Real Analysis Exam: Part I (Spring 1998)

Do all five problems.

1. Suppose $f_n, f \in \mathcal{L}^2([0,1])$ with $f_n \to f$ in measure and with $\|f_n\|_{\mathcal{L}^2} \le 1$ for all $n \ge 1$.
   (a) Prove (i) that $f_n \to f$ weakly in $\mathcal{L}^2$, and (ii) that $\|f_n - f\|_{\mathcal{L}^2} \to 0$ if and only if $\|f_n\|_{\mathcal{L}^2} \to \|f\|_{\mathcal{L}^2}$.
   (b) Prove that $\|f_n - f\|_{\mathcal{L}^p} \to 0$ for each $p \in [1,2)$.

2. Let $f : \mathbf{R} \to \mathbf{R}$ be nowhere continuous. Prove that there is an $\epsilon > 0$ and a nonempty open interval (a,b) such that $\limsup_{t\to x} f(t) - \liminf_{t\to x} f(t) \ge \epsilon$ for every $x \in (a,b)$.

3. Consider the function $f(x) = \sum_{n=1}^{\infty} 10^{-n}\cos(10^{2n}x)$.
   (a) Prove that there is a $C > 0$ such that $|f(x) - f(y)| \le C|x-y|^{1/2}$ for every $x, y \in \mathbf{R}$.
   (b) Prove that f is nowhere differentiable.

Hint: consider the choices $h = 10^{-2m}\pi$ and $h = 10^{-2m}\frac{\pi}{2}$ in $\frac{f(x+h)-f(x)}{h}$, $m = 1, 2, \ldots$.

4. If S is an uncountable subset of $C([0,1])$, prove that there is a uniformly convergent sequence $\{f_n\}_{n=1,2,\ldots}$ of distinct functions in S.

5. Suppose μ is an outer measure on a separable space X such that all Borel sets are μ-measurable and such that $\mu(X) < \infty$. Let

$$S_\infty = \{x \in X : \limsup_{\rho\to 0} \rho^{-1}\mu(\mathbf{B}_\rho(x)) = \infty\}.$$

Prove that S_∞ has 1-dimensional measure 0. That is, prove that for each $\epsilon > 0$, there is a covering of S_∞ by balls $\mathbf{B}_{\rho_j}(x_j)$ such that $\sum_j \rho_j < \epsilon$.

Note: you may assume without proof the "five times covering lemma", which says that if $\mathcal{B}$ is a collection of closed balls in X such that

$$\sup\{\operatorname{diam} B : B \in \mathcal{B}\} < \infty,$$

then there is a countable pairwise disjoint subcollection $\{B_{\rho_j}(x_j)\} \subset \mathcal{B}$ such that

$$\cup_{B\in\mathcal{B}} B \subset \cup_j B_{5\rho_j}(x_j).$$

REAL ANALYSIS EXAM: PART II (SPRING 1998)

Do all five problems.

1. Two quickies:
   (i) Let E be a compact subset of $C([0,1])$, where $C([0,1])$ is equipped with the usual sup norm. Prove that E is equicontinuous.
   (ii) Let $f = f_1 + if_2$ be a complex-valued $\mathcal{L}^1(\mathbf{R})$ function. Prove that $|\int_{\mathbf{R}} f(x)\,dx| \le \int_{\mathbf{R}} |f(x)|\,dx$. Also, if $\int_{\mathbf{R}} |f(x)|\,dx = 1$ and if $\hat{f}(\xi) = \int_{\mathbf{R}} e^{-ix\xi} f(x)\,dx$, prove the strict inequality $|\hat{f}(\xi)| < 1$ for all but possibly one value of $\xi \in \mathbf{R}$.
2. For $t > 0$, let $F(t) = \chi_{[0,t]}$ (the indicator function of the interval $[0,t]$).
   (a) Prove that F, as a map from $(0,\infty)$ to $\mathcal{L}^2(\mathbf{R})$, is nowhere differentiable. (That is, $\lim_{h\downarrow 0} \frac{F(t+h)-F(t)}{h}$ never exists as a limit taken in $\mathcal{L}^2(\mathbf{R})$ for $t \in (0,\infty)$.)
   (b) If $g : \mathcal{L}^2(\mathbf{R}) \to \mathbf{R}$ is a bounded linear functional, prove that $g \circ F$ is differentiable almost everywhere on $(0,\infty)$.
3. In this problem, let $\mathcal{L}^2([0,1])$ denote the complex Hilbert space of square integrable complex-valued functions f on $[0,1]$ with the usual inner product

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g}(t)\,dt.$$

Define $T : \mathcal{L}^2([0,1]) \to \mathcal{L}^2([0,1])$ by $Tf(x) = \int_0^x f(t)\,dt$ for $x \in [0,1]$.
   (a) Prove that T is a compact continuous map.
   (b) Prove that T has no eigenvalues. That is, prove there is no $\lambda \in \mathbf{C}$ such that $T(f) = \lambda f$ for some nonzero $f \in \mathcal{L}^2([0,1])$.
   (c) Prove that the spectrum of T is $\{0\}$. That is, $f \mapsto T(f) - \lambda f$ is an isomorphism of $\mathcal{L}^2([0,1])$ onto $\mathcal{L}^2([0,1])$ for each nonzero λ, and it is not such an isomorphism for $\lambda = 0$.
4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is 2π-periodic and $f \in \mathcal{L}^2(-\pi,\pi)$. Supose also that

$$\sum_{n\in\mathbf{Z}} (|\hat{f}(n)|n^k)^2 < \infty$$

for some integer $k \ge 1$. Prove that f is almost everywhere equal to a $C(\mathbf{R})$ function if $k = 1$ and to a $C^{k-1}(\mathbf{R})$ function if $k \ge 2$.

5.(a) Let $\{x_n\}_{n=1,2,\ldots}$ be a sequence in a Banach space B, and let X be the convex hull. In other words,

$$X = \left\{ \sum_{j=1}^N \lambda_j x_j : N \ge 1,\ \lambda_j \in [0,1],\ \sum_{j=1}^N \lambda_j = 1 \right\}.$$

If x_n converges weakly to some x in B, prove that some sequence $\{y_n\} \subset X$ converges to x strongly (i.e., $\|y_n - x\| \to 0$).

(b) Give an example of a Hilbert space H and a sequence x_n weakly converging to zero in H such that $\|n^{-1}\sum_{j=1}^n x_j\|$ does not converge to zero as $n \to \infty$.