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Introduction

This booklet contains a short presentation of the most important mathematical competitions devoted to high school students in Romania during the year 1997. As it was explained in our previous book - RMC 1996 - the main mathematical competition in Romania is *The National Mathematical Olympiad* (N.M.O.).

In 1997 the final round of the 48[th] N.M.O. was organized by the Ministry of Education in the City of Suceava. The City of Suceava is located in the north of Moldavia, the eastern land of Romania, and it is very rich in history, arts, monuments and etnography. It was in the Middle Ages the capital of Moldavia and in the present days it is the residence of the romanian part of Bucovina.

The 48[th] N.M.O. was also partially supported by *The Romanian Society for Mathematical Sciences* and *Theta Foundation.*

Let say a few words about the competition. The number of participants to the final round was 620. The age of participants was from 13 to 19. Only the students from 7[th] grade to the 12[th] grade are called to take part to the final round and they are selected from the participants to the preliminary district rounds. Each problem was worth 7 points and a half of participants was awarded with first, second and third prizes in the rate 1:2:3.

Forms	Number of participants	First rize	Second prize	Third prize
VII	118	9	20	30
VIII	130	11	23	33
IX	99	8	19	23
X	99	8	16	26
XI	91	8	15	24
XII	83	8	13	22
Total	620	52	106	158

The matrix of scores for each form was the following :

POINTS	0	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5	5,5	6	6,5	7	AVERAGE PROBLEM
The 7th form																
Problem 1		1	2	1	4		6		9		9	1	35	4	46	5,72
Problem 2	61	7	29	5	5		11									0,70
Problem 3	7	7	5	9	9	4	1	16	6	34	3	3	2	2	10	3,5
Problem 4	9	8	14	15	14	5	4	2	25	1	2	2			17	2,92
The 8th form																
Problem 1	18	7	17	3	3	5	6	6	16	2	5	2	3	3	34	3,6
Problem 2	7	5	49	13	31	8	5	1	3		1		1		6	1,82
Problem 3	51	2	37	16			6	2	4		2		2		8	1,45
Problem 4	11	9	36	14	15	6	12	1	1	1	1	4	13	1	5	2,31
The 9th form																
Problem 1	11	4	25	8	25	2	14	2	3	1	1	1	1		1	1,85
Problem 2	14	27	17	10	12	6	5	1	1	1		1	2	1	1	1,44
Problem 3	8	1	10		25	2	10	1	13		4		9	2	14	3,4
Problem 4	23	20	17	14	11	4	4	2	1	1	1		1			1,19

POINTS	0	0,5	1	1,5	2	2,5	3	3,5	4	4,5	5	5,5	6	6,5	7	AVERAGE PROBLEM
The 10th form																
Problem 1	5	3	15	1	4	1	1	5	2	2	7	3	5	3	42	4,65
Problem 2	75	5	3		1				3				5	1	6	0,98
Problem 3	23	12	12	7	8	6	2		2	2	2	9	3	4	7	2,37
Problem 4	11	19	39	5	1	4		1		3	11	1		1	3	1,74
The 11th form																
Problem 1	40	9	8	3	8	1	1		1		2	1	3	2	12	1,90
Problem 2	3	2	18	12	15	8	1	2	2	1	1		2	3	21	3,21
Problem 3	23	6	13	4	3	1	5	7	11	5	2	1		2	8	2,42
Problem 4	18	15	23	8	14	2	4	2	1	1	1		1		1	1,32
The 12th form																
Problem 1	9	9	15	11	9	7	3	4	1	3	2	2		1	7	2,27
Problem 2	32	28	9	3	1	3								2	5	1,02
Problem 3	52	12	4	2		1							1	2	9	1,17
Problem 4	18	3	29	7	11	4	5		1	2		1			2	1,45

For more information we add the four examinations for the selection of the representative team for the 38^{th} I.M.O.

In these examinations were called only 25 students, having the best results in the N.M.O. examination. Finally, the six members of the team for the 38^{th} I.M.O. were selected by adding the points in these four exminations.

In the last part of this booklet we present a short selection of representative problems used in the annual competition of the magazine *Gazeta Matematică* in september 1996. This competition is organized every year, in the autumn, for those students which have obtained the best results in the past academic year, in all national and international mathematical competitions. Also are called the best problem solveres they of the specified magazine. It is the oldest mathematical competition for students in Romania.

SECTION 1

THE 48TH NATIONAL MATHEMATICAL OLYMPIAD

THE FINAL ROUND, SUCEAVA, MARCH 24-30TH, 1997

PROPOSED PROBLEMS

7th Form

7.1. Let $n_1 = \overline{abcabc}$ and $n_2 = \overline{d00d}$ be numbers represented in the decimal system, with $a \neq 0$ and $d \neq 0$.

a) Prove that $\sqrt{n_1}$ cannot be an integer.

b) Find all positive integers n_1 and n_2 such that $\sqrt{n_1 + n_2}$ is an integer number.

c) From all the pairs (n_1, n_2) such that $\sqrt{n_1 n_2}$ is an integer find those for which $\sqrt{n_1 n_2}$ has the greatest possible value.

7.2. Let $a \neq 0$ be a natural number. Prove that a is a perfect square if and only if for every $b \in N^*$ there exists $c \in N^*$ such that $a + bc$ is a perfect square.

7.3. The triangle ABC has $\angle ACB = 30°, BC = 4\text{cm}$ and $AB = 3\text{cm}$. Compute the altitudes of the triangle.

7.4. The quadrilateral $ABCD$ has two parallel sides. Let M and N be the midpoints of $[DC]$ and $[BC]$, and P the common point of the lines AM and DN. If $\frac{PM}{AP} = \frac{1}{4}$, prove that $ABCD$ is a parallelogram.

8th Form

8.1. Let k be an integer number and $P(X)$ be the polynomial

$$P(X) = X^{1997} - X^{1995} + X^2 - 3kX + 3k + 1.$$

Prove that : a) the polynomial has no integer root ;

b) the numbers $P(n)$ and $P(n) + 3$ are relatively prime, for every integer n.

8.2. Let x, y, z be positive real numbers such that $xyz = 1$.

Prove that $$\frac{x^9+y^9}{x^6+x^3y^3+y^6}+\frac{y^9+z^9}{y^6+y^3z^3+z^6}+\frac{z^9+x^9}{z^6+z^3x^3+x^6}\geq 2.$$

8.3. $ABCDA'B'C'D'$ is a rectangular parallelepiped with $AA' = 2AB = 8a$, E is the midpoint of (AB) and M is the point of (DD') for which $DM = a\left(1+\frac{AD}{AC}\right)$.

a) Find the position of the point. F on the segment (AA') for which the sum $CF + FM$ has the minimum possible value.

b) Taking F as above, compute the measure of the angle of the planes (D, E, F) and (D, B', C').

c) Knowing that the straight lines AC' and FD are perpendicular, compute the volume of the parallelepiped $ABCDA'B'C'D'$.

8.4. Let S be a point outside of the plane of the parallelogram $ABCD$, such that the triangles SAB, SBC, SCD and SAD are equivalent.

a) Prove that $ABCD$ is a rhombus.

b) If the distance from S to the plane (A, B, C, D) is 12, $BD = 30$ and $AC = 40$, compute the distance from the projection of the point S on the plane (A, B, C, D) to the plane (S, B, C).

9th Form

9.1. Let $C_1, C_2, \ldots, C_n \ (n \geq 3)$ be circles having a common point M. Three straight lines passing through M intersect again the circles in $A_1, A_2, \ldots, A_n; B_1, B_2, \ldots, B_n$ and $X_1, X_2, \ldots, X_n$ respectively. Prove that if $A_1A_2 = A_2A_3 = \ldots = A_{n-1}A_n$ and $B_1B_2 = B_2B_3 = \ldots = B_{n-1}B_n$ then
$$X_1X_2 = X_2X_3 = \ldots = X_{n-1}X_n.$$

9.2. Find the image of the function $f: \mathbb{R} \to \mathbb{R}$,
$$f(x) = \frac{3+2\sin x}{\sqrt{1+\cos x}+\sqrt{1-\cos x}},$$

9.3. Let a, b, c, $d \in \mathbf{R}$ and $f:\mathbf{R}\to\mathbf{R}$, $f(x)=ax^3+bx^2+cx+d$, such that $f(2)+f(5)<7<f(3)+f(4)$. Prove that there exists u, $v \in \mathbf{R}$ such that $u+v=7$ and $f(u)+f(v)=7$.

9.4. Let a, b, c, $d \in \mathbf{R}$ and the sets $A=\{x\in\mathbf{R}; x^2+a|x|+b=0\}$ and $B=\{x\in\mathbf{R}; [x]^2+c[x]+d=0\}$. Prove that if the set $A\cap B$ has exactly three elements then a cannot be an integer.

10^{th} Form

10.1. Let $f:\mathbf{N}\times\mathbf{N}\to\mathbf{N}$ be a function which fulfils the conditions:

$$f(0,y)=y+1, \forall y\in\mathbf{N},$$
$$f(x+1,0)=f(x,1), \forall x\in\mathbf{N},$$
$$f(x+1,y+1)=f(x,f(x+1,y)), \forall (x,y)\in\mathbf{N}\times\mathbf{N}.$$

Compute $f(3,1997)$.

10.2. Let $n\geq 3$ be an integer and x be a real number such that the numbers x, x^2 and x^n have the same fractional parts. Prove that x is an integer.

10.3. Let d_1, d_2 be two straight lines and A_0 be a point on d_1. For every $n\in\mathbf{N}$ let B_n be the projection of A_n on d_2 and A_{n+1} be the projection of B_n on d_1. Prove that there exists two segments $[A'A'']\subset d_1$ and $[B'B'']\subset d_2$ of lengths 0,001 and a natural number N such that $A_n\in[A'A'']$ and $B_n\in[B'B'']$ for every $n\geq N$.

10.4. Let $a_0, a_1, \ldots, a_n$ be complex numbers such that

$$z\in\mathbf{C}, |z|\leq 1 \Rightarrow |a_n z^n + a_{n-1}z^{n-1}+\ldots+a_1 z + a_0|\leq 1.$$

Prove that $|a_k|\leq 1$ and $|a_0+a_1+\ldots+a_n-(n+1)a_k|\leq n$ for every $k\in\overline{0,n}$.

11^{th} Form

11.1. Let $p \geq 2$ be a natural number and $A = (a_{ij})$ be a square matrix of order n, with integer elements. Prove that for every permutation $\sigma \in S_n$ there exists a function $\varepsilon: \{1,2,\ldots,n\} \to \{0,1\}$ such that by replacing the elements $a_{\sigma(1)1}, a_{\sigma(2)2}, \ldots, a_{\sigma(n)n}$ from the matrix A with

$$a_{\sigma(1)1} + \varepsilon(1), a_{\sigma(2)2} + \varepsilon(2), \ldots, a_{\sigma(n)n} \varepsilon(n)$$

respectively, the determinant of the new matrix is not divisible by p.

11.2. Let A be a square matrix of odd order (at least 3), with integer odd elements. Prove that if A is invertible then it is not possible that all the minors of the elements of a row have the same modulus.

11.3. Let $\Im$ be the set of all the differentiable functions $f: \mathbf{R} \to \mathbf{R}$, which have the property $f(x) \geq f(x + \sin x), \forall x \in \mathbf{R}$.

a) Prove that $\Im$ contains also nonconstant functions.

b) Prove that if $f \in \Im$ then the set of the solutions of the equation $f'(x) = 0$ in infinite.

11.4. Let $f, g: \mathbf{R} \to \mathbf{R}$ be two bijective continuous functions such that $f(g^{-1}(x)) + g(f^{-1}(x)) = 2x, \forall x \in \mathbf{R}$. Prove that if there exists $x_0 \in \mathbf{R}$ such that $f(x_0) = g(x_0)$, then $f = g$.

12^{th} Form

12.1. Let $\alpha \in \mathbb{C} \setminus \mathbb{Q}$ be such that the set $A = \{a + b\alpha | a, b \in \mathbb{Z}\}$ is a ring with respect to the usual operations in $\mathbb{C}$. Prove that if A has exactly 4 invertible elements then $A = \mathbb{Z}[i]$.

12.2. Prove that for every continuous function $f : [-1,1] \to \mathbb{R}$ takes place the inequality $\int_{-1}^{1} f^2(x)\mathrm{d}x \geq \frac{1}{2}\left(\int_{-1}^{1} f(x)\mathrm{d}x\right)^2 + \frac{3}{2}\left(\int_{-1}^{1} xf(x)\mathrm{d}x\right)^2$.

When takes place the equality ?

12.3. Let K be a finite field, $n \in \mathbb{N}$, $n \geq 2$, $f \in K[X]$ an irreducible polynomial of degree n and g the product of all the nonconstant polynomials of $K[X]$ which have the degrees less than n. Prove that f divides $g - 1$.

12.4. Let the sequence $(f_n)_{n\in\mathbb{N}}$ of functions f_n : $[0;1] \to \mathbb{R}$ be such that f_0 is continuous and $f_{n+1}(x) = \int_0^x \frac{1}{1+f_n(t)}\mathrm{d}t, \forall x \in [0;1], \forall x \in \mathbb{N}$. Prove that for every $x \in [0;1]$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ is convergent and compute its limit.

Note: The authors of the problems from this section are the followings : *Adrian Ghioca* (7.1 and 9.4), *Bogdan Enescu* (7.2), *Mircea Fianu* (7.3), *Ştefan Smarandache* (8.1), *Mircea Becheanu* (8.2 and 11.1), *George Turcitu* (8.3), *Dana Radu* (8.4), *Dinu Şerbănescu* (9.1), *Ion Cheşcă* (9.2), *Cristinel Mortici* (9.3), *Călin Burduşel* (10.1), *Laurenţiu Panaitopol* (10.2), *Vasile Pop* (10.3), *Radu Gologan* (10.4), *Mihai Piticari* and *Dan Popescu* (11.2 and 12.4), *Florica Vornicescu* and *Ioan Raşa* (11.3), *Marian Andronache* and *Ion Savu* (11.4 and 12.3), *Marcel Ţena* (12.1), *Dorian Popa* (12.2).

SOLUTIONS

7.1. Let $n_1 = \overline{abcabc}$ and $n_2 = \overline{d00d}$ be numbers represented in the decimal system, with $a \neq 0$ and $d \neq 0$.

a) Prove that $\sqrt{n_1}$ cannot be an integer.

b) Find all positive integers n_1 and n_2 such that $\sqrt{n_1 + n_2}$ is an integer number.

c) From all the pairs (n_1, n_2) such that $\sqrt{n_1 n_2}$ is an integer find those for which $\sqrt{n_1 n_2}$ has the greatest possible value.

Solution. a) $n_1 = 1001 \cdot \overline{abc} = 7 \cdot 11 \cdot 13 \cdot \overline{abc}$. If n_1 is a square then $\overline{abc}$ must be divisible by 7, 11, and 13, and therefore divisible by $7 \cdot 11 \cdot 13 = 1001$, which is impossible.

b) $n_1 + n_2 = 1001(\overline{abc} + d)$ is a square if and only if $\overline{abc} + d = 1001$ which happens if and only if $a = 9$, $b = 9$ and c, d are digits such that $c + d = 11$.

c) $n_1 \cdot n_2 = 1001^2 \cdot \overline{abc} \cdot d$, so $\overline{abc} \cdot d$ is a square not greater than $999 \cdot 9 = 8991$. The greatest square less than 8991 is 94^2, but $94^2 = 2^2 \cdot 47^2 = 4 \cdot 2209$ cannot be written in the form $\overline{abc} \cdot d$. Since $93^2 = 3^2 \cdot 31^2 = 9 \cdot 961$, the required pair is $(961, 9)$.

7.2. Let $a \neq 0$ be a natural number. Prove that a is a perfect square if and only if for every $b \in \mathbf{N}^*$ there exists $c \in \mathbf{N}^*$ such that $a + bc$ is a perfect square.

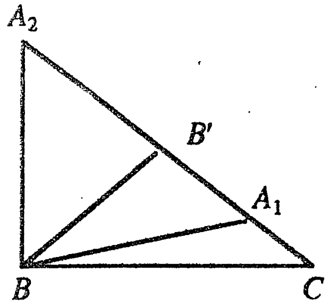
Solution. If $a = k^2, k \in \mathbf{N}^*$ and $b \in \mathbf{N}^*$ then one can take $c = 2k + b$.

For the converse let us choose $b = a^2$ and $c \in \mathbf{N}^*$ such that $a + bc = a(1 + ac)$ is a square. Since a and $1 + ac$ are relatively prime, it follows that a is a square.

7.3. The triangle ABC has $\angle ACB = 30°$, $BC = 4\text{cm}$ and $AB = 3\text{cm}$. Compute the altitudes of the triangle.

Solution. Let BB' be the altitude form B. It follows that $BB' = 2\text{cm}, B'C = 2\sqrt{3}\text{cm}$ and $B'A = \sqrt{5}\text{cm}$. Since $B'A < B'C$, there are possible two cases.

Case 1 : $A = A_1 \in (B'C)$. In this case the area of the triangle is $\frac{1}{2} BB' \cdot A_1C = 2\sqrt{3} - \sqrt{5}$ cm^2 and the altitudes of the triangle are



$$A_1A_1' = \frac{2\sqrt{3} - \sqrt{5}}{2} \text{ cm}, \quad BB' = 2 \text{ cm and}$$

$$CC' = \frac{2(2\sqrt{3} - \sqrt{5})}{3} \text{ cm}.$$

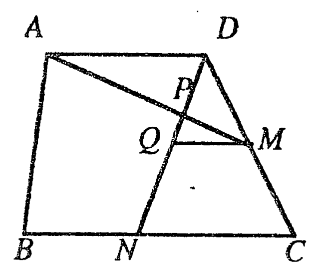
Case 2 : $A = A_2$ such that $B' \in (A_2C)$. Now the area is $2\sqrt{3} + \sqrt{5}$ cm^2 and the altitudes become :

$$A_2A_2' = \frac{2\sqrt{3} + \sqrt{5}}{2} \text{cm}, \quad BB' = 2 \text{ cm and } CC' = \frac{2(2\sqrt{3} + \sqrt{5})}{3} \text{cm}.$$

7.4. The quadrilateral $ABCD$ has two parallel sides. Let M and N be the midpoints of $[DC]$ and $[BC]$, and P the common point of the lines AM and DN. If $\frac{PM}{AP} = \frac{1}{4}$, prove that $ABCD$ is a parallelogram.

Solution. As the problem emphasizes on AM, we must study two cases :

Case 1 : $AD \parallel BC$.

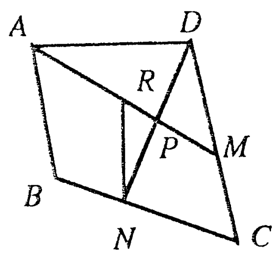


Let Q be the midpoint of DN. It follows that $QM \parallel BC \parallel AD$,

$$QM = \frac{1}{2} NC = \frac{1}{4} BC \text{ and}$$

$$QM = \frac{PM}{PA} \cdot AD = \frac{1}{4} AD,$$

whence $AD=BC$ and therefore $ABCD$ is a parallelogram.



Case 2 : $AB \| CD$. Let R be the midpoint of AM. In the trapezoid $ABCM$:

$$RN = \frac{1}{2}(AB + CM) = \frac{1}{2}AB + \frac{1}{4}CD.$$

On the other hand :

$$RN = \frac{RP}{PM} \cdot MD = \frac{3}{2} \cdot \frac{CD}{2} = \frac{3}{4}CD.$$

It follows that $\frac{1}{2}AB + \frac{1}{4}CD = \frac{3}{4}CD$, whence $AB = CD$.

8.1. Let k be an integer number and $P(X)$ be the polynomial

$$P(X) = X^{1997} - X^{1995} + X^2 - 3kX + 3k + 1.$$

Prove that :

a) the polynomial has no integer root ;

b) the numbers $P(n)$ and $P(n) + 3$ are relatively prime, for every integer n.

Solution. a) If P has an integer root than $P(X) = (X - r) \cdot Q(X)$, where Q is a polynomial with integer coefficients. This leads to

$$P(-1) = (-1 - r)Q(-1) = 6k + 2,$$

$$P(0) = (-r)Q(0) = 3k + 1,$$

$$P(1) = (1 - r)Q(1) = 2.$$

The numbers $-1 - r, -r$ and $1 - r$ are consecutive, so one of them must be divisible by 3. This comes into contradiction with the fact that $Q(-1)$, $Q(0)$, $Q(1)$ are integers and none of the numbers $P(-1)$, $P(0)$, $P(1)$ is disible by 3.

b) The g.c.d. of $P(n)$ and $P(n) + 3$ can be only 1 or 3. But

$$P(n) = n^{1994} \cdot (n - 1) \cdot n(n + 1) - 3k(n - 1) + n^2 + 1,$$

3 is a divisor of $(n - 1)n(n + 1)$ and of $3k(n - 1)$ and 3 is not a divisor of $n^2 + 1$ (consider the remainders of n(mod3)). It follows that $P(n)$ is not divisible by 3, whence the g.c.d. is 1.

8.2. Let x, y, z be positive real numbers such that $xyz = 1$.

Prove that $$\frac{x^9+y^9}{x^6+x^3y^3+y^6}+\frac{y^9+z^9}{y^6+y^3z^3+z^6}+\frac{z^9+x^9}{z^6+z^3x^3+x^6}\geq 2.$$

Solution. It is easy to check that

$$\frac{x^9+y^9}{x^6+x^3y^3+y^6}=x^3+y^3-\frac{2x^3y^3\left(x^3+y^3\right)}{x^6+x^3y^3+y^6}\geq$$

$$\geq x^3+y^3-\frac{2x^3y^3\left(x^3+y^3\right)}{3x^3y^3}=\frac{1}{3}\left(x^3+y^3\right).$$

This gives $\sum\frac{x^9+y^9}{x^6+x^3y^3+y^6}\geq\frac{2}{3}\sum x^3\geq\frac{2}{3}\cdot 3xyz=2$, because

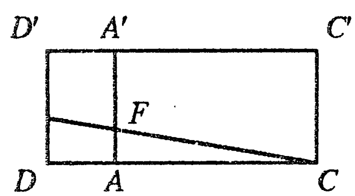
$$\sum x^3-3xyz=\sum x\left(\sum x^2-\sum xy\right)=\left(\sum x\right)\cdot\frac{1}{2}\sum(x-y)^2\geq 0.$$

8.3. $ABCDA'B'C'D'$ is a rectangular parallelepiped with $AA'=2AB=8a$, E is the midpoint of (AB) and M is the point of (DD') for which $DM=a\left(1+\frac{AD}{AC}\right)$.

a) Find the position of the point. F on the segment (AA') for which the sum $CF+FM$ has the minimum possible value.

b) Taking F as above, compute the measure of the angle of the planes (D, E, F) and (D, B', C).

c) Knowing that the straight lines AC' and FD are perpendicular, compute the volume of the parallelepiped $ABCDA'B'C'D'$.

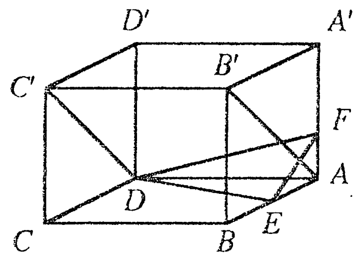


Solution. a) "Unfolding" the parallelepiped so that the half-lines $(AC$ and $(AD$ become opposite, we see that F must be the common point of AA' and CM.

We get :

$$AF=DM\cdot\frac{AC}{CD}=a\left(1+\frac{AD}{AC}\right)\cdot\frac{AC}{AC+AD}=a.$$

b) Since $\frac{AE}{AA'} = \frac{AF}{AB} = \frac{1}{4}$, the triangles AEF and $AA'B$ are similar, therefore $\angle AEF = \angle AA'B = \angle FA'B' =$ $= 90^\circ - \angle EAB'$, whence $EF \perp AB'$. Also $EF \perp AD$ (because $AD \perp (A, E, F)$), so the planes (D, E, F) and (D, B', C') are perpendicular.



c) $EF \perp (A, B', C', D)$ and $AC' \perp FD$ lead to $AC' \perp (D, E, F)$. Fron $CC' \perp DE$ it follows that $AC \perp DE$, therefore triangles ADE and BAC ar similar.

Thus $\frac{AD}{AB} = \frac{AE}{BC}, AD^2 = AB \cdot AE = 8a^2, AD = 2\sqrt{2}a$ and th required volume is $4a \cdot 2\sqrt{2}a \cdot 8a = 64\sqrt{2}a^3$.

8.4. Let S be a point outside of the plane of the parallelogran $ABCD$, such that the triangles SAB, SBC, SCD and SAD are equivalent.

a) Prove that $ABCD$ is a rhombus.

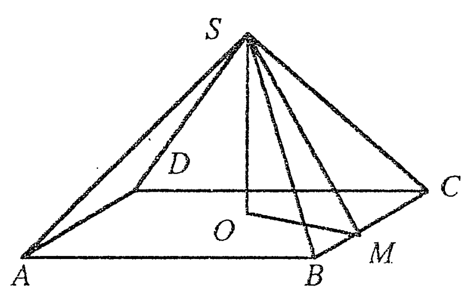
b) If the distance from S to the plane (A, B, C, D) is 12, $BD = 3C$ and $AC = 40$, compute the distance from the projection of the point S oı the plane (A, B, C, D) to the plane (S, B, C).

Solution. a) Let O be the projection of S on the plane (A, B, C, D).

From the equality of the distances $d(S, BC) = d(S, AD)$ anc $d(S, AB) = d(S, CD)$ follow the equalities :

$$d(0, BC) = d(0, AD) \text{ and}$$
$$d(0, AB) = d(0, CD),$$

therefore O is the common point of the diagonals of the parallelogram. Denote :



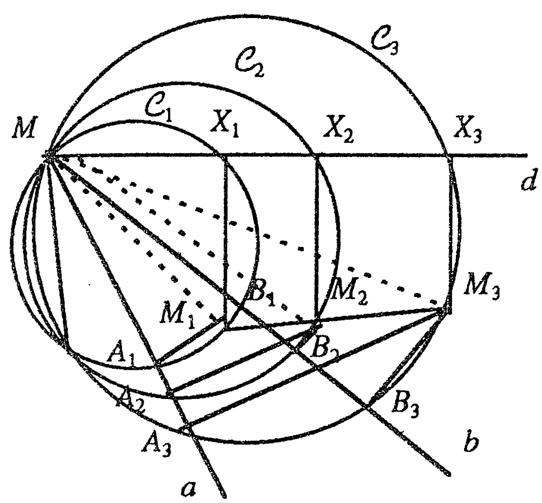
$$d(0, BC) = d(0, AD) = m,$$
$$d(0, AB) = d(0, CD) = n$$

and $SO = h$. This gives $S_{ABCD} = AB \cdot 2n = BC \cdot 2n$ and, from the hypothesis $AB\sqrt{n^2+h^2} = BC\sqrt{m^2+h^2}$, so $AB^2 \cdot n^2 + AB^2 \cdot h^2 = BC^2 \cdot m^2 + BC^2 \cdot h^2$, whence $AB = BC$.

b) It is a common place that the required distance is the altitude from O of the triangle SOM, where M is the projection of O on BC. Since $OM = \dfrac{OB \cdot OC}{BC} = 12$, it follows that $d(O,(S,B,C)) = \dfrac{SO \cdot OM}{SM} = 6\sqrt{2}$.

9.1. Let $C_1, C_2, \ldots, C_n (n \geq 3)$ be circles having a common point M. Three straight lines passing through M intersect again the circles in $A_1, A_2, \ldots, A_n; B_1, B_2, \ldots, B_n$ and $X_1, X_2, \ldots, X_n$ respectively. Prove that if $A_1A_2 = A_2A_3 = \ldots = A_{n-1}A_n$ and $B_1B_2 = B_2B_3 = \ldots = B_{n-1}B_n$ then

$$X_1X_2 = X_2X_3 = \ldots = X_{n-1}X_n.$$



Solution. The problem clearly reduces to the following: if three circles have a common point M and they determine congruent segments on two straight lines passing through M, then they determine congruent segments on every straight line which pass through M.

Denote by:

$$[MM_1], [MM_2], [MM_3]$$

the diameters of the three circles and by N the midpoint of the segment $[M_1M_3]$. Using the rectangular trapezoid $M_1A_1A_3M_3$ one gets $NA_2 \perp MA_2$; analogously $NB_2 \perp MB_2$ and therefore $N = M_2$, so M_2 is the midpoint of $[M_1M_3]$. Since the second common points of any straight line which passes through M, with the three circles are the projections of M_1, M_2 and M_3 on this line, the conclusion follows.

9.2. Find the image of the function $f:\mathbf{R} \to \mathbf{R}$,

$$f(x) = \frac{3+2\sin x}{\sqrt{1+\cos x}+\sqrt{1-\cos x}}.$$

Solution. Clearly $f(\mathbf{R}) = f([0,2\pi])$. The formula of the function can be written :

$$f(x) = \frac{3+2\sin x}{\sqrt{2}\left(\left|\cos\frac{x}{2}\right|+\sin\frac{x}{2}\right)}.$$

For $x \in [0,\pi]$: $$f(x) = \frac{1+2\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{\sqrt{2}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)} = \frac{1+t^2}{t},$$

where $t = \sqrt{2}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right) = 2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)$ describes the interval $\left[\sqrt{2};2\right]$.

The function $g:\left[\sqrt{2};2\right] \to \mathbf{R}, g(t) = \frac{1+t^2}{t}$ is increasing and for every $y \in \left[g\left(\sqrt{2}\right);g(2)\right]$ the equation $\frac{1+t^2}{t} = y$ has a solution in $\left[\sqrt{2};2\right]$, whence $f([0,\pi]) = g\left(\left[\sqrt{2};2\right]\right) = \left[\frac{3\sqrt{2}}{2};\frac{5}{2}\right]$.

For $x \in [\pi,2\pi]$, $$f(x) = \frac{5-2\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}{\sqrt{2}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)} = \frac{5-u^2}{u},$$

where $u=\sqrt{2}\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)=2\sin\left(\frac{x}{2}-\frac{\pi}{4}\right)$ describes the interval $\left[\sqrt{2};2\right]$.

The function $h:\left[\sqrt{2};2\right]\to\mathbf{R}, h(u)=\dfrac{5-u^2}{u}$ is decreasing and for every $y\in\left[h\left(\sqrt{2}\right);h(2)\right]$ the equation $\dfrac{5-u^2}{u}=y$ has a solution in $\left[\sqrt{2};2\right]$, therefore $f([\pi,2\pi])=\left[h(2);h\left(\sqrt{2}\right)\right]=\left[\frac{1}{2};\frac{3\sqrt{2}}{2}\right]$. It follows that :

$$f([0,2\pi])=f([0,\pi])\cup f([\pi,2\pi])=\left[\frac{1}{2};\frac{5}{2}\right].$$

9.3. Let a, b, c, $d\in\mathbf{R}$ and $f:\mathbf{R}\to\mathbf{R}$, $f(x)=ax^3+bx^2+cx+d$, such that $f(2)+f(5)<7<f(3)+f(4)$. Prove that there exists u, $v\in\mathbf{R}$ such that $u+v=7$ and $f(u)+f(v)=7$.

Solution. The conclusion asks to prove that the equation $f(x)+f(7-x)=7$ has a real solution.

Let $g(x)=f(x)+f(7-x)-7$. It easy to see that $g\ (x)$ ia a polynomial function whose degree is at most 2. Since $g(2)<0$ and $g\ (3)>0$, it follows that the equation $g\ (x)=0$ has at least a real solution.

9.4. Let a, b, c, $d\in\mathbf{R}$ and the sets $A=\left\{x\in\mathbf{R};x^2+a|x|+b=0\right\}$ and $B=\left\{x\in\mathbf{R};[x]^2+c[x]+d=0\right\}$. Prove that if the set $A\cap B$ has exactly three elements then a cannot be an integer.

Solution. We see that $A=\left\{x\in\mathbf{R};|x|=p\text{ or }|x|=q\right\}$, where p, q are the real roots of the equation $x^2+ax+b=0$. If $A\cap B$ has three elements then A must have at least three elements, which leads to the following cases:

1) A has three elements. Then $q=0$, $p>0$, $b=0$, $a=-p$ and $A=\{-p;0;p\}\subset B$, therefore $d=0$ and the equation $y^2+cy=0$ has the solutions $-c$ and 0.

These two solutions must be $[p]\geq 0$ and $[-p]<0$, so $[p]=0$, $p\in(0;1)$ and $a=-p$ is not an integer.

2) A has four elements. Then $p > 0$, $q > 0$, $p \neq q$, $a = -(p + q)$ and $A = \{-p, p, -q, q\}$. Since B must contain one of the sets $\{p;-p\}$ and $\{q;-q\}$, it follows that the equation $y^2 + cy + d = 0$ must have as solutions the integers $m < 0$ and $n \geq 0$, whence $B = [m; m+1) \cup [n; n+1)$. Suppose $p \in B$ and $-p \in B$ (so $n = [p]$ and $m = [-p]$).

If p is an integer then $m = -n - p$ and $[q] = p$ or $[-q] = p$, therefore $q \neq p$ cannot be an integer and neither is $a = -(p+q)$.

If p is not an integer then $n = [p], m = -[p] - 1$ and $[q] = [p]$ or $[-q] = -[p] - 1$. If q is not an integer then the last two equalities are equivalent so $A \cap B$ has either two or four elements, and if q is an integer then, again, $a = -(p+q)$ is not an integer.

10.1. Let $f: \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be a function which fulfils the conditions:

$$f(0, y) = y + 1, \forall y \in \mathbf{N},$$

$$f(x+1, 0) = f(x, 1), \forall x \in \mathbf{N},$$

$$f(x+1, y+1) = f(x, f(x+1, y)), \forall (x, y) \in \mathbf{N} \times \mathbf{N}.$$

Compute $f(3, 1997)$.

Solution. Going step by step we get :

- $f(1,0) = f(0,1) = 2; f(1,1) = f(0, f(1,0)) = f(0,2) = 3;$
  $f(1,2) = f(0, f(1,1)) = f(0,3) = 4$ and, using mathematical induction,
  $$f(1, x) = x + 2 \text{ for every natural } x,$$
because $f(1, x+1) = f(0, f(1, x)) = 1 + f(1, x)$.

- $f(2,0) = f(1,1) = 3; f(2,1) = f(1, f(2,0)) = f(1,3) = 5;$
  $f(2,2) = f(1, f(2,1)) = f(1,5) = 7$ and, using mathematical induction,
  $$f(2, x) = 2x + 3 \text{ for every natural } x,$$
because $f(2, x+1) = f(1, f(2, x)) = 2 + f(2, x)$.

- For $x \geq 1$, $f(3,x) = f(2, f(3,x-1)) = 3 + 2f(3,x-1)$, so the sequence $(a_n)_{n\geq 0}$, $a_n = f(3,n)$ satisfies the recurrence relation $a_n = 3 + 2a_{n-1}$ for every $n\geq 1$. This leads to : $a_n = 3 + 2a_{n-1} = 3 + 2(3 + 2a_{n-2}) = 3(1+2) + 2^2(3 + 2a_{n-3}) =$

$$= 3(1+2+2^2) + 2^3(3+2a_{n-4}) = \ldots = 3(1+2+2^2+\ldots+2^{n-1}) + 2^n \cdot a_0 =$$

$$= 3(2^n - 1) + 2^n \cdot 5 = 2^{n+3} - 3,$$

so $f(3,1997) = a_{1997} = 2^{2000} - 3$.

10.2. Let $n\geq 3$ be an integer and x be a real number such that the numbers x, x^2 and x^n have the same fractional parts. Prove that x is an integer.

First solution. We know that $x^2 = x + k$ and $x^n = x + l$ for some integers k, l. It follows that :

$$x^3 = x^2 + kx = (k+1)x + k, \quad x^4 = (k+1)x^2 + kx = (2k+1)x + k^2 + k$$

and, using mathematical induction, $x^p = a_p x + b_p$ for every integer $p \geq 2$, where a_p, b_p are integers and $a_{p+1} = a_p + b_p, b_{p+1} = ka_p$.

Since x is real, and $x^2 - x - k = 0$, the discriminant $\Delta = 1 + 4k$ is non-negative, whence $k \geq 0$ (k is an integer).

If $k = 0$ then $x^2 = x$, so x is an integer.

If $k \geq 1$ then $a_p > 1$ for $p \geq 3$ and $x^n = a_n x + b_n = x + k$ implies that x is rational. In this case Δ must be an odd perfect square, so $x = \dfrac{1 \pm \sqrt{\Delta}}{2}$ is an integer.

Second solution. Let, as above, $x = \dfrac{1 \pm \sqrt{\Delta}}{2}$, where $\Delta = 1 + 4k$ and $k \geq 0$ is an integer. Then :

$$x^n = \frac{1}{2^n}\left[\left(1 + C_n^2\Delta + C_n^4\Delta^2 + C_n^6\Delta^3 + \ldots\right) \pm \sqrt{\Delta}\left(C_n^1 + C_n^3\Delta + C_n^5\Delta^2 + \ldots\right)\right] = a \pm b\sqrt{\Delta}$$

, where $a = \dfrac{1}{2^n}\left(1 + C_n^2\Delta + C_n^4\Delta^2 + \ldots\right)$ and $b = \dfrac{1}{2^n}\left(C_n^1 + C_n^3\Delta + C_n^5\Delta^2 + \ldots\right)$ are rational.

Since $x^n = x + l = \dfrac{2l+1}{2} \pm \dfrac{1}{2}\sqrt{\Delta}$ ($l \in \mathbb{Z}$) it follows that $b = \dfrac{1}{2}$ or $\sqrt{\Delta} \in \mathbb{Q}$.

If $b=\frac{1}{2}$ then $C_n^1+C_n^3\Delta+C_n^5\Delta^2+\ldots=2^{n-1}=C_n^1+C_n^3+C_n^5+\ldots$,
whence $\Delta=1$ and $x\in\{0;1\}$.

If $\sqrt{\Delta}$ is rational then Δ is odd perfect square and $x=\frac{1\pm\sqrt{\Delta}}{2}$ is an integer.

10.3. Let d_1, d_2 be two straight lines and A_0 be a point on d_1. For every $n\in\mathbb{N}$ let B_n be the projection of A_n on d_2 and A_{n+1} be the projection of B_n on d_1. Prove that there exists two segments $[A'A'']\subset d_1$ and $[B'B'']\subset d_2$ of lengths 0,001 and a natural number N such that $A_n\in[A'A'']$ and $B_n\in[B'B'']$ for every $n\geq N$.

Solution. If $d_1 \parallel d_2$ then $A_n = A_0$ and $B_n = B_0$ for every n. The conclusion is reached in this case taking A', $A''\in d_1$ such that $A'A''= 0{,}001$ and $A_0\in(A'A'')$, taking B', $B''\in d_2$ such that $B_0\in(B'B'')$ and $B'B''= 0{,}001$, and taking $N=0$.

If d_1 is not parallel to d_2, let AB ($A\in d_1$, $B\in d_2$) be the common perpendicular of d_1 and d_2 (if $d_1\cap d_2=\{0\}$ then $A=B=0$) and $\alpha\in\left(0,\frac{\pi}{2}\right]$ be the measure of the angle between d_1 and d_2. It follows that $[BB_n]=\text{pr}_{d_2}[AA_n]$ and $[AA_{n+1}]=\text{pr}_{d_1}[BB_n]$, so

$$AA_{n+1}=BB_n\cos\alpha=AA_n\cos^2\alpha$$

$$\text{and } BB_{n+1}=AA_{n+1}\cos\alpha=BB_n\cos^2\alpha\text{, for every } n,$$

whence $AA_n=AA_0\cos^{2n}\alpha$ and $BB_n=BB_0\cos^{2n}\alpha$ for every n.

The inequalities $AA_0\cos^{2n}\alpha<0{,}0005$ and $BB_0\cos^{2n}\alpha<0{,}0005$ are fulfiled if

$$n>M=\max\left\{\log_{\cos^2\alpha}\frac{0{,}0005}{AA_0};\log_{\cos^2\alpha}\frac{0{,}0005}{BB_0}\right\},\ \text{for } \alpha\neq\frac{\pi}{2}$$

and if $n\geq 1$, for $\alpha=\frac{\pi}{2}$.

Now the conclusion is obtained taking A', $A''\in d_1$ such that $AA'=AA''=0{,}005$, taking B', $B''\in d_2$ such that $BB'=BB''=0{,}005$ and taking $N=[M]+1$ for $\alpha\neq\frac{\pi}{2}$ or $N=1$ for $\alpha=\frac{\pi}{2}$.

10.4. Let $a_0,a_1,\dots,a_n$ be complex numbers such that

$$z\in\mathbb{C},\ |z|\le 1 \Rightarrow \left|a_n z^n+a_{n-1}z^{n-1}+\dots+a_1 z+a_0\right|\le 1 .$$

Prove that $|a_k|\le 1$ and $|a_0+a_1+\dots+a_n-(n+1)a_k|\le n$ for every $k\in\overline{0,n}$.

Solution. Let $p(z)=a_n z^n+a_{n-1}z^{n-1}+\dots+a_1 z+a_0$ and $\varepsilon_l, l\in\overline{0,n}$ be the roots of order $n+1$ of the unity. Since

$$\sum_{l=0}^{n}\varepsilon_l^k=\sum_{l=0}^{n}\left(\cos\frac{2l\pi}{n+1}+i\sin\frac{2l\pi}{n+1}\right)^k=\sum_{l=0}^{n}\left(\cos\frac{2k\pi}{n+1}+i\sin\frac{2k\pi}{n+1}\right)^l=$$

$$=\frac{1-\left(\cos\frac{2k\pi}{n+1}+i\sin\frac{2k\pi}{n+1}\right)^{n+1}}{1-\left(\cos\frac{2k\pi}{n+1}+i\sin\frac{2k\pi}{n+1}\right)}=0,$$

if k is not divisible by $n+1$ and $\sum_{l=0}^{n}\varepsilon_l^k=\underbrace{1+1+\dots+1}_{n+1\text{ times}}=n+1$ if k is divisible by $n+1$, it follows that $\sum_{j=0}^{n}\varepsilon_j^k p(\varepsilon_j)=(n+1)a_{n-k}$ for every $k\in\overline{0,n}$. This shows that $(n+1)|a_{n-k}|=\left|\sum_{j=0}^{n}\varepsilon_j^k p(\varepsilon_j)\right|\le\sum_{j=0}^{n}\left|\varepsilon_j^k p(\varepsilon_j)\right|=\sum_{j=0}^{n}|p(\varepsilon_j)|\le\underbrace{1+1+\dots+1}_{n+1\text{ times}}=n+1$,

so $|a_{n-k}|\le 1$ for every $k\in\overline{0,n}$.

For the second part notice that

$$\sum_{j=1}^{n}\varepsilon_j^k p(\varepsilon_j)=\sum_{j=0}^{n}\varepsilon_j^k p(\varepsilon_j)-p(1)=(n+1)a_{n-k}-\sum_{j=0}^{n}a_j$$

and $\left|\sum_{j=1}^{n}\varepsilon_j^k p(\varepsilon_j)\right|\le\sum_{j=1}^{n}\left|\varepsilon_j^k p(\varepsilon_j)\right|\le n$, therefore $\left|(n+1)a_{n-k}-\sum_{j=0}^{n}a_n\right|\le n$ for every $k\in\overline{0,n}$.

11.1. Let $p \geq 2$ be a natural number and $A=(a_{ij})$ be a square matrix of order n, with integer elements. Prove that for every permutation $\sigma \in S_n$ there exists a function $\varepsilon:\{1,2,\ldots,n\} \to \{0,1\}$ such that by replacing the elements $a_{\sigma(1)1}, a_{\sigma(2)2}, \ldots, a_{\sigma(n)n}$ from the matrix A with

$$a_{\sigma(1)1}+\varepsilon(1), a_{\sigma(2)2}+\varepsilon(2), \ldots, a_{\sigma(n)n}\varepsilon(n)$$

respectively, the determinant of the new matrix is not divisible by p.

Solution. We will use induction over n. For $n=1$ it comes to the fact that at least one of the integers a and $a+1$ is not divisible by p, which is obvious.

Suppose now that $\sigma \in S_{n+1}$ and $\varepsilon(1), \varepsilon(2), \varepsilon(3), \ldots, \varepsilon(n+1)$ have been chosen such that the cofactor $\delta_{\sigma(1)1}$ of $a_{\sigma(1)1}$ is not divisible by p.

For the matrix A' thus obtained we have :

$$\det A' = a_{\sigma(1)1}\delta_{\sigma(1)1} + \sum_{\substack{i=1 \\ i\neq j}}^{n+1} a_{i1}\delta_{i1}.$$

Since $(a_{\sigma(1)1}+1)\delta_{11} = a_{\sigma(1)1}\delta_{11}+\delta_{11}$ is not congruent (mod p) with $a_{\sigma(1)1}\delta_{11}$ it follows that at least one of the numbers $\det A'$ and $\delta_{11}+\det A'$ is not 0(mod p) so $\varepsilon(1)$ can be chosen accordingly.

11.2. Let A be a square matrix of odd order (at least 3), with integer odd elements. Prove that if A is invertible then it is not possible that all the minors of the elements of a row have the same modulus.

Solution. Let $A=(a_{ij})_{1\leq i,j\leq 2n+1}$ and δ_{ij} be the cofactor of a_{ij} in A.

If, for some i, every $|\delta_{ij}|$ is d then $d \neq 0$. Taking some $k \neq i$ we get $\sum_{j=1}^{2n+1} a_{kj}\delta_{ij}=0$ and, after dividing by d, $\pm a_{k1} \pm a_{k2} \pm \ldots \pm a_{k,2n+1}=0$, which is impossible because all a_{kj} are odd.

11.3. Let $\Im$ be the set of all the differentiable functions $f:\mathbb{R} \to \mathbb{R}$, which have the property $f(x) \geq f(x+\sin x), \forall x \in \mathbb{R}$.

a) Prove that $\Im$ contains also nonconstant functions.

b) Prove that if $f \in \Im$ then the set of the solutions of the equation $f'(x)=0$ in infinite.

Solution a) We notice that the function $u:\mathbf{R}\to\mathbf{R}, u(x)=x+\sin x$ is increasing, $2k\pi \le x \le x+\sin x \le \pi+2k\pi$ for $x\in[2k\pi;\pi+2k\pi]$ and $2\pi+2k\pi \ge x \ge x+\sin x \ge \pi+2k\pi$ for $x\in[\pi+2k\pi;\pi 2+2k\pi]$. This shows that any nonconstant differentiable function which is decreasing on the intervals $[2k\pi;\pi+2k\pi]$ and increasing on the intervals $[\pi+2k\pi;2\pi+2k\pi]$ would do (for instance "cos").

b) Let $g:\mathbf{R}\to\mathbf{R}, g(x)=f(x)-f(x+\sin x)$. Obviously $g(k\pi)=0$ and $g \ge 0$, so $k\pi$ are minimum points for g. It follows that $g'(k\pi)=0$, whence $f'(k\pi)-f'(k\pi)\cdot(1+\cos k\pi)=0$, therefore $f'(k\pi)=0$ for every integer k.

11.4. Let $f,g:\mathbf{R}\to\mathbf{R}$ be two bijective continuous functions such that $f(g^{-1}(x))+g(f^{-1}(x))=2x, \forall x\in\mathbf{R}$. Prove that if there exists $x_0\in\mathbf{R}$ such that $f(x_0)=g(x_0)$, then $f=g$.

Solution. Let $h=f\circ g^{-1}$. Then h is bijective, continuous and $h(x)+h^{-1}(x)=2x, \forall x\in\mathbf{R}$. Since a bijective continuous function is strictly monotonic, its inverse has the same type of monotony and the function $x\mapsto 2x$ is increasing, it follows that h is strictly increasing.

Suppose now that there exists $a\in\mathbf{R}$ such that $h(a)\ne a$ and denote by r the difference $a-h(a)$. From $h(a)+h^{-1}(a)=2a$ we get $h^{-1}(a)=a+r$, whence $h(a+r)=a$. In the same way $h(a+r)+h^{-1}(a+r)=2(a+r)$ implies $h(a+2r)=a+r$ and, using mathematical induction :

$$h(a+nr)=a+(n-1)r \text{ for every } n\in\mathbf{N}.$$

Also $h(a-r)+h^{-1}(a-r)=2(a-r)$ and $h^{-1}(a-r)=a$ (because $h(a)=a-r$) shows that $h(a-r)=a-2r$; using again mathematical induction one obtains $h(a-nr)=a-(n+1)r$ for every $n\in\mathbf{N}$.

Thus $h(a+nr)=a+(n-1)r$ for every integer n.

Take now the integer n such that $a+nr<x_0<a+(n+1)r$ (the case $r<0$ can be treated in the same way).It follows that $h(x_0)<h[a+(n+1)r]=$ $=a+nr<x_0$, in contradiction with $h(x_0)=x_0$. This shows that $h(a)=a$ for every $a\in\mathbf{R}$.

12.1. Let $\alpha\in\mathbf{C}\setminus\mathbf{Q}$ be such that the set $A=\{a+b\alpha|a,b\in\mathbf{Z}\}$ is a ring with respect to the usual operations in $\mathbf{C}$. Prove that if A has exactly 4 invertible elements then $A=\mathbf{Z}[i]$.

Solution. It is a well known result that if a multiplicative group $G\subset\mathbf{C}$ has n elements then $G=\{z\in\mathbf{C}:z^n=1\}$, so the group $U(A)$ is $\{1,-1,i,-i\}$. Therefore $i=u+v\alpha$ for some integers u, v whence $\alpha=\frac{i-u}{v}$, $v\neq 0$. From $\alpha^2\in A$ it follows that $\alpha^2=a+b\alpha$ for some integers a, b, whence $\frac{u^2-1}{v^2}-\frac{2u}{v^2}i=a-\frac{bu}{v}+\frac{b}{v}i$. This shows that $b=-\frac{2u}{v}$ and

$$a=\frac{u^2-1}{v^2}+\frac{bu}{v}=\frac{1}{4}\left(-b^2-\frac{4}{v^2}\right),\text{ so } v^2\in\{1;4\}.$$

If $v^2=4$ then $4a=-(b^2+1)$, which is impossible, therefore $v^2=1$ and $\{m+n\alpha;m,n\in\mathbf{Z}\}=\{m\pm n(i-u);m,n\in\mathbf{Z}\}\subset\mathbf{Z}[i]$.

Conversely, every complex number $p+qi(p,q\in\mathbf{Z})$ can be written in the form $p+qu+q(i-u)=m\pm n\alpha$, so $\mathbf{Z}[i]\subset A$.

12.2. Prove that for every continuous function $f:[-1,1]\to\mathbf{R}$ takes place the inequality $\int_{-1}^{1}f^2(x)\mathrm{d}x\geq\frac{1}{2}\left(\int_{-1}^{1}f(x)\mathrm{d}x\right)^2+\frac{3}{2}\left(\int_{-1}^{1}xf(x)\mathrm{d}x\right)^2$.

When takes place the equality ?

Solution. If f is even the inequality becomes

$$\int_0^1 f^2(x)\mathrm{d}x\geq\left(\int_0^1 f(x)\mathrm{d}x\right)^2$$

and if f is odd it becomes $\int_0^1 f^2(x)\mathrm{d}x\geq 3\left(\int_{-1}^{1}xf(x)\mathrm{d}x\right)^2$.

These inequalities can be obtained from the *Cauchy-Buniakowski* inequality: $\left(\int_0^1 f(x)g(x)\mathrm{d}x\right)^2 \le \int_0^1 f^2(x)\mathrm{d}x \cdot \int_0^1 g^2(x)\mathrm{d}x$ taking $g(x)=1$ and $g(x)=x$.

If f is a continuous function let $f_1(x)=\dfrac{f(x)+f(-x)}{2}$ and $f_2(x)=\dfrac{f(x)-f(-x)}{2}$. The given inequality becomes :

$$\int_{-1}^{1}\left(f_1^2(x)+f_2^2(x)+2f_1(x)f_2(x)\right)\mathrm{d}x \ge$$

$$\ge \frac{1}{2}\left(\int_{-1}^{1}\left(f_1(x)+f_2(x)\right)\mathrm{d}x\right)^2+\frac{3}{2}\left(\int_{-1}^{1}\left(xf_1(x)+xf_2(x)\right)\mathrm{d}x\right)^2$$

or $\int_0^1 f_1^2(x)\mathrm{d}x+\int_0^1 f_2^2(x)\mathrm{d}x \ge \left(\int_0^1 f_1(x)\mathrm{d}x\right)^2+3\left(\int_0^1 xf_2(x)\mathrm{d}x\right)^2$, which is true because f_1 is even and f_2 is odd.

The equality takes place if and only if the *Cauchy-Buniakowski* relation becomes an equality, that is if and only if $f=\lambda g$ for some real λ, which means $f_1(x)=a$ and $f_2(x)=bx$ for some reals a and b. Therefore the equality takes place if and only if f is a linear function.

12.3. Let K be a finite field, $n \in \mathrm{N}$, $n \ge 2$, $f \in K[X]$ an irreducible polynomial of degree n and g the product of all the nonconstant polynomials of $K[X]$ which have the degrees less than n. Prove that f divides $g-1$.

Solution. Let $L=\{h \in K[X] \mid \deg h \le n-1\}$ and denote by $h_1 \otimes h_2$ the remainder of the division of $h_1 h_2$ by f. It is easy to check that $(L,+,\otimes)$ is a commutative ring. Since f is irreductible then $(f,h)=1$ for every $h \in L, h \ne 0$, so there exists $u,v \in K[X]$ such that $uf+vh=1$, whence $h \otimes v_1=1$, where v_1 is the remainder of $v(\bmod f)$. Thus $(L,+,\otimes)$ is a field.

Let p and q be the cardinals of K^* and L^* respectively. From the fact that the product of the non-nil elements of a finite field is -1 it follows that $h_1 \otimes h_2 \otimes \ldots \otimes h_p \otimes h_{p+1} \otimes \ldots \otimes h_q = -1$, where $h_1, \ldots, h_p$ are the constant non-zero polinomials of $K[X]$. This shows that $f \mid h_1 h_2 \ldots h_p g + 1 = -g + 1$.

12.4. Let the sequence $(f_n)_{n \in \mathbb{N}}$ of functions $f_n : [0;1] \to \mathbf{R}$ be such that f_0 is continuous and $f_{n+1}(x) = \int_0^x \frac{1}{1 + f_n(t)} \mathrm{d}t, \forall x \in [0;1], \forall x \in \mathbb{N}$. Prove that for every $x \in [0;1]$ the sequence $(f_n(x))_{n \in \mathbb{N}}$ is convergent and compute its limit.

Solution. Let us firstly find the probable limit. This should be a continuous function $f:[0;1] \to [0;\infty)$ such that $f(x) = \int_0^x \frac{1}{1 + f(t)} \mathrm{d}t$, whence f is differentiable, $f(0) = 0$ and $f'(x)(1 + f(x)) = 1$.

Therefore $f(x) + \frac{f^2(x)}{2} = x$ so $f(x) = \sqrt{1 + 2x} - 1$.

Let us now prove that $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in [0;1]$.

If $x \in [0;1)$ then :

$$|f_n(x) - f(x)| = \left| \int_0^x \left(\frac{1}{1 + f_{n-1}(t)} - \frac{1}{1 + f(t)} \right) \mathrm{d}t \right| \le \int_0^x \frac{|f_{n-1}(t) - f(t)|}{(1 + f_{n-1}(t))(1 + f(t))} \mathrm{d}t \le$$

$$\le \int_0^x |f_{n-1}(t) - f(t)| \mathrm{d}t = x |f_{n-1}(t_1) - f(t_1)|,$$

for some $t_1 \in [0; x]$. In the same way $|f_{n-1}(t_1) - f(t_1)| \le t_1 |f_{n-2}(t_2) - f(t_2)|$ for some $t_2 \in [0; t_1]$ and, using mathematical induction,

$$|f_n(x) - f(x)| \le x t_1 t_2 \ldots t_{n-1} |f_0(t_n) - f(t_n)|,$$

where $0 \le t_n \le t_{n-1} \le \ldots \le t_1 \le x$.

This shows that $|f_n(x) - f(x)| \le x^n \sup_{t \in [0;1]} |f_0(t) - f(t)|$, whence $\lim_{n \to \infty} f_n(n) = f(x)$ in this case.

If $x = 1$, let $\varepsilon > 0$ and $a \in \left(0; \frac{\varepsilon}{4}\right)$.

Since $\left|f_n(1)-f(1)\right| \le \int_0^1 \left|f_{n-1}(t)-f(t)\right| \mathrm{d}t =$

$$= \int_0^{1-a} \left|f_{n-1}(t)-f(t)\right| \mathrm{d}t + \int_{1-a}^1 \left|f_{n-1}(t)-f(t)\right| \mathrm{d}t \le \int_0^{1-a} \left|f_{n-1}(t)-f(t)\right| \mathrm{d}t + 2a,$$

(because $\left|f_{n-1}(t)\right| \le 1$ and $\left|f(t)\right| \le 1$) and $\lim_{n\to\infty} \int_0^{1-a} \left|f_{n-1}(t)-f(t)\right| \mathrm{d}t = 0$ it follows that there exists $N(\varepsilon)\in \mathbb{N}$ such that $\left|f_{n-1}(1)-f(1)\right| < \varepsilon$ for every $n \ge N(\varepsilon)$, which ends the proof.

SECTION 2

SELECTION EXAMINATIONS FOR THE 38^{TH} I.M.O.

PROPOSED PROBLEMS

The first round, Suceava, 1997, March 28^{th}

Problem 1. We are given in the plane a line Δ and three circles having their centres in the points A, B, C, all tangent to Δ and pairwise externally tangent to one another. Prove that the triangle ABC has an obtuse angle and find all possible values of this angle.

Mircea Becheanu

Problem 2. Find the number of sets containing 9 positive integers

$$A = \{a_1, a_2, \ldots, a_9\}$$

with the following property : for any positive integer n, $n \le 500$, there exist a subset B, $B \subset A$, such that

$$\sum_{b \in B} b = n.$$

Bogdan Enescu and *Dan Ismailescu*

Problem 3. Let n, $n \ge 4$, be an integer number and M be a n-set of points in the plane, every three points being non collinear and not all of them being on the same circle. Find all real functions $f : M \to \mathbf{R}$, such that for any circle $\mathcal{C}$ containing at least three M-points, the following equality holds :

$$\sum_{P \in \mathcal{C} \cap M} f(P) = 0.$$

Dorel Miheţ

Problem 4. Let ABC be a triangle, D be a point on the side BC and $\mathcal{O}$ be the circumcircle of the triangle ABC. Let $\mathcal{B}$, $\mathcal{C}$ be the circles tangent to $\mathcal{O}$, AD, BD and $\mathcal{O}$, AD, DC respectively. Show that $\mathcal{B}$ and $\mathcal{C}$ are tangent if and only if

$$\angle BAD \equiv \angle CAD.$$

Dan Brânzei

The second round, Bucharest, 1997, April 19th

Problem 5. Let $VA_1A_2...A_n$ be a pyramid, where $n \geq 4$. A plane Π intersects the edges VA_1, VA_2, ...; VA_n in the points B_1, B_2, ..., B_n respectively such that the polygons $A_1A_2... A_n$ and $B_1B_2... B_n$ are similar. Show that the plane Π is parallel to the plane contaning the base $A_1A_2... A_n$.

Laurenţiu Panaitopol

Problem 6. Let A be the set of positive integers represented by the form a^2+2b^2, where a, b are integer numbers and $b \neq 0$. Show that if p is a prime number and $p^2 \in A$, then $p \in A$.

Marcel Ţena

Problem 7. Let p be a prime number, $p \geq 5$, and k be a digit in the p-adic representation of positive integers. Find the maximal length of a non constant arithmetic progression whose terms do not contain the digit k in their p-adic representation.

Marian Andronache and *Ion Savu*

Problem 8. Let p, q, r be pairwise distinct prime numbers and A be the set of positive integers,

$$A=\{p^aq^br^c \mid 0 \leq a,b,c \leq 5\}.$$

Find the least number n such that any n-set B, $B \subset A$, contains distinct elements x and y such that x is a divizor of y.

Ioan Tomescu

The third round, Bucharest, 1997, May 16th

Problem 9. Let $ABCDEF$ be a convex hexagon. The lines AB and EF, EF and CD, CD and AB intersect in the points P, Q, R respectively. The lines BC and DE, DE and FA, FA and BC intersect in the point S, T, U respectively. Show that if $\frac{AB}{PR}=\frac{CD}{RQ}=\frac{EF}{QP}$ then $\frac{BC}{US}=\frac{DE}{ST}=\frac{FA}{TU}$.

Remus Nicoară

Problem 10. Let P be the set of the points of the euclidean plane Π and D be the set of the lines of the same plane. Find, with proof, whether there exists a bijective function $f:P \to D$ such that for arbitrary three collinear points A, B, C the lines $f(A), f(B), f(C)$ be either parallel or concurrent.

Gefry Barad

Problem 11. Find the functions $f:\mathbb{R}\to[0,\infty)$ such that

$$f(x^2+y^2)=f(x^2-y^2)+f(2xy),$$

for all real numbers x and y.

Laurenţiu Panaitopol

Problem 12. Let n be an integer number, $n\geq 2$, and

$$P(X)=X^n+a_{n-1}X^{n-1}+\ldots+a_1X+1$$

be a polynomial with positive integer coefficients. Suppose that $a_k=a_{n-k}$ for all k, $k=1, 2, \ldots, n-1$. Prove that there exist infinitely many pairs of positive integers (x, y) such that $x|P(y)$ and $y|P(x)$.

Remus Nicoară

The fourth round, Bucharest, 1997, May 17[th]

Problem 13. Let $P(X)$, $Q(X)$ be monic irreducible polynomials in the ring $Q[X]$. Suppose that $P(X)$ and $Q(X)$ have roots α and β, respectively and that $\alpha+\beta$ is a rational number. Prove that the polynomial $P(X)^2-Q(X)^2$ has a rational root.

Bogdan Enescu

Problem 14. Let a be an integer number, $a>1$. Show that the set of positive integers

$$\{a^2+a-1, a^3+a^2-1,\ldots,a^{n+1}+a^n-1,\ldots\}$$

contains an infinite subset of pairwisely coprime numbers.

Mircea Becheanu

Problems 15. The vertices of a regular dodecagon are colored either blue or red. Find the number of all possible colorings which do not contain monochromatic subpolygons.

Vasile Pop

Problem 16. Let Γ be a circle and AB be a line which do not intersect Γ. For any point P, $P\in\Gamma$, let P' be the second intersection point of the line AP with Γ and $f(P)$ be the second intersection point of the line BP' with Γ. In this way one defines the point sequence $P=P_0,P_1,\ldots,P_n,\ldots$ where $P_{n+1}=f(P_n)$. Show that if k is a positive integer such that $P_0=P_k$, then for any point $Q=Q_0$, the property $Q_0=Q_k$ also holds.

Gheorghe Eckstein

SOLUTIONS

Problem 1. We are given in the plane a line Δ and three circles having their centres in the points A, B, C, all tangent to Δ and pairwise externally tangent to one another. Prove that the triangle ABC has an obtuse angle and find all possible values of this angle.

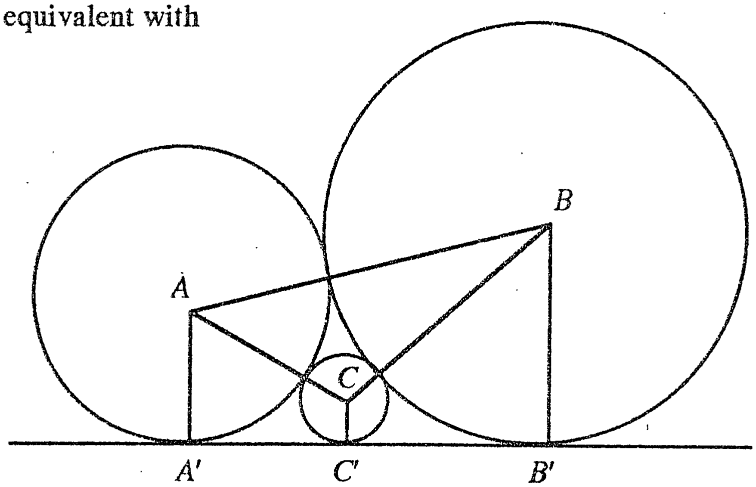
Solution. Let us note the rays of the three circles by a, b, c respectivelly and let A', B', C' be the projections of the centres A, B, C on the line Δ. Suppose $c \le a$, $c \le b$. Then :

$$A'B' = \sqrt{(a+b)^2 - (b-a)^2} = 2\sqrt{ab}\,,\ B'C' = 2\sqrt{bc} \text{ and } A'C' = 2\sqrt{ac}.$$

From the equality $A'B' = A'C' + C'B'$, follows :

$$(1) \qquad \sqrt{ab} = \sqrt{ac} + \sqrt{bc}\,,$$

which is equivalent with



$$(2) \qquad c = \frac{ab}{\left(\sqrt{a} + \sqrt{b}\right)^2}.$$

From the cosine theorem in the triangle ABC we deduce :

$$(3) \qquad \cos C = \frac{c(a+b+c) - ab}{(a+c)(b+c)}$$

It is easy to see that $\angle C$ is obtuse because we have the equivalences :

$$\cos C < 0 \Leftrightarrow c(a+b+c) < ab \Leftrightarrow c\left[\left(\sqrt{a}+\sqrt{b}\right)^2 - 2\sqrt{ab}+c\right] < c\left(\sqrt{a}+\sqrt{b}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow c < 2\sqrt{ab} \Leftrightarrow c < 2\left(\sqrt{ac}+\sqrt{bc}\right) \Leftrightarrow \sqrt{c} < 2\left(\sqrt{a}+\sqrt{b}\right).$$

The measure of $\angle C$ is given by (3), under the form :

$$\cos C = 1 - \frac{2ab}{(a+c)(b+c)}.$$

Equivalently, we obtain :

$$(4) \qquad \sin^2\frac{C}{2} = \frac{ab}{(a+c)(b+c)}.$$

Because $\frac{\pi}{2} < C < \pi$, follows $\frac{\pi}{4} < \frac{C}{2} < \frac{\pi}{2}$. Hence, it is sufficient to find the maximum of the function $\sin^2\frac{C}{2}$, given by the above formula (4).

This formula may be written under the form :

$$\sin^2\frac{C}{2} = \frac{1}{\frac{a+c}{a}\cdot\frac{b+c}{b}} = \frac{1}{\left(1+\frac{c}{a}\right)\left(1+\frac{c}{b}\right)},$$

and the actual problem is to find the maximum of the product

$$P = \left(1+\frac{c}{a}\right)\left(1+\frac{c}{b}\right).$$

Denote $\frac{\sqrt{c}}{\sqrt{a}} = x$, $\frac{\sqrt{c}}{\sqrt{b}} = y$. Then $P = \left(1+x^2\right)\left(1+y^2\right)$, with the supplementary conditions $x+y=1$, and $x, y \geq 0$.

Using Calculus the problem is rather simple. We shall solve it by elementary methods. Note $uv = p$. Then :

$$P = 1+x^2+y^2+x^2y^2 = 2-2xy+x^2y^2 = p^2-2p+2,$$

where $p = xy \leq \left(\frac{x+y}{2}\right)^2 \leq \frac{1}{4}$. The trinomial function $P = p^2-2p+2$ is a decreasing function on the interval $\left(0,\frac{1}{4}\right]$. Hence it takes minimal value for

$p=\dfrac{1}{4}$ and then $P=\dfrac{25}{16}$. The conclusion is : for $x = y$, or $a = b$ we obtain $\hat{C}_{max} = 2\arcsin\dfrac{16}{25}$.

Therefore, the possible values for $\angle C$ are $\angle C \in \left(\dfrac{\pi}{2}, 2\arcsin\dfrac{16}{25}\right]$.

Problem 2. Find the number of sets containing 9 positive integers

$$A = \{a_1, a_2, \ldots, a_9\}$$

with the following property : for any positive integer n, $n \le 500$, there exist a subset B, $B \subset A$, such that

$$\sum_{b\in B} b = n.$$

Solution. First, we observe that such a set A exists : using the binary representation of numbers and the set

$$A = \{1, 2, 2^2, \ldots, 2^8\},$$

it is possible to represent by sums of the elements of subsets $B \subset A$ all the numbers n, $n \le 2^9 - 1 = 511$.

On the other hand, a set A with 9 elements has $2^9 - 1 = 511$ distinct nonempty sets.

Hence, it is useful to construct a set A as the problem asks in such a way that "most of the distinct" subsets B, $B \subset A$, have distinct sums, $\sum_{b\in B} b$.

For example, if A contains the elements x, y, z such that $x = y + z$, for any subsets $B \cup \{x\}$ and $B \cup \{y, z\}$ give rise to the same sum. Hence, the number of sums obtained from the subsets of such a set A is at most :

$$511 - 2^6 = 447.$$

Because we have to obtain 500 different sums, such a set A can't be taken into consideration .

In the same way it can be proven that an element of A can't be the sum of three or four elements of A.

To construct a set A as the problem claims it is obviously necessary to suppose that $1 \in A$ and $2 \in A$. From $3 = 1 + 2$, it follows $3 \notin A$, and therefore $4 \in A$. Using the numbers 1, 2, 4 we can obtain by summing the numbers which do not exceed $2^3 - 1 = 7$. Hence $8 \in A$. Using the numbers

1, 2, 4, 8 we can obtain by summing all the numbers 1, 2, 3, 4, ..., 15, and therefore $16 \in A$. Using 1,2,4,8,16 we can obtainall the numbers 1,2,3,...,31. But, in this case, $31 = 1 + 2 + 4 + 8 + 16$. Because 31 it is the sum of 5 elements from A, it is possible to have $31 \in A$. This means that the sixth element of A can be some number of the form $32 - a$, where $a \geq 0$.

The set $\{1, 2, 4, 8, 16, 32 - a\}$ gives rise by summing to the elements of distinct subsets to the following numbers :

$$1, 2, 3, 4, \ldots, 31, 32 - a, 32 - a + 1, \ldots, 63 - a$$

Therefore, the next element of A is a number of the form $64 - a - b$, where $b \geq 0$. The set $\{1, 2, 4, 8, 16, 32 - a, 64 - a - b\}$ gives rise to the numbers $1, 2, 3, \ldots, 63 - a, 64 - a - b, 64 - a - b + 1, \ldots, 127 - 2a - b$. Hence the next element of A has to be $128 - 2a - b - c$, $c \geq 0$.

The last step of this procedure add to the set A the element

$$256 - 4a - 2b - c - d.$$

In this way, we have obtained that the nine elements of A are :

$$A = \{1, 2, 4, 8, 16, 32 - a, 64 - a - b, 128 - 2a - b - c, 256 - 4a - 2b - c - d\}$$

where a, b, c, d are positive integers and the sums obtained from A are

$$1, 2, \ldots, 511 - 8a - 4b - 2c - d.$$

From the condition $511 - 8a - 4b - 2c - d \geq 500$, it follows

$$8a + 4b + 2c + d \leq 11.$$

Thus, the number of the sets is the number of systems (a, b, c, d) for which $8a + 4b + 2c + d \leq 11$. It is easy to count them, using the cases $a = 0$ and $a = 1$. When $a = 0$, it is possible to have $b = 2$ (6 solutions), $b = 1$ (20 solutions) or $b = 0$ (42 solutions). When $a = 1$, $b = 0$ and $2c + d \leq 3$ gives rise to 6 solutions. We obtain in all 74 solutions.

Problem 3. Let n, $n \geq 4$, be an integer number and M be a n-set of points in the plane, every three points being non collinear and not all of them being on the same circle. Find all real functions $f : M \to \mathbf{R}$, such that for any circle $\mathcal{C}$ containing at least three M-points, the following equality holds :

$$\sum_{P \in \mathcal{C} \cap M} f(P) = 0.$$

Solution. Let A, B be distinct points in the set M and let $\mathcal{C}_{A,B}$ be the set of circles determined by the points A, B and other points of M. By the hypothesis the points of M are not on the same circle and therefore $\mathcal{C}_{A,B}$ has at least two elements.

Let k be the cardinality of $\mathcal{C}_{A,B}$, $k \geq 2$. For any circle $\mathcal{C}$, $\mathcal{C} \in \mathcal{C}_{A,B}$, one has $f(\mathcal{C}) = \sum_{P \in \mathcal{C}} f(P) = 0$.

It is obvious that the sum $\sum_{\mathcal{C} \in \mathcal{C}_{A,B}} f(\mathcal{C}) = 0$ is given by :

$$0 = \sum_{\mathcal{C} \in \mathcal{C}_{A,B}} f(\mathcal{C}) = \sum_{P \in M} f(P) + (k-1)\big(f(A) + f(B)\big).$$

Then the sum $\sum_{P \in M} f(P)$ and $f(A) + f(B)$ have different signs, for any points A, B. Suppose that $\sum_{P \in M} f(P) \geq 0$. Let $M = \{A_1, A_2, \ldots, A_n\}$ be a description of the set M. Because

$$f(A_1) + f(A_2) \leq 0, f(A_2) + f(A_3) \leq 0, \ldots, f(A_n) + f(A_1) \leq 0$$

it follows $\sum_{i=1}^{n} f(A_i) \leq 0$ and therefore $\sum_{P \in M} f(P) = 0$.

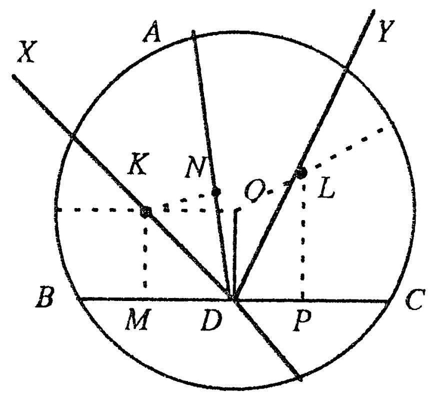
We shall prove now that f is the zero function. Suppose that there exists a point A, $A \in M$, such that $f(A) > 0$. Then, for two different points B, C one has $f(A) + f(B) = f(A) + f(C) = 0$. This gives $f(B) < 0, f(C) < 0$ and hence $f(B) + f(C) < 0$. This is a contradiction.

Problem 4. Let ABC be a triangle, D be a point on the side BC and $\mathcal{O}$ be the circumcircle of the triangle ABC. Let $\mathcal{B}$, $\mathcal{C}$ be the circles tangent to $\mathcal{O}$, AD, BD and $\mathcal{O}$, AD, DC respectively. Show that $\mathcal{B}$ and $\mathcal{C}$ are tangent if and only if

$$\angle BAD \equiv \angle CAD.$$

Solution. Let DX, DY be the bisector lines of the angles $\angle ADB$, $\angle ADC$ respectively. The idea of the solution is the following : we are looking for the centers K, L of the circles $\mathcal{B}$, $\mathcal{C}$ respectively such that $\mathcal{B}$ and $\mathcal{C}$ be tangent to the circle $\mathcal{O}$.

Then let M, N be the projections of the point on the segments BD, AD respectively and let $k = KM = KN$ be the radius of the circle $\mathcal{B}$.



Then $\mathcal{B}$ and $\mathcal{O}$ are tangent if and only i

(1) $\quad OK = R - k$.

Let us denote $\angle ADB = 2\alpha$ and $\angle KDO = \beta$. Considering the length $x = DN = DM$ as a variable length of the problem we get $DK = \dfrac{x}{\cos\alpha}$ and $k = x\,\mathrm{tg}\,\alpha$.

Then the equality (1) becomes

(1') $\quad OK = R - x\,\mathrm{tg}\,\alpha$.

Aplying the cosine theorem in the triangle ΔODK we obtain :

$$OK^2 = DK^2 + DO^2 - 2DK \cdot DO\cos\beta \Leftrightarrow$$

$$\Leftrightarrow (R - x\,\mathrm{tg}\,\alpha)^2 = \left(\frac{x}{\cos\alpha}\right)^2 + DO^2 - 2\frac{x}{\cos\alpha} \cdot DO \cdot \cos\beta.$$

The last equality can be written under the form :

(2) $\quad x^2 + 2\lambda x + DO^2 - R^2 = 0$.

where $\quad \lambda = \dfrac{R\sin\alpha - DO\cos\beta}{\cos\alpha}$, (3).

Using a similar technique for the point L, we obtain the equality :

(2') $\quad y^2 + 2\mu y + DO^2 - R^2 = 0$,

where

(3') $\quad \mu = \dfrac{R\cos\alpha - DO\sin\beta}{\sin\alpha}$

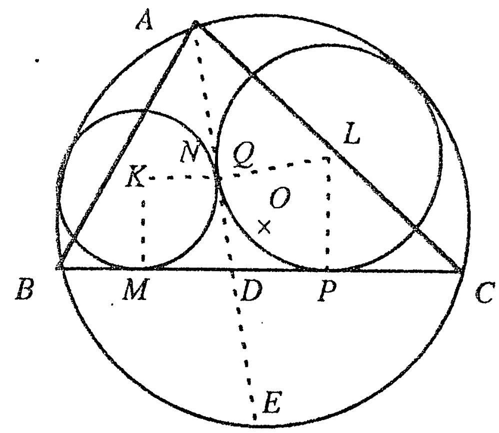
and $DP = y$. We observe that in this case the angles α, β must be replaced by $90° - \alpha$ and $90° - \beta$, respectively.

Now, the condition for the circles $\mathcal{B}$, $\mathcal{C}$ to be tangent is : $x = y$. It means that the equations (2) and (2') have a common root. But this condition is equivalent with $\lambda = \mu$. In this way, the following equality is obtained : $\quad R\cos 2\alpha = DO\sin(\beta - \alpha)$. Let observe that $\beta - \alpha = \angle ADO$. Hence, by the sine theorem in the triangle ΔADO we obtain :

$$R\cos 2\alpha = R\sin\angle DAO.$$

Let H denote the foot of the altitude from A in the triangle ΔABC. Then $\cos 2\alpha = \sin\angle HAD$. It follows that $\angle HAD = \angle DAO$. Taking into account that the lines AH and AO are isogonal conjugates in the vertex A it follows that AD is the bisector line of the angle $\angle BAC$.

Second solution. Let E be the intersection point of the line AD and the circle $\mathcal{O}$. Like in the previous solution, let M, N be the tangent points of the circle $\mathcal{B}$ with BD, AD, respectively and let P, Q be the poins of the circle $\mathcal{C}$ with DC and AD. We apply the *Cassey's* theorem for the circles B, C, E and $\mathcal{B}$, where the points B, C, E are considered as degenerate circles. In this way we obtain :



$$BE \cdot CM + CE \cdot BM = BC \cdot EN .$$

By applying once again *Cassey's* theorem for the circles B, C, E and $\mathcal{C}$, we obtain

$$BE \cdot CP + CE \cdot BP = BC \cdot EQ .$$

The circles $\mathcal{B}$ and $\mathcal{C}$ are tangent iff N and Q coincide. By the above relations, this conditions is equivalent with :

$$BE(CM - CN) = CE(BN - BM) \Leftrightarrow (BE - CE)MN = 0 .$$

It follows $BE = CE$, and this condition is equivalent with $\angle BAE = \angle CAE$.

Remark : After using the problem in the examination, the jury found out the similarity of this problem with the problem IND-4, submitted to the jury of the 33^{th} IMO in Moskow.

Let us recall this problem, but in the present notations. *The circles $\mathcal{O}$, $\mathcal{B}$, $\mathcal{C}$ are related to each other as follows : the circles $\mathcal{B}$ and $\mathcal{C}$ are externally tangent to one another at a point N and both these circles are internally tangent to the circle $\mathcal{O}$. Points A, B, C are located on the circle $\mathcal{O}$ as follows: BC is a direct common tangent to the pair of circles $\mathcal{B}$ and $\mathcal{C}$ and line NA is the transverse common tangent at N to $\mathcal{B}$ and $\mathcal{C}$, with N and A lying on the same side of the line BC. Prove that N is the incenter of the triangle ABC.*

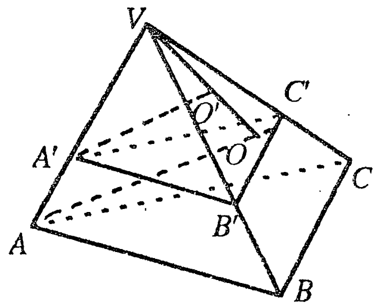
Now, using the solution of our problem, it is not difficult to solve the above problem.

Problem 5. Let $VA_1A_2\ldots A_n$ be a pyramid, where $n \geq 4$. A plane Γ intersects the edges VA_1, VA_2, ..., VA_n in the points B_1, B_2, ..., B_n respectively such that the polygons $A_1A_2\ldots A_n$ and $B_1B_2\ldots B_n$ are similar Show that the plane Π is parallel to the plane contaning the base $A_1A_2\ldots A_n$.

Solution. We shall use a preliminary result given under the form of the following :

Lema: Let $VABC$ be a triangular pyramid and let A', B', C' be points located on the edges VA, VB, VC respectively. Then, the following equality holds :

$$\frac{\mathrm{vol}(VABC)}{\mathrm{vol}(VA'B'C')} = \frac{VA}{VA'} \cdot \frac{VB}{VB'} \cdot \frac{VC}{VC'}.$$



Proof. Let O, O' be the projections of the points A, A' respectively on the plane containing the points V, B, C. Using the fact that AO and $A'O'$ are parallel lines we have :

$$\frac{OA}{O'A'} = \frac{VA}{VA'}.$$

Therefore, the ratio of volumes can be obtained as follows :

$$\frac{\mathrm{vol}(VABC)}{\mathrm{vol}(VA'B'C')} = \frac{\frac{1}{3}S_{VBC} \cdot AO}{\frac{1}{3}S_{VB'C'} \cdot A'O'} = \frac{AO \cdot VB \cdot VC \cdot \sin(\angle BVC)}{A'O' \cdot VB' \cdot VC' \cdot \sin(\angle BVC)} =$$

$$= \frac{VA}{VA'} \cdot \frac{VB}{VB'} \cdot \frac{VC}{VC'}.$$

Turning back to our problem, let us denote the ratio $\frac{VA_i}{VB_i} = x_i$, $\forall i = 1, 2, \ldots, n$ and let K be the ratio of similitude of the two polygons $A_1A_2\ldots A_n$ and $B_1B_2\ldots B_n$. Using the similarity of the triangles $A_iA_jA_k$ and $B_iB_jB_k$ we obtain $\frac{S_{A_iA_jA_k}}{S_{B_iB_jB_k}} = K^2$, and using the lemma we obtain :

$$\frac{\mathrm{vol}\left(VA_iA_jA_k\right)}{\mathrm{vol}\left(VB_iB_jB_k\right)} = x_i x_j x_k.$$

Let H, h be the lengths of the altitudes from V in the pyramids $VA_1A_2\ldots A_n$ and $VB_1B_2\ldots B_n$, respectively. Then

$$\frac{\mathrm{vol}\left(VA_iA_jA_k\right)}{\mathrm{vol}\left(VB_iB_jB_k\right)} = \frac{H}{h}\cdot\frac{S_{A_iA_jA_k}}{S_{B_iB_jB_k}} = \frac{H}{h}K^2.$$

Therefore, for any distinct elements $i, j, k \in \{1, 2, \ldots, n\}$, one has

$$x_i x_j x_k = K^2\frac{H}{h}.$$

Using the fact that $n \geq 4$, it is easy to see that $x_1 = x_2 = \ldots = x_n$.

Remark. The result of the problem does not remain valid for triangular pyramids. It is easy to construct a section $B_1B_2B_3$ such that the base triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar, without the parallel plans. If, in addition, we suppose the pyramid $VA_1A_2A_3$ regular, then the result remains valid.

Problem 6. Let A be the set of positive integers represented by the form a^2+2b^2, where a, b are integer numbers and $b \neq 0$. Show that if p is a prime number and $p^2 \in A$, then $p \in A$.

Solution. It is obvious that $p > 2$; indeed $4 \notin A$. Because p is an odd number, it follows from $p^2 = a^2+2b^2$ that a is an odd number, b an even number and $(a,b)=1$. From the decomposition :

$$(p-a)(p+a) = 2b^2,$$

we obtain the equalities :

$$(1)\ p-a = 2^m A \quad , \quad p+a = 2^n B,$$

where A, B are odd numbers, $m\geq1$, $n\geq1$ and $m+n$ is an odd number.
By adding the equalities (1) we obtain :

$$2p = 2^m A + 2^n B = 2^{\min\{m,n\}} C,$$

where C is again an odd number. It follows that $\min\{m, n\} = 1$ and if one of the two exponents is 1, the other one is an even number. We may consider two cases :

First case : $m=1, n=2r$ with $r \geq 1$. It follows that $p-a=2A$, $p+a=2^{2r}B$ and therefore

$$p^2-a^2=2^{2r+1}AB=2b^2.$$

From this we deduce $2^{2r}AB=b^2$. It is easy to see, using the equalities (1) that $(A,B)=1$. Hence A and B are perfect squares : $A=\alpha^2$, $B=\beta^2$. Using (1) we obtain :

$$(2)\ p-a=2\alpha^2 \quad , \quad p+a=2^{2r}\beta^2$$

Adding the equalities (2), it follows :

$$2p=2\left(\alpha^2+2^{2r-1}\beta^2\right) \text{ and } p=\alpha^2+2\left(2^{r-1}\beta\right)^2.$$

The conclusion follows, from the above equality.

Second case. $n=1, m=2s$ with $s \geq 1$. Then $p-a=2^{2s}A$, $p+a=2B$ and therefore

$$p^2-a^2=2b^2=2^{2s+1}AB.$$

In the same way we obtain $b^2=2^s AB$, $A=\alpha^2$, $B=\beta^2$ and finally

$$p=\beta^2+2\left(2^{s-1}\alpha\right)^2.$$

Remark. The problem arises from the well-known fact : the ring of algebraic integers $\mathbb{Z}\left[i\sqrt{2}\right]=\left\{a+bi\sqrt{2} \mid a,b \in \mathbb{Z}\right\}$ is an euclidian ring with respect to the norm

$$N\left(a+bi\sqrt{2}\right)=a^2+2b^2.$$

It is known that the units of the ring $\mathbb{Z}\left[i\sqrt{2}\right]$ are ± 1. If p is a prime integer number which is not in the set A then p is an irreducible, and hence a prime, element in the ring $\mathbb{Z}\left[i\sqrt{2}\right]$. Indeed, from

$$p=\left(a+bi\sqrt{2}\right)\left(c+di\sqrt{2}\right)$$

we obtain by applying the norm :

$$N(p)=p^2=\left(a^2+2b^2\right)\left(c^2+2d^2\right).$$

Using the factoriality of the ring $\mathbb{Z}$, we get $a^2+2b^2=1$ or $c^2+2d^2=1$.
It follows that, for example, $a+bi\sqrt{2}=1$ and p is irreducible.

Now, let $p^2 = a^2 + 2b^2 = (a + bi\sqrt{2})(a - bi\sqrt{2})$. Using the property of p to be prime, it follows $p \mid a + bi\sqrt{2}$ or $p \mid a - bi\sqrt{2}$. This is a contradiction, because $p \mid a$ and $p \mid b \Rightarrow a = pa_1$, $b = pb_1$, and $p = a_1^2 + 2b_1^2$.

Problem 7. Let p be a prime number, $p \geq 5$, and k be a digit in the p-adic representation of positive integers. Find the maximal length of a non constant arithmetic progression whose terms do not contain the digit k in their p-adic representation.

Solution. We shall prove that an arithmetic progression whose terms do not contain the digit k contains at most $p - 1$ in the case $k \neq 0$ and at most p terms in the case $k = 0$.

For any positive integer N we shall use the notation

$$N = n_s n_{s-1} \ldots n_1 n_0$$

for the p-adic representation of N.

Let $a, a + r, a + 2r, \ldots, a + (p - 1)r$ be an arithmetic progression and let $a = a_s a_{s-1} \ldots a_1 a_0, r = r_t r_{t-1} \ldots r_1 r_0$ be the corresponding p-adic representations. Let r_j be the first non-zero digit in the p-adic representation of r, counting from right to the left. Then the digits j of the terms of the progression are respectively :

$$(1) \qquad a_j, a_j + r_j, a_j + 2r_j, \ldots, a_j + (p-1)r_j,$$

all digits being taken modulo p. Observe that the above sequence is a complete residue system modulo p. It follows that the digit k must appear in the sequence.

If $k = 0$, the failure can be caused if the number of digits in the representation of a is less than j. In this case we are forced to consider in the sequence (1) $a_j = 0$ and then we obtain the longer sequence of digits :

$$(2) \qquad a_j = 0,\ r_j,\ 2\,r_j,\ \ldots,\ (p-1)\,r_j,\ pr_j.$$

This is once again a sequence of digits in which the digit zero appears in the representation of the number $a + pr$.

The conclusion is : the greatest number of terms is $p-1$ for k=1,2, … …, $p-1$ and p for $k = 0$. To complete the proof, we remark that the maximum can be realized in each case.

For $2 \leq k \leq p - 1$ the progression

$$k + 1, k + 2, \ldots, p - 1, p, p + 1, \ldots, p + k - 1$$

has $p-1$ terms ;

for $k = 1$ the progression

$$2p, 2p+(p-1), 2p+2(p-1), \ldots, 2p+(p-2)(p-1)$$

has also $p-1$ terms. Finally, for $k = 0$, the progression

$$1, 1+p, 1+2p, \ldots, 1+(p-1)p$$

has p terms. It is easy to construct the p-adic representation of the terms of those progressions.

Problem 8. Let p, q, r be pairwisely distinct prime numbers and A be the set of positive integers,

$$A = \{p^a q^b r^c \mid 0 \le a,b,c \le 5\}.$$

Find the least number n such that any n-set $B, B \subset A$, contains distinct elements x and y such that x is a divizor of y.

Solution. Let L be the set ordered triples (a, b, c) where $0 \le a,b,c \le 5$. L is itself an ordered set under the relation :

$$\text{“}(a,b,c) \le (a_1,b_2,c_3) \text{ iff } a \le a_1, b \le b_1, c \le c_1\text{”}.$$

It is obvious that $p^a q^b r^c \mid p^{a'} q^{b'} r^{c'}$ iff $(a,b,c) \le (a',b',c')$. Then, our problem is equivalent with the following : find the number of elements of a maximal subset W, $W \subset L$, such that any two elements of W are not comparable and then add to W an extraelement x.

In this way, the set $W \cup \{x\}$ has the required property and it is minimal with this property. We shall call such a set W an antichain.

A subset $V_k \subset L$ such that for any $(a, b, c) \in V$ the sum $a+b+c$ is the constant k is obviously an antichain of L. Geometrically, V_k is the intersection set of the laticial points from the curbe

$$C = \{(a,b,c) \mid a,b,c \in \mathbb{N} \text{ and } 0 \le a,b,c \le 5\}$$

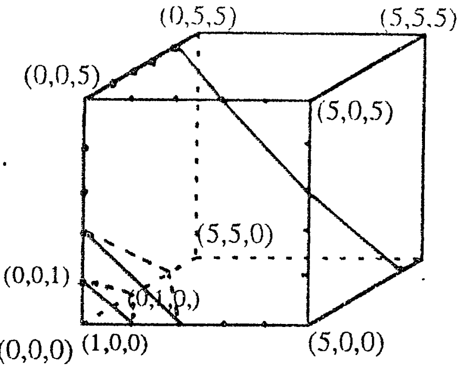
with the plane P_k given by the equation $a+b+c=k$, where $1 \le k \le 14$. For instance V_1 contains three points

$$e_1 = (1,0,0), e_2 = (0,1,0) \text{ and } e_3 = (0,0,1)$$

which are the unitary vectors of a base in $\mathbb{R}^3$.

The antichain V_2 contains six points, which can be obtained in the following way : for any $(a, b, c) \in V_1$, the points :

$$(a,b,c)+e_1=(a+1,b,c)$$
$$(a,b,c)+e_2=(a,b+1,c)$$
$$(a,b,c)+e_3=(a,b,c+1),$$



are the points of V_2. Observe that in this way the elements of V_k are not uniquely generated and then add to W an extraelement x. In this way, the set $W \cup \{x\}$ has the required property and it is minimal with this property.

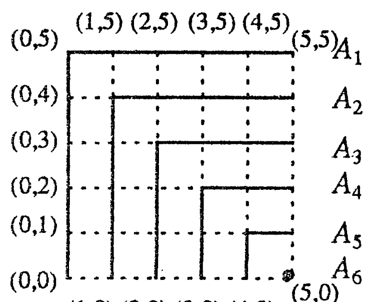
The set V_8 has 27 elements and it is easy to count them : we have 3 ordered triples of the form (0, b, c) ; 4 triples of the form (1, b, c) ; 5 triples of the form (2, b, c) ; 6 triples of the form (3, b, c) ; 5 triples of the form (4, b, c) and finally 4 triples of the form (5, b, c), for which $a+b+c=8$. The total number is $3+4+5+6+5+4=27$ triples.

On the other hand, there exist a partition of the set L with 27 chains such that every chain has a unique representative from the set V_8. In this way, every set with 27 +1 = 28 elements, contains at least two elements which belong to a chain and hence, these two elements are comparable.

The required partition can be obtained as follows. The set

$$A_0=\{(a,b)\in \mathrm{N}^2 \mid 0\le a,b\le 5\}$$

can be partitioned in six chains : A_1, A_2, A_3, A_4, A_5, A_6. That means :



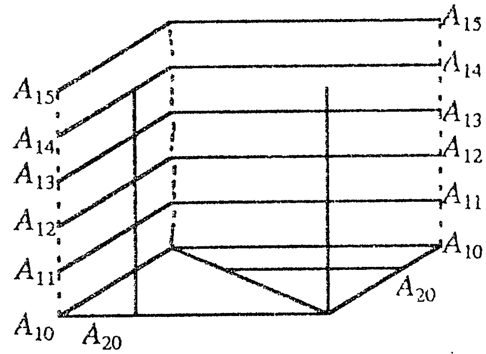
$$A_1=\{(0,0),(0,1),\ldots,(0,5),(1,5),\ldots,(5,5)\}$$
$$A_2=\{(1,0),(1,1),\ldots,(1,4),(2,4),\ldots,(5,4)\}$$
$$A_3=\{(2,0),(2,1),\ldots,(2,3),(3,3),\ldots,(5,3)\}$$
$$A_4=\{(3,0),(3,1),(3,2),(4,2),(5,2)\}$$
$$A_5=\{(4,0),(4,1),(5,1)\}$$
$$A_6=\{(5,0)\}.$$

From the chains A_1, A_2 , A_3 we obtain the new chains A_{1j}, A_{2j} , A_{3j} by adding as a third component the number j, $j\in\{0,1,2,3,4,5\}$; i.e. $A_{1j}=\{(a,b,j) \mid (a,b)\in A_1\}$, etc. In this way we obtain 18 chains, which are orizonthal lines in the cube (see figure).

For each element (a, b) from the chains A_4, A_5, A_6 we obtain a chain $A_{(a,b)}$ in L by adding to (a, b) as a third component, all the numbers j, $j \in \{0,1,2,3,4,5\}$:

i.e. $A_{(a,b)} = \{(a,b,j) | 0 \le j \le 5\}$

is a vertical line in the cube. In this way we obtain $5 + 3 + 1 = 9$ chains.



So, the total number of chains is $18 + 9 = 27$. This finishes the proof.

Problem 9. Let $ABCDEF$ be a convex hexagon. The lines AB anc EF, EF and CD, CD and AB intersect in the points P, Q, R respectively The lines BC and DE, DE and FA, FA and BC intersect in the point S, T, L respectively. Show that if $\frac{AB}{PR} = \frac{CD}{RQ} = \frac{EF}{QP}$ then $\frac{BC}{US} = \frac{DE}{ST} = \frac{FA}{TU}$.

Solution 1 (the author's solution). This solution is founded on a very interesting idea : using the given notations we shall prove that the hypothesis

$$(1) \qquad \frac{AB}{PR} = \frac{CD}{RQ} = \frac{EF}{QP}$$

is equivalent with the following assertion : for every point X interior to the given hexagon $ABCDEF$, the sum of the areas

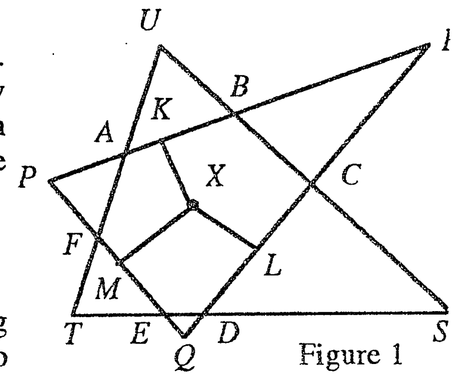


Figure 1

$$S(XAB) + S(XCD) + S(XEF)$$

is constant. Then the sum of the areas

$$S(XBC) + S(XDE) + S(XFA)$$

is also constant, and by the above equivalence and the symmetry of th problem with respect of the choise of three sides, the result comes to b proved.

Let k be the common value of the ratio (1) and let K, L, M be th projections of the point X on the lines PR, PQ, QP respectively. Then w obtain that :

$$S(XAB)+S(XCD)+S(XEF)=kS(PQR).$$

Reciprocally, let us suppose that $S(XAB)+S(XCD)+S(XEF)=\lambda$ is a constant, for any X. By expressing the areas, we obtain

$$(2) \qquad AB\cdot XK+CD\cdot XL+EF\cdot XM=2\lambda$$

is also constant, for any X.

So we are led to the following lemma :

Lemma. Let YOZ be an angle and a, b, ρ be positive real numbers. The loci of the points X which are interior to the angle YOZ and such that

$$a\cdot d(X,OY)+b\cdot d(X,OZ)=\rho$$

is a segment MN where $M\in OY$ and $N\in OZ$. (Here, $d(X, OY)$ represent the distance from the point X to the line OY).

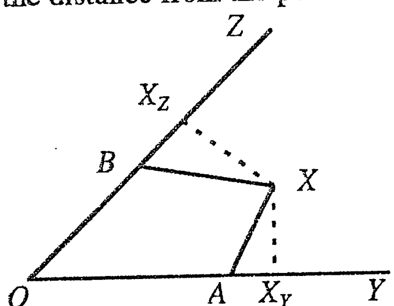


Figure 2

Proof. Take the points A, B on the line OY, OZ respectively such that $OA=a$ and $OB=b$ and let X_Y, X_Z be the projections of the point X on the lines OY, OZ respectively. Then it follows that

$$S(OAX)+S(OBX)=\frac{\rho}{2},$$

is a constant. Because

$$S(OAX)+S(OBX)=S(OAXB)=S(OAB)+S(ABX),$$

follows that $S(ABX)$ is constant, and therefore $d(X, AB)$ is constant. Then the result is: the loci of the points X is a segment MN parallel to the segment AB.

We come back to the problem in the following way : take the points G, H on the segments RP and RQ respectively such that $RG=AB$ and $RH=CD$. Then the condition (2) is equivalent with the condition :

$$S(XGR)+S(XHR)+S(XFE)=2\lambda,$$

for any point X interior to the hexagon. It is also equivalent with the condition:

$$S(XHG)+S(XFE)=2\lambda-S(GHR),$$

that is a constant, for any X. By applying twice the lemma, it follows that EF and GH, FG and EH are parallel. Therefore, the quadrilater $EFGH$ is a parallelogram. Using the similitude of the triangles ΔRPQ and ΔRGH we obtain :

$$\frac{RG}{RP}=\frac{RH}{RQ}=\frac{GH}{PQ}\Rightarrow\frac{AB}{PR}=\frac{CD}{RQ}=\frac{EF}{QP}.$$

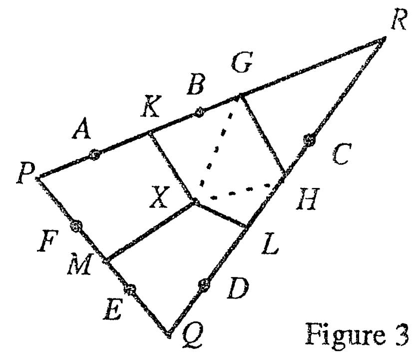


Figure 3

An alternative solution.

Consider the point M such that the segments AM and FE, BM and CD are equal and parallel in the same time. Such a point M exists because the vectors $\overrightarrow{PR},\overrightarrow{RQ}$ and $\overrightarrow{QP}$ define the triangle PRQ and hence $\overrightarrow{PR}+\overrightarrow{RQ}+\overrightarrow{QP}=0$. Therefore

$$k\overrightarrow{AB}+k\overrightarrow{CD}+k\overrightarrow{EF}=0$$

implies succesively :

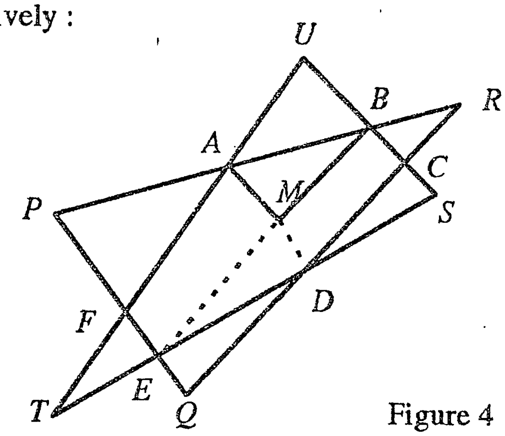


Figure 4

$$\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{EF}=0\Rightarrow\overrightarrow{AB}+\overrightarrow{BM}+\overrightarrow{MA}=0$$

Then the triangles PRQ and ABM are similar and the quadrilaterals $AMEF$ and $BCDM$ are parallelograms. The triangles EMD and TUS have parallel sides and therefore they are similar. We obtain the ratio :

$$\frac{MD}{US}=\frac{DE}{ST}=\frac{EM}{TU}.$$

Using $MD = BC$ and $EM = AF$, we obtain $\frac{BC}{US} = \frac{DE}{ST} = \frac{AF}{TU}$.

Problem 10. Let P be the set of the points of the euclidean plane Π and D be the set of the lines of the same plane. Find, with proof, whether there exists a bijective function $f: P \to D$ such that for arbitrary three collinear points A, B, C the lines $f(A), f(B), f(C)$ are either parallel or concurrent.

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Solution. We shall prove that such a function doesn't exist. Let us suppose contrary, that the required bijective function $f: P \to D$ exists. First, we shall prove the following :

Lemma : If the lines d_1, d_2, d_3 have a common point M and $d_i = f(B_i)$, $B_i \in P$, $i = 1, 2, 3$, then the points B_1, B_2, B_3 are collinear. Indeed, if $B_1B_2B_3$ is a triangle then the image of any point C, $C \in B_iB_j$, $i \neq j$, is a line $d = f(C)$ through M. It is clear that any point C of the plane is collinear with two different points C_1, C_2 belonging to the sides B_iB_j of the triangle.

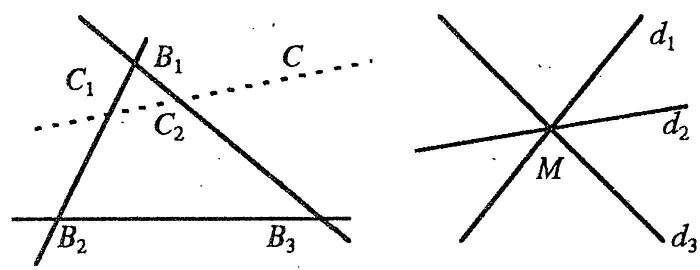

Figure 1

Then, for any point C of the plane its image must be a line $d = f(C)$ through M. This contradicts the supposition that the map f is onto.

In the same way it can be proved that if the lines d_1, d_2, d_3 are parallel and $d_i = f(B_i)$, then the points B_1, B_2, B_3 are collinear. From the above two results it follows that f defines a bijection between the points of a line d and the lines of a pencil of lines through a point or a pencil of parallel lines (hence, a pencil of lines).

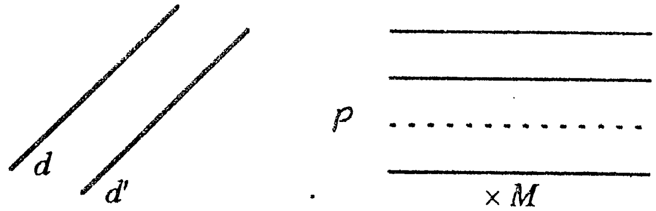

Figure 2

Let us consider two parallel lines d, d' such that the images of the points $B \in d$ are the lines of a pencil P of parallel lines. The images of the points $B' \in d'$ are the lines of a pencil P'.

If P' is a pencil of lines through a point M then it contains a line p belonging to the pencil P, $p \in P$. It follows that p is the image of a point belonging to d and of another point belonging to d' ; this is a contradiction. Therefore P' is also a pencil of parallel lines.

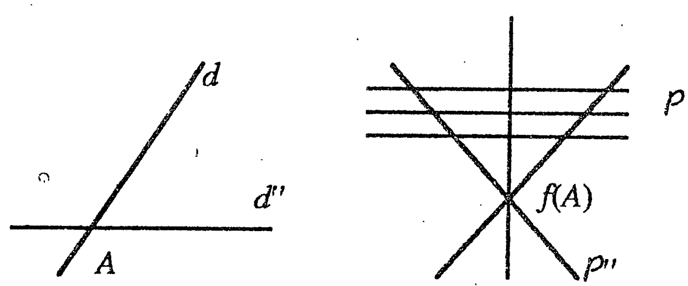

Figure 3

Now, let d'' be a line which intersects d and such that the images of the points belonging to d'' are a pencil P'' of lines through a point. For any line δ, $\delta \parallel d''$, the images of the points of δ are a pencil Π of lines through a point. The pencils P'' and Π have a common line l and hence $l = f(B)$ where $B \in d'' \cap \delta$; this is a contradiction.

Problem 11. Find the functions $f:\mathbb{R} \to [0,\infty)$ such that

$$f(x^2+y^2) = f(x^2-y^2) + f(2xy),$$

for all real numbers x and y.

Solution. For $x = y = 0$ we obtain $f(0) = 0$. For $x = 0$ we obtain $f(y^2) = f(-y^2)$; hence f is an even function and it is sufficient to find it for positive numbers. Let a, b positive real numbers. Then the algebraic system

$$x^2 - y^2 = a \text{ and } 2xy = b$$

has always real solutions. It is sufficient, for proving this, to observe that the solution can be obtained as the intersection point of two hyperbolas: one is reported to the axis and the other to its asymptotes.

Taking a solution (x, y) of the above system, one obtains $x^2 + y^2 = \sqrt{a^2 + b^2}$. Then, for any $a, b \geq 0$, the function f satisfies

$$f(a) + f(b) = f\left(\sqrt{a^2 + b^2}\right).$$

Let $g:[0,\infty) \to [0,\infty)$ be defined $g(a) = f\left(\sqrt{a}\right)$. Then $g(a^2) + g(b^2) =$ $= g(a^2 + b^2)$, for any a, $b \in \mathbf{R}$. For $x \geq 0$, $y \geq 0$ and by taking $a = \sqrt{x}, b = \sqrt{y}$ we obtain $g(x + y) = g(x) + g(y)$, for all positive numbers x, y. Using *Cauchy* equation we obtain $g(x) = kx$ and $f(x) = kx^2$, for all $x \geq 0$.

Problem 12. Let n be an integer number, $n \geq 2$, and

$$P(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_1 X + 1$$

be a polynomial with positive integer coefficients. Suppose that $a_k = a_{n-k}$ for all k, $k = 1, 2, \ldots, n - 1$. Prove that there exist infinitely many pairs of positive integers (x, y) such that $x \mid P(y)$ and $y \mid P(x)$.

Solution. Observe that the pair $(1, P(1))$ verifies the conditions :

$$1 \mid P(P(1)) \text{ and } P(1) \mid P(1).$$

Suppose that only a finite number of pairs (x, y) verify the condition. Then it is possible to consider the ordered pair (x, y) with $x \leq y$ and such that y has the greatest value. In this conditions, we shall prove that the pair $\left(y, \frac{P(y)}{x}\right)$ also verifies the hypothesis.

Firstly, because $x \mid P(y)$ the number $\frac{P(y)}{x}$ is an integer. It is obvious that $\frac{P(y)}{x} \mid P(y)$. We have to prove only that $y \mid P\left(\frac{P(y)}{x}\right)$.

From $y \mid P(x)$, i.e. $y \mid x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + 1$ it follows that x and y are coprime numbers. Hence, there exists a solution z of the congruence $x z \equiv 1 \pmod y$. From $P(y) \equiv 1 \pmod y$ we obtain

$$P\left(\frac{P(y)}{x}\right) \equiv P(zP(y)) \equiv P(z) \pmod y.$$

Using the property of P to be a reciprocal polynomial we also have :

$$x^n P\left(\frac{1}{x}\right) = P(x).$$

From $y \mid P(x)$ follows : $x^n P\left(\frac{1}{x}\right) \equiv 0 \pmod y$, and hence $P(z) \equiv 0 \pmod y$. In this way the pair $\left(y, \frac{P(y)}{x}\right)$ verifies the condition. Moreover :

$$P(y) \geq y^n + 1 > y^2 \geq xy$$

and therefore $\frac{P(y)}{x} > y$. This contradicts the maximality of y in the pair (x,y).

Problem 13. Let $P(X)$, $Q(X)$ be monic irreducible polynomials in the ring $Q[X]$. Suppose that $P(X)$ and $Q(X)$ have roots α and β, respectively and that $\alpha + \beta$ is a rational number. Prove that the polynomial $P(X)^2 - Q(X)^2$ has a rational root.

Solution. Let us denote $\alpha + \beta = q, q \in \mathbf{Q}$.

Then $P(\alpha) = P(q - \beta) = 0$. It follows that β is a root of the polynomial $P(q-X)$. The polynomials $P(q - X)$ and $Q(X)$ have rational coefficients,are irreducible and have the common root β. Therefore they differ by a rational multiplicative constant :

$$P(q - X) = cQ(X).$$

Taking into account that the polynomials P and Q are monic, it follows $c = \pm 1$, depending on the parity of deg P. From the equality

$$P(q - X) = \pm Q(X)$$

we obtain

$$P^2(q - X) = \pm Q^2(X).$$

Then, it is obvious that, $P^2\left(q-\frac{q}{2}\right)=\pm Q^2\left(\frac{q}{2}\right)$. This proves that $\frac{q}{2}$ is a root of the polynomial $P^2(X)-Q^2(X)$.

Problem 14. Let a be an integer number, $a > 1$. Show that the set of positive integers

$$\{a^2+a-1, a^3+a^2-1, \ldots, a^{n+1}+a^n-1, \ldots\}$$

contains an infinite subset of pairwisely coprime numbers.

Solution. We shall indicate how to produce an infinite set of pairwisely coprime numbers from the given set.

Firstly we observe that

$$(a^2+a-1, a^3+a^2-1)=1,$$

and, more generally, any two conssecutive numbers from the given set are coprime. Hence, there exists in the set two coprime numbers. Suppose that :

$$a^{n_1+1}+a^{n_1}-1, a^{n_2+1}+a^{n_2}-1, \ldots, a^{n_k+1}+a^{n_k}-1$$

are pairwisely coprime numbers and let N be their product :

$$N=\prod_{i=1}^{k}\left(a^{n_i+1}+a^{n_i}-1\right).$$

The numbers $a^0 = 1$, a, a^2, ..., a^N are distinct integers and then, using the pigeonhole principle, there exist distinct exponents i, j, $i > j$, such that

$$a^i \equiv a^j (\bmod N).$$

Then $N \mid a^j (a^{i-j} - 1)$. It is easy to see that N and a are coprime numbers (using the definition of N). It follows that

$$(N, a^{i-j+1}+a^{i-j}-1)=1.$$

Alternative solution. Let P be the set of all prime divisors of the number N and let n be the product :

$$n=\prod_{p\in P}(p-1).$$

As before, it is easy to see that $(p, a) = 1$, for any $p\in P$. From the *Fermat*'s theorem we obtain $a^{p-1}\equiv 1(\bmod p), \forall p\in P$ and therefore

$$a^{n+1}+a^n-1\equiv a(\bmod p), \forall p\in P.$$

This proves that $a^{n+1}+a^n-1$ and N are coprime numbers.

A second alternative solution. Like in the first solution $(a, N) = 1$ and then, using *Euler*'s theorem obtain :

$$a^{\varphi(N)} \equiv 1(\bmod N) \text{ and}$$

$$a^{\varphi(N)+1} + a^{\varphi(N)} - 1 \equiv a(\bmod N).$$

This proves that $a^{\varphi(N)+1} + a^{\varphi(N)} - 1$ and N are coprime numbers.

Problems 15. The vertices of a regular dodecagon are coloured either blue or red. Find the number of all possible colourings which it does not contain monochromatic subpolygons.

Solution. Firstly, we observe that it is sufficient to take into account only equilateral triangles and squares ; indeed, if the dodecagon do not contain monochromatic equilateral triangles, then do not contain monochromatic regular hexagons too.

Let us denote the vertices of the dodecagon by the numbers 1,2,3,..., ..., 12. The dodecagon contains the equilateral triangles $T_1 = \{1, 5, 9\}$, $T_2=\{2, 6, 10\}$, $T_3=\{3, 7, 11\}$, $T_4 = \{4, 8, 12\}$ and the squares $P_1=\{1,4,7,10\}$, $P_2= \{2, 5, 8, 11\}$, $P_3= \{3, 6, 9, 12\}$. The families of subsets $\{T_1, T_2, T_3, T_4\}$ and $\{P_1, P_2, P_3\}$ are both partitions of the set $\{1, 2, 3, ..., 12\}$ of vertices of the dodecagon.

To obtain a colouring without monochromatic triangles it is sufficient to take care to assign two colours to the vertices of every triangle. The number of such colouring of a triangle T_i is $2^3 - 2 = 6$. Because we have 4 triangles, taking into account all possible combinations, we obtain $6^4 = 1296$ colourings without monochromatic triangles.

It is possible that a colouring without monochromatic triangles contains a monochromatic square. Let S_i be the set of colourings without monochromatic triangles for which the squareP_i is monochromatic, $i= 1,2,3$. So, we are going to find the cardinality $| S_1 \cup S_2 \cup S_3 |$. By the inclusion-exclusion principle, we have the formula :

$$| S_1 \cup S_2 \cup S_3 | = | S_1 | + | S_2 | + | S_3 | - | S_1 \cap S_2 | - | S_1 \cap S_3 | -$$
$$- | S_2 \cap S_3 | + | S_1 \cap S_2 \cap S_3 |.$$

In order to obtain the cardinal number $| S_i |$, observe that every square P_i has its vertices in the four different triangles T_1, T_2, T_3, T_4. If the vertices of P_i have the same colour, say red, then in every triangle the two resting vertices can have the combinations {blue, blue}, {blue, red} or {red, blue}.

The total number of such combinations is 3^4, and it is necessary to count twice, for the case of P_i - vertices coloured in blue.

Then $|S_i| = 2\cdot 3^4 = 162$.

In the same way, $|S_i \cap S_j| = 34$ and $|S_1 \cap S_2 \cap S_3| = 6$.

Then the required number is $1296 - 3\cdot 162 + 3\cdot 34 - 6 = 906$.

Problem 16. Let Γ be a circle and AB be a line which does not intersect Γ. For any point P, $P \in \Gamma$, let P' be the second intersection point of the line AP with Γ and $f(P)$ be the second intersection point of the line BP' with Γ. In this way one defines the point sequence $P = P_0, P_1, \ldots, P_n, \ldots$ where $P_{n+1} = f(P_n)$. Show that if k is a positive integer such that $P_0 = P_k$, then for any point $Q = Q_0$, the property $Q_0 = Q_k$ also holds.

First solution. Let O be the centre and R be the radius of Γ ; i.e. $\Gamma = \mathcal{C}(O,R)$. The function $P \mapsto P'$ corresponds to the inversion with respect to the circle $\Gamma_1 = \mathcal{C}\left(A;\sqrt{OA^2 - R^2}\right)$ and the function $P' \mapsto f(P)$ corresponds to the inversion with respect to the circle $\Gamma_2 = \mathcal{C}\left(B;\sqrt{OB^2 - R^2}\right)$. Note that by taking $O' = \mathrm{pr}_{AB}O$ we get :

$$AB \le AO' + O'B = \sqrt{OA^2 - OO'^2} + \sqrt{OB^2 - OO'^2} < \sqrt{OA^2 - R^2} + \sqrt{OB^2 - R^2}$$

and

$$AB \ge |AO' - O'B| = \left|\sqrt{OA^2 - OO'^2} - \sqrt{OB^2 - OO'^2}\right| > \left|\sqrt{OA^2 - R^2} - \sqrt{OB^2 - R^2}\right|$$

So $\mathcal{C}_1$ and $\mathcal{C}_2$ have two common points C and D.

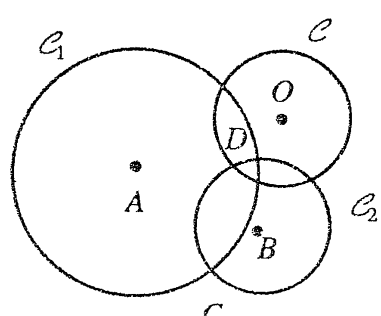

Figure 1

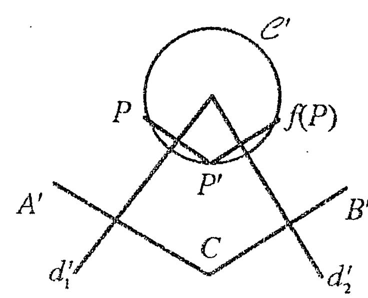

Figure 2

Consider now an inversion of center C. This inversion transforms the circle Γ into a circle Γ' and the circles Γ_1 and Γ_2, which are orthogonal to Γ'' into the straight lines d_1', d_2' which are orthogonal to Γ''; that is d_1' and d_2' pass through the center of Γ'.

The inversion also transforms A and B into A' and B', the reflections of C in d_1' and d_2', therefore the straight lines AP and BP' become circles symetrical about d_1' and d_2'. This shows that the functions $P \mapsto P'$ and $P' \mapsto f(P)$ become the reflections in d_1' and d_2', so the function f becomes the rotation about the center of $\mathcal{C}'$ of angle $2\angle(d_1', d_2')$.

The assumption $P_k = P_0$ is translated now into $k\angle(d_1', d_2') \in \mathbb{Z}\pi$ and obviously, this condition does not depend of the position of P_0.

Second solution. Consider the projective transformation $\mathfrak{I}$ which throws AB at infinity ; $\mathfrak{I}$ transforms Γ into an ellipse ε. Take now an affine transformation $\mathcal{A}$ which transforms ε into a circle Γ'.

The transformation $\tau = \mathcal{A} \circ \mathfrak{I}$ is projective, throws AB at infinity and transforms the circle Γ into a circle Γ'.

The straight lines which pass through A and B become straight lines which are parallel to fixed directions a and b and the function f becomes a function g which assigns to every $P \in \mathcal{C}'$ the point $g(P) \in \mathcal{C}'$ such that $PP' \| a$, $P' \in \mathcal{C}'$ and $P'g(P) \| b$. This shows that

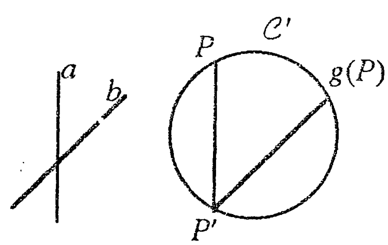

Figure 3

the oriented arc $\overset{\frown}{Pg(P)}$ is twice the oriented angle $\angle(a, b)$, so g is a rotation about the center of the circle Γ'. The proof can now be finished like in the previous solution.

Advanced alternative solution. The union of Γ and the line Δ is a plane cubic curve X. It is known that for a fixed point O, $O \in X$, there exists a group structure G on X defined as follows : for any points A, B on X, $C = A*B$ is the third intersection point of the line $l = l_{AB}$ with X and $A+B$ is the third intersection point of X with the line $l' = l_{OC}$ (see figure 4).

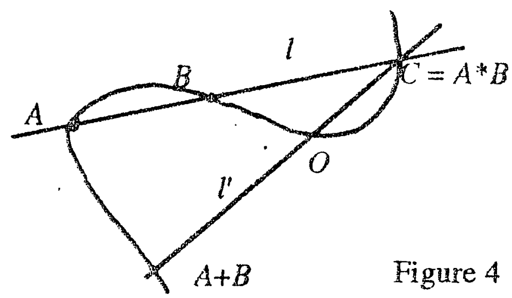

Figure 4

It is easy to verify that the composition law is commutative and the point O is the zero element. It is not easy to verify the associativity and the main fact that by changing the point O with another point O', the two groups such obtained are isomorphic.

In this new context, our problem claims that :

$$P + nA = P.$$

Obviously, this is equivalent with the property : the point A is a torsion element of the group G. It is evident that this property does not depend on P.

SECTION III

The annual competition of the journal "Gazeta Matematică", September 1997*)

The 7th and the 8th Forms

First day

1. Five integers have the following property : if we add in different ways four of these numbers we obtain the numbers 21, 25, 28 and 30. Find the five numbers.

2. We are given 21 distinct numbers selected from the set

$$\{1,2,3, \ldots, 2046\}.$$

Show that there exists three numbers, say a, b, c, among the selected numbers such that

$$bc < 2a^2 < 4bc.$$

3. A rectangular sheet of dimensions 12×10 is cut with a pair of scissors into two parts having equal areas. Show that the length of the cutting is at least 10.

Second day

4. Let a, b, c be positive numbers. Prove the inequality :

$$\frac{bc}{a^2+2bc}+\frac{ca}{b^2+2ca}+\frac{ab}{b^2+2ab} \leq 1 \leq \frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ac}+\frac{c^2}{c^2+2ab}.$$

5. Let ABC be an equilateral triangle. A line divides the triangle into two parts having the same perimeter and areas S_1, S_2. Show that :

$$\frac{7}{9} \leq \frac{S_1}{S_2} \leq \frac{9}{7}.$$

6. Let $ABCD$ be a square and M be a point on the side AD such that $\frac{AM}{MD}=\frac{1}{2}$. Let P be the intersection point of the line MB with the circle circumscribed to the square. Show that the line PC passes through the midpoint of the side AD.

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*) The solutions of the problems from this section are available in *Gazeta Matematică*, vol.101 (1996), no. 10, pp. 476-496.

The 9[th] and the 10[th] Forms

First day

1. Let $ABCD$ be a convex quadrilateral and M, N, P, Q be points on the segments AB, CD, AD, BC respectively such that :

$$\frac{AM}{MB}=\frac{DN}{NC}=x \text{ and } \frac{AP}{PD}=\frac{BQ}{QC}=y.$$

Show that if I is the intersection point of the segments MN and PQ then

$$\frac{PI}{IQ}=x \text{ and } \frac{MI}{IN}=y.$$

2. Find all polynomial functions f:R→R with integers coefficients which are bijective and such that

$$f^2(x)=f\left(x^2\right)-2f(x)+a \text{ ,for all } x\in \mathbb{R}.$$

3. Let $A_1A_2\ldots A_n$ be a convex polygon and Γ be a circle interior to the polygon. Let t_i be the length of the tangent from the point A_i to Γ and p be the semiperimeter of the polygon. Show that $\sum_{i=1}^{n} t_i \geq p$.

Second day

4. Find positive integers n such that

$$\sqrt{x-1}+\sqrt{x-2}+\ldots+\sqrt{x-n}<x,$$

for all $x \geq n$.

5. Let ABC be a triangle and M be a point in the plane but not on the sides of the triangle. A_1, B_1, C_1 are the intersection points of the incircle with the sides BC, CA, AB respectively and A_2, B_2, C_2 are the intersection points of the incircle with the line A_1M, B_1M, C_1M respectively, such that $A_1\neq A_2$, $B_1\neq B_2$, $C_1\neq C_2$. Prove that the lines AA_2, BB_2, CC_2 are concurrent.

6. We are given a triangle ABC such that

$$\frac{a}{b}=\frac{l_b}{l_a}=2,$$

where l_a, l_b are the bisector segments of the angles A, B respectively. Find the angles of the triangle ABC.

The 11th and the 12th Forms

First day

1. Let a, b be positive real numbers. Show that the sequence $(x_n)_{n\geq 0}$ defined by :

$$x_0 = a, x_1 = b, x_{n+1} = \frac{x_n^2 + x_{n-1}^2}{x_n + x_{n-1}}, \forall n \geq 1, \text{ is convergent.}$$

2. Let $f:[0,a] \to \mathbf{R}$ be a derivable function, with a continuous derivative, such that $f'(0) > 0$ and $f(a) = f(0) + \frac{a^2}{2}$. Show that there exists a convergent sequence $(x_n)_{n\geq 0}$ such that $f'(x_n) = (n+1)x_n$. Then find the limit $\lim\limits_{n\to\infty} x_n$.

3. Let A be a real nonsingular square matrix of dimensions $n \times n$ and let $f_A : \mathrm{M}_n(\mathbf{R}) \to \mathbf{R}$ be the function

$$f_A(X) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_{ij},$$

where $B = (b_{ij})$ is the matrix $B = AA'$. Show that :

(i) if $X \in \mathrm{M}_n(\mathbf{R})$ and $f_A(X \cdot X') = 0$ then $X = O_n$;

ii) for all matrices $X, Y \in \mathrm{M}_n(\mathbf{R})$,

$$(f_A(XY))^2 \leq f_A(X \cdot X') \cdot f_A(Y' \cdot Y).$$

Second day

4. Let $P(X)$ be a rational polynomial,

$$P(X) = aX^3 + bX^2 + cX + d,$$

and let x_1, x_2, x_3 be the roots of f. Show that if $\frac{x_1}{x_2}$ is a rational number different from 0 and 1, then all the roots of $P(X)$ are rational numbers.

5. Let $(x_n)_{n\geq 0}$ be a real sequence given by the conditions : $x_0 > 0$, $x_1 > 0$ and $x_{n+2} = \frac{2+x_{n+1}}{2+x_n}$, for all $n \geq 0$. Show that $(x_n)_{n\geq 0}$ is a convergent sequence.

6. Prove that there exists an unique function $f:[1,\infty) \to [0,\infty)$ such that :

$$e^{f(x)} = \frac{f(x)}{x} + x, \text{ for all } x \geq 1.$$

Show that this function satifies the conditions :

b) The sequence $a_n = f(1) + f(2) + \ldots + f(n) - \ln n!$ is bounded.

c) $\lim\limits_{x\to\infty} x(f(x) - \ln x) = 0$.

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