USA Mathematical Talent Search

Round 1 Solutions

Year 22 — Academic Year 2010–2011

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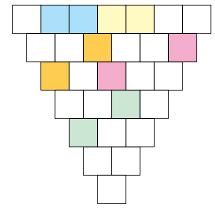
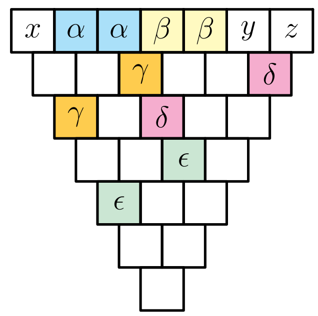
1/2/22. Show that there is a unique way to place positive integers in the grid to the right following these three rules:



1. Each entry in the top row is one digit.
2. Each entry in any row below the top row is the sum of the two entries immediately above it.
3. Each pair of same-color squares contain the same integer. These five distinct integers are used exactly twice and no other integer is used more than once.

Label additional squares as shown at right. Comparing the two pink boxes, we have



$$\delta = \alpha + 3\beta = y + z \leq 17,$$

since y and z are distinct and each a single digit. Thus $\beta \leq 5$. Next, comparing the two orange boxes, we see that

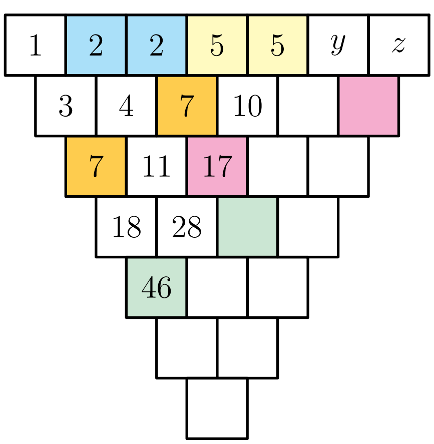
$$\gamma = x + 3\alpha = \alpha + \beta,$$

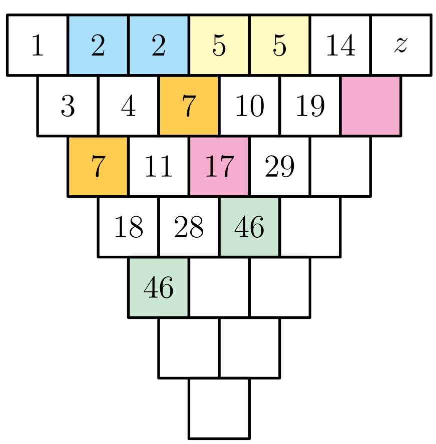
so $x = \beta - 2\alpha$. But $\beta \leq 5$, so in order to have $x > 0$ we must have α equal to 1 or 2. The only possibilities are thus

$$(\alpha, \beta) \in \{(1,3), (1,4), (1,5), (2,5)\}.$$

The first of these gives $x = 1$, which is disallowed since then x and α are not distinct. The second gives $x = 2$, but this is disallowed because then the box directly below the two blue boxes would also be 2. So we conclude that $\beta = 5$ and that $(\alpha, x) \in \{(1,3), (2,1)\}$.

If we attempt to use $\alpha = 2$ and $x = 1$, we can sum downwards to get the grid on the left below. However, we then also have 46 in the green box to the right of the box with 28, and subtracting upwards gives the grid on the right below; this makes $y = 14$ in the top row, which violates condition 1. So this case cannot occur.





The only remaining possibility is the case $\alpha = 1$, $x = 3$, which will fill in at left below. Summing downwards gives $\epsilon = 38$. We can place 38 in the other green box and subtract upwards, and then complete the picture as shown at right below. This completed picture satisfies all the requirements, and it is the only possibility.

3	1	1	5	5	y	z
4	2	6	10			
6	8	16				
14	24					
38						

3	1	1	5	5	7	9
4	2	6	10	12	16	
6	8	16	22	28		
14	24	38	50			
38	62	88				
100	150					
250						

2/2/22. A sequence is called **tworrific** if its first term is 1 and the sum of every pair of consecutive terms is a positive power of 2. One example of a tworrific sequence is 1, 7, −5, 7, 57.

(a) Find the shortest possible length of a tworrific sequence that contains the term 2011.

(b) Find the number of tworrific sequences that contain the term 2011 and have this shortest possible length.

(a) We claim that the shortest possible length of a tworrific sequence that contains the term 2011 is 5. First, the sequence

$$1, 7, -5, 37, 2011$$

contains the term 2011 and has length 5. The sums of pairs of consecutive terms are $1+7 = 8$, $7 + (-5) = 2$, $(-5) + 37 = 32$, and $37 + 2011 = 2048$, so this sequence is tworrific.

Let

$$a_1, a_2, a_3, \dots, a_k$$

be a tworrific sequence of length k ending with the term 2011, so $a_1 = 1$ and $a_k = 2011$. Then

$$\begin{aligned} a_1 + a_2 &= 2^{e_1}, \\ a_2 + a_3 &= 2^{e_2}, \\ &\dots, \\ a_{k-1} + a_k &= 2^{e_{k-1}} \end{aligned}$$

for some nonnegative integers e_1, e_2, ..., e_k, so

$$\begin{aligned} a_k &= 2^{e_{k-1}} - a_{k-1} \\ &= 2^{e_{k-1}} - 2^{e_{k-2}} + a_{k-2} \\ &= 2^{e_{k-1}} - 2^{e_{k-2}} + 2^{e_{k-3}} - a_{k-3} \\ &= \cdots \\ &= 2^{e_{k-1}} - 2^{e_{k-2}} + 2^{e_{k-3}} - \cdots + (-1)^k 2^{e_1} + (-1)^{k+1} a_1 \\ &= 2^{e_{k-1}} - 2^{e_{k-2}} + 2^{e_{k-3}} - \cdots + (-1)^k 2^{e_1} + (-1)^{k+1}. \end{aligned}$$

We claim that that there are no tworrific sequences of length 1, 2, 3, or 4 that contain the term 2011.

Case 1: The tworrific sequence has length 1.

This case is trivial, because the first term is always 1.

Case 2: The tworrific sequence has length 2.

In this case, we seek a nonnegative integer e_1 such that

$$2^{e_1} - 1 = 2011.$$

Then $2^{e_1} = 2012$. But $2012 = 2^2 \cdot 503$ is not a power of 2, so there is no such nonnegative integer e_1.

Case 3: The tworrific sequence has length 3.

In this case, we seek nonnegative integers e_1 and e_2 such that

$$2^{e_2} - 2^{e_1} + 1 = 2011.$$

Then

$$2^{e_2} - 2^{e_1} = 2010.$$

Clearly $e_2 > e_1$, so we can write

$$2^{e_1}(2^{e_2-e_1} - 1) = 2010 = 2 \cdot 1005.$$

The factor 2^{e_1} is a power of 2, and the factor $2^{e_2-e_1} - 1$ is odd, so we must have $2^{e_2-e_1} - 1 = 1005$, which means $2^{e_2-e_1} = 1006$. But $1006 = 2 \cdot 1003$ is not a power of 2, so there are no such nonnegative integers e_1 and e_2.

Case 4: The tworrific sequence has length 4.

In this case, we seek nonnegative integers e_1, e_2, and e_3 such that

$$2^{e_3} - 2^{e_2} + 2^{e_1} - 1 = 2011.$$

Then

$$2^{e_3} + 2^{e_1} = 2^{e_2} + 2012.$$

Note that a nonnegative integer (that is at least 2) can be written in the form $2^{e_3} + 2^{e_1}$ if and only if it has exactly one or two 1s in its binary representation. (We get one 1 if $e_3 = e_1$, which makes $2^{e_3} + 2^{e_1} = 2^{e_1+1}$.) Thus, we seek a nonnegative integer e_2 such that the binary representation of $2^{e_2} + 2012$ has one or two 1s.

But the binary representation of 2012 is

$$2012 = 11111011100_2.$$

It is easy to check that no value $0 \le e_2 \le 10$ will work, and for $e_2 \ge 11$, the binary representation of $2^{e_2} + 2012$ is

$$1\underbrace{00\ldots0}_{e_2-11\text{ 0s}}11111011100_2.$$

Hence, there are no such nonnegative integers e_1, e_2, and e_3.

Therefore, the shortest possible length of a tworrific sequence that contains the term 2011 is 5.

(b) Finding all tworrific sequences of length 5 that contain the term 2011 is equivalent to finding all nonnegative integers e_4, e_3, e_2, and e_1 such that

$$2^{e_4} - 2^{e_3} + 2^{e_2} - 2^{e_1} + 1 = 2011,$$

or

$$2^{e_4} + 2^{e_2} = 2^{e_3} + 2^{e_1} + 2010.$$

Hence, the binary representation of $2^{e_3} + 2^{e_1} + 2010$ must have exactly one or two 1s.

The binary representation of 2010 is

$$2010 = 11111011010_2.$$

Checking all values where $0 \le e_1$, $e_3 \le 10$, we find that the binary representation of $2^{e_3} + 2^{e_1} + 2010$ has exactly one or two 1s only for $(e_1, e_3) = (3, 5)$ and $(5, 3)$, for which

$$2^{e_3} + 2^{e_1} + 2010 = 100000000010_2.$$

Hence, $(e_2, e_4) = (1, 11)$ or $(11, 1)$.

For e_1, $e_3 \ge 11$, the binary representation of $2^{e_3} + 2^{e_1} + 2010$ is

$$1\underbrace{00\ldots0}_{e_1-10\text{ 0s}}11111011010_2$$

if $e_1 = e_3$, and

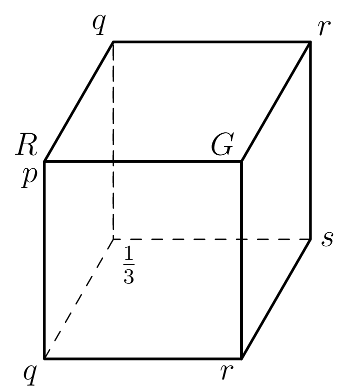
$$1\underbrace{00\ldots0}_{e_1-e_3-1\text{ 0s}}1\underbrace{00\ldots0}_{e_3-11\text{ 0s}}11111011010_2$$

if $e_1 > e_3$ (and similarly for $e_1 < e_3$).

Hence, there are four quadruples of nonnegative integers (e_1, e_2, e_3, e_4), namely $(3, 1, 5, 11)$, $(3, 11, 5, 1)$, $(5, 1, 3, 11)$, and $(5, 11, 3, 1)$. This gives us the four tworiffic sequences

$$1, 7, -5, 37, 2011,$$
$$1, 7, 2041, -2009, 2011,$$
$$1, 31, -29, 37, 2011,$$
$$1, 31, 2017, -2009, 2011.$$

3/2/22. Richard, six of his friends, and a Gortha beast are standing at different vertices of a cube-shaped planet. Richard has a potato and is a neighbor to the Gortha. On each turn, whoever has the potato throws it at random to one of his three neighbors. If the Gortha gets the potato he eats it. What is the probability that Richard is the one who feeds the Gortha?



The planet is shown at right, where R is Richard and G is the Gortha. Let p be the probability that Richard feeds the Gortha given that he is holding the potato. For each non-Gortha neighbor of Richard, let q be the probability that Richard feeds the Gortha given that the neighbor is holding the potato. (By symmetry, it is the same probability for each neighbor.) For each non-Richard neighbor of the Gortha, let r be the probability that Richard feeds the Gortha given that the neighbor is holding the potato. (Again, by symmetry, it is the same probability for each of the Gortha's other neighbors.) Finally, let s be the probability that Richard feeds the Gortha given that the person opposite from Richard is holding the potato. Note also that if the person who is opposite the Gortha has the potato, then the probability that Richard feeds the Gortha is $\frac{1}{3}$, by symmetry, since each of the Gortha's three neighbors is then equally likely to be the feeder. These probabilities are labeled on the diagram above.

When Richard has the potato, he has probability $\frac{1}{3}$ of immediately feeding the Gortha, and probability $\frac{2}{3}$ of passing it to a neighbor. Therefore,

$$p = \frac{1}{3} + \frac{2}{3}q.$$

When one of Richard's non-Gortha neighbors has the potato, she has probabilty $\frac{1}{3}$ of throwing the potato back to Richard, probability $\frac{1}{3}$ of throwing the potato to one of the Gortha's other neighbors, and probability $\frac{1}{3}$ of throwing the potato to the person opposite the Gortha. Therefore,

$$q = \frac{1}{3}p + \frac{1}{3}r + \frac{1}{9}.$$

When one the Gortha's other neighbors has the potato, he has probability $\frac{1}{3}$ of feeding the Gortha (in which case Richard does not feed the Gortha), probability $\frac{1}{3}$ of throwing the potato to one of Richard's neighbors, and probability $\frac{1}{3}$ of throwing the potato to the person opposite Richard. Therefore,

$$r = \frac{1}{3}q + \frac{1}{3}s.$$

Finally, when the person opposite Richard has the potato, she has probability $\frac{2}{3}$ of throwing it to one of the Gortha's non-Richard neighbors, and probability $\frac{1}{3}$ of throwing it to the person opposite the Gortha. Therefore,

$$s = \frac{2}{3}r + \frac{1}{9}.$$

Thus we have the system of equations:

$$p = \frac{1}{3} + \frac{2}{3}q, \tag{1}$$
$$q = \frac{1}{3}p + \frac{1}{3}r + \frac{1}{9}, \tag{2}$$
$$r = \frac{1}{3}q + \frac{1}{3}s, \tag{3}$$
$$s = \frac{2}{3}r + \frac{1}{9}. \tag{4}$$

Substituting equation (4) into equation (3) gives:

$$p = \frac{1}{3} + \frac{2}{3}q, \tag{5}$$
$$q = \frac{1}{3}p + \frac{1}{3}r + \frac{1}{9}, \tag{6}$$
$$r = \frac{1}{3}q + \frac{2}{9}r + \frac{1}{27}. \tag{7}$$

Equation (7) rearranges to give $r = \frac{3}{7}q + \frac{1}{21}$, so substituting this into (6) gives:

$$p = \frac{1}{3} + \frac{2}{3}q, \tag{8}$$
$$q = \frac{1}{3}p + \frac{1}{7}q + \frac{8}{63}. \tag{9}$$

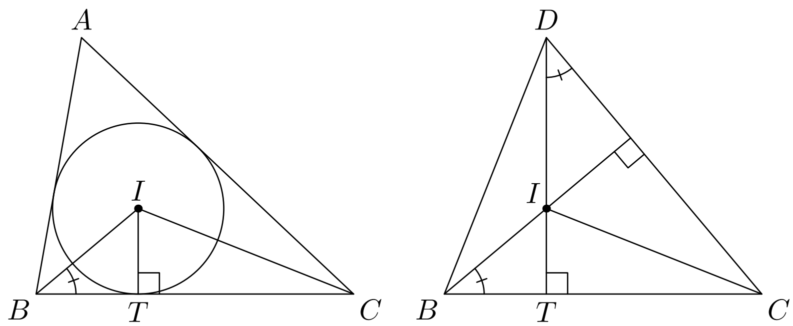
Clearing denominators gives

$$3p = 1 + 2q, \tag{10}$$
$$54q = 21p + 8. \tag{11}$$

Solving this system gives $p = \frac{7}{12}$ and $q = \frac{3}{8}$, so the probability that Richard feeds the Gortha is $\boxed{\frac{7}{12}}$.

4/2/22. Let A, B, C, and D be points in the plane such that $\overline{AD} \parallel \overline{BC}$. Let I be the incenter of $\triangle ABC$ and assume that I is also the orthocenter of $\triangle DBC$. Show that $AB + AC = 2BC$.

Let $a = BC$, $b = AC$, and $c = AB$. Let s, r, and K denote the semiperimeter, inradius, and area of triangle ABC, respectively. Let h be the common heights of triangles ABC and DBC, with respect to base $\overline{BC}$. Let T be the foot of the perpendicular from I to $\overline{BC}$.



Since I is the orthocenter of triangle DBC, $\angle CDT = 90^\circ - \angle DCB = \angle IBT$, so triangles IBT and CDT are similar. Hence,

$$\frac{CT}{IT} = \frac{DT}{BT}.$$

Since BT is the tangent from B to the incircle of triangle ABC, $BT = s - b$. Similarly, $CT = s - c$, so

$$\frac{s-c}{r} = \frac{h}{s-b}.$$

Cross-multiplying, we get

$$(s-b)(s-c) = rh.$$

But $r = K/s$ and $h = 2K/a$, so

$$(s-b)(s-c) = \frac{2K^2}{as},$$

or

$$as(s-b)(s-c) = 2K^2.$$

By Heron's formula, $K^2 = s(s-a)(s-b)(s-c)$, so

$$as(s-b)(s-c) = 2s(s-a)(s-b)(s-c).$$

Dividing both sides by $s(s-b)(s-c)$, we get

$$a = 2(s-a) = 2s - 2a = b + c - a,$$

so $b + c = 2a$, as desired.

5/2/22. Zara and Ada are playing a game. Ada begins by picking an integer from 1 to 2011 (inclusive). On each turn Zara tries to guess Ada's number. Ada then tells Zara whether her guess is too high, too low, or correct. If Zara's guess is not correct, Ada adds or subtracts 1 from her number (always constructing a new number from 1 to 2011). Assuming Zara plays optimally, what is the minimum number of turns she needs to guarantee that she will guess Ada's number?

Zara needs a minimum of $\boxed{1008}$ moves to guarantee a win.

Before demonstrating the formal proof, let us sketch the algorithm that Zara should follow. At each guess, we will assume that Ada reacts in a way so as to maximize the number of guesses Zara will need.

1. Zara guesses 1006 (the middle number). Without loss of generality assume Ada says "lower" (if she says "higher" the remaining strategy is essentially the same).
2. Before Ada changes her number, Ada's number must be in the set $\{1, 2, \ldots, n\}$, where $n = 1005$ initially.
3. After Ada changes her number, Ada's number must be in the set $\{1, 2, \ldots, n + 1\}$.
4. Zara guesses n.
5. If Ada says "lower", then Ada's number must be in the set $\{1, 2, \ldots, n - 1\}$ (before she changes her number), so we go back to Step 2 with n reduced by 1.
6. If Ada says "higher", then continue to step 7 below with $m = n + 1$.
7. Before Ada changes her number, Ada's number must be m.
8. After Ada changes her number, Ada's number must be either $m - 1$ or $m + 1$.
9. Zara guesses $m + 1$. We assume this is the wrong guess (otherwise Zara wins immediately).
10. Ada's number must be $m - 1$. If $m - 1 > 1$, go back to Step 7 (with m reduced by 1).
11. Ada's number must be 1, so when she changes it, her new number must be 2.
12. Zara guesses 2 and wins.

Notice that the above algorithm has four basic parts:

- The initial guess of 1006.

- A guess for each time we loop through steps 2-5. Each guess (except the last guess in the loop) reduces n by 1, where n starts at 1005.
- A guess for each time we loop through steps 7-10. Each guess (except the last guess in the loop) reduces m by 1, where m starts at $n+1$ for whatever value of n causes us to end the first loop
- The final guess of 2, which occurs after we have reduced to $m = 2$.

We run the two loops for a total of 1006 guesses, as we reduce from $n = 1005$ to $m = 2$: each guess decreases the current variable (n or m by 1), except for the guess that shifts to $m = n+1$ somewhere in the middle, and the final guess of the second loop at $m = 2$. Adding these 1006 guesses to the start and end guess gives us our total of 1008 guesses necessary.

Now we prove that this algorithm is the best we can do.

Let $X = \{1, 2, 3, \ldots, 2011\}$. Call a subset $S \subseteq X$ an *interval* if it is of the form

$$S = \{n \mid a \le n \le b\}$$

for some $a, b \in X$ with $a \le b$. We will denote this set as $S = [a, b]$ (using the same notation as we would use for an interval on the real line). For any $n \in X$, let

$$e(n) = \min\{x, 2012 - x\}$$

denote the distance from n from the boundary of X; that is, $e(n)$ is the distance from n to the closer of 0 or 2012. Note that $e(1) = e(2011) = 1$ and $e(1006) = 1006$.

For any nonempty subset $S \subseteq X$, define

$$E(S) = \begin{cases} e(n) & \text{if } S = \{n\} \text{ (that is, if } S \text{ is the single element } n\text{)}, \\ 2 + \max\limits_{n \in S}(e(n)) & \text{if } S \text{ has more than 1 element.} \end{cases}$$

(This value E will measure the least number of turns it takes for Zara to guess Ada's number from any of the sets A_t which interest us.)

Let $t > 0$ denote the turn (i.e. the number of guesses Zara has made). On each turn:

- Zara guesses an integer $z_t \in X$. If z_t is Ada's number, the game is over; otherwise Ada replies "higher" or "lower."
- Zara gains information about Ada's numbers. Let A_t denote the set of Ada's possible numbers *before* she adds or subtracts 1 to her number.
- Ada adds or subtract 1 to her number. Let B_t denote the set of Ada's possible numbers *after* she adds or subtracts 1 to her number.

Note that if A_t is an interval with more than one element, then B_t is also an interval; in particular if $A_t = [a, b]$ then $B_t = [a-1, b+1] \cap X$. Also note that if B_t is an interval, then either

$$A_{t+1} = B_t \cap \{x \in X \mid x < z_t\}$$

or

$$A_{t+1} = B_t \cap \{x \in X \mid x > z_t\}.$$

In either case A_{t+1} is an interval. So, since $B_0 = X$ is an interval (that is, Ada's number could be anything at the beginning of the game), we have that A_t and B_t will always be intervals until A_t becomes a set with a single element.

Lemma 1: If A_t is an interval with more than one element, then for any guess z_t Ada can reply so that $E(A_{t+1}) \geq E(A_t) - 1$.

Proof of Lemma 1:

If $1006 \in A_t$ then $1005 \in A_t$ or $1007 \in A_t$, and $E(A_t) = 1008$. Also note that $[1005, 1007] \subseteq B_t$ and at least one of 1004 and 1008 is also in B_t. Without loss of generality (by symmetry) we may assume B_t contains $[1004, 1007]$. Guessing anything other $z_t = 1006$ may result in $[1005, 1006]$ or $[1006, 1007]$ being a subset of A_{t+1}. In this case $E(A_{t+1}) = 1008 = E(A_t)$. Guessing $z_t = 1006$ may result in $[1004, 1005] \subseteq A_{t+1}$. In this case, $E(A_{t+1}) = 1007 = E(A_t) - 1$.

Otherwise we have $1006 \notin A_t$. Without loss of generality assume that $A_t = [a, b]$ with $b < 1006$, so that $E(A_t) = b + 2$. (The case where $a > 1006$ is symmetric.) Then $B_t = [a-1, b+1]$ (or $[a, b+1]$ if we have $a = 1$, but this does not affect our argument). If z_t is anything other than b or $b+1$, then we may have $[b, b+1] \subseteq A_{t+1}$, in which case $E(A_{t+1}) \geq (b+1) + 2 = b + 3 > E(A_t)$.

If $z_t = b+1$ then (assuming this is not the correct guess) we have $A_{t+1} = [a-1, b]$, and then $E(A_{t+1}) = b + 2 = E(A_t)$.

If $z_t = b$ then there are two possibilities. If Ada says "lower" then $A_{t+1} = [a-1, b-1]$, and $E(A_{t+1}) = 2 + (b-1) = b + 1 = E(A_t) - 1$. If Ada says "higher" then $A_{t+1} = \{b+1\}$ and $E(A_{t+1}) = b + 1 = E(A_t) - 1$.

In all cases $E(A_{t+1}) \geq E(A_t) - 1$, so this proves the Lemma.

□

Lemma 2: If $A_t = \{a\}$ for some a, then Zara needs at least $e(a)$ more guesses to guarantee a win.

Proof of Lemma 2:

Zara can only win immediately if B_t consists of a single element. But the only possible A_t that produce a single element are $A_t = \{1\}$ and $A_t = \{2011\}$; if any $1 < n < 2011$ are in A_t, then both $n-1$ and $n+1$ are in B_t.

Suppose without loss of generality that $1 < a \leq 1006$ (the proof for $a > 1006$ is symmetric). Then $B_t = \{a-1, a+1\}$, and in particular no element less than $a-1$ is in A_{t+1}. Similarly, no element less than $a-2$ is in A_{t+2}, so we cannot have $1 \in A_{t+k}$ until $k \geq a-1$. Thus we require at least $a-1$ guesses to have a possible winning move, plus 1 move (for the winning guess itself). Thus Zara needs at least $a = e(a)$ more guesses.

□

Now we are ready to complete the proof. The game starts in an *interval phase* for which A_t is always an interval with at least 2 elements until some guess causes A_t to be a single element. Specifically, by Lemma 1, A_t is the single $\{k\}$ after at least $E(A_1) - k$ guesses (since $E(A_t)$ can decrease by at most 1 each guess), and then by Lemma 2 another k guesses are required to guarantee Zara a win. So we require at least $1 + E(A_1)$ guesses to win (the extra 1 is the first guess). If z_1 is anything other than 1006, then $1006 \in A_1$ and then $E(A_1) = 1008$. If $z_1 = 1006$, then either $A_1 = [1, 1005]$ or $A_1 = [1007, 2011]$. In either case $E(A_1) = 1007$, so 1007 in the minimum possible value of $E(A_1)$. Therefore we need at least $1 + 1007 = 1008$ guesses.

The algorithm at the beginning describes how 1008 guesses will guarantee a win, so we are done.

6/2/22. The roving rational robot rolls along the rational number line. On each turn, if the robot is at $\frac{p}{q}$, he selects a positive integer n and rolls to $\frac{p+nq}{q+np}$. The robot begins at the rational number 2011. Can the roving rational robot ever reach the rational number 2?

Solution 1. Let n_j be the integer the robot chooses for step j. Let a_j and b_j the sequences defined recursively by $a_0 = 2011$, $b_0 = 1$, and

$$\begin{aligned} a_j &= a_{j-1} + n_j b_{j-1}, \\ b_j &= b_{j-1} + n_j a_{j-1}. \end{aligned}$$

Then at step j, the robot's location is $\frac{a_j}{b_j}$. Notice now that

$$\begin{aligned} a_j + b_j &= (1 + n_j)(a_{j-1} + b_{j-1}), \\ a_j - b_j &= (1 - n_j)(a_{j-1} - b_{j-1}). \end{aligned}$$

Let $\Pi_+ = \prod_{j=1}^{k}(1 + n_j)$ and $\Pi_- = \prod_{j=1}^{k}(1 - n_j)$, so

$$\begin{aligned} a_k + b_k &= \Pi_+(a_0 + b_0), \\ a_k - b_k &= \Pi_-(a_0 - b_0). \end{aligned}$$

Also notice that $a_0 + b_0 = 2012$ and $a_0 - b_0 = 2010$. Therefore

$$\begin{aligned} 2a_k &= 2012\Pi_+ + 2010\Pi_-, \\ 2b_k &= 2012\Pi_+ - 2010\Pi_-. \end{aligned}$$

At step k the robot's location is

$$\frac{2a_k}{2b_k} = \frac{2012\Pi_+ + 2010\Pi_-}{2012\Pi_+ - 2010\Pi_-}.$$

If this is equal to 2, then

$$2012\Pi_+ + 2010\Pi_- = 2(2012\Pi_+ - 2010\Pi_-),$$

or

$$3 \cdot 2010\Pi_- = 2012\Pi_+.$$

We need to determine whether there is some set of positive integers $n_1, \ldots, n_k$ for some k such that

$$3 \cdot (2 \cdot 3 \cdot 5 \cdot 67) \prod_{j=1}^{k}(1 - n_j) = (2 \cdot 2 \cdot 503) \prod_{j=1}^{k}(1 + n_j).$$

Clearly k must be even so that the left side of the above is positive. First we eliminate the powers of two by choosing $n_1 = 5$. Then we want to solve

$$3 \cdot (2 \cdot 3 \cdot 5 \cdot 67)(1-5) \prod_{j=2}^{n} (1-n_j) = (2 \cdot 2 \cdot 503)(1+5) \prod_{j=2}^{n} (1+n_j).$$

or

$$-3 \cdot 2 \cdot 3 \cdot 5 \cdot 67 \cdot 4 \prod_{j=2}^{n} (1-n_j) = 2 \cdot 2 \cdot 503 \cdot 6 \prod_{j=2}^{n} (1+n_j).$$

Cancellation gives

$$-3 \cdot 5 \cdot 67 \prod_{j=2}^{n} (1-n_j) = 503 \prod_{j=2}^{n} (1+n_j)$$

Now we choose our n_j to reduce the size of the largest primes. For example if $n_2 = 504$, then we reduce to solving

$$-3 \cdot 5 \cdot 67(1-504) \prod_{j=3}^{n} (1-n_j) = 503(1+504) \prod_{j=3}^{n} (1+n_j)$$

which simplifies to

$$3 \cdot 5 \cdot 67 \prod_{j=3}^{n} (1-n_j) = 5 \cdot 101 \prod_{j=3}^{n} (1+n_j)$$

Choosing the sequence for the n_j

$$5, 504, 102, 104, 66, 12, 10, 8$$

gives the products

$$\begin{aligned} 3 \cdot (2 \cdot 3 \cdot 5 \cdot 67) \prod (1-c_j) &= (-1)^8 \quad 3 \cdot (2 \cdot 3 \cdot 5 \cdot 67) \cdot (4 \cdot 503 \cdot 101 \cdot 103 \cdot 65 \cdot 11 \cdot 9 \cdot 7) \\ (2 \cdot 2 \cdot 503) \prod (1+c_j) &= \qquad\qquad (2 \cdot 2 \cdot 503) \cdot (6 \cdot 505 \cdot 103 \cdot 105 \cdot 67 \cdot 13 \cdot 11 \cdot 9). \end{aligned}$$

Since these are equal, this sequence gives a valid solution. Explicitly the rationals the roving rational robot sees are

$$\frac{2011}{2} \xrightarrow{5} \frac{84}{419} \xrightarrow{504} \frac{84}{17} \xrightarrow{102} \frac{18}{85} \xrightarrow{104} \frac{86}{19} \xrightarrow{66} \frac{4}{17} \xrightarrow{12} \frac{16}{5} \xrightarrow{10} \frac{2}{5} \xrightarrow{8} \frac{2}{1}$$

Solution 2. Suppose the robot is at the rational number p, where p is a positive integer, $p \geq 2$. Taking $n = p^2 - p - 1$, the next number is

$$\frac{p+n}{1+np} = \frac{p^2-1}{p^3-p^2-p+1} = \frac{p^2-1}{(p^2-1)(p-1)} = \frac{1}{p-1}.$$

Next, suppose the robot is at the rational number $1/q$, where q is a positive integer, $q \geq 2$. Taking $n = q^2 - q - 1$, the next number is

$$\frac{1+nq}{q+n} = \frac{q^3-q^2-q+1}{q^2-1} = \frac{(q^2-1)(q-1)}{q^2-1} = q-1.$$

Hence, the robot can take the path

$$2011 \to \frac{1}{2010} \to 2009 \to \frac{1}{2008} \to \cdots \to \frac{1}{4} \to 3.$$

When the robot is at 3, taking $n = 9$, the next number is

$$\frac{3+9\cdot 1}{1+9\cdot 3} = \frac{12}{28} = \frac{3}{7},$$

and taking $n = 11$, the next number is

$$\frac{3+11\cdot 7}{7+11\cdot 3} = \frac{80}{40} = 2.$$

Credits: Problem 1/2/22 was proposed by Palmer Mebane.
All other problems and solutions by USAMTS staff.