Stanford Mathematics PhD Qualifying Exam
Algebra – Fall 2005
Morning Session

1. Let p and q be primes with $p, q \neq 2$ and suppose that p divides $q+1$.

(a) Show that there exists a nonabelian group G of order pq^2 whose Sylow q-subgroup is not cyclic.

(b) Show that if G is a nonabelian group of order pq^2 then it has a normal q-Sylow subgroup Q, and if Q is not cyclic then a p-Sylow subgroup of $\mathrm{Aut}(Q)$ is cyclic.

(c) Show that any two nonabelian groups of order pq^2 with noncyclic Sylow q-subgroups are isomorphic.

2. Suppose that $f(t) \in \mathbb{Q}[t]$ is an irreducible polynomial of degree 5 with exactly 3 real roots. Let K be the splitting field of f over $\mathbb{Q}$. Show that $\mathrm{Gal}(K/\mathbb{Q}) \cong S_5$. Prove any nonobvious facts you use about S_5.

3. Let A be a commutative ring. The ring A is called *Artinian* if it satisfies the *decreasing chain condition:* if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ is a sequence of ideals then for some N we have $I_N = I_{N+1} = I_{N+2} = \dots$.

(a) If A is an Artinian integral domain show that A is a field.

(b) If A is an Artinian commutative ring, show that every prime ideal in A is maximal.

4. Let G be a group of odd order. Prove that if χ is a complex irreducible character of G and $\chi(g)$ is real for all $g \in G$ then $\chi = 1$. (**Hint:** Consider the value of $\sum_{g \neq 1} \chi(g)$ and the fact that $g \neq g^{-1}$ when $g \neq 1$. Think about algebraic integers.)

5. Let T, $U \in \mathrm{Mat}_n(F)$ where F is any field. Prove that if T and U are nilpotent matrices and $\mathrm{rank}(T^k) = \mathrm{rank}(U^k)$ for all k, then $T = AUA^{-1}$ for some $A \in \mathrm{Mat}_n(F)$.

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Afternoon Session

1. Let G be the group of order 18 with generators x, y, z subject to relations

$$x^3 = y^3 = z^2 = 1, \qquad xy = yx, \qquad zxz^{-1} = y, \qquad zyz^{-1} = x.$$

Determine the conjugacy classes of G and compute its character table.

2. Let p be an odd prime and $\zeta = e^{2\pi i/p}$. Show that there exists a unique subfield K of $\mathbb{Q}(\zeta)$ such that $[K\colon \mathbb{Q}] = 2$. Let $\chi\colon (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \{\pm 1\}$ be the unique nontrivial homomorphism and let

$$\alpha = \sum_{a=1}^{p-1} \chi(a)\zeta^a.$$

Show that $\alpha^2 = (-1)^{(p-1)/2} p$ and conclude that $K = \mathbb{Q}(\alpha)$.

3. Let A be an Noetherian integral domain with field of fractions F. If $f \in A$ is not a unit, prove that the ring $A[f^{-1}]$ generated by f^{-1} and A is not a finitely-generated A-module.

4. Let G be a finite group of odd order. Prove that if $g \in G$ is conjugate to g^{-1} then $g = 1$.

5. Let $n > 1$ be odd. Let A and B be matrices in $\mathrm{GL}_2(\mathbb{C})$ such that $A^n = 1$, $BAB^{-1} = A^{-1}$ and $A \neq I$. Suppose that X commutes with both A and B. Prove that X is a scalar matrix.