^2 + A^2 \leqslant n$. Equality in this case requires all $x_i - \lambda a_i$ equal, and this leads to

$$x_i = \varepsilon\frac{1 - Aa_i}{\sqrt{n - A^2}},\ 1 \leqslant i \leqslant n, \text{ with } \varepsilon = \pm 1,$$

therefore exactly two such solutions $\mathbf{x}$.

(In fact, for $n = 2$, all possible, admissible, vectors $\mathbf{x}$, only two in number anyhow (!), yield equality $X^2 + A^2 = 2$.)

Another yet *alternative solution*, by Lagrange multipliers method, is also available (and not too taxing, provided the method is fully mastered). This provides all information garnered in the above, including the exact expressions of the equality cases.

Remarks. Quite similar methods allow a generalization.

For $\mathbf{a_k} = (a_{k1}, a_{k2}, \dots, a_{kn}) \in \mathbb{R}^n$, such that $\|\mathbf{a_k}\|^2 = 1$, $k = 1, 2, \dots, m$, $m \leqslant n$, and $\langle \mathbf{a_k}, \mathbf{a_l}\rangle = 0$ for all $k \neq l$, if we denote $A_k := \sum_{i=1}^n a_{ki}$, then

$$\sum_{k=1}^m A_k^2 \leqslant n.$$

For $m = n$ one even gets true equality, and this in turn implies $\sum_{k=1}^n A_k a_{ki} = 1$, for all $i = 1, 2, \dots, n$.

Also, under the notations above, one can prove that, for $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$, one has

$$\left\langle \frac{\mathbf{a}}{\|\mathbf{a}\|}, \mathbf{x}\right\rangle^2 + \left\langle \frac{\mathbf{b}}{\|\mathbf{b}\|}, \mathbf{x}\right\rangle^2 \leqslant \left(1 + \frac{|\langle \mathbf{a}, \mathbf{b}\rangle|}{\|\mathbf{a}\| \cdot \|\mathbf{b}\|}\right) \cdot \|\mathbf{x}\|^2.$$

When $\langle \mathbf{a}, \mathbf{b}\rangle = 0$, this comes to

$$\left(\frac{\langle \mathbf{a}, \mathbf{x}\rangle}{\|\mathbf{a}\|}\right)^2 + \left(\frac{\langle \mathbf{b}, \mathbf{x}\rangle}{\|\mathbf{b}\|}\right)^2 \leqslant \|\mathbf{x}\|^2,$$

and for $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$, this is $\langle \mathbf{a}, \mathbf{x}\rangle^2 + \langle \mathbf{b}, \mathbf{x}\rangle^2 \leqslant \|\mathbf{x}\|^2$.

Finally, for $\mathbf{x} = \mathbf{1}$, it comes to $\langle \mathbf{a}, \mathbf{1}\rangle^2 + \langle \mathbf{b}, \mathbf{1}\rangle^2 \leqslant n$, i.e.

$$\left(\sum_{i=1}^n a_i\right)^2 + \left(\sum_{i=1}^n b_i\right)^2 \leqslant n.$$

Back to the result above, since

$$\left\langle \frac{\mathbf{a}}{\|\mathbf{a}\|}, \mathbf{x} \right\rangle^2 + \left\langle \frac{\mathbf{b}}{\|\mathbf{b}\|}, \mathbf{x} \right\rangle^2 \geqslant 2\left|\left\langle \frac{\mathbf{a}}{\|\mathbf{a}\|}, \mathbf{x} \right\rangle \left\langle \frac{\mathbf{b}}{\|\mathbf{b}\|}, \mathbf{x} \right\rangle\right|,$$

simple computations yield

$$|\langle \mathbf{a}, \mathbf{x}\rangle \langle \mathbf{b}, \mathbf{x}\rangle| \leqslant \frac{1}{2}(\|\mathbf{a}\| \cdot \|\mathbf{b}\| + |\langle \mathbf{a}, \mathbf{b}\rangle|) \cdot \|\mathbf{x}\|^2,$$

also true for $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, a related inequality, generalizing Cauchy-Schwartz.

Problem 24. Let ABC be a triangle, let E, F be the tangency points of the incircle $\Gamma(I)$ to the sides AC, respectively AB, and let M be the midpoint of the side BC. Let $N = AM \cap EF$, let $\gamma(M)$ be the circle of diameter BC, and let X, Y be the other (than B, C) intersection points of BI, respectively CI, with γ. Prove that

$$\frac{NX}{NY} = \frac{AC}{AB}.$$

Solution. We will assume $AB \leqslant AC$, so the solution matches a possible drawing. Let $T = EF \cap BC$ (for $AB = AC$, $T = \infty$), and D the tangency point of Γ to BC. We will start with a couple of lemmata.

LEMMA 1. *In the configuration described above, for $X' = BI \cap EF$, one has $BX' \perp CX'$.*

Proof. The fact that BI effectively intersects EF follows from $\angle DFE = \frac{1}{2}(\angle ABC + \angle BAC) = \frac{1}{2}\pi - \frac{1}{2}\angle ACB < \frac{1}{2}\pi$, and $BI \perp DF$ (similarly, CI effectively intersects EF).

The division $(TBDC)$ is harmonic, and triangles BFX' and BDX' are congruent, therefore $\angle TX'B = \angle DX'B$, which is equivalent to $BX' \perp CX'$ (similarly, for $Y' = CI \cap EF$, one has $CY' \perp BY'$).

LEMMA 2. *In the configuration described above, one has $N = DI \cap EF$.*

Proof. It is enough to prove that $NI \perp BC$. Let d be the line through A, parallel to BC. Since the pencil $A(BMC\infty)$ is harmonic, it follows the division $(FNEZ)$ is harmonic, where $Z = d \cap EF$. Therefore N lies on the polar of Z relative to circle Γ, and as $N \in EF$ (the polar of A), it follows that AZ is the polar of N relative to circle Γ, hence $NI \perp d$, so $NI \perp BC$. In conclusion, since $DI \perp BC$, one has $N \in DI$.

It follows, according to Lemma 1, that $X \equiv X'$ and $Y \equiv Y'$, therefore $X, Y \in EF$. Since the division $(TBDC)$ is harmonic, it follows that D lies on the polar p of T relative to circle γ. But $TM \perp p$, so $BC \perp p$, and since $DI \perp BC$, it follows that p is, in fact, DI.

Now, according to Lemma 2, it follows that D, I, N are collinear. Since DN is the polar, it means the division $(TYNX)$ is harmonic, thus the pencil $D(TYNX)$ is harmonic. But $DT \perp DN$, so DN is the angle bisector of $\angle XDY$, hence

$$\frac{NX}{NY} = \frac{DX}{DY} = \frac{\sin \angle DYX}{\sin \angle DXY}.$$

As quadrilaterals $BDIY$ and $CDIX$ are cyclic (since pairs of opposing angles are right angles), it follows that $\frac{1}{2}\angle ABC = \angle DBI = \angle DYI = \frac{1}{2}\angle DYX$ (triangles CDY and CEY are congruent), so $\angle DYX = \angle ABC$. Similarly, $\angle DXY = \angle ACB$. Therefore,

$$\frac{NX}{NY} = \frac{DX}{DY} = \frac{\sin \angle DYX}{\sin \angle DXY} = \frac{\sin \angle ABC}{\sin \angle ACB} = \frac{AC}{AB}.$$

This concludes our solution.

Remarks. Lemma 2 above could also be used towards an alternative solution to Problem 2, as foreseen in the corresponding write-up, providing a projective alternative to its synthetic solution.

Problem 25. i) Prove that a real polynomial function f cannot be a sum of (at most) $\deg f$ real periodical functions.

ii) For $\deg f = 1$, show that f can effectively be represented as the sum of two real periodical functions.

iii) For $\deg f = 1$, show that if f is the sum of two real periodical functions, they must be unbounded in any interval.

iv) Show that a real, not null, polynomial function f can effectively be represented as the sum of $\deg f + 1$ real periodical functions.

v) Exhibit a real function that cannot be represented as a (finite) sum of real periodical functions.

Solution. For any $f \in \mathcal{F}(\mathbb{R}) := \{\varphi\,;\,\varphi : \mathbb{R} \to \mathbb{R}\}$, and any $a \in \mathbb{R}^*$, define the operator $\Delta_a : \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ through $\Delta_a f(x) := f(x+a) - f(x)$, for all $x \in \mathbb{R}$. The operator Δ_a is clearly linear.

i) We will start by proving a couple of lemmata.

LEMMA 1. *For p periodical of period t, so is $\Delta_a p$.*

Proof. A simple computation yields $\Delta_a p(x+t) = p((x+t)+a) - p(x+t) = p((x+a)+t) - p(x+t) = p(x+a) - p(x) = \Delta_a p(x)$.

LEMMA 2. *For $f \in \mathbb{R}[X]$, $\deg f \geqslant 1$, then $\Delta_a f \in \mathbb{R}[X]$, and $\deg \Delta_a f = \deg f - 1$.*

Proof. Let $f(x) = \sum_{k=0}^{\deg f} a_k x^k \in \mathbb{R}[X]$. Then the coefficient of $x^{\deg f}$ in $\Delta_a f$ is 0, while the coefficient of $x^{\deg f - 1}$ is $(\deg f) a a_{\deg f} \neq 0$.

A nice formula (provable by induction) is

$$\Delta_a f = \sum_{k=1}^{\deg f} \frac{a^k}{k!} f^{(k)},$$

also yielding the claimed result.

Assume $f = \sum_{i=1}^{k} p_i$, with p_i periodical of period t_i, $i = 1, 2, \ldots, k$. We will prove by induction that then $\deg f < k$. For $k = 1$ this is trivial, since a periodical polynomial can only be constant (or the null polynomial). For $k > 1$, $\Delta_{t_k} f = \sum_{i=1}^{k-1} \Delta_{t_k} p_i$, where $\Delta_{t_k} p_i$ is periodical of period t_i, according to Lemma 1. The induction hypothesis yields $\deg \Delta_{t_k} f < k - 1$, but $\deg \Delta_{t_k} f = \deg f - 1$, according to Lemma 2, hence $\deg f < k$.

ii) We urge the reader to adapt the approach presented below, as clearly point iv) generalizes point ii). This is readily done for $\deg f = 1$, and, in fact, will help understanding the method taken.

iii) Assume $f = p_1 + p_2$, with p_i periodical of period t_i, $i = 1, 2$. Some classical result (Dirichlet, Kronecker, Weyl) states that, for $\lambda \notin \mathbb{Q}$, the sequence $(\{n\lambda\})_{n \geqslant 1}$ is dense in $(0,1)$. Now, if $\lambda = \frac{t_1}{t_2} \in \mathbb{Q}$, so $\frac{t_1}{t_2} = \frac{n_2}{n_1}$, i.e. $n_1 t_1 = n_2 t_2 = t$, then t is a common period for p_1, p_2, hence for f, absurd. Therefore $\lambda \notin \mathbb{Q}$, so for any $\varepsilon > 0$ there exist infinitely many n such that $\{n\lambda\} < \frac{\varepsilon}{t_2}$, and for each such n there exists an integer m_n such that $|nt_1 - m_n t_2| < \varepsilon$. But then consider an interval I, and $x_0 \in I$ such that $[x_0 - \varepsilon, x_0 + \varepsilon] \subset I$. We have $|p_1(x_0 + m_n t_2)| = |p_1(x_0 + m_n t_2 - nt_1)| = |p_1(x_n)|$, with $x_n \in I$, and $|p_2(x_0 + m_n t_2)| = |p_2(x_0)|$.

Therefore, $|f(x_0 + m_n t_2)| \leqslant |p_1(x_0 + m_n t_2)| + |p_2(x_0 + m_n t_2)| = |p_1(x_n)| + |p_2(x_0)|$. Since $(x_0 + m_n t_2)_{n \geqslant 1}$ is unbounded, so must be $(p_1(x_n))_{n \geqslant 1}$, therefore p_1, and, of course, p_2 too, are unbounded on I. As such, they are of course discontinuous at any real value!

This part is meant to help intuition, and revive erudition, towards solving the following point. One may be reminded of the "wild" solutions to the Cauchy functional equation $F(x+y) = F(x) + F(y)$, first found by Hamel, and thus be led to considerations like those in the sequel.

iv) We will start with

LEMMA. *Assume that h_i, $i = 1, 2, \ldots, n+1$, are positive real numbers, and that we can express $x = \sum_{i=1}^{n+1} q_i(x)$, with q_i periodical, admitting as periods all h_j, $j \neq i$. Then $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[X]$ can be expressed $f = \sum_{j=1}^{n+1} p_j$, with p_j periodical of period h_j.*

Proof. Any monomial $a_k x^k$, with $0 \leqslant k \leqslant n$, will be a sum of terms made of products of $q_i(x)$'s, homogeneous of degree k, hence never containing all $n+1$ $q_i(x)$'s. But each such term is periodical of periods h_j of index not present in that term. We will then addition it towards a function $p_j(x)$, of index least among those not present in it, so p_j will be periodical of period h_j. At the end of this process, we get $f = \sum_{j=1}^{n+1} p_j$, as required.

Consider a Hamel basis $\mathcal{H}$ of $\mathbb{R}/\mathbb{Q}$, and a partition of it into $\deg f + 1$ non-empty classes

$$\mathcal{H} = \bigcup_{i=1}^{\deg f + 1} \mathcal{H}_i.$$

The sets $\langle \mathcal{H}_i \rangle$, linearly generated by $\mathcal{H}_i$, only overlap over 0, hence any $x \in \mathbb{R}$ can be written uniquely as $x = \sum x_i$, with $x_i \in \langle \mathcal{H}_i \rangle$, therefore x_i, regarded as function of x, is periodical of any period $h_j \in \mathcal{H}_j$, $j \neq i$, since then $x + h_j =$

$(x_j+h_j)+\sum_{i\neq j} x_i$. Taking $q_i(x) := x_i$, the conditions of the lemma are fulfilled, whence the result.

Alternatively, we can avoid using Hamel bases, using instead the equivalence relation relative to an additive subgroup H of $\mathbb{R}$, generated by $\deg f + 1$ elements, for example

$$H = \mathbb{Q}[2^{\frac{1}{n+1}}] = \left\{ \sum_{i=1}^{n+1} r_i 2^{\frac{i-1}{n+1}};\ r_i \in \mathbb{Q},\ 1 \leqslant i \leqslant n+1 \right\}.$$

Take $h_i = 2^{\frac{i-1}{n+1}}$, $1 \leqslant i \leqslant n+1$. Since the polynomial $X^{n+1} - 2$ is irreducible in $\mathbb{Q}[X]$ (Eisenstein's criterium), and $2^{\frac{1}{n+1}}$ is a root for it, it follows that h_i, $1 \leqslant i \leqslant n+1$, are linearly independent over $\mathbb{Q}$. Defining $x \approx y$ iff $x - y \in H$, $\approx$ is trivially seen as being an equivalence relation over $\mathbb{R}$. Consider the equivalence classes of $\mathbb{R}/\approx$, and $\{\hat{x}; x \in \mathbb{R}\}$ a system of representants. Then $x = \hat{x} + \sum_{i=1}^{n+1} r_{i,x} h_i$, with $r_{i,x} \in \mathbb{Q}$ uniquely determined. Define $q_1(x) := \hat{x} + r_{1,x}$, $q_i(x) := r_{i,x} h_i$, for $2 \leqslant i \leqslant n+1$. Clearly, $q_i(x+h_j) = q_i(x)$, for any $j \neq i$, so the conditions of the lemma are fulfilled, whence the result.

Another way is to inductively find h_i, each linearly independent over $\mathbb{Q}$ with the previously taken ones, possible since $\mathbb{Q}^k$ is countable, while $\mathbb{R}$ is not. Now build H as being generated by the h_i, and proceed as above.

v) We claim that $\exp(x) := e^x$ is such a function. Assume it can be expressed as $\exp = \sum_{i=1}^n p_i$, with p_i periodical of period t_i, and n minimal with this property. Then, as in the solution for point i), since $\Delta_{t_n} \exp = (e^{t_n} - 1)\exp$, we have $\exp = \sum_{i=1}^{n-1} q_i$, with $q_i := (e^{t_n} - 1)^{-1} \Delta_{t_n} p_i$ periodical of period t_i (according to Lemma 1), contradicting the minimality of n.

(Notice that over $\mathbb{C}$ this isn't true anymore, as $\exp(z)$ is periodical of period $2\pi i$.)

In fact, for $f = \sum_{i=1}^n p_i$, with p_i periodical of period t_i, denoting by $[\mathbf{n}] := \{1, 2, \ldots, n\}$, and by T_i any integer multiple of t_i, simple induction yields

$$\sum_{J\subseteq[\mathbf{n}]} (-1)^{|J|} f\Big(x + \sum_{j\in J} T_j\Big) \equiv 0.$$

This offers another example: consider the characteristic function χ_a, taking value 0 over all reals, except $\chi_a(a) = 1$. Then the equation above (for $T_i = t_i$) cannot hold, since, for $x = a$, all the terms in the equation are equal to 0, except $\chi_a(a) = 1$.

Alternative solution. (E. Dobriban) v) We will prove that any real function f such that $\lim\limits_{x\to\infty} f(x) = C$, C a constant, and $f \not\equiv C$, is such a function (then χ_a is shown to be such, and also functions like $f(x) = \frac{1}{x}$ for $x \neq 0$, while $f(0)$ is arbitrary). Take such a function, that can be expressed as $f = \sum_{i=1}^n p_i$, with p_i periodical of period t_i, and n minimal (over all such functions) with this property. Clearly $n > 1$, since $n = 1$ would mean f is periodical, while $\lim\limits_{x\to\infty} f(x) = C$ for a periodical function forces it to be constant: $f \equiv C$, contradiction. Then, as in the solution for point i), $\Delta_{t_n} f = \sum_{i=1}^{n-1} \Delta_{t_n} p_i$, with $\Delta_{t_n} p_i$ periodical of period t_i (according to Lemma 1). Now, $\lim\limits_{x\to\infty} \Delta_{t_n} f(x) = \lim\limits_{x\to\infty} f(x+t_n) - \lim\limits_{x\to\infty} f(x) = C - C = 0$, so, by the minimality of n, we need have $\Delta_{t_n} f \equiv 0$. But then $f(x+t_n) \equiv f(x)$, so f periodic, hence, by the remark above, f would be forced to be constant: $f \equiv C$, contradiction. (Obviously, the same is true for the case $\lim\limits_{x\to-\infty} f(x) = C$, etc.)

Alternative solution. (A. Zahariuc) iv) The following is kind of a reciprocal for Lemma 1.

LEMMA 1′. *For p periodical of period t, and $a \in \mathbb{R}^*$, incommensurable with t, there exists a function q periodical of period t, such that $p = \Delta_a q$.*

Proof. Consider the set $H = \{ka + lt\ ;\ k, l \in \mathbb{Z}\}$. Clearly, H is closed to addition and subtraction (i.e. H is an additive subgroup of $\mathbb{R}$), and for an element of H, its representation is unique. Consider the equivalence classes of $\mathbb{R}/H$, and $\{\hat{x}; x \in \mathbb{R}\}$ a system of representants. Then $x = \hat{x} + k_x a + l_x t$, with $k_x, l_x \in \mathbb{Z}$ uniquely determined. Define $q(\hat{x})$ arbitrarily, and prolong q to all $\mathbb{R}$ by $q(x) = q(\hat{x} + k_x a + l_x t) := q(\hat{x} + k_x a) = q(\hat{x}) + \sum_{k=0}^{k_x - 1} p(\hat{x} + ka)$ for $k_x > 0$, or $q(\hat{x}) - \sum_{k_x}^{-1} p(\hat{x} + ka)$ for $k_x < 0$. Clearly, $q(x+t) = q(x)$, since $\widehat{x+t} = \hat{x}$ and $k_{x+t} = k_x$, and also $q(x+a) - q(x) = p(x)$, since $\widehat{x+a} = \hat{x}$ and $k_{x+a} = k_x + 1$, hence $\Delta_a q = p$.

Now, for $\Delta_a f = \sum_{i=1}^n p_i$, with p_i periodical of period t_i, such that a is incommensurable with all t_i, $i = 1, 2, \ldots, n$, apply Lemma 1′ to find functions

q_i periodical of period t_i, such that $p_i = \Delta_a q_i$. Since the operator Δ_a is linear, we have $0 = \Delta_a f - \sum_{i=1}^n p_i = \Delta_a f - \sum_{i=1}^n \Delta_a q_i = \Delta_a (f - \sum_{i=1}^n q_i)$, hence $q_{n+1} := f - \sum_{i=1}^n q_i$ is periodical of period $t_{n+1} := a$, and therefore $f = \sum_{i=1}^{n+1} q_i$, with q_i periodical of period t_i.

The same induction argument warrants the possibility to choose, at any step, incommensurable (with any predefined countable set) values, acting as periods for the periodical functions considered.

Remarks. The combination of these two alternative solutions with the method presented at point i) offers a powerful tool for determining other exhibits for point v). For example, the function $f(x) = \ln(|x|)$ for $x \neq 0$, while $f(0)$ is arbitrary, is acceptable, since $\Delta_1 f(x) = \ln(|x+1|) - \ln(|x|) = \ln(|1 + \frac{1}{x}|)$, and so $\lim_{x\to\infty} \Delta_1 f(x) = 0$, therefore $\Delta_1 f$ is acceptable, which in turn forces f to be acceptable. Same for the function $f(x) = \ln(1 + |x|)$.

The equation

$$\sum_{J\subseteq[n]} (-1)^{|J|} f\Big(x + \sum_{j\in J} T_j\Big) \equiv 0.$$

is seen to be fulfilled by real polynomial functions f of degree at most $n-1$, providing further support to points i) and iv), and suggesting how to look for a function at point v). For $n = 2$, the equation writes as

$$f(x + A + B) + f(x) = f(x + A) + f(x + B),$$

reminiscent of Cauchy's equation, and of Hamel's "wild" solutions, useful for finding the ideas behind solving points ii) and iv).

It is interesting, and educational, how parts i) and iv) complement each other: a term made of $\prod q_i(x)$, with all indices present, does not follow to be periodical, inducing the need of (at least) $\deg f + 1$ such $q_i(x)$'s, which is counterbalanced by the proof that indeed (at least) $\deg f + 1$ periodical functions are required.

The result of the problem leads to the statement:

The linear space $\mathbb{R}[X]$, of real polynomial functions, is a subspace of the one spanned by the periodical real functions, which, in turn, is a proper subspace of $\mathcal{F}(\mathbb{R})$, the linear space of all real functions.

PROBLEMS AND SOLUTIONS

JUNIOR BMO SELECTION TESTS

Problem 1. Let a and b be integer numbers. Show that there exists a unique pair of integers x, y so that

$$(x + 2y - a)^2 + (2x - y - b)^2 \leqslant 1.$$

Solution. Solving for a and b the system of equations

$$\begin{cases} x + 2y - a = s \\ 2x - y - b = t, \end{cases}$$

one has

$$\begin{cases} x = \dfrac{(a+2b) + (s+2t)}{5} \\ y = \dfrac{(2a-b) + (2s-t)}{5}. \end{cases}$$

Restating the claim, one has to prove that there exists a unique pair of integers $s, t \in \mathbb{Z}$ with $s^2 + t^2 \leqslant 1$ so that both numbers $(a+2b)+(s+2t), (2a-b)+(2s-t)$ are divisible by 5.

Notice that $(a + 2b) + (s + 2t) + 2[(2a - b) + (2s - t)] = 5(a + s)$, so

$$x \in \mathbb{Z} \Leftrightarrow y \in \mathbb{Z}.$$

Since $s^2+t^2 \leqslant 1 \Leftrightarrow (s,t) \in \{(0,0),(1,0),(0,1),(-1,0),(0,-1)\}$, it follows that $(a+2b)+(s+2t) \in \{a+2b-2, a+2b-1, a+2b, a+2b+1, a+2b+2\}$, so there is exactly one pair (s,t) with $5 \mid (a+2b)+(s+2t)$, as needed.

Problem 2. Consider a trapezoid $ABCD$ with the bases AB and CD so that the circles with the diameters AD and BC are secant; denote by M and N their common points. Prove that the intersection point of the diagonals AC and BD belongs to the line MN.

Solution. Let Q be the intersection point of the diagonals, T the second point of intersection of the line AC with the circle C_1 of diameter AD and S the second point of intersection of the line BD with the circle C_2 of diameter BC. Then $\angle ATD = \angle BSC = 90°$, so DT and SC meet in the orthocenter H of the triangle DQC. Denote by C_3 the circle $DCTS$.

The radical axis of the circles C_1, C_2 is MN, the radical axis of the circles C_1, C_3 is DT, while the pair of circles C_2, C_3 has SC as radical axis, hence the radical center of the three circles is H.

The line segment MN is the common chord of the circles C_1 and C_2, thus perpendicular to the line passing through the centers, which is in fact the middle line of the trapezoid. As $H \in MN$, then $MH \| DC$, and since $QH \| DC$ the conclusion follows.

Problem 3. A rectangular cardboard is divided successively into smaller pieces by a straight cut; at each step, only one single piece is divided in two. Find the smallest number of cuts required in order to obtain — among others — 251 polygons with 11 sides.

Solution. Let n be the required number. We claim that $n = 2007$.

With 7 cuts, from the given rectangular piece one can obtain an 11-sided polygon and some triangles. From a triangle,with 8 cuts one can get an 11-sided polygon and some extra pieces, sufficiently enough to continue the same procedure. Hence, using $7 + 8 \cdot 250 = 2007$ cuts one can obtain the 251 requested 11-sided polygons.

Denote by k the number of pieces left at the end which are not 11-sided polygons and notice that each has at least 3 sides. Now, observe that with each cut the number of pieces increases by 1 and total the number of vertices increases with at most 4 — actually, with 2, 3 or 4, according to the number of existent vertices through which the cutting line passes.

Then $n = k + 250$ and $4n + 4 \geqslant v \geqslant 11 \cdot 251 + 3k$, where v is the total number of vertices of all polygons at the end. Hence $4n + 4 \geqslant 11 \cdot 251 + 3(n - 250) = 2011 + 3n$, so $n \geqslant 2007$, as claimed.

Problem 4. Find all integers n, $n \geqslant 4$ such that $[\sqrt{n}] + 1$ divides $n - 1$ and $[\sqrt{n}] - 1$ divides $n + 1$.

Solution. Let $m = [\sqrt{n}]$. Since $n \geqslant 4$, then $m \geqslant 2$ is an integer. We have $m^2 \leqslant n < (m+1)^2$, so $m^2 \leqslant n \leqslant m^2 + 2m$.

Set $n = m^2 + k$, $k = 0, 1, 2, \ldots, 2m$. From $m - 1 \mid m^2 + k - 1$ we get $m - 1 \mid k$. On the other hand $k < 2(m+1)$, thus $k = 0$ or $k = m + 1$.

If $k = 0$, from $m - 1 \mid m^2 + 1$ follows $m - 1 \mid 2$, so $m = 2$ or $m = 3$, hence $n = 4$ or $n = 9$.

If $k = m + 1$, then $m - 1 \mid m^2 + m + 2 = m^2 - 1 + m - 1 + 4$, so $m - 1 \mid 4$. We obtain $m = 2, 3$ or 5, hence $n = 7, 13$ or 31.

Therefore, $n \in \{4, 7, 9, 13, 31\}$.

Problem 5. Let $ABCD$ be a convex quadrilateral. The incircle ω_1 of triangle ABD touches the sides AB, AD at points M, N respectively, while the incircle ω_2 of triangle CBD touches the sides CD, CB at points P, Q respectively. Given that ω_1 and ω_2 are tangent, show that:

a) the quadrilateral $ABCD$ is circumscriptible;

b) the quadrilateral $MNPQ$ is cyclic;

c) the incircles of triangles ABC and ADC are tangent.

Solution. Let $T \in BD$ be the tangency point of the incircles of the triangles ABD and CBD. Notice that $BM = BT = BN$ and $DN = DT = DM$.

a) We have $AB + CD = AM + MB + CP + PD = AN + BQ + CQ + DN = AD + BC$, so the quadrilateral $ABCD$ is circumscriptible.

b) Triangles AMN, DNP, CQP, BQM are isosceles, so $\angle QMN + \angle NPQ = 360° - (\angle AMN + \angle BMQ + \angle QPC + \angle NPC) = 360° - \frac{1}{2}(4 \cdot 180° - A - B - C - D) = 180°$, hence the quadrilateral $MNPQ$ is cyclic.

c) Let U be the point where the side AC touches the incircle of triangle ABC. Since $AB - BC = AD - DC$, then $AU = \frac{AB+AC-BC}{2} = \frac{AD+AC-DC}{2}$, so

U is also the tangency point of the side AC with the incircle of triangle ADC, as needed.

Problem 6. Let ABC be an acute-angled triangle with $AB = AC$. For any point P inside the triangle ABC consider the circle centered at A with radius AP and let M and N be the intersection points of the sides AB and AC with the circle. Determine the position of the point P so that $MN + BP + CP$ is minimum.

Solution. For a fixed point P inside the given triangle consider the point Q on the bisector line of BC so that $AQ = AP$. The parallel line d from Q to BC separates the arc MN and the side BC, so D meets the line segment $[BP]$ at a point, say S. The triangle's inequality gives $SP + PC \geqslant SC$, so $BP + PC \geqslant BS + SC$. On the other hand, with an argument frequently refer to as Heron's problem we have $BS + SC \geqslant BQ + QC$, so $BP + PC$ is minimum if $P = Q$.

Let T be the midpoint of the segment MN. Notice that triangle AMQ is isosceles and MT is an altitude in this triangle, hence $MT = QZ$, where Z is the foot of the altitude from Q onto AC. Then $MN + BQ + QC = 2(MT + CQ) = 2(CQ + QZ)$ is minimum when $CZ \perp AC$. Consequently, the required point is the orthocenter of the triangle ABC, which belongs to the interior of the triangle, since it is an acute-angled one.

Problem 7. Let ABC be a triangle. Points M, N, P are given on the sides AB,BC, CA respectively so that $CPMN$ is a parallelogram. Lines AN and MP intersect at point R, lines BP and MN intersect at point S, while Q is the intersection point of the lines AN and BP. Show that $S[MRQS] = S[NQP]$.

Solution. Let $k = \frac{AM}{AB}$. Using Thales Theorem, $MP \| BC$ yields $\frac{AP}{AC} = k$, while $MN \| AC$ implies $\frac{MS}{MN} = k$. On the other hand, $\frac{CN}{BC} = \frac{PR}{PM} = k$.

Setting $S = \text{area}[MNP]$, we have

$$\frac{\text{area}[MSP]}{S} = \frac{MS}{MN} = \frac{RP}{PM} = \frac{\text{area}[NPR]}{S},$$

hence area$[MSP]$ = area$[NPR]$. Subtracting area$[RPQ]$ from both sides of the latter equality we get the conclusion.

Problem 8. Solve in positive integers the equation:

$$(x^2 + 2)(y^2 + 3)(z^2 + 4) = 60xyz.$$

Solution. At first, notice that $(x-1)(x-2) \geqslant 0$ for all $x \in \mathbb{N}$, $(y-1)(y-3) \geqslant 0$ if $y \in \mathbb{N} \setminus \{2\}$ and $(z-1)(z-4) \geqslant 0$ when $z \in \mathbb{N} \setminus \{2, 3\}$. In other words, if $y \neq 2$ and $z \notin \{2, 3\}$, then $x^2 + 2 \geqslant 2x$, $y^2 + 3 \geqslant 3y$ and $z^2 + 4 \geqslant 4z$. Multiplying the above inequalities yields $(x^2 + 2)(y^2 + 3)(z^2 + 4) \geqslant 60xyz$, so in all three inequalities the equality must occur. Until now we have the solutions:

$$(x, y, z) = (1, 1, 1), (1, 1, 4), (1, 3, 1), (2, 1, 1), (2, 3, 1), (2, 1, 4), (1, 3, 4), (2, 3, 4).$$

We claim the there are no more solutions. For this, we will show that if $z = 2$ or $z = 3$ or $y = 2$, there are no integers satisfying the given equation.

The quadratic residues modulo 5 are 0, 1, 4, so 5 do not divide neither $x^2 + 2$ nor $y^2 + 3$. Since 5 divides $60\,xyz$, it follows that 5 divides $z^2 + 4$, hence $z \in \{5k \pm 1 \mid k \in \mathbb{Z}\}$. As a consequence, $z \neq 2$ and $z \neq 3$.

If $y = 2$, the equation rewrites as $120xz = 7(x^2 + 2)(z^2 + 4)$, from which we may notice that 8 divides $(x^2 + 2)(z^2 + 4)$. If x, z are even integers, then $x^2 + 2$ is even and $z^2 + 4$ is divisible by 4, but $4 \nmid x^2 + 2$ and $16 \nmid z^2 + 4$, so the power of 2 in the right-hand side is at most 4, while in the left hand-side is at least 5, a contradiction. If only one of the numbers x and z is even, the contradiction is reached similarly. Hence $y \neq 2$ and the only solutions of the equation are the ones previously obtained.

Problem 9. Consider a $n \times n$ array divided into unit squares which are randomly colored in black or white. Three of the four corner squares are colored in white and the fourth is colored in black. Prove that there exists a 2×2 square which contains an odd number of white squares.

Solution. Assign the number 0 to each white square and the number 1 to each black square. The claim is achieved if we prove the existence of a 2×2 square with an odd sum of the 4 numbers inside.

Assume the contrary, so each sum of the 4 numbers inside a 2×2 square is even. Summing over all squares we get an even number S. Notice that each square not sharing a common side with the given array occurs 4 times in S, the squares with only a common side occurs twice, while the 4 squares in the corners only once. But in the four corners there are three 0's and one 1, so the sum S is even, a contradiction.

Remark. The given array may have a rectangular form, and the above solution requires no alteration. However, this remark can easily lead to alternative solutions using induction. Here is a sketch: choose a row with 0 and 1 at endpoints and call it the first row. Suppose that the number below 0 is also 0; arguing by contradiction, we notice that all "doubletons" formed vertically from the first two rows have equal numbers inside, so the second row — which starts with 0 — ends with 1. Deleting the first row of the given rectangular array, the claim is reached by induction. The same line of reasoning is applied to the case when below 0 the number is 1.

Problem 10. Suppose a, b, c are positive real numbers satisfying:

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \geqslant 1.$$

Show that

$$a+b+c \geqslant ab+bc+ca.$$

Solution. The Cauchy-Schwarz inequality gives

$$(a+b+1)(a+b+c^2) \geqslant (a+b+c)^2,$$

so

$$\sum_{cyc} \frac{a+b+c^2}{(a+b+c)^2} \geqslant \sum_{cyc} \frac{1}{a+b+1} \geqslant 1.$$

Then

$$2\sum_{cyc} a + \sum_{cyc} a^2 \geqslant (a+b+c)^2 = \sum_{cyc} a^2 + 2\sum_{cyc} ab,$$

and the claim follows immediately.

Problem 11. Find all non-empty subsets A of the set $\{2,3,4,5,\ldots\}$ so that for any $n \in A$, both n^2+4 and $[\sqrt{n}]+1$ also belong to A.

Solution. We claim that $A = \{2,3,4,5,\ldots\}$.

Let m be the smallest element of the set A. Since $[\sqrt{m}]+1 \in A$, we have $m \leqslant [\sqrt{m}]+1 \leqslant \sqrt{m}+1$, which gives $m = 2$.

Notice that $[\sqrt{n^2+4}] = n$ for all $n \geqslant 2$. Indeed, $n^2 \leqslant n^2+4 < (n+1)^2 = n^2+2n+1$, for all $n \geqslant 2$. Using both hypothesis, we have

$$n \in A \Rightarrow n^2+4 \in A \Rightarrow [\sqrt{n^2+4}]+1 \in A \Rightarrow n+1 \in A.$$

The conclusion follows by induction.

Problem 12. Circles ω_1 and ω_2 intersect at points A and B. A third circle ω_3, which intersects ω_1 at points D and E, is internally tangent to ω_2 at point C and tangent to the line AB at point F, and lines DE and AB meet at point G. Let H be the mirror image of F across G. Calculate the measure of the angle $\angle HCF$.

Solution. Line AB is the radical axis of the circles ω_1 and ω_2, and line DE is the radical axis of the circles ω_1 and ω_3, hence point G is the radical center of the three circles. Since the radical axis of the circles ω_3 and ω_2 is the tangent line at C to these circles, it follows that the tangents from G to $\omega - 3$ are GF and GC. Then $GF = GC$ and $GH = GF$, so the triangle HCF is right-angled at C. Therefore, $\angle HCF = 90°$.

Problem 13. Consider the numbers from 1 to 16. A *solitaire* game is played in the following manner: the numbers are paired and each pair is replaced by the greatest prime divisor of the sum of the numbers in that pair — for example, $(1,2); (3,4); (5,6); \ldots; (15,16)$ produces the sequence 3,7,11,5,19,23,3,31. The game continues similarly until one single number is left. Find the greatest possible value of the number which ends the game.

Solution. Let $a \heartsuit b$ be the greatest prime divisor of $a+b$.

At first, notice that from the initial 16 numbers we obtain 8 primes. The largest prime that can be obtained is $31 = 15 \heartsuit 16$; if this number occurs, the second largest can be $23 = 11 \heartsuit 12$. Otherwise, 29 may occurs twice, from $16 \heartsuit 13$ and $15 \heartsuit 14$, followed by 19 — or lower.

From the stage when we are left with 8 primes, and after pairing them we get 4 primes. If a prime is obtained from two odd primes a and b, then $a \heartsuit b \leqslant \frac{a+b}{2}$.

If else, at least one is 2 and let p be the other. The number $p+2$ is prime only if $p \in \{3, 11, 17, 29\}$. Therefore, if p and q are prime with $p \leqslant q$, then $p \heartsuit q \leqslant q+2$.

We will prove the the largest number which can end the game is 19. One example to obtain it is exhibit below:

$$(1,8);(2,7);(9,16);(10,15);(3,14);(4,13);(5,12);(6,11)$$

$$\to 3,3,5,5,17,17,17,17$$

$$(3,3);(5,5);(17,17);(17,17) \to 3,5,17,17$$

$$(3,5);(17,17) \to 2,17$$

$$(2,17) \to 19.$$

Now, we have to show that the game cannot end with a number strictly greater than 19. Since from the second stage the number cannot increase with more than 2, and since $31 \heartsuit 2 = 11$, we derive that the game will end with a prime $p \leqslant 31$. Suppose by contradiction that $p \in \{23, 29, 31\}$.

If $p = 29$, as 29 is not sum of two primes, then p is obtained from two of 29. In the previous stage four 29's are needed, then in the second stage eight 29's are required, in contradiction with an initial observation. Moreover, we have obtained a stronger result: 29 cannot end the game and cannot occur even in the last pair, since after 2 steps at most one 29 may occur.

Suppose that $p = 31$. Two cases are possible: $31 = 2 \heartsuit 29$ or $31 = 31 \heartsuit 31$. The latter result forbids the first case, while the second case requires that the last four numbers are 31,31,31,31 or 31,31,29,2. But among the 8 primes obtained after the first step we have at most two 29's or one 31, not enough to produce three 31's or two 31's and one 29.

Assume that $p = 23$. Again two cases are possible: $23 = 29 \heartsuit 17$ or $23 = 23 \heartsuit 23$. The first case is impossible as shown above, while the second case is allowed if the last four primes are 23,23,23,23 or 29,17,23,23. If all primes are 23, the previous step has eight numbers with the average of 23, which is a contradiction with

$$8 \cdot 23 < 1+2+3+\cdots+16 = 8 \cdot 17.$$

The second case lead similarly to contradiction, since 29 requires two 29's and the pair of 23's are given by four numbers with the sum $4 \cdot 23$:

$$2 \cdot 29 + 4 \cdot 23 = 150 < 1+2+\cdots+16 = 136.$$

The solution is now complete.

Problem 14. Determine all positive integers n which can be represented in the form

$$n = [a,b] + [b,c] + [c,a],$$

where a, b, c are positive integers.

Note: $[p,q]$ is the lowest common multiple of the integers p and q.

Solution. Any integer which can be represented as described in the problem will be called *good*.

Setting $b = c = 1$ yields $[a,b]+[b,c]+[c,a] = a+1+a = 2a+1$, hence any odd integer is good.

Notice that $[2x, 2y] = 2 \cdot [x,y]$. Therefore, if n can be represented as $[a,b] + [b,c]+[c,a]$, then $2n$ writes as $[2a,2b]+[2b,2c]+[2c,2a] = 2([a,b]+[b,c]+[c,a])$, thus all integers with are not powers of 2 are good.

We claim that all numbers of the form 2^k, $k \in \mathbb{N}$ are not good. For $k = 0$ and $k = 1$ this is obvious, as $[a,b]+[b,c]+[c,a] \geqslant 1+1+1 = 3$. If $k \geqslant 2$, suppose by contradiction that there exist a, b, c as needed. Let $a = 2^A \cdot a_1, b = 2^B \cdot b_1, c = 2^C \cdot c_1$, where a_1, b_1, c_1 are odd. Without loss of generality, assume that $A \geqslant B \geqslant C$. Then $2^k = [a,b]+[b,c]+[c,a] = 2^A \cdot ([a_1,b_1]+[a_1,c_1]) + 2^B \cdot [b_1, c_1]$. Dividing by $2^B, k > B$ yields $2^{k-B} = 2^{A-B} \cdot ([a_1,b_1]+[a_1,c_1]) + [b_1,c_1]$. But $[a_1,b_1]+[a_1,c_1]$ is even and $[b_1,c_1]$ is odd, contradiction.

Problem 15. Let ρ be a semicircle of diameter AB. A parallel line to AB intersects the semicircle in C and D so that points B and C lie on opposite sides of the line AD. The parallel line from C to AD meets ρ again at point E Lines BE and CD meet at point F and the parallel line from F to AD intersects AB at point P. Prove that the line PC is tangent to the semicircle ρ.

Solution. A "special" position occurs when $\angle CAB = 60°$, when $C = E = F$. In this case the claim is obvious.

Consider the case $\angle CAB > 60°$, where E belongs to the small arc $\overset{\frown}{CD}$ and F lie on the segment CD. Notice that $\angle PFC = \angle ADC = \angle BCD$, hence the trapezoid $PBFC$ is isosceles. On the other hand, as $CD \| AB$ and $CE \| AD$, it follows that the arcs AC, BD, DE are equal. Then $\angle EFC = \overset{\frown}{CE} + \overset{\frown}{BD} = \overset{\frown}{CE} + \overset{\frown}{DE} = \overset{\frown}{CD} = \angle P'CD$, where $P' \in PC$, $C \in (PP')$. The last equality proves the conclusion.

Slight changes in notations are required for the case $\angle CAB < 60°$.

Problem 16. Prove that

$$\frac{x^3 + y^3 + z^3}{3} \geqslant xyz + \frac{3}{4}|(x-y)(y-z)(z-x)|,$$

for any real numbers $x, y, z \geqslant 0$.

Solution. Let $p = |(x-y)(y-z)(z-x)|$. Recall the identities:

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

and

$$x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}[(x-y)^2 + (y-z)^2 + (z-x)^2].$$

Using AM-GM inequality, we have

$$(1) \qquad x^2 + y^2 + z^2 - xy - yz - zx \geqslant \frac{3}{2}\sqrt[3]{p^2}.$$

On the other hand, since $|x - y| \leqslant x + y$, $|y - z| \leqslant y + z$ and $|z - x| \leqslant z + x$, it follows that

$$2(x+y+z) \geqslant |x-y| + |y-z| + |z-x|.$$

Applying again the AM-GM inequality gives

$$(2) \qquad 2(x+y+z) \geqslant 3\sqrt[3]{p},$$

and the claim follows from the inequalities (1) and (2).

Problem 17. Eight persons attend a party, and each participant has at most three others to whom he/she cannot speak. Show that the persons can be grouped in 4 pairs so that each pair can converse.

Solution. Consider an arbitrary grouping in pairs. A pair in which the persons cannot speak will be called "bad". If there are bad pairs, we prove that some changes can be made to decrease to number of bad pairs. Applying this at most 4 times exhibit a grouping with no bad pairs, and we are done.

Label the pairs A, B, C, D and the persons in pair X by X_1 and X_2. Two persons that cannot converse are called "enemies", otherwise "friends". Assume that A is a bad pair. Beside A_1, the person A_2 has at most two other enemies. Two cases arise:

a) If the other enemies of A_2 belong to the same pair — call it B, then A_1 has at least a friend among C_1, C_2, D_1, D_2.

Choose C_1 as a friend of A_1 and swap A_1 with C_2. The new pairs A and C are good, and the claim is satisfied.

b) If not, in at least one of the pairs B, C, D there are only friends of A_1. Wlog, say that this pair is B. One of the persons in this pair must be a friend of A_2; call this person B_1. Now swap A_1 with B_1 and the new pairs A and B are good, as desired.

Remark. Consider the graph with vertices in the eight persons and edges corresponding to each pair of friends. The degree of each vertex is at least 4, so, according to Dirac's theorem there exists a hamiltonian cycle. Taking 4 edges with no common endpoint from this cycle, we get 4 good pairs, as needed.

Problem 18. A set of points is called *free* if there is no equilateral triangle whose vertices are among the points in the set. Show that any set of n points in the plane contains a free subset of at least $\sqrt{n}$ points.

Solution. Given a set X of n points in the plane, consider a maximal free subset Y made of m elements, hence such that any point in $X \setminus Y$ completes an equilateral triangle with (at least) a pair of points from Y. (Any X contains free subsets, since any subset with 1 or 2 elements is obviously free.) But for any pair of points from Y there exist only two points in the plane which complete an equilateral triangle, so

$$n - m \leqslant 2\binom{m}{2}, \text{ that is } n \leqslant m^2, \text{ or } m \geqslant \sqrt{n}.$$

One checks the validity of this result for small values $(1, 2, 3)$ of n, too (while the coplanarity restriction is obvious).

Problem 19. A 8×8 square board is divided into 64 unit squares. A "skew-diagonal" of the board is a set of 8 unit squares with the property that each row or column of the board contains only one unit square of the set. Checkers are placed in some of the unit squares so that each "skew-diagonal" has exactly 2 squares occupied by checkers. Prove that there exist two rows or two columns which contain all the checkers.

Solution. Label the rows from 1 to 8 and the columns from 1 to 8. The unit square which lies on the row i and the column j will be referred as (i, j).

On the skew-diagonal $\{(i, i) \mid i = 1, 2, \ldots, 8\}$ there are exactly 2 squares in which checkers were placed; wlog, assume that the squares are $(1, 1), (2, 2)$. Looking at the 6×6 sub-array Q determined by the rows 3-8 and the columns 3-8, we see that any "skew-diagonal" of Q, togheter with $(1, 1), (2, 2)$, is a skew-diagonal of the initial array. In view of the given conditions, no checkers are placed in the squares of Q. Now take any skew-diagonal of Q with the squares $(2, 1), (1, 2)$; this is a skew-diagonal of the initial array, and the two checkers are placed inside $(2, 1), (1, 2)$.

Up to the point, we know that checkers are placed in the squares on the rows 1-2 or on the columns 1-2. Suppose by way of contradiction that there exist a square located on the first two rows — say $(i, m), i = 1, 2, m \geqslant 3$ — and a square on the first two columns — say $(s, j), j = 1, 2, s \geqslant 3$ — that hold checkers. Then squares $(i, m), (s, j), (3 - i, 3 - j)$ belongs to a skew-diagonal, contradiction.

Problem 20. Let $1 \leqslant m < n$ be positive integers, and consider the set $M = \{(x, y); x, y \in \mathbb{N}^*, 1 \leqslant x, y \leqslant n\}$. Determine the least value $v(m, n)$ with the property that for any subset $P \subseteq M$ with $|P| = v(m, n)$ there exist $m+1$ elements

$A_i = (x_i, y_i) \in P, i = 1, 2, \ldots, m + 1$, for which the values x_i are all distinct, and y_i are also all distinct.

Solution. We claim $v(m, n) = mn + 1$.

Partition M into n sets $P_k = \{(x, y); n \mid x + y - k\}, k = 1, 2, \ldots, n$. The pigeonhole principle now forces (at least) $m + 1$ elements from P, be them $A_i = (x_i, y_i)$, to belong to a same P_k. Now, if we assume $x_i = x_j$, then from $x_i + y_i - k \equiv x_j + y_j - k \pmod{n}$ it follows $n \mid y_i - y_j$, but as $y_i, y_j \in \{1, 2, \ldots, n\}$, it follows $y_i = y_j$, i.e. $A_i = A_j$.

Conversely, $mn + 1$ is the least cardinality of P to warrant the claimed result; for $|P| = mn$, one can pick $P = \{(x, y); 1 \leqslant x \leqslant m, 1 \leqslant y \leqslant n\}$; then any $m + 1$ elements from P, be them $A_i = (x_i, y_i)$, will share at least one $x_i = x_j$ (pigeonhole principle again).

Problem 21. Let ABC be a triangle right-angled at A and let D be a point on the side AC. Point E is the mirror image of A across BD and point F is the intersection of the line CE with the perpendicular line from D to CB. Show that the lines AF, DE and CB are concurrent.

Solution. Line BC meet DF, AE at points T, G respectively. Using Ceva's theorem, it suffices to prove that

$$\frac{CF}{FE} \cdot \frac{EG}{GA} \cdot \frac{AD}{DC} = 1,$$

since only one or all points D, F, G lies on the sides of the triangle AEC.

Observe that $\angle BAD = \angle BED = \angle BTD = 90°$, so points A, C, E, T, D lies on the circle of diameter BD. Then $\angle FDE = \angle TBE \overset{\text{not}}{=} \alpha$ and $\angle TDC = \angle ABC \overset{\text{not}}{=} \beta$. Moreover, $DE = DA$ and $AB = BE$. The law of sinuses gives

$$\frac{FE}{\sin \alpha} = \frac{DE}{\sin \angle EFD}, \quad \frac{DC}{\sin \angle CFD} = \frac{FC}{\sin \beta},$$

and since $\sin \angle EFD = \sin \angle CFD$ we have

(1) $$\frac{CF}{FE} = \frac{DC}{DA} \cdot \frac{\sin \beta}{\sin \alpha}.$$

On the other hand,

$$\frac{EG}{\sin \alpha} = \frac{BE}{\sin \angle EGB} = \frac{BA}{\sin \angle AGB} = \frac{AG}{\sin \beta},$$

hence

$$\text{(2)} \qquad \frac{EG}{GA} = \frac{\sin\alpha}{\sin\beta}.$$

Multiplying the inequalities (1) and (2) concludes the proof.

Alternative solution. Let H be the intersection point of the lines DF and AE. The claim is equivalent to $\frac{AG}{GE} = \frac{AH}{HE}$, in other words the pairs of points A, E and G, H are harmonical conjugates.

Since I is the midpoint of the segment AE, the claim reduces to $IG \cdot IH = IA^2$.

The segment AI is an altitude in the right-angled triangle ABD, so $AI^2 = ID \cdot IB$.

Angles $\angle HTB$ and $\angle HIB$ are right, so the points H, T, I, B are cocyclic. It follows that $\angle DHI = \angle IBG$ and further, $\Delta DHI \sim \Delta IBG$. Hence $\frac{ID}{IH} = \frac{IG}{IB}$, so $ID \cdot IB = IH \cdot IG$. the conclusion is now obvious.

Problem 22. An irrational number x, $0 < x < 1$ is called *suitable* if its first 4 decimals in the decimal representation are equal. Find the smallest positive integer n such that any real number t, $0 < t < 1$ may be written as a sum of n distinct *suitable* numbers.

Solution. At first, we look for a lower margin of n. The number $0,1111$ can be written as a sum of n distinct suitable numbers, all starting with $0,0000\ldots$, therefore lower than 0.0001. Hence if $0.1111 = a_1 + a_2 + \cdots + a_n$, with $a_k < 0.0001$, then $0.1111 < n \cdot 0.0001$, or $n > 1111$. Thus n is at least 1112.

We claim that 1112 is the requested number. Let $t \in (0,1)$ be a real number. If $t > 0.1111$, choose a suitable number of the form $y = 0.xxxx\ldots$ so that $y < t \leqslant y + 0.1111$. The other suitable numbers will have the form $0,0000\ldots$, so they are different from y. We have $0 < t - y \leqslant 0,1111$. Because $0 < \frac{t-y}{1112} \leqslant \frac{0.1111}{1112} < 0.0001$, the first four decimals of the number $u = \frac{t-y}{1112}$ are all equal to 0. Choose an irrational number e, small enough not to change the first four decimals of the numbers $u + e, u + 2e, \ldots, u + 1111e$, and such that all — plus $y + u + e$ — are left irrationals. Then

$$t - y = (u+e) + (u+2e) + (u+3e) + \cdots + (u+1111e) + \left(u - \frac{1111 \cdot 1112 \cdot e}{2}\right).$$

As previously stated, number e was selected so that all summands are irrational, suitable numbers with the first four decimals 0 and paiwisely distinct — in other words, different from the last one. Consequently,

$$t = (y+u+e) + (u+2e) + (u+3e) + \cdots + (u+1111e) + \left(u - \frac{1111 \cdot 1112 \cdot e}{2}\right).$$

The number $y + u + e$ starts with $0.xxxx$, while the others with 0.0000, so they are suitable. Since t is now represented as required, the proof is concluded.

PROBLEMS AND SOLUTIONS

BALKAN MATHEMATICAL OLYMPIAD

Problem 1. Let $ABCD$ be a convex quadrilateral with $AB = BC = CD$, $AC \neq BD$, and let E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120^\circ$.

Solution. (Official) Let us first denote $\angle BAC = \angle BCA = \alpha$, $\angle CBD = \angle CDB = \beta$. Assume $AE = DE$. The triangles BAE and CDE have two pairs of equal sides, and their angles $\angle AEB$ and $\angle DEC$ are also equal (to $\alpha + \beta$, as vertex opposite). By a simple argument, the angles $\angle ABE$ and $\angle DCE$ are then equal or supplementary, so either $2\alpha + \beta = \alpha + 2\beta$, or $(2\alpha + \beta) + (\alpha + 2\beta) = 180^\circ$. From the first one, we get $\alpha = \beta$, so $\angle BAD = \angle CDA$, hence $AC = BD$, contradiction. From the second one, we get $\alpha + \beta = 60^\circ$. Then $\angle BAD + \angle ADC = (\alpha + \angle EAD) + (\beta + \angle EDA) = 2(\alpha + \beta) = 120^\circ$.

Conversely, assume $\angle BAD + \angle ADC = 120^\circ$. Let $S = AB \cap DC$. We have $\angle AEB = \alpha + \beta$, and also $\angle AEB = \angle EAD + \angle EDA$, hence $2\angle AEB = \angle BAD + \angle CDA = 120^\circ$, therefore $\angle AEB = 60^\circ$. But $\angle ASD = 60^\circ$ also, hence the quadrilateral $SBEC$ is cyclic, so $\angle BSE = \angle BCA = \alpha = \angle SAE$, therefore $EA = ES$. Similarly, $ED = ES$, so $AE = DE$.

Problem 2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y$$

for all $x, y \in \mathbb{R}$.

Solution. (N. Păpară) The identically null function $f \equiv 0$ is a solution of the equation.

For $y = f(x)$ one gets $f(2f(x)) = f(0) + 4f(x)^2$. Replacing y by $2f(y) - f(x)$ one gets $f(2f(x) - 2f(y)) = f(2f(y)) - 4f(x)(2f(y) - f(x)) = f(0) + 4f(y)^2 - 4f(x)(2f(y) - f(x)) = f(0) + (2f(x) - 2f(y))^2$.

When there exists x_0 with $f(x_0) \neq 0$, then for any $x \in \mathbb{R}$ one has

$$x = 2f\Big(f(x_0) + \frac{x}{8f(x_0)}\Big) - 2f\Big(f(x_0) - \frac{x}{8f(x_0)}\Big),$$

hence $f(x) = x^2 + f(0)$.

Alternative solution. (A. Zahariuc) Let $A = \operatorname{Im} f$. We start with the following

LEMMA. $f \equiv 0$ *or* $A - A = \mathbb{R}$ *(Minkowski difference).*

Proof. If f is not identically null, then there exists $x_0 \in \mathbb{R}$ such that $f(x_0) \neq 0$. We have $f(f(x_0) + y) - f(f(x_0) - y) = 4f(x_0)y$, for any $y \in \mathbb{R}$, and as $4f(x_0)y$ is onto as a mapping in y, it follows that any real number may be written as $f(\beta) - f(\alpha)$, that is $A - A = \mathbb{R}$.

As the identically null function trivially fulfills the equation, we are left with $f \not\equiv 0$. Let $g(t) = f(t) - t^2$. It immediately follows $g(f(x) + y) = g(f(x) - y)$, so $g(y) = g(2f(x) - y)$, for all $x, y \in \mathbb{R}$. According to $A - A = \mathbb{R}$, there exist $\alpha, \beta \in \mathbb{R}$ such that $f(\beta) - f(\alpha) = \frac{1}{2}t$, for any t, and then $g(0) = g(2f(\alpha)) = g(2f(\beta) - 2f(\alpha)) = g(t)$, therefore g is constant. It follows that $f(x) = x^2 + f(0)$, readily checked as solutions.

Problem 3. Find all positive integers n such that there exists a permutation σ of the set $\{1, 2, \ldots, n\}$ for which

$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}} \in \mathbb{Q}.$$

Solution. (Official, adapted) Let us denote, for $k = 1, 2, \ldots, n$,

$$E_k = \sqrt{\sigma(k) + \sqrt{\sigma(k+1) + \sqrt{\cdots + \sqrt{\sigma(n)}}}}.$$

It immediately follows that we need have $E_k \in \mathbb{N}^*$, because the square root of an integer is either integer, or irrational. Now, let us denote, for $k = 1, 2, \ldots, n$,

$$N_k = \sqrt{n + \sqrt{n + \sqrt{\cdots + \sqrt{n}}}},$$

where $n - k + 1$ square root signs are used. It may easily be proven (through simple induction, or otherwise) that $E_k \leqslant N_k < \sqrt{n} + 1$, for $k = 1, 2, \ldots, n$.

Let $p^2 \leqslant n < (p+1)^2$, hence $\sqrt{n} + 1 < p + 2$. For $p > 1$ we have $p^2 - 1 \in \{1, 2, \ldots, n\}$, and $p^2 - 1$ not a perfect square, therefore $p^2 - 1 = \sigma(m)$, for some $m < n$. But then $p < E_m < p + 2$, so $E_m = p + 1$, hence $p^2 - 1 + E_{m+1} = p^2 + 2p + 1$, that is $E_{m+1} = 2p + 2 > p + 2 > \sqrt{n} + 1$, absurd (unless $m = n - 1$ and $1 = \sigma(n)$, when $p^2 = \sigma(l)$, for some $l < n - 1$, and an even easier similar contradiction is obtained).

It follows the only possibility remains $p = 1$, hence $n \in \{1, 2, 3\}$, for which it is trivial to check that the sole solutions are $n = 1$, with $\sqrt{1} = 1$, and $n = 3$, with $\sqrt{2 + \sqrt{3 + \sqrt{1}}} = 2$.

Problem 4. For a given positive integer $n > 2$, let C_1, C_2, C_3 be the boundaries of three convex n-gons in the plane, such that all three sets $C_1 \cap C_2$, $C_2 \cap C_3, C_3 \cap C_1$, are finite. Find the maximum number of points of the set $C_1 \cap C_2 \cap C_3$.

Solution. (A. Zahariuc) We claim the answer is $\lfloor \frac{3n}{2} \rfloor$.

Let $V = C_1 \cap C_2 \cap C_3$, $|V| = m$. Let $\mathcal{P}$ be the polygon of vertices V, and $\mathcal{P}_k$ be the polygon of boundary C_k, $k = 1, 2, 3$. It is readily seen that $\mathcal{P}$ turns to be convex, with any three of its vertices non-collinear. It is clear that every side of $\mathcal{P}$ is included in one side of some $\mathcal{P}_k$ (and one only). Let a_k be the number of sides of $\mathcal{P}$ included in sides of $\mathcal{P}_k$. It follows $a_1 + a_2 + a_3 = m$. Let us count the vertices of $\mathcal{P}_k$. On one hand, they are exactly n, while on the other hand, there are exactly a_k sides of $\mathcal{P}_k$ including sides of $\mathcal{P}$, and for each of the remaining $m - 2a_k$ vertices of $\mathcal{P}$, a side each containing the corresponding vertex (these sides being always different). It follows that $a_k + (m - 2a_k) \leqslant n$, hence $m - a_k \leqslant n$. By summation over k follows $3m - (a_1 + a_2 + a_3) \leqslant 3n$, hence $2m \leqslant 3n$, that is $m \leqslant \lfloor \frac{3n}{2} \rfloor$.

Let us build a model in order to prove the bound found in the above is best. We will present the case n even; the odd case is similar. Let $n = 2l$. We start with a convex polygon $\mathcal{P}$ with $3l$ sides. We index its sides, sequentially, with $1, 2, \ldots, 3l$. For each $k \in \{1, 2, 3\}$ we choose $\mathcal{P}_k$ to be the polygon determined by the l sides

of $\mathcal{P}$, indexed $\equiv k$ modulo 3, and other more l lines "tangent" to $\mathcal{P}$ at its remaining l vertices.

Remarks. This solution may be readily extended to intersecting boundaries of $p \geqslant 2$ convex n-gons in the plane, pairwise finite. The answer is $\lfloor \frac{pn}{p-1} \rfloor$, which, for $p = 2$ is (almost) trivial, for $p = 3$ is the contest problem, while for $p > 3$ it makes a complete extension. Let us notice that asymptotically, for p growing to infinity, the answer is n. It is remarkable that the boundaries of $n + 1$ n-gons may intersect in as many points as $n + 1$.

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