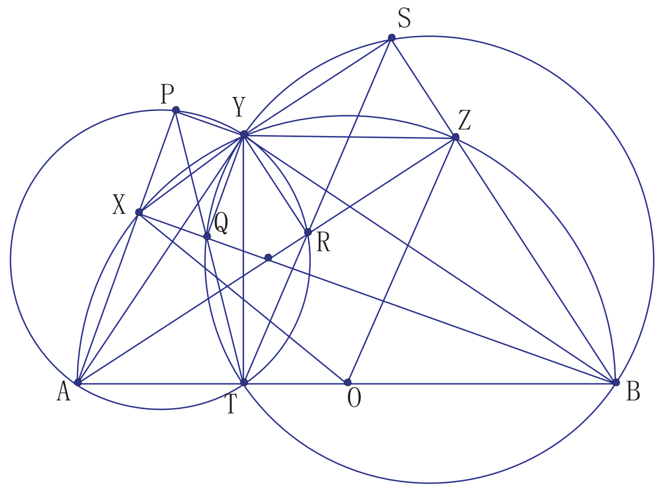
39$^{\text{th}}$ **United States of America Mathematical Olympiad 2010**

1. **Solution by Titu Andreescu:** Let T be the foot of the perpendicular from Y to line AB. We note the P, Q, T are the feet of the perpendiculars from Y to the sides of triangle ABX. Because Y lies on the circumcircle of triangle ABX, points P, Q, T are collinear, by Simson's theorem. Likewise, points S, R, T are collinear.



We need to show that $\angle XOZ = 2\angle PTS$ or

$$\angle PTS = \frac{\angle XOZ}{2} = \frac{\overset{\frown}{XZ}}{2} = \frac{\overset{\frown}{XY}}{2} + \frac{\overset{\frown}{YZ}}{2}$$
$$= \angle XAY + \angle ZBY = \angle PAY + \angle SBY.$$

Because $\angle PTS = \angle PTY + \angle STY$, it suffices to prove that

$$\angle PTY = \angle PAY \quad \text{and} \quad \angle STY = \angle SBY;$$

that is, to show that quadrilaterals $APYT$ and $BSYT$ are cyclic, which is evident, because $\angle APY = \angle ATY = 90^\circ$ and $\angle BTY = \angle BSY = 90^\circ$.

Alternate Solution from Lenny Ng and Richard Stong: Since YQ, YR are perpendicular to BX, AZ respectively, $\angle RYQ$ is equal to the acute angle between lines BX and AZ, which is $\frac{1}{2}(\widehat{AX} + \widehat{BZ}) = \frac{1}{2}(180^\circ - \widehat{XZ})$ since X, Z lie on the circle with diameter AB. Also, $\angle AXB = \angle AZB = 90^\circ$ and so $PXQY$ and $SZRY$ are rectangles, whence $\angle PQY = 90^\circ - \angle YXB = 90^\circ - \widehat{YB}/2$ and $\angle YRS = 90^\circ - \angle AZY = 90^\circ - \widehat{AY}/2$. Finally, the angle between PQ and RS is

$$\begin{aligned}\angle PQY + \angle YRS - \angle RYQ &= (90^\circ - \widehat{YB}/2) + (90^\circ - \widehat{AY}/2) - (90^\circ - \widehat{XZ}/2)\\ &= \widehat{XZ}/2\\ &= (\angle XOZ)/2,\end{aligned}$$

as desired.

This problem was proposed by Titu Andreescu.

2. **Solution from Kiran Kedlaya:** Let h_i also denote the student with height h_i. We prove that for $1 \le i < j \le n$, h_j can switch with h_i at most $j-i-1$ times. We proceed by induction on $j-i$, the base case $j-i=1$ being evident because h_i is not allowed to switch with h_{i-1}.

   For the inductive step, note that h_i, h_{j-1}, h_j can be positioned on the circle either in this order or in the order h_i, h_j, h_{j-1}. Since h_{j-1} and h_j cannot switch, the only way to change the relative order of these three students is for h_i to switch with either h_{j-1} or h_j. Consequently, any two switches of h_i with h_j must be separated by a switch of h_i with h_{j-1}. Since there are at most $j-i-2$ of the latter, there are at most $j-i-1$ of the former.

   The total number of switches is thus at most

$$\begin{aligned}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(j-i-1) &= \sum_{i=1}^{n-1}\sum_{j=0}^{n-i-1} j \\ &= \sum_{i=1}^{n-1}\binom{n-i}{2} \\ &= \sum_{i=1}^{n-1}\left(\binom{n-i+1}{3}-\binom{n-i}{3}\right) \\ &= \binom{n}{3}.\end{aligned}$$

   Note: One can also ask to prove that the number of switches before no more are possible depends only on the original ordering, or to find all initial positions for which $\binom{n}{3}$ switches are possible (the only one is when the students are sorted in increasing order).

   Alternative Solution from Warut Suksompong: For $i = 1, 2, \ldots, n-1$, let s_i be the number of students with height no more than h_{i+1} standing (possibly not directly) behind the student with height h_i and (possibly not directly) in front of the one with height h_{i+1}. Note that $s_i \le i-1$ for all i.

   Now we take a look what happens when two students switch places.

   - If the student with height h_n is involved in the switch, s_{n-1} decreases by 1, while all the other s_i's remain the same.

- Otherwise, suppose the students with heights h_a and h_b are switched, with $a+1 < b < n$, then s_{b-1} decreases by 1, while s_b increases by 1. All the other s_i's remain the same.

Since $s_i \leq i-1$ for all $i = 1, 2, \ldots, n-1$, the maximal number of switches is no more than the number of switches in the case where initially $s_i = i-1$ for all i. In that case, the number of switches is $\sum_{i=1}^{n-2} i(n-1-i) = \binom{n}{3}$.

Note: With this solution, it is also easy to see that the number of switches until no more are possible depends only on the original ordering.

This problem was proposed by Kiran Kedlaya jointly with Travis Schedler and David Speyer.

3. **Solution from Gabriel Carroll:** Multiplying together the inequalities $a_{2i-1}a_{2i} \leq 4i-1$ for $i = 1, 2, \ldots, 1005$, we get

$$a_1 a_2 \cdots a_{2010} \leq 3 \cdot 7 \cdot 11 \cdots 4019. \tag{1}$$

The tricky part is to show that this bound can be attained.

Let

$$a_{2008} = \sqrt{\frac{4017 \cdot 4018}{4019}}, \qquad a_{2009} = \sqrt{\frac{4019 \cdot 4017}{4018}}, \qquad a_{2010} = \sqrt{\frac{4018 \cdot 4019}{4017}},$$

and define a_i for $i < 2008$ by downward induction using the recursion

$$a_i = (2i+1)/a_{i+1}.$$

We then have

$$a_i a_j = i + j \qquad \text{whenever } j = i+1 \qquad \text{or} \qquad i = 2008, j = 2010. \tag{2}$$

We will show that (2) implies $a_i a_j \leq i + j$ for all $i < j$, so that this sequence satisfies the hypotheses of the problem. Since $a_{2i-1}a_{2i} = 4i-1$ for $i = 1, \ldots, 1005$, the inequality (1) is an equality, so the bound is attained.

We show that $a_i a_j \leq i + j$ for $i < j$ by downward induction on $i + j$. There are several cases:

- If $j = i+1$, or $i = 2008, j = 2010$, then $a_i a_j = i + j$, from (2).

- If $i = 2007, j = 2009$, then

$$a_i a_{i+2} = \frac{(a_i a_{i+1})(a_{i+2} a_{i+3})}{(a_{i+1} a_{i+3})} = \frac{(2i+1)(2i+5)}{2i+4} < 2i+2.$$

  Here the second equality comes from (2), and the inequality is checked by multiplying out: $(2i+1)(2i+5) = 4i^2 + 12i + 5 < 4i^2 + 12i + 8 = (2i+2)(2i+4)$.

- If $i < 2007$ and $j = i + 2$, then we have

$$a_i a_{i+2} = \frac{(a_i a_{i+1})(a_{i+2} a_{i+3})(a_{i+2} a_{i+4})}{(a_{i+1} a_{i+2})(a_{i+3} a_{i+4})} \leq \frac{(2i+1)(2i+5)(2i+6)}{(2i+3)(2i+7)} < 2i+2.$$

  The first inequality holds by applying the induction hypothesis for $(i+2, i+4)$, and (2) for the other pairs. The second inequality can again be checked by multiplying out: $(2i+1)(2i+5)(2i+6) = 8i^3 + 48i^2 + 82i + 30 < 8i^3 + 48i^2 + 82i + 42 = (2i+2)(2i+3)(2i+7)$.

- If $j - i > 2$, then

$$a_i a_j = \frac{(a_i a_{i+1})(a_{i+2} a_j)}{a_{i+1} a_{i+2}} \leq \frac{(2i+1)(i+2+j)}{2i+3} < i+j.$$

  Here we have used the induction hypothesis for $(i+2, j)$, and again we check the last inequality by multiplying out: $(2i+1)(i+2+j) = 2i^2 + 5i + 2 + 2ij + j < 2i^2 + 3i + 2ij + 3j = (2i+3)(i+j)$.

This covers all the cases and shows that $a_i a_j \leq i + j$ for all $i < j$, as required.

Variant Solution by Paul Zeitz: It is possible to come up with a semi-alternative solution, after constructing the sequence, by observing that when the two indices differ by an even number, you can divide out precisely. For example, if you wanted to look at $a_3 a_8$, you would use the fact that $a_3 a_4 a_5 a_6 a_7 a_8 = (7)(11)(15)$ and $a_4 a_5 a_6 a_7 = (9)(13)$. Hence we need to check that $(7)(11)(15)/((9)(13)) < 11$, which is easy AMGM/ Symmetry.

However, this attractive method requires much more subtlety when the indices differ by an odd number. It can be pulled off, but now you need, as far as I know, either to use the precise value of a_{2010} or establish inequalities for $(a_k)^2$ for all values of k. It is ugly, but it may be attempted.

This problem was suggested by Gabriel Carroll.

4. **Solution from Zuming Feng:** The answer is *no*, it is not possible for segments AB, BC, BI, ID, CI, IE to all have integer lengths.

Assume on the contrary that these segments do have integer side lengths. We set $\alpha = \angle ABD = \angle DBC$ and $\beta = \angle ACE = \angle ECB$. Note that I is the incenter of triangle ABC, and so $\angle BAI = \angle CAI = 45^\circ$. Applying the Law of Sines to triangle ABI yields

$$\frac{AB}{BI} = \frac{\sin(45^\circ + \alpha)}{\sin 45^\circ} = \sin\alpha + \cos\alpha,$$

by the addition formula (for the sine function). In particular, we conclude that $s = \sin\alpha + \cos\alpha$ is rational. It is clear that $\alpha + \beta = 45^\circ$. By the subtraction formulas, we have

$$s = \sin(45^\circ - \beta) + \cos(45^\circ - \beta) = \sqrt{2}\cos\beta,$$

from which it follows that $\cos\beta$ is not rational. On the other hand, from right triangle ACE, we have $\cos\beta = AC/EC$, which is rational by assumption. Because $\cos\beta$ cannot not be both rational and irrational, our assumption was wrong and not all the segments AB, BC, BI, ID, CI, IE can have integer lengths.

Alternate Solution from Jacek Fabrykowski: Using notations as introduced in the problem, let $BD = m$, $AD = x$, $DC = y$, $AB = c$, $BC = a$ and $AC = b$. The angle bisector theorem implies

$$\frac{x}{b - x} = \frac{c}{a}$$

and the Pythagorean Theorem yields $m^2 = x^2 + c^2$. Both equations imply that

$$2ac = \frac{(bc)^2}{m^2 - c^2} - a^2 - c^2$$

and since $a^2 = b^2 + c^2$ is rational, a is rational too (observe that to reach this conclusion, we only need to assume that b, c, and m are integers). Therefore, $x = \frac{bc}{a+c}$ is also rational, and so is y. Let now (similarly to the notations above from the solution by Zuming Feng) $\angle ABD = \alpha$ and $\angle ACE = \beta$ where $\alpha + \beta = \pi/4$. It is obvious that $\cos\alpha$ and $\cos\beta$are both rational and the above shows that also $\sin\alpha = x/m$ is rational. On the other hand, $\cos\beta = \cos(\pi/4 - \alpha) = (\sqrt{2}/2)(\sin\alpha + \sin\beta)$, which is a contradiction. The solution shows that a stronger statement holds true: There is no right triangle with both legs and bisectors of acute angles all having integer lengths.

Alternate Solution from Zuming Feng: Prove an even stronger result: there is no such right triangle with AB, AC, IB, IC having rational side lengths. Assume on the contrary, that AB, AC, IB, IC have rational side lengths. Then $BC^2 = AB^2 + AC^2$ is rational. On the other hand, in triangle BIC, $\angle BIC = 135^\circ$. Applying the law of cosines to triangle BIC yields

$$BC^2 = BI^2 + CI^2 - \sqrt{2}BI \cdot CI$$

which is irrational. Because BC^2 cannot be both rational and irrational, we conclude that our assumption was wrong and that not all of the segments AB, AC, IB, IC can have rational lengths.

This problem was proposed by Zuming Feng.

5. **Solution by Titu Andreescu:** We have

$$\begin{aligned}\frac{2}{k(k+1)(k+2)} = \frac{(k+2)-k}{k(k+1)(k+2)} &= \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} \\ &= \frac{1}{k} - \frac{1}{k+1} - \left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\ &= \frac{1}{k} + \frac{1}{k+1} + \frac{1}{k+2} - \frac{3}{k+1}.\end{aligned}$$

Hence

$$\begin{aligned}2S_q &= \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{q} + \frac{1}{q+1} + \frac{1}{q+2}\right) - 3\left(\frac{1}{3} + \frac{1}{6} + \ldots + \frac{1}{q+1}\right) \\ &= \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{\frac{3p-1}{2}}\right) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{\frac{p-1}{2}}\right),\end{aligned}$$

and so

$$\begin{aligned}1 - \frac{m}{n} = 1 + 2S_q - \frac{1}{p} &= \frac{1}{\frac{p+1}{2}} + \ldots + \frac{1}{p-1} + \frac{1}{p+1} + \ldots + \frac{1}{\frac{3p-1}{2}} \\ &= \left(\frac{1}{\frac{p+1}{2}} + \frac{1}{\frac{3p-1}{2}}\right) + \ldots + \left(\frac{1}{p-1} + \frac{1}{p+1}\right) \\ &= \frac{p}{\left(\frac{p+1}{2}\right)\left(\frac{3p-1}{2}\right)} + \ldots + \frac{p}{(p-1)(p+1)}.\end{aligned}$$

Because all denominators are relatively prime with p, it follows that $n - m$ is divisible by p and we are done.

This problem was suggested by Titu Andreescu.

6. **Solution by Zuming Feng and Paul Zeitz:** The answer is 43.

We first show that we can always get 43 points. Without loss of generality, we assume that the value of x is positive for every pair of the form (x, x) (otherwise, replace every occurrence of x on the blackboard by $-x$, and every occurrence of $-x$ by x). Consider the ordered n-tuple $(a_1, a_2, \ldots, a_n)$ where $a_1, a_2, \ldots, a_n$ denote all the distinct absolute values of the integers written on the board.

Let $\phi = \frac{\sqrt{5}-1}{2}$, which is the positive root of $\phi^2 + \phi = 1$. We consider 2^n possible underlining strategies: Every strategy corresponds to an ordered n-tuple $s = (s_1, \ldots, s_n)$ with $s_i = \phi$

or $s_i = 1-\phi$ $(1 \le i \le n)$. If $s_i = \phi$, then we underline all occurrences of a_i on the blackboard. If $s_i = 1-\phi$, then we underline all occurrences of $-a_i$ on the blackboard. The weight $w(s)$ of strategy s equals the product $\prod_{i=1}^n s_i$. It is easy to see that the sum of weights of all 2^n strategies is equal to $\sum_s w(s) = \prod_{i=1}^n [\phi + (1-\phi)] = 1$.

For every pair p on the blackboard and every strategy s, we define a corresponding cost coefficient $c(p,s)$: If s scores a point on p, then $c(p,s)$ equals the weight $w(s)$. If s does not score on p, then $c(p,s)$ equals 0. Let $c(p)$ denote the the sum of of coefficients $c(p,s)$ taken over all s. Now consider a fixed pair $p=(x,y)$:

(a) In this case, we assume that $x = y = a_j$. Then every strategy that underlines a_j scores a point on this pair. Then $c(p) = \phi \prod_{i\ne j}^n [\phi + (1-\phi)] = \phi$.

(b) In this case, we assume that $x \ne y$. We have

$$c(p) = \begin{cases} \phi^2 + \phi(1-\phi) + (1-\phi)\phi = 3\phi - 1, & (x,y) = (a_k, a_\ell); \\ \phi(1-\phi) + (1-\phi)\phi + (1-\phi)^2 = \phi, & (x,y) = (-a_k, -a_\ell); \\ \phi^2 + \phi(1-\phi) + (1-\phi)^2 = 2 - 2\phi, & (x,y) = (\pm a_k, \mp a_\ell). \end{cases}$$

By noting that $\phi \approx 0.618$, we can easily conclude that $c(p) \ge \phi$.

We let C denote the sum of the coefficients $c(p,s)$ taken over all p and s. These observations yield that

$$C = \sum_{p,s} c(p,s) = \sum_p c(p) \ge \sum_p \phi = 68\phi > 42.$$

Suppose for the sake of contradiction that every strategy s scores at most 42 points. Then every s contributes at most $42w(s)$ to C, and we get $C \le 42\sum_s w(s) = 42$, which contradicts $C > 42$.

To complete our proof, we now show that we cannot always get 44 points. Consider the blackboard contains the following 68 pairs: For each of $m = 1, \ldots, 8$, five pairs of (m,m) (for a total of 40 pairs of type (a)); For every $1 \le m < n \le 8$, one pair of $(-m,-n)$ (for a total of $\binom{8}{2} = 28$ pairs of type (b)). We claim that we cannot get 44 points from this initial stage. Indeed, assume that exactly k of the integers $1, 2, \ldots, 8$ are underlined. Then we get at most $5k$ points on the pairs of type (a), and at most $28 - \binom{k}{2}$ points on the pairs of type (b). We can get at most $5k + 28 - \binom{k}{2}$ points. Note that the quadratic function $5k + 28 - \binom{k}{2} = -\frac{k^2}{2} + \frac{11k}{2} + 28$ obtains its maximum 43 (for integers k) at $k = 5$ or $k = 6$. Thus, we can get at most 43 points with this initial distribution, establishing our claim and completing our solution.

This problem was suggested by Zuming Feng.