Mathematics Department, Stanford University
Real Analysis Qualifying Exam, Autumn 1998—Part I

DO ALL FIVE PROBLEMS
(USE A DIFFERENT BLUE BOOK FOR EACH PROBLEM)

1. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable at 0, and suppose a_n and b_n are sequences converging to 0 with $a_n < b_n$ for all n.

(a) If $a_n < 0 < b_n$ for all n, prove that

$$* \qquad \frac{f(b_n) - f(a_n)}{b_n - a_n} \to f'(0).$$

(b) Prove that if f is differentiable in a neighborhood of 0 and if f' is continuous at 0, then (*) holds for all sequences a_n and b_n converging to 0 with $b_n > a_n$.

(c) Give an example of a function f such that (*) holds for all sequences $a_n < b_n$ tending to 0, but for which there are points arbitrarily close to 0 at which f is not differentiable.

2. Suppose $f : [0,1] \to \mathbf{R}$ is a bounded Lebesgue measurable function. Suppose for every $x \in [0,1]$ there is a function g_x such that

$$f = g_x \qquad a.e.$$

and such that

$$\lim_{t \to x} g_x(t) \text{ exists.}$$

Prove that there is a continuous function g such that $g = f$ almost everywhere.

3. Let $f : \mathbf{R} \to \mathbf{R}$ be an L^1 function. Show that f and its Fourier transform cannot both have compact support (unless $f = 0$ a.e.).

4. Let X be an infinite-dimensional Banach space.

(a) Let S be a subset of X such that the linear span of $S \subset X$ (that is, the set of all linear combinations of finite subsets of X) is all of X. Prove that S is uncountable.

(b) Suppose the dual space X^* of X is separable. Prove that X is separable.

(c) Let P be a finite-dimensional subspace of X. Prove that there is a bounded linear projection $\pi : X \to P$ (in other words, prove that there is a bounded linear operator $\pi : X \to P$ such that $\pi(x) = x$ if $x \in P$.)

5. Let $f_n : [0,1] \to \mathbf{R}$ be a sequence of continuous functions converging pointwise to a continuous function g. Suppose $f_n(x) \geq f_{n+1}(x)$ for every n and every $x \in [0,1]$. Prove that $f_n \to f$ uniformly.

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Do all five problems
(Use a different blue book for each problem)

1. Let $f : \mathbf{R} \to \mathbf{R}$ be a convex function. That is, suppose

$$* \qquad f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $t \in (0,1)$ and for all x and y.

(a) Prove that f is continuous everywhere.

(b) Prove that f is differentiable except at a countable set of points.

(c) Suppose f is strictly convex. (That is, suppose the inequality (*) is strict whenever $x \ne y$ and $0 < t < 1$.) If $u : [0,1] \to \mathbf{R}$ is an L^1 function, Jensen's theorem says

$$f\left(\int u\right) \le \int f(u).$$

Prove that if we have equality, then u is equal a.e. to a constant function.

2. Suppose $f : (0,\infty) \to \mathbf{R}$ is a continuous function such that

$$\lim_{n\to\infty} f(n^2 x) = a$$

for every x. (Of course here n is an integer.) Prove that $\lim_{x\to\infty} f(x) = a$.

3. Suppose $f : [0,1] \to \mathbf{R}$ is a Lebesgue measurable function.

(a) Show that the image $\{f(x) : x \in [0,1]\}$ need not be a Lebesgue measurable set.

(b) Show that there is a function g which is equal to f almost everywhere and such that the image under g of any closed subset of $[0,1]$ is an F_σ set (i.e., a countable union of closed sets).

4. (a) If f and g are in $\mathcal{L}^2(\mathbf{T})$, prove that $f * g$ is continuous.
(b) Construct a continuous function g on $\mathbf{T}$ such that $g * g * \cdots * g$ (k times) is not in $C^1(\mathbf{T})$ for any k.

5. Let $-\infty < a < b < \infty$ and suppose $\mathcal{B}$ is a countable collection of closed subintervals of (a,b). Give the proof that there is a countable pairwise-disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ such that $\cup_{I\in\mathcal{B}'}\widetilde{I} \supset \cup_{I\in\mathcal{B}} I$. Here $\widetilde{I}$ denotes the "5-times enlargment" of I; thus if $I = [x-\rho, x+\rho]$ then $\widetilde{I} = [x - 5\rho, x + 5\rho]$.