Real Analysis Exam: Part I (Spring 1999)

Do all five problems.

1. Write the Fourier series of the function $f(x) = x^2$ on the interval $[-\pi, \pi]$, and use it to compute $\sum_{n=1}^{\infty} n^{-2}$.

2. Let B be a Banach space and $L : B \to B$ be a linear (but not necessarily bounded) operator such that $\|Lx\| \geq \|x\|$ for all $x \in B$. Let $f : B \to \mathbf{R}$ be a bounded linear functional. Prove that there is a bounded linear functional $g : B \to \mathbf{R}$ such that $g(L(x)) = f(x)$ for all $x \in B$.

3. Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be the circle group. Let $k \in \mathcal{L}^1(\mathbf{T})$ and let $\mathbf{K}$ be the integral operator on $\mathcal{L}^2(\mathbf{T})$ defined by

$$\mathbf{K} : f \mapsto \frac{1}{2\pi} \int_{t \in \mathbf{T}} k(x-t) f(t)\, dt$$

(a) Prove that $\mathbf{K}$ is a compact operator (i.e., that the image of the unit ball under $\mathbf{K}$ has compact closure.)
(b) Find the eigenvalues and the corresponding eigenfunctions of $\mathbf{K}$. Hint: use Fourier series.

4. If $f \in \mathcal{L}^1(\mathbf{R})$, let M_f denote the Hardy-Littlewood maximal function of f:

$$M_f(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|\, dt$$

Prove that

$$\mu\{x : M_f(x) > \lambda\} \leq C \frac{\|f\|_1}{\lambda}$$

where $\|f\|_1$ is the $\mathcal{L}^1$ norm of f and C is a fixed constant.

5. Let $F : [0,1] \to \mathbf{R}$ be a continuous function. The arclength L of the graph of F is defined to be

$$\sup \sum_{i=1}^{n} \sqrt{|\Delta x_i|^2 + |\Delta y_i|^2}$$

where $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = F(x_i) - F(x_{i-1})$, and where the supremum is taken over all partitions $0 = x_0 \leq x_1 \leq \cdots \leq x_n = 1$.

In this problem, F is increasing, $F(0) = 0$, and $F(1) = 1$.

(a) Prove that $\sqrt{2} \leq L \leq 2$.
(b) Let μ be the measure such that $\mu((a,b]) = F(b) - F(a)$. Suppose μ is singular with respect to Lebesgue measure. Prove that $L = 2$.

Real Analysis Exam: Part II (Spring 1999)

Do all five problems.

1. Let S be the set of real numbers $x \in [0,1]$ such that for every $\epsilon > 0$, there is a rational number p/q with
$$\left| x - \frac{p}{q} \right| < \frac{\epsilon}{q^3}$$

(a) Prove that S is uncountable.
(b) Prove that S has measure 0.

2. Let X be a compact Hausdorff space and J be an ideal in $C(X)$, the space of continuous real-valued functions on X. Assume that for each point $x \in X$, there is a function $f \in J$ such that $f(x) \neq 0$. Prove that $J = C(X)$.

3. Suppose $f : [0,1] \to \mathbf{R}$ is a function such that
$$\limsup_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| < \infty$$
for almost every x. Prove that for each $\epsilon > 0$, there is a Lipschitz function $g : [0,1] \to \mathbf{R}$ such that
$$\{x : f(x) \neq g(x)\} \text{ has Lebesgue measure } < \epsilon.$$

NOTE: You may use (without proof) the following fact. If $S \subset \mathbf{R}$ and $h : S \to \mathbf{R}$ is Lipschitz, then there is a Lipschitz function $H : \mathbf{R} \to \mathbf{R}$ such that $H(x) = h(x)$ for all $x \in S$.

4. Let X and Y be Hilbert spaces and $L : X \to Y$ be a linear operator. Prove that the following two conditions are equivalent:

(a) The image $L(\mathbf{B})$ of the unit ball in X has compact closure in Y,
(b) There is a sequence of bounded linear maps $L_n : X \to Y$ such that the image $L_n(X)$ is finite dimensional and such that $\|L_n - L\| \to 0$. (Here $\|\cdot\|$ is the operator norm.)

5. Let f be a continuous real-valued function on $[0,1]$ and let $g(y)$ be the number of points x such that $f(x) = y$. Prove that if g is in $\mathcal{L}^1$, then f has bounded variation.