Ph. D Qualifying Examination
Complex analysis–Spring 1999

Work all 6 problems. All problems have equal weight. Write each solution in a separate bluebook.

1. Let $f(x):(-\frac{1}{2},\infty)\to \mathbf{C}$ be a continuous function. Suppose f is analytic in a neighborhood of the origin and that there is a positive constant N so that

$$\lim_{x\to\infty} f(x)e^{Nx}=0.$$

For the complex variable s we define

$$F(s)=\int_0^\infty f(x)x^s\,dx.$$

(a) Show that the integral converges for $\mathrm{Re}\,(s)>-1$ and that $F(s)$ has a meromorphic continuation to all s with possible poles only at $s=-1,-2,\cdots$.
(b) Determine the exact location of the poles of F and its *singular parts* at poles.

Hint: The answer to the second part depends on the coefficients of the Taylor expansion of f at $x=0$.

2. Let D be a bounded region in $\mathbf{C}$ whose boundary consists of n-smooth disjoint Jordan arcs. Thus D is n-connected. We denote by $\overline{D}$ the closure of D.

(a) Suppose $f(z)$ is a non-constant continuous function on $\overline{D}$ and is analytic in D. Suppose further that

(*) $$|f(w)|=1 \quad \text{for all}\, w\in\partial D.$$

Show that f has at least n zeros (counting multiplicities) in D.
(b) For any $n>0$, find an n-connected region D and an analytic function $f:D\to\mathbf{C}$ such that f satisfies (*) and has exactly n zeros in D.

Remark: You will receive partial credits if you work out the special case where $D=\{1<|z|<c\}$ in part (a) and/or (b).

3. Show that

$$\int_0^\infty \frac{x^{-c}}{1+x}dx=\frac{\pi}{\sin\pi c} \qquad \text{if } 0<c<1.$$

Remark: You need to provide details to justify each step in your computation.

4. Let D be the open unit disk and $f:D\to\mathbf{C}$ be a *bounded* analytic function.

(a) Let $\{a_n\}_{n\geq 1}$ be the non-zero zeros of f in D counted according to multiplicity. We assume $\{a_n\}$ is an infinite sequence. Then prove that

$$\sum_{n=1}^\infty (1-|a_n|)<\infty.$$

(b) Let f and $\{a_n\}$ be as above. We define

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n}\Big(\frac{a_n - z}{1 - \bar{a}_n z}\Big).$$

Show that $B(z)$ is a *bounded* analytic function on D with zeros $\{a_n\}$. Show further that there is an integer m and a bounded non-vanishing holomorphic function $h(z)$ so that

$$f(z) = z^m B(z) h(z).$$

5. Let D be the open unit disk and let f be a non-constant analytic function in D.

(a) Suppose for every $a \in \partial D \setminus \{1\}$ we have

$$* \qquad \lim_{z \to a} |f(z)| \leq 1$$

and for any $\delta > 0$ we have

$$** \qquad \lim_{z \to 1} |f(z)||z-1|^{\delta} = 0.$$

Show that $|f(z)| < 1$ in D.

(b) Construct an analytic function on the unit disk D that is *not* bounded in D and that satisfies $(*)$ but not $(**)$.

Hint: Consider $u(z) = (z-1)^{\delta} f(z)$.

6. Let ω_1 and ω_2 be two non-zero complex numbers with non-real ratio ω_1/ω_2. Let Λ be the lattice $\Lambda = \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$ and let a and b be two complex numbers not congruent to each other. We form the linear space V of all elliptic functions of period Λ with *at most* simple poles at a and b.

(a) Prove that $\dim_{\mathbf{C}} V$ is at most 2.

(b) Using the method of infinite series, construct explicitly a two dimensional families of elliptic functions in V, thereby proving that $\dim V = 2$.

Remark: In (b) one needs to provide details to why the series converges, why they have period Λ and why they provide a two dimensional family.