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567 Nice And Hard Inequalities

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1.

a) if a, b, c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \sqrt{\frac{a^2+1}{b^2+1}}+\sqrt{\frac{b^2+1}{c^2+1}}+\sqrt{\frac{c^2+1}{a^2+1}}.$$

b)Let a, b, c, d be positive real numbers.Prove that

$$\frac{a^2-bd}{b+2c+d}+\frac{b^2-ca}{c+2d+a}+\frac{c^2-db}{d+2a+b}+\frac{d^2-ac}{a+2b+c} \geq 0.$$

Solution:

a)By Cauchy-Schwarz's inequality, We have:

$$\left(a^2+b^2\right)\sqrt{\left(a^2+1\right)\left(b^2+1\right)} \geq \left(a^2+b^2\right)(ab+1)$$

$$= ab\left(a^2+b^2\right)+a^2+b^2 \geq ab\left(a^2+b^2+2\right)$$

$$\Rightarrow \sum \frac{a}{b}+\sum \frac{b}{a}=\sum \frac{a^2+b^2}{ab}$$

$$\geq \sum \frac{a^2+b^2+2}{\sqrt{\left(a^2+1\right)\left(b^2+1\right)}}=\sum \sqrt{\frac{a^2+1}{b^2+1}}+\sum \sqrt{\frac{b^2+1}{a^2+1}}$$

By Chebyshev's inequality, We have

$$\sum \frac{a^2}{b^2}=\sum \frac{a^2}{b^2+1}+\sum \frac{a^2}{b^2\left(b^2+1\right)} \geq \sum \frac{a^2}{b^2+1}+\sum \frac{b^2}{b^2\left(b^2+1\right)}=\sum \frac{a^2+1}{b^2+1}.$$

Therefore

$$\left(1+\sum \frac{a}{b}\right)^2=1+2\left(\sum \frac{a}{b}+\sum \frac{b}{a}\right)+\frac{a^2}{b^2}$$

$$\geq 1+2\left(\sum \sqrt{\frac{a^2+1}{b^2+1}}+\sum \sqrt{\frac{b^2+1}{a^2+1}}\right)+\sum \frac{a^2+1}{b^2+1}$$

$$=\left(1+\sum \sqrt{\frac{a^2+1}{b^2+1}}\right)^2.$$

Therefore

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \sqrt{\frac{a^2+1}{b^2+1}}+\sqrt{\frac{b^2+1}{c^2+1}}+\sqrt{\frac{c^2+1}{a^2+1}}$$

as require.

b)Notice that

$$\frac{2(a^2-bd)}{b+2c+d}+b+d=\frac{2a^2+b^2+d^2+2c(b+d)}{b+2c+d}$$

$$=\frac{(a-b)^2+(a-d)^2+2(a+c)(b+d)}{b+2c+d}(1)$$

And similarly,

$$\frac{2(c^2-db)}{d+2a+b}+b+d=\frac{(c-d)^2+(c-b)^2+2(a+c)(b+d)}{d+2a+b}(2)$$

Using Cauchy-Schwarz's inequality,we get

$$\frac{(a-d)^2}{b+2c+d}+\frac{(c-d)^2}{d+2a+b} \geq \frac{[(a-b)^2+(c-d)^2]}{(b+2c+d)+(d+2a+b)}(3)$$

$$\frac{(a-d)^2}{b+2c+d}+\frac{(c-b)^2}{d+2a+b}\geq\frac{[(a-d)^2+(c-b)^2]^2}{(b+2c+d)+(d+2a+b)}(4)$$

$$\frac{2(a+c)(b+d)}{b+2c+d}+\frac{2(a+c)(b+d)}{d+2a+b}\geq\frac{8(a+c)(b+d)}{(b+2c+d)+(d+2a+b)}(5)$$

From (1),(2),(3),(4) and (5), we get

$$2(\frac{a^2-bd}{b+2c+d}+\frac{c^2-db}{d+2a+b})+b+d\geq\frac{(a+c-b-d)^2+4(a+c)(b+d)}{a+b+c+d}=a+b+c+d.$$

or

$$\frac{a^2-bd}{b+2c+d}+\frac{c^2-db}{d+2a+b}\geq\frac{a+c-b-d}{2}$$

In the same manner,we can also show that

$$\frac{b^2-ca}{c+2d+a}+\frac{d^2-ac}{a+2b+c}\geq\frac{b+d-a-c}{2}$$

and by adding these two inequalities,we get the desired result.
Enquality holds if and only if $a=c$ and $b=d$.

2,

Let a,b,c be positive real numbers such that

$$a+b+c=1$$

Prove that the following inequality holds

$$\frac{ab}{1-c^2}+\frac{bc}{1-a^2}+\frac{ca}{1-b^2}\leq\frac{3}{8}$$

Solution: From the given condition The inequality is equivalent to

$$\sum\frac{4ab}{a^2+b^2+2(ab+bc+ca)}\leq\frac{3}{2}$$

but from Cauhy Shwarz inequality

$$\sum\frac{4ab}{a^2+b^2+2(ab+bc+ca)}$$

$$\leq\sum\left(\frac{ab}{a^2+ab+bc+ca}+\frac{ab}{b^2+ab+bc+ca}\right)$$

$$=\sum\frac{ab}{(a+b)(a+c)}+\sum\frac{ab}{(b+c)(a+b)}$$

$$=\sum\frac{a(b+c)^2}{(a+b)(b+c)(c+a)}$$

Thus We need prove that

$$3(a+b)(b+c)(c+a)\geq 2\sum a(b+c)^2$$

which reduces to the obvious inequality

$$\sum ab(a+b)\geq 6abc$$

The **Solution** is completed.with equality if and only if

$$a=b=c=\frac{1}{3}$$

Or We can use the fact that

$$\begin{aligned}\sum \frac{4ab}{a^2+b^2+2(ab+bc+ca)} &\le \sum \frac{4ab}{(2ab+2ac)+(2ab+2bc)}\\ &\le \sum \frac{ab}{2a(b+c)} + \sum \frac{ab}{2b(a+c)}\\ &= \frac{1}{2}\sum \left(\frac{b}{b+c}+\frac{a}{a+c}\right)\\ &= \frac{1}{2}\sum \left(\frac{b}{b+c}+\frac{c}{b+c}\right) = \frac{3}{2}\end{aligned}$$

3, Let a, b, c be the positive real numbers. Prove that

$$1+\frac{ab^2+bc^2+ca^2}{(ab+bc+ca)(a+b+c)} \ge \frac{4.\sqrt[3]{(a^2+ab+bc)(b^2+bc+ca)(c^2+ca+ab)}}{(a+b+c)^2}$$

Solution: Multiplying both sides of the above inequality with $(a+b+c)^2$ it's equivalent to prove that

$$\begin{aligned}&(a+b+c)^2+\frac{(a+b+c)(ab^2+bc^2+ca^2)}{ab+bc+ca}\\ &\ge 4.\sqrt[3]{(a^2+ab+bc)(b^2+bc+ca)(c^2+ca+ab)}\end{aligned}$$

We have

$$(a+b+c)^2+\frac{(a+b+c)(ab^2+bc^2+ca^2)}{ab+bc+ca} = \sum \frac{(a^2+ab+bc)(c+a)(c+b)}{ab+bc+ca}$$

By using AM-GM inequality We get

$$\sum \frac{(a^2+ab+bc)(c+a)(c+b)}{ab+bc+ca} \ge 3.\frac{\sqrt[3]{(a^2+ab+bc)(b^2+bc+ca)(c^2+ca+ab)[(a+b)(b+c)(c+a)]^2}}{ab+bc+ca}$$

Since it's suffices to show that

$$\sqrt{3}.\sqrt[3]{(a+b)(b+c)(c+a)} \ge 2.\sqrt{ab+bc+ca}$$

which is clearly true by AM-GM inequality again. The **Solution** is completed. Equality holds for $a=b=c$

4,

Let $a_0, a_1, \ldots, a_n$ be positive real numbers such that $a_{k+1}-a_k \ge 1$ for all $k = 0, 1, \ldots, n-1$. Prove that

$$1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_n-a_0}\right) \le \left(1+\frac{1}{a_0}\right)\left(1+\frac{1}{a_1}\right)\cdots\left(1+\frac{1}{a_n}\right)$$

Solution: We will prove it by induction.

For $n=1$ We need to check that

$$1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right) \le \left(1+\frac{1}{a_0}\right)\left(1+\frac{1}{a_1}\right)$$

which is equivalent to $a_0(a_1-a_0-1) \ge 0$, which is true by given condition.

Let

$$1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_k-a_0}\right) \le \left(1+\frac{1}{a_0}\right)\left(1+\frac{1}{a_1}\right)\cdots\left(1+\frac{1}{a_k}\right)$$

it remains to prove that:

$$1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_{k+1}-a_0}\right)\le$$
$$\le\left(1+\frac{1}{a_0}\right)\left(1+\frac{1}{a_1}\right)\cdots\left(1+\frac{1}{a_{k+1}}\right)$$

By our hypothesis

$$\left(1+\frac{1}{a_0}\right)\left(1+\frac{1}{a_1}\right)\cdots\left(1+\frac{1}{a_{k+1}}\right)\ge$$
$$\ge\left(1+\frac{1}{a_{k+1}}\right)\left(1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_k-a_0}\right)\right)$$

id est, it remains to prove that:

$$\left(1+\frac{1}{a_{k+1}}\right)\left(1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_k-a_0}\right)\right)\ge$$
$$\ge 1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_{k+1}-a_0}\right)$$

But

$$\left(1+\frac{1}{a_{k+1}}\right)\left(1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_k-a_0}\right)\right)\ge$$
$$\ge 1+\frac{1}{a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_{k+1}-a_0}\right)\Leftrightarrow$$
$$\Leftrightarrow\frac{1}{a_{k+1}}+\frac{1}{a_{k+1}a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_k-a_0}\right)\ge$$
$$\ge\frac{1}{(a_{k+1}-a_0)a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_k-a_0}\right)\Leftrightarrow$$
$$\Leftrightarrow 1\ge\frac{1}{a_{k+1}-a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_k-a_0}\right)$$

But by our conditions We obtain:

$$\frac{1}{a_{k+1}-a_0}\left(1+\frac{1}{a_1-a_0}\right)\cdots\left(1+\frac{1}{a_k-a_0}\right)\le$$
$$\le\frac{1}{k}\left(1+\frac{1}{1}\right)\cdots\left(1+\frac{1}{k-1}\right)=1.$$

Thus, the inequality is proven.

5,

Given $a,b,c>0$. Prove that

$$\sum\sqrt[3]{\frac{a^2+bc}{b^2+c^2}}\ge 9.\frac{\sqrt[3]{abc}}{(a+b+c)}$$

Solution : This ineq is equivalent to:

$$\sum\frac{a^2+bc}{\sqrt[3]{abc(a^2+bc)^2(b^2+c^2)}}\ge\frac{9}{(a+b+c)^3}$$

By AM-GM ineq , We have

$$\frac{a^2+bc}{\sqrt[3]{abc(a^2+bc)^2(b^2+c^2)}}=$$

$$= \frac{a^2+bc}{\sqrt[3]{(a^2+bc)c(a^2+bc)b(b^2+c^2)a}} \ge \frac{3(a^2+bc)}{\sum\limits_{sym} a^2b}$$

Similarly, this ineq is true if We prove that:

$$\frac{3(a^2+b^2+c^2+ab+bc+ca)}{\sum\limits_{sym} a^2b} \ge \frac{9}{(a+b+c)^3}$$

$$a^3+b^3+c^3+3abc \ge \sum_{sym} a^2b$$

Which is true by Schur ineq. Equality holds when $a=b=c$

6,

Let a,b,c be nonnegative real numbers such that $ab+bc+ca>0$. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \ge \frac{2}{ab+bc+ca}.$$

The inequality is equivalent to

$$\sum \frac{ab+bc+ca}{2a^2+bc} \ge 2, (1)$$

or

$$\sum \frac{a(b+c)}{2a^2+bc} + \sum \frac{bc}{bc+2a^2} \ge 2.(2)$$

Using the Cauchy-Schwarz inequality, We have

$$\sum \frac{bc}{bc+2a^2} \ge \frac{\left(\sum bc\right)^2}{\sum bc(bc+2a^2)} = 1.(3)$$

Therefore, it suffices to prove that

$$\sum \frac{a(b+c)}{2a^2+bc} \ge 1.(4)$$

Since

$$\frac{a(b+c)}{2a^2+bc} \ge \frac{a(b+c)}{2(a^2+bc)}$$

it is enough to check that

$$\sum \frac{a(b+c)}{a^2+bc} \ge 2, (5)$$

which is a known result.

Remark:

$$\frac{2ca+bc}{2a^2+bc}+\frac{2bc+ca}{2b^2+ca} \ge \frac{4c}{a+b+c}.$$

7,

Let a,b,c be non negative real numbers such that $ab+bc+ca>0$. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}+\frac{1}{ab+bc+ca} \ge \frac{12}{(a+b+c)^2}.$$

Solution: 1) We can prove this inequality using the following auxiliary result if $0 \le a \le \min\{a,\ b\}$, then

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca} \ge \frac{4}{(a+b)(a+b+c)}.$$

in fact, this is used to replaced for "no two of which are zero", so that the fractions

$$\frac{1}{2a^2+bc}, \frac{1}{2b^2+ca}, \frac{1}{2c^2+ab}, \frac{1}{ab+bc+ca}$$

have meanings.

Besides, the iaker also works for it:

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \geq \frac{2(ab+bc+ca)}{\sum a^2b^2+abc(a+b+c)}$$

But our **Solution** for both of them is expand

Let a, b, c be non negative real numbers such that $ab+bc+ca>0$. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}+\frac{1}{ab+bc+ca} \geq \frac{12}{(a+b+c)^2}.$$

2) Consider by AM-GM inequality, We have

$$2\left(a^2+ab+b^2\right)(a+b+c)$$

$$=(2b+a)\left(2a^2+bc\right)+(2a+b)\left(2b^2+ca\right)$$

$$\geq 2\sqrt{(2a+b)(2b+a)\left(2a^2+bc\right)\left(2b^2+ca\right)}.$$

And by AM-GM inequality, We have

$$\frac{c^2(2a+b)}{2a^2+bc}+\frac{c^2(2b+a)}{2b^2+ca}$$

$$\geq 2\sqrt{\frac{c^4(2a+b)(2b+a)}{\left(2a^2+bc\right)\left(2b^2+ca\right)}}$$

$$\geq \frac{2c^2(2a+b)(2b+a)}{\left(a^2+ab+b^2\right)(a+b+c)}$$

$$=\frac{4c^2}{a+b+c}+\frac{6abc}{a+b+c}\left(\frac{c}{a^2+ab+b^2}\right)$$

$$\sum \frac{2c^2a+bc^2+2ab^2+b^2c}{2a^2+bc}$$

$$=\sum\left(\frac{c^2(2a+b)}{2a^2+bc}+\frac{c^2(2b+a)}{2b^2+ca}\right)$$

$$\geq \sum\left(\frac{4c^2}{a+b+c}+\frac{6abc}{a+b+c}\left(\frac{c}{a^2+ab+b^2}\right)\right)$$

$$=\frac{4\left(a^2+b^2+c^2\right)}{a+b+c}+\frac{6abc}{a+b+c}\left(\sum \frac{c}{a^2+ab+b^2}\right)$$

$$\geq \frac{4\left(a^2+b^2+c^2\right)}{a+b+c}+\frac{6abc}{a+b+c}\left(\frac{(a+b+c)^2}{\sum c\left(a^2+ab+b^2\right)}\right)$$

$$=\frac{4\left(a^2+b^2+c^2\right)}{ab+c}+\frac{6abc}{ab+bc+ca}$$

$$\Rightarrow \sum \frac{2a^2b+2ab^2+2b^2c+2bc^2+2c^2a+2ca^2}{2a^2+bc}$$

$$= \sum (b+c) + \sum \frac{2c^2a + bc^2 + 2ab^2 + b^2c}{2a^2+bc}$$

$$\geq \sum (b+c) + \frac{4\left(a^2+b^2+c^2\right)}{a+b+c} + \frac{6abc}{ab+bc+ca}$$

$$= \frac{8\left(a^2+b^2+c^2+ab+bc+ca\right)}{a+b+c} - \frac{2\sum\left(a^2b+ab^2\right)}{ab+bc+ca}$$

$$\Rightarrow \sum \frac{1}{2a^2+bc} + \frac{1}{ab+bc+ca}$$

$$\geq \frac{4\left(a^2+b^2+c^2+ab+bc+ca\right)}{(a+b+c)\left(\sum\left(a^2b+ab^2\right)\right)}$$

$$\geq \frac{12}{(a+b+c)^2}.$$

$$<=> \sum \frac{(a+b)(a+c)}{2a^2+bc} + \sum \frac{a^2+bc}{2a^2+bc} - 2 \geq \frac{12(ab+bc+ca)}{(a+b+c)^2}$$

From

$$\sum \frac{2a^2+2bc}{2a^2+bc} - 3 = \frac{bc}{2a^2+bc} \geq 1$$

We get

$$\sum \frac{a^2+bc}{2a^2+bc} - 2 \geq 0$$

Now, We will prove the stronger

$$\sum \frac{(a+b)(a+c)}{2a^2+bc} \geq \frac{12(ab+bc+ca)}{(a+b+c)^2}$$

From cauchy-scharzt, We have

$$\sum \frac{(a+b)(a+c)}{2a^2+bc} = (a+b)(b+c)(c+a)(\sum \frac{1}{(2a^2+bc)(b+c)} \geq \frac{3(a+b)(b+c)(c+a)}{ab(a+b)+bc(b+c)+ca(c+a)}$$

Finally, We only need to prove that

$$\frac{(a+b)(b+c)(c+a)}{ab(a+b)+bc(b+c)+ca(c+a)} \geq \frac{4(ab+bc+ca)}{(a+b+c)^2}$$

$$\frac{(a+b+c)^2}{ab+bc+ca} \geq \frac{4[ab(a+b)+bc(b+c)+ca(c+a)}{(a+b)(b+c)(c+a)} = 4 - \frac{8abc}{(a+b)(b+c)(c+a)}$$

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

which is old problem. Our **Solution** are completed equality occur if and if only

$$a = b = c, a = b, c = 0$$

or any cyclic permution.

8, Let a, b, c be positive real numbers such that $16(a+b+c) \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Prove that

$$\sum \frac{1}{\left[a+b+\sqrt{2(a+c)}\right]^3} \leq \frac{8}{9}.$$

Solution: This problem is rather easy. Using the AM-GM inequality, We have:

$$a+b+\sqrt{2(c+a)} = a+b+\sqrt{\frac{c+a}{2}} + \sqrt{\frac{c+a}{2}} \geq 3\sqrt[3]{\frac{(a+b)(c+a)}{2}}.$$

So that:

$$\sum \frac{1}{\left[a+b+\sqrt{2(c+a)}\right]^3} \le \sum \frac{2}{27(a+b)(c+a)}.$$

Thus, it's enough to check that:

$$\sum \frac{1}{3(a+b)(c+a)} \le 4 \iff 6(a+b)(b+c)(c+a) \ge a+b+c,$$

which is true since

$$9(a+b)(b+c)(c+a) \ge 8(a+b+c)(ab+bc+ca)$$

and

$$16abc(a+b+c) \ge ab+bc+ca \Rightarrow \frac{16(ab+bc+ca)^2}{3} \ge ab+bc+ca \iff ab+bc+ca \ge \frac{3}{16}.$$

The **Solution** is completed. Equality holds if and only if $a=b=c=\frac{1}{4}$.

9, Let x, y, z be positive real numbers such that $xyz=1$. Prove that

$$\frac{x^3+1}{\sqrt{x^4+y+z}} + \frac{y^3+1}{\sqrt{y^4+z+x}} + \frac{z^3+1}{\sqrt{z^4+x+y}} \ge 2\sqrt{xy+yz+zx}.$$

Solution: Using the AM-GM inequality, We have

$$\begin{aligned}
2\sqrt{(x^4+y+z)(xy+yz+zx)} &= 2\sqrt{[x^4+xyz(y+z)](xy+yz+zx)} \\
&= 2\sqrt{(x^3+y^2z+yz^2)(x^2y+x^2z+xyz)} \\
&\le (x^3+y^2z+yz^2)+(x^2y+x^2z+xyz) \\
&= (x+y+z)(x^2+yz) = \frac{(x+y+z)(x^3+1)}{x}.
\end{aligned}$$

it follows that

$$\frac{x^3+1}{\sqrt{x^4+y+z}} \ge \frac{2x\sqrt{xy+yz+zx}}{x+y+z}.$$

Adding this and it analogous inequalities, the result follows.

10, Let a, b, c be nonnegative real numbers satisfying $a+b+c=\sqrt{5}$. Prove that

$$(a^2-b^2)(b^2-c^2)(c^2-a^2) \le \sqrt{5}$$

Solution: For this one, We can assume WLOG that $c \ge b \ge a$ so that We have

$$P = (a^2-b^2)(b^2-c^2)(c^2-a^2) = (c^2-b^2)(c^2-a^2)(b^2-a^2) \le b^2c^2(c^2-b^2).$$

Also note that $\sqrt{5} = a+b+c \ge b+c$ since $a \ge 0$. Now, using the AM-GM inequality We have

$$\begin{aligned}
&(c+b)\cdot\left(\left(\frac{\sqrt{5}}{2}-1\right)\cdot c\right)^2\cdot\left(\left(\frac{\sqrt{5}}{2}+1\right)b\right)^2\cdot(c-b) \\
&\qquad\le (c+b)\left\{\frac{\sqrt{5}(b+c)}{5}\right\}^5 \le \sqrt{5};
\end{aligned}$$

So that We get $P \le \sqrt{5}$. And hence We are done. Equality holds if and only if $(a,b,c) = \left(\frac{\sqrt{5}}{2}+1; \frac{\sqrt{5}}{2}-1; 0\right)$ and all its cyclic permutations. □

11, Let $a, b, c > 0$ and $a+b+c=3$. Prove that

$$\frac{1}{3+a^2+b^2}+\frac{1}{3+b^2+c^2}+\frac{1}{3+c^2+a^2} \leq \frac{3}{5}$$

Solution: We have:

$$\frac{1}{3+a^2+b^2}+\frac{1}{3+b^2+c^2}+\frac{1}{3+c^2+a^2} \leq \frac{3}{5}$$

$$<=> \frac{3}{3+a^2+b^2}+\frac{3}{3+b^2+c^2}+\frac{3}{3+c^2+a^2} \leq \frac{9}{5}$$

$$\sum \frac{a^2+b^2}{3+a^2+b^2} \geq \frac{6}{5}$$

Using Cauchy-Schwarz's inequality:

$$\left(\sum \frac{a^2+b^2}{3+a^2+b^2}\right)(\sum 3+a^2+b^2) \geq (\sum \sqrt{a^2+b^2})^2$$

That means We have to prove

$$(\sum \sqrt{a^2+b^2})^2 \geq \frac{6}{5}(\sum (3+a^2+b^2))$$

$$\sum (a^2+b^2)+2\sum \sqrt{(a^2+b^2)(a^2+c^2)} \geq \frac{54}{5}+\frac{12}{5}\sum a^2$$

$$8\sum a^2+10\sum ab \geq 54 <=> 5(a+b+c)^2+3\sum a^2 \geq 54$$

it is true with $a+b+c=3$.

12,

Given $a, b, c > 0$ such that $ab+bc+ca=1$. Prove that

$$\frac{1}{4a^2-bc+1}+\frac{1}{4b^2-ca+1}+\frac{1}{4c^2-ab+1} \geq 1$$

Solution: in fact, the sharper inequality holds

$$\frac{1}{4a^2-bc+1}+\frac{1}{4b^2-ca+1}+\frac{1}{4c^2-ab+1} \geq \frac{3}{2}.$$

The inequality is equivalent to

$$\frac{1}{a(4a+b+c)}+\frac{1}{b(4b+c+a)}+\frac{1}{c(4c+a+b)} \geq \frac{3}{2}.$$

Using the Cauchy-Schwarz inequality, We have

$$\left[\sum \frac{1}{a(4a+b+c)}\right]\left(\sum \frac{4a+b+c}{a}\right) \geq \left(\sum \frac{1}{a}\right)^2 = \frac{1}{a^2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{2}{3a^2b^2c^2} \geq \frac{4a+b+c}{a}+\frac{4b+c+a}{b}+\frac{4c+a+b}{c}.$$

Since

$$\sum \frac{4a+b+c}{a} = \sum \left(3+\frac{a+b+c}{a}\right) = 9+\frac{(a+b+c)(ab+bc+ca)}{abc}$$
$$= 9+\frac{a+b+c}{abc},$$

this inequality can be written as

$$9a^2b^2c^2 + abc(a+b+c) \le \frac{2}{3},$$

which is true because

$$a^2b^2c^2 \le \left(\frac{ab+bc+ca}{3}\right)^3 = \frac{1}{27},$$

and

$$abc(a+b+c) \le \frac{(ab+bc+ca)^2}{3} = \frac{1}{3}.$$

13, Given $a, b, c \ge 0$ such that $ab + bc + ca = 1$. Prove that

$$\frac{1}{4a^2 - bc + 2} + \frac{1}{4b^2 - ca + 2} + \frac{1}{4c^2 - ab + 2} \ge 1$$

Solution: Notice that the case $abc = 0$ is trivial so let us consider now that $abc > 0$. Using the AM-GM inequality, We have

$$4a^2 - bc + 2(ab + bc + ca) = (2a+b)(2a+c) \le \frac{[c(2a+b) + b(2a+c)]^2}{4bc}$$
$$= \frac{(ab+bc+ca)^2}{bc} = \frac{1}{bc}.$$

it follows that

$$\frac{1}{4a^2 - bc + 2} \ge bc.$$

Adding this and its analogous inequalities, We get the desired result.

14, Given a, b, c are positive real numbers. Prove that

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}\right) \ge \frac{9}{1+abc}.$$

Solution: The original inequality is equivalent to

$$\left(\frac{abc+1}{a} + \frac{abc+1}{b} + \frac{abc+1}{c}\right)\left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right) \ge 9$$

or

$$\left(\sum_{cyc} \frac{1+a^2c}{a}\right)\left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right) \ge 9$$

By Cauchy Schwarz ineq and AM-GM ineq,

$$\sum_{cyc} \frac{1+a^2c}{a} \ge \sum_{cyc} \frac{c(1+a)^2}{a(1+c)} \ge 3\sqrt[3]{(1+a)(1+b)(1+c)}$$

and

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \ge \frac{3}{\sqrt[3]{(1+a)(1+b)(1+c)}}$$

Multiplying these two inequalities, the conclusion follows. Equality holds if and only if $a = b = c = 1$.

15. Given a, b, c are positive real numbers. Prove that:

$$\sqrt{a(b+1)} + \sqrt{b(c+1)} + \sqrt{c(a+1)} \le \frac{3}{2}\sqrt{(a+1)(b+1)(c+1)}$$

Solution: Case1.if $a+b+c+ab+bc+ca\le 3abc+3 <=> 4(ab+bc+ca+a+b+c)\le 3(a+1)(b+1)(c+1)$ Using Cauchy-Schawrz's inequality ,We have:

$$(\sqrt{a(b+1)}+\sqrt{b(c+1)}+\sqrt{c(a+1)})^2\le 3(ab+bc+ca+a+b+c)\le \frac{9(a+1)(b+1)(c+1)}{4}$$

The inequality is true. Case2. if$a+b+c+ab+bc+ca\le 3abc+3$.

$$<=> \frac{9(a+1)(b+1)(c+1)}{4}\ge 2(a+b+c+ab+bc+ca)+3abc+3$$

By AM-GM's inequality :

$$2\sum\sqrt{ab(b+1)(c+1)}\le\sum[ab(c+1)+(b+1)]=a+b+c+ab+bc+ca+3abc+3$$

$$=> ab+bc+ca+a+b+c+2\sum\sqrt{ab(b+1)(c+1)}\le\frac{9}{4(a+1)(b+1)(c+1)}$$

$$=> (\sqrt{a(b+1)}+\sqrt{b(c+1)}+\sqrt{c(a+1)})^2\le[\frac{3}{2}\sqrt{(a+1)(b+1)(c+1)}]^2$$

$=> Q.E.D$
Enquality holds when $a=b=c=1$.
16, Given a,b,c are positive real numbers. Prove that:

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\ge\frac{10}{(a+b+c)^2}$$

Solution: Assume $c=\min\{a,b,c\}$. Then

$$\frac{1}{a^2+c^2}+\frac{1}{b^2+c^2}\ge\frac{2}{ab+c^2}\iff (ab-c^2)(a-b)^2\ge 0$$

And by Cauchy-schwarz

$$((a^2+b^2)+8(ab+c^2))\left(\frac{1}{a^2+b^2}+\frac{2}{ab+c^2}\right)\ge 25$$

Hence We need only to prove:

$$5(a+b+c)^2\ge 2((a^2+b^2)+8(ab+c^2))\iff$$

$$3(a-b)^2+c(10b+10a-11c)\ge 0$$

Equality for $a=b,c=0$ or permutations.

17, Let a,b and c are non-negative numbers such that $ab+ac+bc\neq 0$. Prove that

$$\frac{a^2(b+c)^2}{a^2+3bc}+\frac{b^2(a+c)^2}{b^2+3ac}+\frac{c^2(a+b)^2}{c^2+3ab}\le a^2+b^2+c^2$$

Solution:
By Cauchy-Schwarz ineq , We have

$$\frac{a^2(b+c)^2}{a^2+bc}=\frac{a^2(b+c)^3}{(a^2+bc)(b+c)}=\frac{a^2(b+c)^3}{b(a^2+c^2)+c(a^2+b^2)}$$

$$\le\frac{a^2(b+c)}{4}\left(\frac{b^2}{b(a^2+c^2)}+\frac{c^2}{c(a^2+b^2)}\right)=\frac{a^2(b+c)}{4}\left(\frac{b}{a^2+c^2}+\frac{c}{a^2+b^2}\right)$$

Similarly, We have

$$LHS\le\sum a^2(b+c)\left(\frac{b}{a^2+c^2}+\frac{c}{a^2+b^2}\right)=\sum\frac{c(a^2(b+c)+b^2(c+a))}{a^2+b^2}$$

$= a^2+b^2+c^2+\sum \frac{abc(a+b)}{a^2+b^2} \le a^2+b^2+c^2+\sum \frac{abc(a+b)}{a^2+b^2} \le a^2+b^2+c^2+ab+bc+ca$
which is true by AM-GM ineq

The original inequality can be written as

$$\sum \frac{(a+b)^2(a+c)^2}{a^2+bc} \le \frac{8}{3}(a+b+c)^2.$$

Since $(a+b)(a+c) = (a^2+bc)+a(b+c)$ We have

$$\begin{aligned}\frac{(a+b)^2(a+c)^2}{a^2+bc} &= \frac{(a^2+bc)^2+2a(b+c)(a^2+bc)+a^2(b+c)^2}{a^2+bc}\\ &= a^2+bc+2a(b+c)+\frac{a^2(b+c)^2}{a^2+bc},\end{aligned}$$

and thus the above inequality is equivalent to

$$\sum \frac{a^2(b+c)^2}{a^2+bc} \le \frac{8}{3}(a+b+c)^2 - \sum a^2 - 5\sum ab,$$

or

$$\sum \frac{a^2(b+c)^2}{a^2+bc} \le \frac{5(a^2+b^2+c^2)+ab+bc+ca}{3}.$$

Since

$$\frac{5(a^2+b^2+c^2)+ab+bc+ca}{3} \ge a^2+b^2+c^2+ab+bc+ca$$

it is enough show that

$$\sum \frac{a^2(b+c)^2}{a^2+bc} \le a^2+b^2+c^2+ab+bc+ca.$$

Q.E.D

18, Given

$$a_1 \ge a_2 \ge \ldots \ge a_n \ge 0, b_1 \ge b_2 \ge \ldots \ge b_n \ge 0$$
$$\sum_{i=1}^{n} a_i = 1 = \sum_{i=1}^{n} b_i$$

Find the maxmium of

$$\sum_{i=1}^{n} (a_i-b_i)^2$$

W**Solution**:ithout loss of generality, assume that

$$a_1 \ge b_1$$

Notice that for

$$a \ge x \ge 0, b, y \ge 0$$

We have

$$(a-x)^2+(b-y)^2-(a+b-x)^2-y^2 = -2b(a-x+y) \le 0.$$

According to this inequality, We have

$$(a_1-b_1)^2+(a_2-b_2)^2 \le (a_1+a_2-b_1)^2+b_2^2,$$

$$(a_1+a_2-b_1)^2+(a_3-b_3)^2 \le (a_1+a_2+a_3-b_1)^2+b_3^2, \cdots\cdots$$

$$(a_1+a_2+\cdots+a_{n-1}-b_1)^2+(a_n-b_n)^2 \le (a_1+a_2+\cdots+a_n-b_1)^2+b_n^2.$$

Adding these inequalities, We get

$$\begin{aligned}\sum_{i=1}^{n}(a_i-b_i)^2 &\le (1-b_1)^2+b_2^2+b_3^2+\cdots+b_n^2\\ &\le (1-b_1)^2+b_1(b_2+b_3+\cdots+b_n)\\ &= (1-b_1)^2+b_1(1-b_1)=1-b_1\le 1-\frac{1}{n}.\end{aligned}$$

Equality holds for example when

$$a_1=1, a_2=a_3=\cdots=a_n=0$$

and

$$b_1=b_2=\cdots=b_n=\frac{1}{n}$$

19, Given

$$a,b,c\ge 0$$

such that

$$a^2+b^2+c^2=1$$

Prove that

$$\frac{1-ab}{7-3ac}+\frac{1-bc}{7-3ba}+\frac{1-ca}{7-3cb}\ge\frac{1}{3}$$

Solution: First, We will show that

$$\frac{1}{7-3ab}+\frac{1}{7-3bc}+\frac{1}{7-3ca}\le\frac{1}{2}.$$

Using the Cauchy-Schwarz inequality, We have

$$\frac{1}{7-3ab}=\frac{1}{3(1-ab)+4}\le\frac{1}{9}\left[\frac{1}{3(1-ab)}+1\right].$$

it follows that

$$\sum\frac{1}{7-3ab}\le\frac{1}{27}\sum\frac{1}{1-ab}+\frac{1}{3},$$

and thus, it is enough to show that

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\le\frac{9}{2},$$

which is Vasc's inequality. Now, We write the original inequality as

$$\frac{3-3ab}{7-3ac}+\frac{3-3bc}{7-3ba}+\frac{3-3ca}{7-3cb}\ge 1,$$

or

$$\frac{7-3ab}{7-3ac}+\frac{7-3bc}{7-3ba}+\frac{7-3ca}{7-3cb}\ge 1+4\left(\frac{1}{7-3ab}+\frac{1}{7-3bc}+\frac{1}{7-3ca}\right).$$

Since

$$4\left(\frac{1}{7-3ab}+\frac{1}{7-3bc}+\frac{1}{7-3ca}\right)\le 2$$

it is enough to show that

$$\frac{7-3ab}{7-3ac}+\frac{7-3bc}{7-3ba}+\frac{7-3ca}{7-3cb}\geq 3,$$

which is true according to the AM-GM inequality.

21, Let

$$a,b,c\geq 0$$

such that

$$a+b+c>0$$

and

$$b+c\geq 2a$$

For

$$x,y,z>0$$

such that

$$xyz=1$$

Prove that the following inequality holds

$$\frac{1}{a+x^2(by+cz)}+\frac{1}{a+y^2(bz+cx)}+\frac{1}{a+z^2(bx+cy)}\geq\frac{3}{a+b+c}$$

Solution: Setting

$$u=\frac{1}{x},v=\frac{1}{y}$$

and

$$w=\frac{1}{z}$$

and using the condition

$$uvw=1$$

the inequality can be rewritten as

$$\sum\frac{u}{au+cv+bw}=\sum\frac{u^2}{au^2+cuv+bwu}\geqslant\frac{3}{a+b+c}.$$

Applying Cauchy, it suffices to prove

$$\frac{(u+v+w)^2}{a\sum u^2+(b+c)\sum uv}\geqslant\frac{3}{a+b+c}$$

$$\frac{1}{2}\cdot(b+c-2a)\left(\sum(x-y)^2\right)\geqslant 0,$$

which is obvious due to the condition for

$$a,b,c$$

22, Given

$$x,y,z>0$$

such that

$$xyz=1$$

Prove that
$$\frac{1}{(1+x^2)(1+x^7)}+\frac{1}{(1+y^2)(1+y^7)}+\frac{1}{(1+z^2)(1+z^7)}\geq\frac{3}{4}$$
Solution: First We prove this ineq easy
$$\frac{1}{(1+x^2)(1+x^7)}\geq\frac{3}{4(x^9+x^{\frac{9}{2}}+1)}$$
And this ineq became:
$$\frac{1}{x^9+x^{\frac{9}{2}}+1}+\frac{1}{y^9+y^{\frac{9}{2}}+1}+\frac{1}{z^9+z^{\frac{9}{2}}+1}\geq 1$$
with
$$xyz=1$$
it's an old result

23, Let
$$a,b,c$$
be positive real numbers such that
$$3(a^2+b^2+c^2)+ab+bc+ca=12$$
Prove that
$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}}\leq\frac{3}{\sqrt{2}}.$$
Solution: Let
$$A=a^2+b^2+c^2, B=ab+bc+ca$$
$$2A+B=2\sum a^2+\sum ab\leq\frac{3}{4}\left(3\sum a^2+\sum ab\right)=9.$$
By Cauchy Schwarz inequality, We have
$$\sum\frac{a}{\sqrt{a+b}}=\sum\sqrt{a}\sqrt{\frac{a}{a+b}}$$
$$\leq\sqrt{a+b+c}\sqrt{\sum\frac{a}{a+b}}.$$
By Cauchy Schwarz inequality again, We have
$$\sum\frac{b}{a+b}=\sum\frac{b^2}{b(a+b)}$$
$$\geq\frac{(a+b+c)^2}{\sum b(a+b)}$$
$$=\frac{A+2B}{A+B}$$
$$\sum\frac{a}{a+b}=3-\sum\frac{b}{a+b}\leq 3-\frac{A+2B}{A+B}=\frac{2A+B}{A+B}$$
hence, it suffices to prove that
$$(a+b+c)\cdot\frac{2A+B}{A+B}\leq\frac{9}{2}$$

Consider

$$(a+b+c)\sqrt{2A+B}$$
$$=\sqrt{(A+2B)\,(2A+B)}$$
$$\leq \frac{(A+2B)+(2A+B)}{2}$$
$$=\frac{3}{2}\,(A+B)$$
$$\Rightarrow (a+b+c)\cdot\frac{2A+B}{A+B}\leq \frac{3}{2}\sqrt{2A+B}\leq \frac{9}{2}$$

as require.

By AM-GM ineq easy to see that

$$3\leq a^2+b^2+c^2\leq 4$$

By Cauchy-Schwarz ineq, We have

$$LHS^2=(\sum \frac{a\sqrt{a+c}}{\sqrt{(a+b)(a+c)}})\leq (a^2+b^2+c^2+ab+bc+ca)(\sum \frac{a}{(a+b)(a+c)})$$

Using the familiar ineq

$$9(a+b)(b+c)(c+a)\geq 8(a+b+c)(ab+bc+ca)$$

We have

$$\sum \frac{a}{(a+b)(a+c)}=\frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)}\leq \frac{9}{4(a+b+c)}$$

And We need to prove that

$$\frac{9(a^2+b^2+c^2+ab+bc+ca)}{4(a+b+c)}\leq \frac{9}{2}\Leftrightarrow \frac{6-(a^2+b^2+c^2)}{\sqrt{24-5(a^2+b^2+c^2)}}\leq 1$$
$$\Leftrightarrow (6-(a^2+b^2+c^2))^2\leq 24-5(a^2+b^2+c^2)$$
$$\Leftrightarrow (3-(a^2+b^2+c^2))(4-(a^2+b^2+c^2))\leq 0$$

Which is true We are done equality holds when

$$a=b=c=1$$

24.
Given

$$a,b,c\geq 0$$

Prove that

$$\sum \frac{1}{(a^2+bc)(b+c)^2}\leq \frac{8(a+b+c)^2}{3(a+b)^2(b+c)^2(c+a)^2}$$

Solution: in fact, the sharper and nicer inequality holds:

$$\frac{a^2(b+c)^2}{a^2+bc}+\frac{b^2(c+a)^2}{b^2+ca}+\frac{c^2(a+b)^2}{c^2+ab}\leq a^2+b^2+c^2+ab+bc+ca.$$

$$\frac{a^2(b+c)^2}{a^2+bc}+\frac{b^2(c+a)^2}{b^2+ca}+\frac{c^2(a+b)^2}{c^2+ab}\leq a^2+b^2+c^2+ab+bc+ca$$

25.
Given
$$a,b,c\ge 0$$
such that
$$ab+bc+ca=1$$
Prove that
$$\frac{1}{\frac{8}{5}a^2+bc}+\frac{1}{\frac{8}{5}b^2+ca}+\frac{1}{\frac{8}{5}c^2+ab}\ge\frac{9}{4}$$
Assume WLOG
$$a\ge b\ge c$$
this ineq
$$\frac{1}{\frac{8}{5}a^2+bc}-\frac{5}{8}+\frac{1}{\frac{8}{5}b^2+ca}-\frac{5}{8}+\frac{1}{\frac{8}{5}c^2+ab}-1\ge 0$$
$$\frac{8-8a^2-5bc}{8a^2+5bc}+\frac{8-8b^2-5ca}{8b^2+5ca}+\frac{1-\frac{8}{5}c^2-ab}{c^2+\frac{8}{5}ab}\ge 0$$
$$\frac{8a(b+c-a)+3bc}{8a^2+5bc}+\frac{8b(a+c-b)+5ac}{8b^2+5ca}+\frac{c(a+b-\frac{8}{5}c)}{c^2+\frac{8}{5}ab}\ge 0$$
Notice that We only need to prove this ineq when
$$a\ge b+c$$
by the way We need to prove that
$$\frac{8b}{8b^2+5ca}\ge\frac{8a}{8a^2+5bc}$$
$$(a-b)(8ab-5ac-5bc)\ge 0$$
Easy to see that: if
$$a\ge b+c$$
then
$$8ab=5ab+3ab\ge 5ac+6bc\ge 5ac+5ac$$
So this ineq is true, We have q.d.e , equality hold when
$$(a,b,c)=(1,1,0)$$
26, Give
$$a,b,c\ge 0$$
Prove that:
$$\frac{a}{b^2+c^2}+\frac{b}{a^2+c^2}+\frac{c}{a^2+b^2}\ge\frac{a+b+c}{ab+bc+ca}+\frac{abc(a+b+c)}{(a^3+b^3+c^3)(ab+bc+ca)}$$
$$\sum\frac{a}{b^2+c^2}=\sum\frac{a^2}{ab^2+c^2a}$$
$$\ge\frac{(a+b+c)^2}{\sum(ab^2+c^2a)},$$
it suffices to prove that
$$\frac{a+b+c}{\sum(ab^2+c^2a)}\ge\frac{1}{ab+bc+ca}+\frac{abc}{(ab+bc+ca)(a^3+b^3+c^3)},$$

because

$$\frac{a+b+c}{\sum\left(ab^2+c^2a\right)}-\frac{1}{ab+bc+ca}$$
$$=\frac{3abc}{(ab+bc+ca)\sum\left(ab^2+ca^2\right)},$$

it suffices to prove that

$$3\left(a^3+b^3+c^3\right)\geq\sum\left(ab^2+c^2a\right),$$

which is true because

$$2\left(a^3+b^3+c^3\right)\geq\sum\left(ab^2+c^2a\right).$$

Remark:

$$\frac{a}{b^2+c^2}+\frac{b}{c^2+a^2}+\frac{c}{a^2+b^2}\geq\frac{a+b+c}{ab+bc+ca}+\frac{3abc(a+b+c)}{2(a^3+b^3+c^3)(ab+bc+ca)}.$$

Give

$$a,b,c\geq 0$$

Prove that

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}\geq\frac{3}{ab+bc+ca}+\frac{81a^2b^2c^2}{2(a^2+b^2+c^2)^4}$$

Equality occur if and if only

$$a=b=c,a=b,c=0$$

or any cyclic permution.

it is true because

(1)

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}\geq\frac{3\left(a^2+b^2+c^2\right)}{a^3b+ab^3+b^3c+bc^3+c^3a+ca^3}$$

and

$$(2)\ \frac{3\left(a^2+b^2+c^2\right)}{a^3b+ab^3+b^3c+bc^3+c^3a+ca^3}\geq\frac{3}{ab+bc+ca}+\frac{81a^2b^2c^2}{2(a^2+b^2+c^2)^4}.$$

Because

$$\frac{\sum a^2}{\sum\left(a^3b+ab^3\right)}-\frac{1}{ab+bc+ca}$$
$$=\frac{abc(a+b+c)}{(ab+bc+ca)\left(\sum\left(a^3b+ab^3\right)\right)},$$

it suffices to prove that

$$2(a+b+c)\left(a^2+b^2+c^2\right)^4\geq 27abc(ab+bc+ca)\left(\sum\left(a^3b+ab^3\right)\right),$$

which is true because

$$(a)\ (a+b+c)\left(a^2+b^2+c^2\right)\geq 9abc,$$

$$(b)\ a^2+b^2+c^2\geq ab+bc+ca,$$

$$(c)\ 2\left(a^2+b^2+c^2\right)^2\geq 3\sum\left(a^3b+ab^3\right),$$

which

$$(c)$$

is equivalent to

$$\sum \left(a^2-ab+b^2\right)(a-b)^2 \geq 0,$$

which is true.

27, Let

$$a, b, c$$

be nonnegative numbers, no two of which are zero. Prove that

$$\frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2} \geqslant \frac{2(a^2+b^2+c^2)}{a+b+c}.$$

Solution:

$$\begin{aligned}
&\sum \frac{a^2(b+c)}{b^2+bc+c^2} \\
&= \sum \frac{4a^2(b+c)(ab+bc+ca)}{\left(b^2+bc+c^2\right)(ab+bc+ca)} \\
&\geq \sum \frac{4a^2(b+c)(ab+bc+ca)}{\left(b^2+bc+c^2+ab+bc+ca\right)^2} \\
&= \sum \frac{4a^2(ab+bc+ca)}{(b+c)(a+b+c)^2},
\end{aligned}$$

it suffices to prove

$$\sum \frac{a^2}{b+c} \geq \frac{(a+b+c)\left(a^2+b^2+c^2\right)}{2(ab+bc+ca)},$$

or

$$\sum\left(\frac{a^2}{b+c}+a\right) \geq \frac{(a+b+c)^3}{2(ab+bc+ca)},$$

or

$$\sum \frac{a}{b+c} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)},$$

which is true by Cauchy-Schwarz inequality

$$\begin{aligned}
\sum \frac{a}{b+c} &= \sum \frac{a^2}{a(b+c)} \\
&\geq \frac{(a+b+c)^2}{2(ab+bc+ca)}.
\end{aligned}$$

We just want to give a little note here. Notice that

$$\frac{a^2(b+c)}{b^2+bc+c^2}+\frac{a(b+c)}{a+b+c}=\frac{a(b+c)(a^2+b^2+c^2+ab+bc+ca)}{(b^2+bc+c^2)(a+b+c)},$$

and

$$\frac{2(a^2+b^2+c^2)}{a+b+c}+\sum \frac{a(b+c)}{a+b+c}=\frac{2(a^2+b^2+c^2+ab+bc+ca)}{a+b+c}.$$

Therefore, the inequality can be written in the form

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2} \geq 2,$$

Note that

$$\sum_{\text{cyc}} \frac{a(b+c)}{b^2+bc+c^2}=\sum_{\text{cyc}} \frac{4a(b+c)(ab+bc+ca)}{4(b^2+bc+c^2)(ab+bc+ca)} \geqslant \sum_{\text{cyc}} \frac{4a(ab+bc+ca)}{(b+c)(a+b+c)^2}.$$

So that We have to prove:

$$\sum_{\text{cyc}} \frac{4a(ab+bc+ca)}{(b+c)(a+b+c)^2} \geqslant 2,$$

or

$$\sum_{\text{cyc}} \frac{a}{b+c} \geqslant \frac{(a+b+c)^2}{2(ab+bc+ca)},$$

which is obviously true due to the Cauchy-Schwarz inequality.

This is another new **Solution**. First, We will prove that

$$\sqrt{(a^2+ac+c^2)(b^2+bc+c^2)} \leq \frac{ab(a+b)+bc(b+c)+ca(c+a)}{a+b}.(1)$$

indeed, using the Cauchy-Schwarz inequality, We have

$$\begin{aligned}\sqrt{ac}\cdot\sqrt{bc}+\sqrt{a^2+ac+c^2}\cdot\sqrt{b^2+bc+c^2} &\leq \sqrt{(ac+a^2+ac+c^2)(bc+b^2+bc+c^2)}\\ &= (a+c)(b+c).\end{aligned}$$

it follows that

$$\begin{aligned}\sqrt{(a^2+ac+c^2)(b^2+bc+c^2)} \leq ab+c^2+c\left(a+b-\sqrt{ab}\right) &\leq ab+c^2+c\left(a+b-\frac{2ab}{a+b}\right)\\ &= \frac{ab(a+b)+bc(b+c)+ca(c+a)}{a+b}.\end{aligned}$$

Now, from (1), using the AM-GM inequality, We get

$$\begin{aligned}\frac{1}{a^2+ac+c^2}+\frac{1}{b^2+bc+c^2} &\geq \frac{2}{\sqrt{(a^2+ac+c^2)(b^2+bc+c^2)}}\\ &\geq \frac{2(a+b)}{ab(a+b)+bc(b+c)+ca(c+a)}.\end{aligned} \tag{2}$$

From

$$(2)$$

We have

$$\begin{aligned}\sum \frac{a(b+c)}{b^2+bc+c^2} &= \sum ab\left(\frac{1}{a^2+ac+c^2}+\frac{1}{b^2+bc+c^2}\right)\\ &\geq \sum \frac{2ab(a+b)}{ab(a+b)+bc(b+c)+ca(c+a)} = 2.\end{aligned}$$

29, if

$$a,b,c>0$$

then the following inequality holds:

$$\frac{a^2(b+c)}{b^2+bc+c^2}+\frac{b^2(c+a)}{c^2+ca+a^2}+\frac{c^2(a+b)}{a^2+ab+b^2} \geq 2\sqrt{\frac{a^3+b^3+c^3}{a+b+c}}$$

This inequality is equivalent to

$$\sum \frac{a^2(b+c)(a+b+c)}{b^2+bc+c^2} \geq 2\sqrt{\left(a^3+b^3+c^3\right)(a+b+c)}$$

or

$$\sum\left(a^2+\frac{a^2(ab+bc+ca)}{b^2+bc+c^2}\right) \geq 2\sqrt{\left(a^3+b^3+c^3\right)(a+b+c)},$$

because

$$2\sqrt{(a^3+b^3+c^3)(a+b+c)} \le \left(a^2+b^2+c^2\right) + \frac{\left(a^3+b^3+c^3\right)(a+b+c)}{a^2+b^2+c^2},$$

it suffices to prove that

$$\sum \frac{a^2}{b^2+bc+c^2} \ge \frac{\left(a^3+b^3+c^3\right)(a+b+c)}{(a^2+b^2+c^2)(ab+bc+ca)},$$

by Cauchy-Schwarz inequality, We have

$$\sum \frac{a^2}{b^2+bc+c^2} \ge \frac{\left(a^2+b^2+c^2\right)^2}{\sum a^2(b^2+bc+c^2)} = \frac{\left(a^2+b^2+c^2\right)^2}{2\sum a^2b^2 + \sum a^2bc},$$

it suffices to prove that

$$\left(a^2+b^2+c^2\right)^3(ab+bc+ca) \ge \left(a^3+b^3+c^3\right)(a+b+c)\left(2\sum a^2b^2 + \sum a^2bc\right).$$

Let

$$A = \sum a^4, B = \frac{1}{2}\sum\left(a^3b+ab^3\right), C = \sum a^2b^2, D = \sum a^2bc,$$

We have

$$\left(a^2+b^2+c^2\right)^2 = A + 2C,$$

$$\left(a^2+b^2+c^2\right)(ab+bc+ca) = 2B + D,$$

$$\left(a^3+b^3+c^3\right)(a+b+c) = A + 2B,$$

and

$$2\sum a^2b^2 + \sum a^2bc = 2C + D.$$

Therefore, it suffices to prove that

$$(A+2C)(2B+D) \ge (A+2B)(2C+D),$$

or

$$2(A-D)(B-C) \ge 0,$$

which is true because

$$A \ge D$$

and

$$B \ge C$$

30, Given

$$a, b, c \ge 0$$

such that

$$a+b+c=1$$

Prove that

$$2\sqrt{a^2b+b^2c+c^2a} + ab + bc + ca \le 1$$

Rewrite the inform inequality as

$$2\sqrt{a^2b+b^2c+c^2a} + ab + bc + ca \le (a+b+c)^2$$

$$2\sqrt{\left(a^2b+b^2c+c^2a\right)(a+b+c)} \le a^2+b^2+c^2+ab+bc+ca$$

Assume that b is the number betien a and c. Then, by applying the AM-GM inequality, We get

$$2\sqrt{\left(a^2b+b^2c+c^2a\right)(a+b+c)} \le \frac{a^2b+b^2c+c^2a}{b}+b(a+b+c)$$

it is thus sufficient to prove the stronger inequality

$$a^2+b^2+c^2+ab+bc+ca \ge \frac{a^2b+b^2c+c^2a}{b}+b(a+b+c)$$

This inequality is equivalent to

$$\frac{c(a-b)(b-c)}{b} \ge 0,$$

which is obviously true according to the assumption of

$$b$$

How to prove

$$\sum a^4+2\sum a^3c \ge \sum a^2b^2+2\sum a^3b$$

only by AM-GM Equivalent to prove

$$\sum (a-b)^2(a+b)^2 \ge 4(a-b)(b-c)(a-c)(a+b+c)$$

WLOG We can assume that

$$a \ge b \ge c, a-b=x, b-c=y$$

then We need to prove that

$$x^2(2c+2y+x)^2+y^2(2c+y)^2+(x+y)^2(2c+x+y)^2 \ge xy(x+y)(3c+2x+y)$$

by

$$(x+y)^4 \ge xy(x+y)(x+2y)$$

and

$$(x+y)^3 \ge 3xy(x+y)$$

We have completed the **Solution**

31, Let

$$a, b, c$$

be positive numbers such that

$$a^2b^2+b^2c^2+c^2a^2 \ge a^2b^2c^2$$

Find the minimum of A

$$A=\frac{a^2b^2}{c^3(a^2+b^2)}+\frac{b^2c^2}{a^3(b^2+c^2)}+\frac{c^2a^2}{b^3(c^2+a^2)}$$

No one like this problem? Setting

$$x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$$

We have

$$x^2 + y^2 + z^2 \geq 1$$

We will prove that

$$\frac{x^3}{y^2+z^2} + \frac{y^3}{x^2+z^2} + \frac{z^3}{x^2+y^2} \geq \frac{\sqrt{3}}{2}$$

Using Cauchy-Schwarz:

$$LHS \geq \frac{(x^2+y^2+z^2)^2}{x(y^2+z^2)+y(x^2+z^2)+z(x^2+y^2)}$$

By AM-GM We have:

$$x(y^2+z^2)+y(x^2+z^2)+z(x^2+y^2) \leq \frac{2}{3}(x^2+y^2+z^2)(x+y+z) \leq \frac{2}{\sqrt{3}}(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}$$

Because

$$x^2 + y^2 + z^2 \geq 1$$

So

$$\frac{(x^2+y^2+z^2)^2}{\frac{2}{\sqrt{3}}(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}} \geq \frac{\sqrt{3}}{2}$$

We done!

32.

Let x,y,z be non negative real numbers such that $x^2 + y^2 + z^2 = 1$

. find the minimum and maximum of $f = x + y + z - xyz$.

Solution 1.

First We fix z and let $m = x+y = x+\sqrt{1-x^2-z^2} = g(x)(0 \leq x \leq \sqrt{1-z^2})$, then We have

$$g'(x) = 1 - \frac{x}{\sqrt{1-x^2-z^2}},$$

We get

$$g'(x) > 0 \Leftrightarrow 0 \leq x < \sqrt{\frac{1-z^2}{2}}$$

and

$$g'(x) < 0 \Leftrightarrow \sqrt{\frac{1-z^2}{2}} < x \leq \sqrt{1-z^2},$$

so We have

$$m_{\min} = \min\{g(0), g(\sqrt{1-z^2})\} = \sqrt{1-z^2}$$

and

$$m_{\max} = g\left(\sqrt{\frac{1-z^2}{2}}\right) = \sqrt{2-2z^2}.$$

Actually, f and written as

$$f = f(m) = -\frac{z}{2}m^2 + m + \left(1-z^2\right)\frac{z}{2} + z,$$

easy to prove that the axis of symmetry

$$m = \frac{1}{z} > \sqrt{2-2z^2}$$

so $f(m)$ is increasing in the interval of m, thus, We have

$$f(m) \geq f(\sqrt{1-z^2}) = \sqrt{1-z^2} + z$$

and

$$f(m) \leq f(\sqrt{2-2z^2}) = \frac{z^3}{2} + \frac{z}{2} + \sqrt{2-2z^2}.$$

Since

$$(\sqrt{1-z^2} + z)^2 = 1 + 2z\sqrt{1-z^2} \geq 1$$

We get $f(m) \geq 1$ and when two of x,y,z are zero We have $f = 1, soWegetf_{\min} = 1$.
Let

$$h(z) = \frac{z^3}{2} + \frac{z}{2} + \sqrt{2-2z^2},$$

easy to prove that

$$h'(z) > 0 \Leftrightarrow 0 \leq z < \frac{1}{\sqrt{3}} andh'(z) < 0 \Leftrightarrow \frac{1}{\sqrt{3}} < z \leq 1$$

then We get

$$f(m) \leq h\left(\frac{1}{\sqrt{3}}\right) = \frac{8\sqrt{3}}{9},$$

when $x = y = z = \frac{1}{\sqrt{3}} Wehavef = \frac{8\sqrt{3}}{9}$, so We get$f_{\max} = \frac{8\sqrt{3}}{9}$.
Done.
Solution 2.
When two of x,y,z are zero We have$f = 1$, and We will prove that $f \geq 1$ then We can get $f_{\min} = 1$. Actually, We have

$$f \geq 1 \Leftrightarrow x+y+z-xyz \geq 1 \Leftrightarrow (x+y+z)\left(x^2+y^2+z^2\right) - xyz \geq \left(\sqrt{x^2+y^2+z^2}\right)^3 \Leftrightarrow \left((x+y+z)\left(x^2+y^2+z^2\right) - xyz\right)^2 \geq \left(x^2+y^2+z^2\right)^3 \Leftrightarrow x^2y^2z^2 + 2\sum_{sym}\left(x^5y + x^3y^3 + x^3y^2z\right) \geq 0,$$

the last inequality is obvious true, so We got $f \geq 1$; When$x = y = z = \frac{1}{\sqrt{3}}$We have $f = \frac{8\sqrt{3}}{9}$,
and We will prove that

$$f \leq \frac{8\sqrt{3}}{9}$$

then We can get

$$f_{\max} = \frac{8\sqrt{3}}{9}$$

Actually, We have

$$f \leq \frac{8\sqrt{3}}{9} \Leftrightarrow x+y+z-xyz \leq \frac{8\sqrt{3}}{9} \Leftrightarrow (x+y+z)\left(x^2+y^2+z^2\right) - xyz \leq \frac{8\sqrt{3}}{9}\left(\sqrt{x^2+y^2+z^2}\right)^3 \Leftrightarrow 27\left((x+y+z)\left(x^2+y^2+z^2\right) - xyz\right)^2 \leq 64\left(x^2+y^2+z^2\right)^3 \Leftrightarrow \frac{1}{4}\sum_{cyc} S(x,y,z)(y-z)^2 \geq 0,$$

where

$$S(x,y,z) = 17y^2(2y-x)^2+17z^2(2z-x)^2+56y^2(z-x)^2++56z^2(y-x)^2+24x^4+6y^4+6z^4+57x^2(y^2+z^2)+104y^2z^2$$

is obvious positive, so the last inequality is obvious true, so We got$f_{\max} = \dfrac{8\sqrt{3}}{9}$.

33, For positive real numbers, show that

$$\frac{a^3(b+c-a)}{a^2+bc} + \frac{b^3(c+a-b)}{b^2+ca} + \frac{c^3(a+b-c)}{c^2+ab} \le \frac{ab+bc+ca}{2}$$

ineq

$$a^2+b^2+c^2+\frac{ab+bc+ca}{2} \ge \sum \frac{a^3(b+c-a)}{a^2+bc} + a^2$$

$$a^2+b^2+c^2+\frac{ab+bc+ca}{2} \ge (ab+bc+ca)(\sum \frac{a^2}{a^2+bc})$$

$$a^2+b^2+c^2+(ab+bc+ca)(\sum \frac{bc}{a^2+bc}) \ge \frac{5}{2}(ab+bc+ca)$$

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \sum \frac{bc}{a^2+bc} \ge \frac{5}{2}$$

Use two ineq

$$\frac{bc}{a^2+bc} + \frac{ab}{c^2+ab} + \frac{ac}{b^2+ac} \ge \frac{4abc}{(a+b)(b+c)(c+a)} + 1(1)$$

it is easy to prove.

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \ge 2(2)$$

So easy to see that

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \sum \frac{bc}{a^2+bc} \ge \frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{4abc}{(a+b)(b+c)(c+a)} + 1$$

$$\ge \frac{a^2+b^2+c^2}{2(ab+bc+ca)} + 2 \ge \frac{5}{2}$$

We have done !

$$\frac{a^3(b+c-a)}{a^2+bc} + \frac{b^3(c+a-b)}{b^2+ca} + \frac{c^3(a+b-c)}{c^2+ab} \le \frac{3abc(a+b+c)}{2(ab+bc+ca)}$$

Solution

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \sum \frac{bc}{a^2+bc} + \frac{3abc(a+b+c)}{2(ab+bc+ca)^2} \ge 3$$

And We prove that

$$\frac{3abc(a+b+c)}{2(ab+bc+ca)^2} \ge \frac{4abc}{(a+b)(b+c)(c+a)}$$

$$3(a+b+c)(a+b)(b+c)(c+a) \ge 8(ab+bc+ca)^2$$

This ineq is true because

$$3(a+b+c)(a+b)(b+c)(c+a) \ge \frac{8}{3}(a+b+c)^2(ab+bc+ca) \ge 8(ab+bc+ca)^2$$

So

$$LHS \geq \frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{4abc}{(a+b)(b+c)(c+a)}+\frac{4abc}{(a+b)(b+c)(c+a)}+1 \geq 3$$

Let

$$a,b,c>0$$

Show that

$$\frac{a^3(b+c-a)}{a^2+bc}+\frac{b^3(c+a-b)}{b^2+ca}+\frac{c^3(a+b-c)}{c^2+ab} \leq \frac{9abc}{2(a+b+c)}$$

First,We prove this lenma:

$$\frac{a^2}{a^2+bc}+\frac{b^2}{b^2+ca}+\frac{c^2}{c^2+ab} \leq \frac{(a+b+c)^2}{2(ab+bc+ca)}$$

$$\frac{bc}{a^2+bc}+\frac{ac}{b^2+ac}+\frac{ab}{c^2+ab}+\frac{a^2+b^2+c^2}{2(ab+bc+ca)} \geq 2$$

which is true from

$$\frac{bc}{a^2+bc}+\frac{ac}{b^2+ac}+\frac{ab}{c^2+ab} \geq 1+\frac{4abc}{(a+b)(b+c)(c+a)}$$

$$\frac{a^2+b^2+c^2}{2(ab+bc+ca)}+\frac{4abc}{(a+b)(b+c)(c+a)} \geq 1$$

equality occur if and if only

$$a=b=c$$

or

$$a=b, c=0$$

or any cyclic permution.

Return to your inequality,We have

$$\sum\left(\frac{a^3(b+c-a)}{a^2+bc}+a^2\right) \leq a^2+b^2+c^2+\frac{9abc}{2(a+b+c)}$$

or

$$(ab+bc+ca)\sum\frac{a^2}{a^2+bc} \leq a^2+b^2+c^2+\frac{9abc}{2(a+b+c)}$$

From

$$\frac{a^2}{a^2+bc}+\frac{b^2}{b^2+ca}+\frac{c^2}{c^2+ab} \leq \frac{(a+b+c)^2}{2(ab+bc+ca)}$$

We only need to prove that

$$\frac{(a+b+c)^2}{2} \leq a^2+b^2+c^2+\frac{9abc}{2(a+b+c)} or$$

$$a^2+b^2+c^2+\frac{9abc}{a+b+c} \geq 2(ab+bc+ca)$$

Which is schur inequality. Our **Solution** are completed equality occur if and if only

$$a=b=c, a=b, c=0$$

or any cyclic permution.

33, Let

$$a,b,c>0$$

such that

$$a+b+c=1$$

Then

$$\frac{a^3+bc}{a^2+bc}+\frac{b^3+ca}{b^2+ca}+\frac{c^3+ab}{c^2+ab}\geq 2$$

From the condition

$$a-1=-(b+c)$$

it follows that

$$\sum \frac{a^3+bc}{a^2+bc}=\sum\left(-\frac{a^2(b+c)}{a^2+bc}+1\right)$$

Thus it suffices to prove that

$$a+b+c\geq \sum \frac{a^2(b+c)}{a^2+bc}$$

For

$$a,b,c$$

positive reals prove that

$$\sum \frac{ab(a+b)}{c^2+ab}\geq \sum a$$

$$<=> a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2\geq 0$$

$$\sum \frac{ab(a+b)}{c^2+ab}+\sum \frac{c^2(a+b)}{c^2+ab}=2\sum a$$

and our inequality becomes

$$\sum \frac{c^2(a+b)}{(c^2+ab)}\leq \sum a$$

but

$$\sum \frac{c^2(a+b)}{(c^2+ab)}=\sum \frac{c^2(a+b)^2}{(c^2+ab)(a+b)}=\sum \frac{(ca+cb)^2}{a(b^2+c^2)+b(a^2+c^2)}\leq$$

$$\sum \frac{c^2a^2}{a(b^2+c^2)}+\sum \frac{c^2b^2}{b(a^2+c^2)}=\sum a$$

34

Let

$$a,b,c\geq 0$$

such that

$$a+b+c=1$$

Then

$$6(a^2+b^2+c^2)\geq \frac{a^3+bc}{a^2+bc}+\frac{b^3+ca}{b^2+ca}+\frac{c^3+ab}{c^2+ab}$$

Solution

$$6(a^2+b^2+c^2)+\sum \frac{a^2(b+c)}{a^2+bc}\geq 3$$

$$6(a^2+b^2+c^2)-2(a+b+c)^2\geq \sum (a-\frac{a^2(b+c)}{a^2+bc})$$

$$4\sum (a-b)(a-c)\geq \sum \frac{a(a-b)(a-c)}{a^2+bc}$$

$$\sum (a-b)(a-c)(4-\frac{a}{a^2+bc})\geq 0$$

Assuming WLOG

$$a \geq b \geq c$$

then easy to see that

$$4 - \frac{a}{a^2 + bc} \geq 0$$

and

$$4 - \frac{c}{c^2 + ab} \geq 0$$

$$(c-a)(c-b)(4 - \frac{c}{c^2+ab}) \geq 0 and (a-b)(a-c)(4 - \frac{a}{a^2+bc}) \geq 0$$

We have two cases

Case 1

$$4 - \frac{b}{b^2 + ac} \leq 0$$

then

$$(b-c)(b-a)(4 - \frac{b}{b^2+ac}) \geq 0$$

so this ineq is true

Case 2

$$4 - \frac{b}{b^2 + ac} \leq 0$$

easy to see that

$$4 - \frac{c}{c^2+ab} \geq 4 - \frac{b}{b^2+ac}$$

So

$$LHS \geq (c-b)^2(4 - \frac{c}{c^2+ab}) + (a-b)(b-c)(\frac{b}{b^2+ac} - \frac{c}{c^2+ab}) \geq 0$$

Q.E.D

35. Let

$$x, y, z$$

be real numbers satisfy:

$$x^2y^2 + 2yx^2 + 1 = 0$$

Find the maximum and minimum values of:

$$f(x,y) = \frac{2}{x^2} + \frac{1}{x} + y(y + \frac{1}{x} + 2)$$

Solution: Put

$$t = \frac{1}{x}; k = y + 1$$

, We have:

$$t^2 + k^2 = 1$$

$$f(x,y) = t^2 + tk$$

Put

$$t = \cos\alpha; k = \sin\alpha$$

then

$$f(x,y) = \cos\alpha^2 + \cos\alpha \ sin\alpha =$$

$$\sin 2\alpha 2 = \frac{1}{2} + \frac{1}{\sqrt{2}} \cos(2\alpha - \frac{\pi}{4})$$

$$\max f(x,y) = \frac{1}{2} + \frac{1}{\sqrt{2}}$$

$$\min f(x,y) = \frac{1}{2} - \frac{1}{\sqrt{2}}$$

Q.E.D

.

36.

Suppose a,b,c,d are positive integers with $ab + cd = 1$.

Then, For $We = 1, 2, 3, 4$,let $(x_i)^2 + (y_i)^2 = 1$, where x_i and y_i are real numbers.

Show that

$$(ay_1 + by_2 + cy_3 + dy_4)^2 + (ax_4 + bx_3 + cx_2 + dx_1)^2 \leq 2(\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c}).$$

Solition:

Use Cauchy-Schwartz , We have

$$(ay_1 + by_2 + cy_3 + dy_4)^2 \leq$$
$$(ab + cd)(\frac{(ay_1 + by_2)^2}{ab} + \frac{(cy_3 + dy_4)^2}{cd}) =$$
$$\frac{(ay_1 + by_2)^2}{ab} + \frac{(cy_3 + dy_4)^2}{cd}$$

Similar:

$$(ax_4 + bx_3 + cx_2 + dx_1)^2 \leq$$
$$(ab + cd)(\frac{(ax_4 + bx_3)^2}{ab} + \frac{(cx_2 + dx_1)^2}{cd})$$
$$= \frac{(ax_4 + bx_3)^2}{ab} + \frac{(cx_2 + dx_1)^2}{cd}$$

But:

$$(ay_1 + by_2)^2 \leq (ay_1 + by_2)^2 + (ax_1 - bx_2)^2 = a^2 + b^2 + 2ab(y_1y_2 - x_1x_2)$$

Similar.

$$(cx_2 + dx_1)^2 \leq c^2 + d^2 + 2cd(x_1x_2 - y_1y_2)$$

, then We get:

$$\frac{(ay_1 + by_2)^2}{ab} + \frac{(cx_2 + dx_1)^2}{cd} \leq$$
$$\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c}$$

(1)

The same argument show that:

$$\frac{(cy_3 + dy_4)^2}{cd} + \frac{(ax_4 + bx_3)^2}{ab} \leq$$
$$\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c}$$

(2)

Combining (1);(2) We get . Q.E.D

37.

in any convex quadrilateral with sides

$$a \leq b \leq c \leq d$$

and area F
Prove that :

$$F \leq \frac{3\sqrt{3}}{4} c^2$$

Solution:
The inequality is rewritten as:
$(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d) \leq 27c^4$.
We substitute $x = -a+b+c+d$, $y = a-b+c+d$, $z = a+b-c+d$, $t = a+b+c-d$.
Then $\dfrac{x+y-z+t}{4} = c$ and $x \geq y \geq z \geq t$.
Thus We have: $xyzt \leq 27(\dfrac{x+y-z+t}{4})^4$.
The left side of the inequality is maximum when $z = y$
while the right side of the inequality is minimum (We have fixed x,y and t).
Then We just prove that $xy^2t \leq 27(\dfrac{x+t}{4})^4$.
Because $xy^2t \leq x^3t$, We just have to prove

$$x^3t \leq (\frac{x+t}{4})^4$$

And then it follows that the above inequality is also true.

$$\frac{x}{3} + \frac{x}{3} + \frac{x}{3} + t \geq$$
$$4\sqrt[4]{\frac{x}{3} \cdot \frac{x}{3} \cdot \frac{x}{3} \cdot t}$$

hence

$$27(\frac{x+y}{4})^4 \geq x^3t$$

38.
Let ABC be a triangle. Prove that:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{a+b-c} + \frac{1}{b+c-a} + \frac{1}{c+a-b}$$

Solution:
1.

$$\frac{1}{a+b-c} + \frac{1}{b+c-a} = \frac{2b}{(a+b-c)(b+c-a)} = \frac{2b}{b^2-(c-a)^2} \geq \frac{2b}{b^2} = \frac{2}{b}$$

Similarly, We have

$$\frac{1}{b+c-a} + \frac{1}{c+a-b} \geq \frac{2}{c}$$
$$\frac{1}{c+a-b} + \frac{1}{a+b-c} \geq \frac{2}{a}$$

Add three inequalities together and divide by 2 to get the desired result.
2.
use Karamata for the number arrays $(b+c-a;\ c+a-b;\ a+b-c) \succ (a;\ b;\ c)$
and the convex function

$$f(x) = \frac{1}{x}$$

Or make the substitution $x = \frac{1}{2}(b+c-a)$, $y = \frac{1}{2}(c+a-b)$,
$z = \frac{1}{2}(a+b-c)$ and get

$$a = y+z, b = z+x, c = x+y,$$

so that the inequality in question can be rewritten as

$$\frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y} \le \frac{1}{2x} + \frac{1}{2y} + \frac{1}{2z}$$

,
what directly follows from AM-HM:

$$\frac{2}{y+z} \le \frac{1}{2y} + \frac{1}{2z}, \frac{2}{z+x} \le \frac{1}{2z} + \frac{1}{2x}, \frac{2}{x+y} \le \frac{1}{2x} + \frac{1}{2y}$$

39.
Let a, b, c be nonnegative real numbers. Prove that

$$a^3 + b^3 + c^3 + 3abc \ge \frac{(a^2b + b^2c + c^2a)^2}{ab^2 + bc^2 + ca^2} + \frac{(ab^2 + bc^2 + ca)^2}{a^2b + b^2c + c^2a}$$

Solution:
if $a = 0$ or $b = 0$ or $c = 0$,it's true.if

$$abc > 0$$

Put $x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$.We need prove

$$\frac{x}{z} + \frac{y}{x} + \frac{z}{y} + 3 \ge \frac{(xy+yz+zx)^2}{xyz(x+y+z)} + \frac{(x+y+z)^2}{xy+yz+zx}$$

$$\frac{x}{z} + \frac{y}{x} + \frac{z}{y} \ge \frac{x^2y^2 + y^2z^2 + z^2x^2}{xyz(x+y+z)} + \frac{x^2+y^2+z^2}{xy+yz+zx}$$

$$\frac{x^2}{z} + \frac{z^2}{y} + \frac{y^2}{x} \ge \frac{(x^2+y^2+z^2)(x+y+z)}{xy+yz+zx}$$

$$\frac{x^3y}{z} + \frac{y^3z}{x} + \frac{z^3x}{y} \ge \ x^2y + y^2z + z^2x$$

By using AM GM'inequality, We have:

$$\frac{x^3y}{z} + xyz \ge \ 2x^2y, \frac{y^3z}{x} + xyz \ge \ 2y^2z$$

$$\frac{z^3x}{y} + xyz \ge \ 2z^2y, \ x^2y + y^2z + z^2x \ge \ 3xyz$$

We have done
40.
Let x, y, z be positive real numbers. Prove that:

$$\sum \left(x + \frac{1}{y} - 1\right)\left(y + \frac{1}{z} - 1\right) \ge 3.$$

Solution.
We rewrite the inequality as

$$\sum \frac{y}{x} + \left(\frac{1}{xyz} - 2\right)\sum x + \left(1 - \frac{2}{xyz}\right)\sum xy + 3 \ge 0.$$

Putting $xyz = k^3$, then there exist $a, b, c > 0$ such that $x = \frac{ka}{b}, y = \frac{kb}{c}, z = \frac{kc}{a}$.
The inequality becomes

$$\sum \frac{a^2}{bc} + \left(\frac{1}{k^2} - 2k\right) \sum \frac{a}{b} + \left(k^2 - \frac{2}{k}\right) \sum \frac{b}{a} + 3 \geq 0$$

$$f(k) = \sum a^3 + \left(k^2 - \frac{2}{k}\right) \sum a^2 b + \left(\frac{1}{k^2} - 2k\right) \sum ab^2 + 3abc \geq 0$$

We have that

$$f'(k) = \frac{2(k^3+1)}{k^3} \left(k \sum a^2 b - \sum ab^2\right)$$

$$f'(k) = 0 \Leftrightarrow k = \frac{\sum ab^2}{\sum a^2 b}.$$

From now, according to the Variation Board, We can deduce our inequality to show that

$$f\left(\frac{\sum ab^2}{\sum a^2 b}\right) \geq 0$$

or equivalently,

$$a^3 + b^3 + c^3 + 3abc \geq \frac{(a^2b + b^2c + c^2a)^2}{ab^2 + bc^2 + ca^2} + \frac{(ab^2 + bc^2 + ca^2)^2}{a^2b + b^2c + c^2a}.$$

Q.E.D

41.
Given $a, b, c \geq\ 0$.Prove that:

$$\frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{a^2}{a^2+bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab}$$

Solution:
We have

$$\sum \frac{2a^2}{(a+b)(a+c)} - \sum \frac{a^2}{a^2+bc} = \sum \frac{a^2(a-b)(a-c)}{(a+b)(a+c)(a^2+bc)} \geq 0$$

(easy to check by Vornicu Schur) it suffices to prove that

$$\frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \sum \frac{2a^2}{(a+b)(a+c)} = \frac{2\sum ab(a+b)}{(a+b)(b+c)(c+a)}$$

Assume that $a + b + c = 1$ and put $q = ab + bc + ca, r = abc$, then the inequality becomes

$$\frac{1}{4q} \geq \frac{q-3r}{q-r}$$

$$\Leftrightarrow \frac{q-r}{q-3r} \geq 4q$$

$$\Leftrightarrow \frac{2r}{q-3r} \geq 4q - 1$$

By Schur's inequality for third degree, We have $r \geq \frac{4q-1}{9}$, then

$$\frac{2r}{q-3r} \geq \frac{2r}{q - \frac{4q-1}{3}} = \frac{6r}{1-q}$$

it suffices to show that

$$6r \geq (4q-1)(1-q)$$

But this is just Schur's inequality for fourth degree

$$\sum a^4 + abc\sum a \geq \sum ab(a^2+b^2)$$

We have done.
2.
Suppose $a+b+c=3$. We need to prove:

$$f(r) = 4q^4 - 9q^3 + 24qr^2 - 54q^2r - 72r^2 - 243r + 216qr \leq\ 0$$

$$f'(r) = 48qr - 54q^2 - 144r - 243 + 216q$$

$$f''(r) = 48(q-3) \leq\ 0, so f'(r) \leq\ f'(0) = -54q^2 - 144 + 216q \leq\ 0$$

So, with $q \leq\ \frac{9}{4}, f(r) \leq\ f(0) = q^3(4q-9) \leq\ 0$
With $q \geq\ \frac{9}{4}$, We have: $f(r) \leq\ f(\frac{4q-9}{3}) \leq\ 0$ (trues with $q \geq\ \frac{9}{4}$)
42.
Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\frac{a^3}{b^2-bc+c^2} + \frac{b^3+c^3}{a^2} \geq \sqrt{2}$$

Solution:

$$\frac{a^3}{b^2-bc+c^2} + \frac{b^3+c^3}{a^2} \geq \frac{(a^2+b^2+c^2)^2}{a[b^2-bc+c^2+a(b+c)]}$$

$$\geq \frac{1}{2.a[\frac{3-2a^2}{4}]} = \frac{1}{2\sqrt{a^2(\frac{\frac{3}{2}-a^2}{2})^2}} \geq \sqrt{2}$$

43.
Let ΔABC and $max(A,B,C) \leq 90$. Prove that :

$$\frac{cosAcosB}{sin2C} + \frac{cosBcosC}{sin2A} + \frac{cosCcosA}{sin2B} \geq \frac{\sqrt{3}}{2}$$

Solution:
But if $A = 90^\circ$ the left side does not exist.
if $\max\{A,B,C\} < 90^\circ$. Let $a^2+b^2-c^2 = z$, $a^2+c^2-b^2 = y$ and $b^2+c^2-a^2 = x$.
Hence, x, y and z are positive and

$$\sum_{cyc} \frac{\cos A\cos B}{\sin 2C} = \sum_{cyc} \frac{\frac{b^2+c^2-a^2}{2bc}\cdot\frac{a^2+c^2-b^2}{2ac}}{2\cdot\frac{2S}{ab}\cdot\frac{a^2+b^2-c^2}{2ab}} =$$

$$= \sum_{cyc} \frac{ab(b^2+c^2-a^2)(a^2+c^2-b^2)}{8c^2S(a^2+b^2-c^2)} = \sum_{cyc} \frac{xy}{(x+y)z}\sqrt{\frac{a^2b^2}{2(a^2b^2+a^2c^2+b^2c^2)-a^4-b^4-c^4}} =$$

$$= \sum_{cyc} \frac{xy}{(x+y)z}\sqrt{\frac{(x+z)(y+z)}{4(xy+xz+yz)}}.$$

Thus, it remains to prove that

$$\sum_{cyc} \frac{xy}{(x+y)z}\sqrt{\frac{(x+z)(y+z)}{xy+xz+yz}} \geq \sqrt{3},$$

which is equivalent to

$$\sum_{cyc} \frac{x^2y^2\sqrt{(x+z)(y+z)}}{x+y} \geq xyz\sqrt{3(xy+xz+yz)}.$$

By Cauchy-Schwartz We obtain:

$$\sum_{cyc} \frac{x^2y^2\sqrt{(x+z)(y+z)}}{x+y} \cdot \sum_{cyc} \frac{x+y}{\sqrt{(x+z)(y+z)}} \geq (xy+xz+yz)^2.$$

Hence, We need to prove that

$$(xy+xz+yz)^2 \geq \sum_{cyc} \frac{x+y}{\sqrt{(x+z)(y+z)}} \cdot xyz\sqrt{3(xy+xz+yz)}.$$

We obtain:

$$(xy+xz+yz)^2 \geq \sum_{cyc} \frac{x+y}{\sqrt{(x+z)(y+z)}} \cdot xyz\sqrt{3(xy+xz+yz)} \Leftrightarrow$$

$$\Leftrightarrow \left(\sqrt{xy+xz+yz}\right)^3 \sqrt{(x+y)(x+z)(y+z)} \geq \sum_{cyc} xyz(x+y)\sqrt{3(x+y)} \Leftrightarrow$$

$$\Leftrightarrow (xy+xz+yz)^3(x+y)(x+z)(y+z) \geq 3x^2y^2z^2\sum_{cyc}(x+y)^3+$$

$$+6x^2y^2z^2\sum_{cyc}(x+y)(x+z)\sqrt{(x+y)(x+z)} \Leftrightarrow$$

$$\Leftrightarrow (xy+xz+yz)^3(x+y)(x+z)(y+z) \geq 3x^2y^2z^2\sum_{cyc}(2x^3+3x^2y+3x^2z)+$$

$$+3x^2yz\sum_{cyc}(x+y)(x+z)2\sqrt{z^2(x+y)y^2(x+z)}.$$

By AM-GM $2\sqrt{z^2(x+y)y^2(x+z)} \leq y^2x+y^2z+z^2x+z^2y$.
Hence, it remains to prove that

$$\Leftrightarrow (xy+xz+yz)^3(x+y)(x+z)(y+z) \geq 3x^2y^2z^2\sum_{cyc}(2x^3+3x^2y+3x^2z)+$$

$+3x^2yz\sum_{cyc}(x+y)(x+z)(y^2x+y^2z+z^2x+z^2y)$, which is equivalent to $\sum_{sym}(x^5y^4+x^5y^3z-5x^4y^3z^2+x^4y^4z+2x^3y^3z^3) \geq 0$, which is true by AM-GM because

$$\sum_{sym}(x^5y^4+x^5y^3z-5x^4y^3z^2+x^4y^4z+2x^3y^3z^3) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{sym}(x^5y^4+x^5y^3z+\frac{1}{3}x^4y^4z+\frac{2}{3}x^4z^4y+2x^3y^3z^3) \geq \sum_{sym} 5x^4y^3z^2.$$

Q.E.D

44.
Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{a^2b+b^2c+c^2a}{ab^2+bc^2+ca^2} \geq \frac{5}{2}$$

Solution

1...asume $p=1$ and Lemma $ab^2+bc^2+ca^2 \le \frac{4}{27}-r$

$$\Leftrightarrow \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{\sum ab(a+b)}{ab^2+bc^2+ca^2} \ge \frac{7}{2}$$

We have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{\sum ab(a+b)}{ab^2+bc^2+ca^2} \ge \frac{1-2q+3r}{q-r}+\frac{27q-81r}{4-27r}$$

We need prove that

$$\frac{1-2q+3r}{q-r}+\frac{27q-81r}{4-27r} \ge \frac{7}{2}$$

$$\Leftrightarrow \frac{1+r}{q-r}-2+\frac{27q-12}{4-27r}+3 \ge \frac{7}{2}$$

$$\Leftrightarrow \frac{1+r}{q-r}+\frac{27q-12}{4-27r} \ge \frac{5}{2}$$

$$\Leftrightarrow -135r^2+r(81q+2)+(54q^2+8-44q) \ge 0$$

$$f(r)=-135r^2+r(81q+2)+(54q^2+8-44q)$$

$f'(r)=-270r+81q+2 \ge 0$ (because $q \ge 9r$)

$$\Rightarrow f(r) \ge f(\frac{4q-1}{9}) = \frac{570q^2-349q+55}{9} \ge 0$$

2........

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{a^2b+b^2c+c^2a}{ab^2+bc^2+ca^2} \ge \frac{5}{2} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{a}{b+c}-\frac{1}{2}\right) \ge \frac{\sum(a^2c-a^2b)}{\sum a^2c} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\frac{a-b-(c-a)}{2(b+c)} \ge \frac{(a-b)(b-c)(c-a)}{a^2c+b^2a+c^2b} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\frac{a-b}{2}\left(\frac{1}{b+c}-\frac{1}{a+c}\right) \ge \frac{(a-b)(b-c)(c-a)}{a^2c+b^2a+c^2b} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\frac{(a-b)^2}{(a+c)(b+c)} \ge \frac{2(a-b)(b-c)(c-a)}{a^2c+b^2a+c^2b}.$$

if $(a-b)(b-c)(c-a) \le 0$ then the inequality holds.
Let $(a-b)(b-c)(c-a)>0$ and $\frac{a^2c+b^2a+c^2b}{a^2b+b^2c+c^2a}=t$. Then $t>1$.
By AM-GM We obtain:

$$\sum_{cyc}\frac{(a-b)^2}{(a+c)(b+c)} \ge 3\sqrt[3]{\frac{(a-b)^2(b-c)^2(c-a)^2}{(a+b)^2(a+c)^2(b+c)^2}}.$$

Thus, it remains to prove that

$$27(a^2c+b^2a+c^2b)^3 \ge 8(a+b)^2(a+c)^2(b+c)^2(a-b)(b-c)(c-a).$$

But
$$(a+b)(a+c)(b+c)=\sum_{cyc}(a^2b+a^2c)+2abc\le\frac{4}{3}\sum_{cyc}(a^2b+a^2c).$$
id est, it remains to prove that $27t^3\ge 8\cdot\frac{16}{9}(t+1)^2(t-1)$, which obvious.
45.
For all nonnegative real numbers a,b and c, no two of which are zero,
$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\ge\frac{3\sqrt{3abc(a+b+c)}(a+b+c)^2}{4(ab+bc+ca)^3}$$
Solution Replacing a,b,c by $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ respectively, We have to prove that
$$\sum\frac{a^2b^2}{(a+b)^2}\ge\frac{3\sqrt{3(ab+bc+ca)}(ab+bc+ca)^2}{4(a+b+c)^3}.$$
Now, using Cauchy Schwarz inequality, We have
$$\sum\frac{a^2b^2}{(a+b)^2}\ge\frac{(ab+bc+ca)^2}{(a+b)^2+(b+c)^2+(c+a)^2}=\frac{(ab+bc+ca)^2}{2(a^2+b^2+c^2+ab+bc+ca)}.$$
it suffices to prove that
$$\frac{(ab+bc+ca)^2}{2(a^2+b^2+c^2+ab+bc+ca)}\ge\frac{3\sqrt{3(ab+bc+ca)}(ab+bc+ca)^2}{4(a+b+c)^3}$$
or equivalently,
$$2(a+b+c)^3\ge 3\sqrt{3(ab+bc+ca)}(a^2+b^2+c^2+ab+bc+ca),$$
that is
$$4(a+b+c)^6\ge 27(ab+bc+ca)(a^2+b^2+c^2+ab+bc+ca)^2$$
By AM-GM, We see that
$$27(ab+bc+ca)(a^2+b^2+c^2+ab+bc+ca)^2\le\frac{1}{2}$$
$$\big(2(ab+bc+ca)+(a^2+b^2+c^2+ab+bc+ca)+(a^2+b^2+c^2+ab+bc+ca)\big)^3=4(a+b+c)^6.$$
Therefore, our **Solution** is completed
46.
$$\frac{2}{3}(\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab})\ge\frac{1}{ab+bc+ca}+\frac{2}{a^2+b^2+c^2}$$
Solution:
Rewrite our inequality as:
$$\sum\frac{1}{a^2+bc}\ge\frac{3(a+b+c)^2}{2(a^2+b^2+c^2)(ab+bc+ca)}.$$
We will consider 2 cases: Case 1. $a^2+b^2+c^2\le 2(ab+bc+ca)$, then applying Cauchy Schwarz inequality, We can reduce our inequality to
$$\frac{6}{a^2+b^2+c^2+ab+bc+ca}\ge\frac{(a+b+c)^2}{(a^2+b^2+c^2)(ab+bc+ca)},$$
$(a^2+b^2+c^2-ab-bc-ca)(2ab+2bc+2ca-a^2-b^2-c^2)\ge 0$, which is true.
Case 2. $a^2+b^2+c^2\ge 2(ab+bc+ca)$, then $(a+b+c)^2\le 2(a^2+b^2+c^2)$, which yields that
$$\frac{3(a+b+c)^2}{2(a^2+b^2+c^2)(ab+bc+ca)}\le\frac{3}{ab+bc+ca},$$

and We just need to prove that

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}\geq\frac{3}{ab+bc+ca},$$

which is just your very known (and nice) inequality.
47.
Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$(a)\qquad \frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\geq\frac{1}{ab+bc+ca}+\frac{2}{a^2+b^2+c^2}$$

Solution:
1st **Solution**
By Cauchy inequality,

$$\sum_{cyc}(b+c)^2(2a^2+bc)\sum_{cyc}\frac{1}{2a^2+bc}\geq 4(a+b+c)^2$$

it remains to show that

$$\sum_{cyc}(b+c)^2(2a^2+bc)\leq 4(a^2+b^2+c^2)(ab+bc+ca)$$

which is easy. 2nd **Solution**.
Since

$$\sum_{cyc}\frac{ab+bc+ca}{2a^2+bc}=\sum_{cyc}\frac{bc}{2a^2+bc}+\sum_{cyc}\frac{a(b+c)}{2a^2+bc}$$

We need to show

$$\sum_{cyc}\frac{bc}{2a^2+bc}\geq 1$$

and

$$\sum_{cyc}\frac{a(b+c)}{2a^2+bc}\geq\frac{2(ab+bc+ca)}{a^2+b^2+c^2}$$

The former is ill-known: if $x,y,z\geq 0$ such that $xyz=1$, then

$$\frac{1}{2x+1}+\frac{1}{2y+1}+\frac{1}{2z+1}\geq 1$$

The later: by Cauchy inequality,

$$\sum_{cyc}a(b+c)(2a^2+bc)\sum_{cyc}\frac{a(b+c)}{2a^2+bc}\geq 4(ab+bc+ca)^2$$

The result then follows from the following identity

$$\sum_{cyc}a(b+c)(2a^2+bc)=2(ab+bc+ca)(a^2+b^2+c^2)$$

3rd **Solution**.

$$\frac{LHS-RHS}{a+b+c}=\frac{2(a+b+c)(a-b)^2(b-c)^2(c-a)^2+3abc\sum_{cyc}(a^2+ab+b^2)(a-b)^2}{(2a^2+bc)(2b^2+ca)(2c^2+ab)(ab+bc+ca)(a^2+b^2+c^2)}$$

3rd **Solution**.
Assume that $c=\min\{a,b,c\}$, then the Cauchy Schwarz inequality yields

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}\geq\frac{4}{2(a^2+b^2)+c(a+b)},$$

then We just need to prove that

$$\frac{4}{2(a^2+b^2)+c(a+b)}+\frac{1}{ab+2c^2}\geq\frac{1}{ab+bc+ca}+\frac{2}{a^2+b^2+c^2},$$

or equivalently

$$\frac{c(a+b-2c)}{(ab+2c^2)(ab+bc+ca)}\geq\frac{2c(a+b-2c)}{(a^2+b^2+c^2)(2a^2+2b^2+ac+bc)},$$

that is

$$(a^2+b^2+c^2)(2a^2+2b^2+ac+bc)\geq 2(ab+2c^2)(ab+bc+ca),$$

which is true since $a^2+b^2+c^2\geq ab+bc+ca$ and $2a^2+2b^2+ac+bc\geq 2(ab+2c^2)$. Done.
48.
Let a,b,c be positive real number . Prove that:

$$(c)\qquad 2\left(\frac{1}{a^2+8bc}+\frac{1}{b^2+8ca}+\frac{1}{c^2+8ab}\right)\geq\frac{1}{ab+bc+ca}+\frac{1}{a^2+b^2+c^2}$$

Solution:
Replacing a,b,c by $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ respectively, We can rewrite our inequality as

$$4(a+b+c)\left(\frac{a}{8a^2+bc}+\frac{b}{8b^2+ca}+\frac{c}{8c^2+ab}\right)\geq 2+\frac{2abc(a+b+c)}{a^2b^2+b^2c^2+c^2a^2}.$$

Now, assume that $c=mina,b,c$, then We have the following estimations:

$$\frac{a(4a+4b+c)}{8a^2+bc}+\frac{b(4a+4b+c)}{8b^2+ca}-2=\frac{(a-b)^2(32ab-12ac-12bc+c^2)}{(8a^2+bc)(8b^2+ca)}\geq 0,$$

and

$$\frac{2abc(a+b+c)}{a^2b^2+b^2c^2+c^2a^2}\leq\frac{2c(a+b+c)}{ab+2c^2}.$$

With these estimations, We can reduce our inequality to

$$\frac{3ac}{8a^2+bc}+\frac{3bc}{8b^2+ca}+\frac{4c(a+b+c)}{8c^2+ab}\geq\frac{2c(a+b+c)}{ab+2c^2},$$

or

$$\frac{3a}{8a^2+bc}+\frac{3b}{8b^2+ca}\geq\frac{2(a+b+c)(4c^2-ab)}{(ab+2c^2)(ab+8c^2)}.$$

According to Cauchy Schwarz inequality, We have

$$\frac{3a}{8a^2+bc}+\frac{3b}{8b^2+ca}\geq\frac{12}{8(a+b)+c\left(\frac{a}{b}+\frac{b}{a}\right)}.$$

it suffices to show that

$$\frac{6}{8(a+b)+c\left(\frac{a}{b}+\frac{b}{a}\right)}\geq\frac{(a+b+c)(4c^2-ab)}{(ab+2c^2)(ab+8c^2)}.$$

if $4c^2\leq ab$, then it is trivial. Otherwise, We have
$a+b\leq c+\frac{ab}{c}$, and $\frac{a}{b}+\frac{b}{a}\leq\frac{ab}{c^2}+\frac{c^2}{ab}$. We need to prove

$$\frac{6}{8\left(c+\frac{ab}{c}\right)+c\left(\frac{ab}{c^2}+\frac{c^2}{ab}\right)}\geq\frac{\left(2c+\frac{ab}{c}\right)(4c^2-ab)}{(ab+2c^2)(ab+8c^2)},$$

which is, after expanding, equivalent to

$$\frac{(9ab-4c^2)(ab-c^2)^2}{c(ab+8c^2)(c^4+8abc^2+9a^2b^2)} \geq 0,$$

which is true as $c = mina, b, c$.
Our **Solution** is completed.
49.
Let $a, b, c > 0$.Prove that:

$$\frac{5}{3}\left(\frac{1}{4a^2+bc}+\frac{1}{4b^2+ca}+\frac{1}{4c^2+ab}\right) \geq \frac{2}{ab+bc+ca}+\frac{1}{a^2+b^2+c^2}$$

Solution:
Assume that $c = min(a, b, c)$, then We have the following estimations:

$$\frac{1}{4a^2+bc}+\frac{1}{4b^2+ca}-\frac{4}{8ab+ac+bc} = \frac{(a-b)^2(32ab-12ac-12bc+c^2)}{(4a^2+bc)(4b^2+ca)(8ab+ac+bc)} \geq 0,$$

and

$$\frac{1}{a^2+b^2+c^2} \leq \frac{1}{2ab+c^2}.$$

Using these, We can reduce our inequality to

$$\frac{20}{8ab+ac+bc}+\frac{5}{ab+4c^2} \geq \frac{6}{ab+ac+bc}+\frac{3}{2ab+c^2}.$$

Denote $x = a+b \geq 2\sqrt{ab}$ then this inequality can be rewritten as

$$f(x) = \frac{20}{cx+8ab}-\frac{6}{cx+ab}+\frac{5}{ab+4c^2}-\frac{3}{2ab+c^2} \geq 0.$$

We have

$$f'(x) = \frac{6c}{(cx+ab)^2}-\frac{20}{(cx+8ab)^2} \geq \frac{20c}{(cx+ab)(cx+8ab)}-\frac{20c}{(cx+8ab)^2} = \frac{140abc}{(cx+ab)(cx+8ab)^2} \geq 0.$$

This shows that $f(x)$ is increasing, and We just need to prove that $f(2\sqrt{ab}) \geq 0$, which is equivalent to

$$\frac{7c(13t^2+6tc+8c^2)(t-c)^2}{t(t+2c)(4t+c)(2t^2+c^2)(t^2+4c^2)} \geq 0,$$

Where

$$t = \sqrt{ab}$$

This is obviously nonnegative, so our **Solution** is completed.
50.
Let a, b and c real numbers such that $a+b+c+d=e=0$. Prove that:

$$30(a^4+b^4+c^4+d^4+e^4) \geq 7(a^2+b^2+c^2)^2$$

Solution:
Notice that there exitst three numbers among a, b, c, d, e havinh the same sing. Let these number be a, b, c, d, e .Without loss of generality,We may assume that $a, b, c \geq 0$(it not,We can take $-1, -b, -c$).
Now ,using the Cauchy-Schawrz inequality,We have:

$$[(9(a^4+b^4+c^4)+2(d^4+e^4))+7d^4+7e^4)](84+63+63) \geq [2\sqrt{21(9(a^4+b^4+c^4)+2(d^4+e^4))}+21d^2+21e^2]^2.$$

And thus,it suffices to prove that:

$$2\sqrt{9(a^4+b^4+c^4)+2(d^4+e^4))} \geq \sqrt{21}(a^2+b^2+c^2).$$

Or

$$36(a^4+b^4+c^4)+8(d^4+e^4) \geq 21(a^2+b^2+c^2)^2.$$

Since

$$d^4+e^4 \geq \frac{(d^2+e^2)^2}{2} \geq \frac{(d+e)^4}{8} = \frac{(a+b+c)^4}{8},$$

it is enough to show that

$$36(a^4+b^+c^4)+(a+b+c+d)^4 \geq 21(a^2+b^2+c^2)^2$$

Which is true and it is easy to prove.

51.

Let $a, b, c > 0$.Prove that:

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab} \leq \frac{1}{2}\sqrt{27+(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

Solution

The inequality is equivalent to

$$\sum \frac{a^2(b+c)^2}{(a^2+bc)^2}+2\sum \frac{ab(b+c)(c+a)}{(a^2+bc)(b^2+ca)} \leq \frac{15}{2}+\frac{1}{4}\left(\sum \frac{b+c}{a}\right)$$

Notice that

$$(a^2+bc)(b^2+ca)-ab(b+c)(c+a) = c(a+b)(a-b)^2$$

then

$$2\sum \frac{ab(b+c)(c+a)}{(a^2+bc)(b^2+ca)} \leq 6$$

(1) Other hand,

$$\sum \frac{a^2(b+c)^2}{(a^2+bc)^2} \leq \sum \frac{a^2(b+c)^2}{4a^2bc} = \frac{1}{4}\sum\left(\frac{b}{c}+\frac{c}{b}+2\right)$$

(2) From (1) and (2) We have done!

Besides, by the sam ways, We have a nice **Solution** for an old problem:

$$\sum \sqrt{\frac{a(b+c)}{a^2+bc}} \leq \sqrt{\left(\sum \sqrt{a}\right)\left(\sum \frac{1}{\sqrt{a}}\right)}$$

52.

For any positive real numbers a, b and c,

$$\sqrt{\frac{a(b+c)}{a^2+bc}}+\sqrt{\frac{b(c+a)}{b^2+ca}}+\sqrt{\frac{c(a+b)}{c^2+ab}} \leq \sqrt{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)}$$

Solution:

We have the inequality is equivalent to

$$\left(\sum \sqrt{\frac{a(b+c)}{a^2+bc}}\right)^2 \leq \left(\sum \sqrt{a}\right)\left(\sum \frac{1}{\sqrt{a}}\right)$$

$$<=> \sum \frac{a(b+c)}{a^2+bc} + 2\sum \sqrt{\frac{ab(a+c)(b+c)}{(a^2+bc)(b^2+ca)}} \le \left(\sum \sqrt{a}\right)\left(\sum \frac{1}{\sqrt{a}}\right)$$

We can easily prove that

$$\sum \sqrt{\frac{ab(a+c)(b+c)}{(a^2+bc)(b^2+ca)}} \le 3$$

So, it suffices to prove that

$$<=> \sum \frac{a(b+c)}{a^2+bc} + 6 \le \left(\sum \sqrt{a}\right)\left(\sum \frac{1}{\sqrt{a}}\right)$$

To prove this ineq, We only need to prove that

$$\frac{a+b}{\sqrt{ab}} - \frac{c(a+b)}{c^2+ab} - 1 \ge 0$$

But this is trivial, because

$$\frac{a+b}{\sqrt{ab}} - \frac{c(a+b)}{c^2+ab} - 1 = (a+b)\left(\frac{1}{\sqrt{ab}} - \frac{c}{c^2+ab}\right) - 1$$

$$\ge 2\sqrt{ab}\left(\frac{1}{\sqrt{ab}} - \frac{c}{c^2+ab}\right) - 1 = \frac{\left(c-\sqrt{ab}\right)^2}{c^2+ab} \ge 0$$

We are done.

53.

Let $a, b, c > 0$. Prove that:

$$\frac{(a+b+c)^3}{3abc} + \frac{ab^2+bc^2+ca^2}{a^3+b^3+c^3} \ge 10$$

Solution

$$\left(\frac{a^3+b^3+c^3}{3abc} + \frac{ab^2+bc^2+ca^2}{a^3+b^3+c^3}\right) + \frac{(a+b)(b+c)(c+a)}{abc} \ge 10$$

Using AM-GM's inequality ,We have:

$$\frac{a^3+b^3+c^3}{3abc} + \frac{ab^2+bc^2+ca^2}{a^3+b^3+c^3} \ge 2\sqrt{\frac{ab^2+bc^2+ca^2}{3abc}} \ge 2$$

$$\frac{(a+b)(b+c)(c+a)}{abc} \ge 8$$

54.

in any triangle ABC show that

$$am_a + bm_b + cm_c \le \sqrt{bc}m_a + \sqrt{ca}m_b + \sqrt{ab}m_c$$

Solution:

We have to prove the inequality

$$am_a + bm_b + cm_c \le \sqrt{bc}m_a + \sqrt{ca}m_b + \sqrt{ab}m_c$$

, where m_a, m_b, m_c are the medians of a triangle ABC.

Since $\frac{2bc}{b+c} \le \sqrt{bc}$, $\frac{2ca}{c+a} \le \sqrt{ca}$ and $\frac{2ab}{a+b} \le \sqrt{ab}$

by the HM-GM inequality, it will be enough to show the stronger inequality

$$am_a + bm_b + cm_c \leq \frac{2bc}{b+c}m_a + \frac{2ca}{c+a}m_b + \frac{2ab}{a+b}m_c$$

,

since then We will have

$$\begin{aligned} am_a + bm_b + cm_c &\leq \frac{2bc}{b+c}m_a + \frac{2ca}{c+a}m_b + \frac{2ab}{a+b}m_c \\ &\leq \sqrt{bc}m_a + \sqrt{ca}m_b + \sqrt{ab}m_c \end{aligned}$$

and the initial inequality will be proven.
So in the following, We will concentrate on proving this stronger inequality.
Because the inequality We have to prove is symmetric, We can WLOG assume that $a \geq b \geq c$.
Then, clearly, $bc \leq ca \leq ab$.
On the other hand, using the formulas $m_a^2 = \frac{1}{4}\left(2b^2 + 2c^2 - a^2\right)$ and $m_b^2 = \frac{1}{4}\left(2c^2 + 2a^2 - b^2\right)$, We can get as a result of a straightforward computation.

$$\left(\frac{m_a}{b+c}\right)^2 - \left(\frac{m_b}{c+a}\right)^2 = \frac{\left(3ac + 3bc + a^2 + b^2 + 4c^2\right)(a+b-c)(b-a)}{4(b+c)^2(c+a)^2}$$

Now, the fraction on the right hand side is ≤ 0, since $3ac + 3bc + a^2 + b^2 + 4c^2 \geq 0$ (this is trivial),

$a + b - c > 0$ (in fact, $a + b > c$ because of the triangle inequality) and $b - a \leq 0$ (since $a \geq b$). Hence,

$$\left(\frac{m_a}{b+c}\right)^2 - \left(\frac{m_b}{c+a}\right)^2 \leq 0$$

what yields $\left(\frac{m_a}{b+c}\right)^2 \leq \left(\frac{m_b}{c+a}\right)^2$ and thus $\frac{m_a}{b+c} \leq \frac{m_b}{c+a}$. Similarly, using $b \geq c$, We can find $\frac{m_b}{c+a} \leq \frac{m_c}{a+b}$.

Thus, We have

$$\frac{m_a}{b+c} \leq \frac{m_b}{c+a} \leq \frac{m_c}{a+b}$$

Since We have also $bc \leq ca \leq ab$, the sequences

$$\left(\frac{m_a}{b+c}; \frac{m_b}{c+a}; \frac{m_c}{a+b}\right)$$

and $(bc;\ ca;\ ab)$ are equally sorted. Thus, the Rearrangement inequality yields

$$\frac{m_a}{b+c} \cdot bc + \frac{m_b}{c+a} \cdot ca + \frac{m_c}{a+b} \cdot ab \geq \frac{m_a}{b+c} \cdot ca + \frac{m_b}{c+a} \cdot ab + \frac{m_c}{a+b} \cdot bc$$

and

$$\frac{m_a}{b+c} \cdot bc + \frac{m_b}{c+a} \cdot ca + \frac{m_c}{a+b} \cdot ab \geq \frac{m_a}{b+c} \cdot ab + \frac{m_b}{c+a} \cdot bc + \frac{m_c}{a+b} \cdot ca$$

Summing up these two inequalities, We get

$$2\frac{m_a}{b+c} \cdot bc + 2\frac{m_b}{c+a} \cdot ca + 2\frac{m_c}{a+b} \cdot ab$$

$$\geq \frac{m_a}{b+c} \cdot (ca+ab) + \frac{m_b}{c+a} \cdot (ab+bc) + \frac{m_c}{a+b} \cdot (bc+ca)$$

This simplifies to

$$\frac{2bc}{b+c} m_a + \frac{2ca}{c+a} m_b + \frac{2ab}{a+b} m_c$$

$$\geq \frac{m_a}{b+c} \cdot a\,(b+c) + \frac{m_b}{c+a} \cdot b\,(c+a) + \frac{m_c}{a+b} \cdot c\,(a+b)$$

i. e. to

$$\frac{2bc}{b+c} m_a + \frac{2ca}{c+a} m_b + \frac{2ab}{a+b} m_c \geq am_a + bm_b + cm_c$$

Thus, We have

$$am_a + bm_b + cm_c \leq \frac{2bc}{b+c} m_a + \frac{2ca}{c+a} m_b + \frac{2ab}{a+b} m_c$$

and the **Solution** is complete. Note that in each of the inequalities

$$am_a + bm_b + cm_c \leq \sqrt{bc}m_a + \sqrt{ca}m_b + \sqrt{ab}m_c$$

and

$$am_a + bm_b + cm_c \leq \frac{2bc}{b+c} m_a + \frac{2ca}{c+a} m_b + \frac{2ab}{a+b} m_c$$

equality holds only if the triangle ABC is equilateral.

55.

For a, b, c positive reals prove that

$$\left(a^2+3\right)\left(b^2+3\right)\left(c^2+3\right) \geq \left(\frac{4}{3}\right)^3 \sqrt{abc}\,(ab+bc+ca)$$

Solution:

Divide abc for both term and take $x = \sqrt{\frac{bc}{a}}; y = \sqrt{\frac{ac}{b}}; z = \sqrt{\frac{ab}{c}}$ and We must prove that: $\prod(xy + \frac{3}{xy}) \geq (\frac{4}{3})^3(x+y+z)$ Note that:

$$LHS \geq 3(x^2+y^2+z^2) + x^2y^2z^2 + \frac{4}{x^2y^2z^2} \geq (x+y+z)^2 + 4 \geq 4(x+y+z) \geq (\frac{4}{3})^3(x+y+z).$$

56. Let$a, b, c > 0$ .Prove that:

$$\frac{a+b}{c\sqrt{a^2+b^2}} + \frac{b+c}{a\sqrt{b^2+c^2}} + \frac{c+a}{b\sqrt{c^2+a^2}} \geq \frac{3\sqrt{6}}{\sqrt{a^2+b^2+c^2}}.$$

Solution

1...Alternatively, using Chebyshev and Cauchy,

$$\sum_{cycl} \frac{a+b}{c\sqrt{a^2+b^2}} \geq \frac{2(a+b+c)}{3} \cdot \frac{9}{\sum_{cycl} c\sqrt{a^2+b^2}} = \frac{6(a+b+c)}{\sum_{cycl} c\sqrt{a^2+b^2}}$$

and

$$\sum_{cycl} c\sqrt{a^2+b^2} \leq \frac{a+b+c}{3} \sum_{cycl} \sqrt{a^2+b^2} \leq \frac{a+b+c}{3} \sqrt{6(a^2+b^2+c^2)}$$

Combining We get the desired result.

57.

Let $a, b, c > 0$ such that $a^2+b^2+c^2+abc = 4$

Prove that

$$a^2b^2 + b^2c^2 + c^2a^2 \leq a^2+b^2+c^2$$

Solution:

Let $a = 2\sqrt{\frac{yz}{(x+y)(x+z)}}, b = 2\sqrt{\frac{xz}{(x+y)(y+z)}}$ and $c = 2\sqrt{\frac{xy}{(x+z)(y+z)}}$,

where x, y and z are positive numbers (easy to check that it exists).

Thus, it remains to prove that

$$\sum_{cyc} \frac{xy}{(x+z)(y+z)} \geq \sum_{cyc} \frac{4x^2yz}{(x+y)(x+z)(y+z)^2},$$

which equivalent to $\sum_{cyc}(x^4y^2+x^4z^2-2x^4yz+2x^3y^3-2x^2y^2z^2) \geq 0$, which true by AM-GM.

58.

Let $a, b, c > 0$ such that $a+b+c = 1$.Prove that

$$\frac{b^2}{a+b^2} + \frac{c^2}{b+c^2} + \frac{a^2}{c+a^2} \geq \frac{3}{4}$$

Solution

We have

$$\frac{b^2}{a+b^2} + \frac{c^2}{b+c^2} + \frac{a^2}{c+a^2} \geq \frac{\left(a^2+b^2+c^2\right)^2}{(a^4+b^4+c^4)+(ab^2+bc^2+ca^2)}$$

Hence it suffices to prove that

$$\frac{\left(a^2+b^2+c^2\right)^2}{(a^4+b^4+c^4)+(ab^2+bc^2+ca^2)} \geq \frac{3}{4}$$

$$\Leftrightarrow 4\left(\sum a^2\right)^2 \geq 3\left(\sum ab^2\right)\left(\sum a\right) + 3\sum a^4$$

$$\Leftrightarrow 4\sum a^4 + 8\sum a^2b^2 \geq 3\sum a^4 + 3\sum\left(a^2b^2+abc^2+a^3c\right)$$

$$\Leftrightarrow \sum a^4 + 5\sum a^2b^2 \geq 3abc\left(\sum a\right) + 3\sum a^3c$$

Since We always have

$$3\left(a^3c+b^3a+c^3b\right) \leq \left(a^2+b^2+c^2\right)^2 = \left(a^4+b^4+c^4\right) + 2\left(a^2b^2+b^2c^2+c^2a^2\right)$$

Therefor it suffices to prove that

$$3\left(a^2b^2+b^2c^2+c^2a^2\right) \geq 3abc\left(a+b+c\right)$$

which obviously true.

59.

Let $a; b; c > 0$. Prove that

$$a^{b+c} + b^{a+c} + c^{a+b} \geq 1$$

Solution

if $a \geq 1$ or $b \geq 1$ or $c \geq 1$ then the inequality is true

if $0 \leq a, b, c \leq 1$ then suppose $c = mina, b, c$

$+$ if $a+b < 1$ We have $b+c < 1$,$c+a < 1$

Apply BernoullWe ' inequality

$$(\frac{1}{a})^{b+c}) = (1 + \frac{1-a}{a})^{b+c} < 1 + \frac{(b+c)(1-a)}{a} < \frac{a+b+c}{a}$$

Therefore $a^{b+c} > \frac{a}{a+b+c}$
Similar for b^{c+a} and c^{a+b} deduce $a^{b+c} + b^{a+c} + c^{a+b} > 1$

+ if $a+b > 1$ then $a^{b+c} + b^{a+c} + c^{a+b} > a^{b+c} + b^{a+c} \geq a^{a+b} + b^{a+b}$

Apply BernoullWe ' inequality We have : D$a^{a+b} = (1 + (a-1))^{a+b} > 1 + (a+b)(a-1)$

Similar forb^{a+b} whence $a^{b+c} + b^{a+c} + c^{a+b} > 2 + (a+b)(a+b-2) = (a+b-1)^2 + 1 \geq 1$

60.

Let a, b, c be the sidelengths of triangle with perimeter 2 ($\Rightarrow a+b+c=2$). Prove that

$$\left|\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} - \frac{a^3}{c} - \frac{b^3}{a} - \frac{c^3}{b}\right| < 3$$

Solution:

This ineq is equivalent to:

$$|a^4c + c^4b + b^4a - a^4b - b^c - c^4a| \leq 3abc$$

$$<=> |(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)| \leq 3abc$$

By RavWe Substitution , denote: $a = x+y, b = y+z, c = z+x$, so $x+y+z=1$, this ineq becomes:

$$|(x-y)(y-z)(z-x)(3(x^2+y^2+z^2)+5(xy+yz+zx)| \leq 3(x+y)(y+z)(z+x)$$

Easy to see that $|(x-y)(y-z)(z-x)| \leq (x+y)(y+z)(z+x)$
So We need to prove $(3(x^2+y^2+z^2)+5(xy+yz+zx) \leq 3 = 3(x+y+z)^2$

$$<=> xy+yz+zx \geq 0$$

which is obvious true
Q.E.D
.

61.

Given x,y,z>0.Prove that

$$\frac{x(y+z)^2}{2x+y+z} + \frac{y(x+z)^2}{x+2y+z} + \frac{z(x+y)^2}{x+y+2z} = \sqrt(3xyz(x+y+z))$$

Solution:

$$\sum_{cyc} \frac{x(y+z)^2}{2x+y+z} - \sqrt{3xyz(x+y+z)} =$$

$$= \sum_{cyc} \left(\frac{x(y+z)^2}{2x+y+z} - yz\right) + xy + xz + yz - \sqrt{3xyz(x+y+z)} =$$

$$= \sum_{cyc} \frac{z^2(x-y) - y^2(z-x)}{2x+y+z} + \sum_{cyc} \frac{z^2(x-y)^2}{2\left(xy+xz+yz+\sqrt{3xyz(x+y+z)}\right)} =$$

$$= \sum_{cyc}(x-y)\left(\frac{z^2}{2x+y+z}-\frac{z^2}{2y+x+z}\right)+$$

$$+\sum_{cyc}\frac{z^2(x-y)^2}{2\left(xy+xz+yz+\sqrt{3xyz(x+y+z)}\right)}=$$

$$=\sum_{cyc}(x-y)^2\left(\frac{z^2}{2\left(xy+xz+yz+\sqrt{3xyz(x+y+z)}\right)}-\frac{z^2}{(2x+y+z)(2y+x+z)}\right).$$

Thus, it remains to prove that

$$(2x+y+z)(2y+x+z)\geq 2\left(xy+xz+yz+\sqrt{3xyz(x+y+z)}\right).$$

But

$$(2x+y+z)(2y+x+z)\geq 2\left(xy+xz+yz+\sqrt{3xyz(x+y+z)}\right)\Leftrightarrow$$

$$\Leftrightarrow 2x^2+2y^2+z^2+3xy+xz+yz\geq 2\sqrt{3xyz(x+y+z)},$$

which is true because

$$x^2+y^2+z^2\geq xy+xz+yz\geq\sqrt{3xyz(x+y+z)}.$$

it seems that the following inequality is true too.
Let x, y and z are positive numbers. Prove that:

$$\frac{x(y+z)^2}{3x+2y+2z}+\frac{y(x+z)^2}{2x+3y+2z}+\frac{z(x+y)^2}{2x+2y+3z}\geq\frac{4}{7}\sqrt{3xyz(x+y+z)}$$

62.
Let $a, b\in R$ such that $9a^2+8ab+7b^2\leq 6$ Prove that :$7a+5b+12ab\leq 9$
Solution:
1...
By AM-GM inequality, We see that

$$7a+5b+12ab\leq 7\left(a^2+\frac{1}{4}\right)+5\left(b^2+\frac{1}{4}\right)+12ab$$

$$=(9a^2+8ab+7b^2)-2(a-b)^2+3\leq(9a^2+8ab+7b^2)+3\leq 6+3=9.$$

Equality holds if and only if $a=b=\frac{1}{2}$.
63.
Let x, y and z are positive numbers such that $x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.
Prove that $xyz+yz+zx+xy\geq 4$.
Solution:

$$\sqrt{\frac{xyz(x+y+z)}{yz+zx+xy}}+x+y+z-\frac{4xyz(x+y+z)^2}{(yz+zx+xy)^2}$$

$$\geq\frac{3xyz}{yz+zx+xy}+x+y+z-\frac{4xyz(x+y+z)^2}{(yz+zx+xy)^2}$$

$$=\frac{x^3(y-z)^2+y^3(z-x)^2+z^3(x-y)^2}{(yz+zx+xy)^2}\geq 0\Longrightarrow$$

$$1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{4}{xyz}\Longleftrightarrow xyz+yz+zx+xy\geq 4$$

64.
Let a,b,c=0 satisfy $a+b+c=1$
Prove that $(a^2+b^2)(b^2+c^2)(c^2+a^2)$=$\frac{1}{32}$
Solution:
let $f(a,b,c)=(a^2+b^2)(b^2+c^2)(c^2+a^2)$
let $c=max(a,b,c)$;
We have
$f(a,b,c)\le f(a+b,0,c)$(which is equivalent $ab(-4abc^2+a^3b+ab^3-4a^2c^2-4b^2c^2-2c^4)\le 0$(true) We will prove that $f(a+b,0,c)=f(1-c,0,c)\le\frac{1}{2}$ which is equivalent to

$$\frac{1}{32}*(16c^4-32c^3+20c^2-4c-1))(-1+2c)^2\le 0$$

remember that
$16c^4-32c^3+20c^2-4c-1=4(2c^2-2c+\frac{1-\sqrt{5}}{4})(2c^2-2c+\frac{1+\sqrt{5}}{4})\ge 0$ for every $c\in[0,1]$
65.
Let a,b,c be the sides of triangle. Prove that:

$$\frac{a}{2a-b+c}+\frac{b}{2b-c+a}+\frac{c}{2c-a+b}\ge\frac{3}{2}$$

Solution:
the inequality is equivalent to

$$\sum\frac{1}{1+\frac{a}{a+c-b}}\le\frac{3}{2}$$

By Cauchy We have :

$$\frac{a}{a+c-b}+1\ge 2\sqrt{\frac{a}{a+c-b}}$$

So We need to prove

$$\sum\sqrt{\frac{a+c-b}{a}}\le 3$$

Because a,b,c be the sides of a triangle so We have:

$$\frac{a+c-b}{a}=\frac{\sin A+\sin C-\sin B}{\sin A}=\frac{2\sin\frac{C}{2}\cos\frac{B}{2}}{\cos\frac{A}{2}}$$

it's following that

$$\sum\sqrt{\frac{a+c-b}{a}}=\frac{\cos\frac{B}{2}\sqrt{\sin C}+\cos\frac{C}{2}\sqrt{\sin A}+\cos\frac{A}{2}\sqrt{\sin B}}{\sqrt{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}}$$

$$\le\sqrt{\frac{\left(\cos^2\frac{B}{2}+\cos^2\frac{C}{2}+\cos^2\frac{A}{2}\right)(\sin C+\sin A+\sin B)}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}}$$

$$=\sqrt{2(\cos B+\cos C+\cos A)+6}\le\sqrt{2.\frac{3}{2}+6}=3$$

66.
Let $a,b,c,d>0$.Prove the following inequality.When does the equality hold?

$$\frac{3}{1}+\frac{5}{1+a}+\frac{7}{1+a+b}+\frac{9}{1+a+b+c}+\frac{36}{1+a+b+c+d}\leqq 4\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Solution:
We can have

$$(1+a+b+c+d)(\frac{4}{25}+\frac{16}{25a}+\frac{36}{25b}+\frac{64}{25c}+\frac{4}{d}) \ge (\frac{2}{5}+\frac{4}{5}+\frac{6}{5}+\frac{8}{5}+2)^2 = 9$$

so

$$(\frac{4}{25}+\frac{16}{25a}+\frac{36}{25b}+\frac{64}{25c}+\frac{4}{d}) \ge 36\frac{1}{1+a+b+c+d}$$

and

$$(1+a+b+c)(\frac{9}{100}+\frac{9}{25a}+\frac{81}{100b}+\frac{36}{25c}) \ge (\frac{3}{10}+\frac{3}{5}+\frac{9}{10}+\frac{6}{5})$$

so We have

$$(\frac{9}{100}+\frac{9}{25a}+\frac{81}{100b}+\frac{36}{25c}) \ge 9\frac{1}{1+a+b+c}$$

and

$$(1+a+b)(\frac{7}{36}+\frac{7}{9a}+\frac{7}{4b}) \ge 7$$

We get

$$(\frac{7}{36}+\frac{7}{9a}+\frac{7}{4b}) \ge \frac{7}{1+a+b}$$

and

$$(1+a)(\frac{5}{9}+\frac{20}{9a}) \ge 5$$

then

$$\frac{5}{9}+\frac{20}{9a} \ge \frac{5}{1+a}$$

and add these inequality up We can solve the problem.
67.
Let a,b,c be positive real number such that $9+3abc = 4(ab+bc+ca)$
Prove that $a+b+c \ge 3$
Solution:
Take $a = x+1; b = y+1; c = z+1$,then We must prove that:
$x+y+z \ge 0$ when $5(x+y+z)+xy+xz+yz = 3xyz$
We consider three case:
Case 1:$xyz \ge 0 \Rightarrow \frac{(x+y+z)^2}{3}+5(x+y+z)$

$$\ge 5(x+y+z)+xy+xz+yz = 3xyz \ge 0 \Rightarrow x+y+z \ge 0$$

Case 2: $x \ge 0; y \ge 0; z \le 0$
Assume that $x+y+z \le 0 \Rightarrow -x \ge y+z$.

$$5(x+y+z) = yz(3x-1)-x(y+z) \ge -4yz+(y+z)^2 = (y-z)^2 \ge 0$$

Case 3: $x \le 0; y \le 0; z \le 0$.Observe that $-x, -y, -z \in [0,1]$ then:

$$0 = 5(x+y+z)+xy+yz+xz-3xyz \le 5(x+y+z)+2(xy+xz+yz) \le 5(x+y+z)+\frac{2(x+y+z)^2}{3} \Rightarrow x+y+z \ge 0$$

So We have done.
68.
if a, b, c, d are non-negative real numbers such that $a+b+c+d = 4$, then

$$\frac{a^2}{b^2+3}+\frac{b^2}{c^2+3}+\frac{c^2}{d^2+3}+\frac{d^2}{a^2+3} \ge 1.$$

SOLUTiON:
By Cauchy-Schwarz

$$[a^2(b^2+3)+b^2(c^2+3)+c^2(d^2+3)](\frac{a^2}{b^2+3}+\frac{b^2}{c^2+3}+\frac{c^2}{d^2+3}+\frac{d^2}{a^2+3}) \geq (a^2+b^2+c^2)^2.$$

From Cauchy We see that it is sufficient to prove that

$$(a^2+b^2+c^2+d^2)^2 \geq 3(a^2+b^2+c^2+d^2)+a^2b^2+b^2c^2+c^2d^2+d^2a^2$$

which can be rewritten as

$$(a^2+b^2+c^2+d^2)(a^2+b^2+c^2+d^2-3) \geq a^2b^2+b^2c^2+c^2d^2+d^2a^2$$

Now you must homogeneize to have HarazWe form

$$(a^2+b^2+c^2+d^2)(a^2+b^2+c^2+d^2-3(\frac{a+b+c+d}{4})^2) \geq a^2b^2+b^2c^2+c^2d^2+d^2a^2$$

which follows from

$$a^2+b^2+c^2+d^2 \geq 4$$

and

$$(x+y+z+t)^2 \geq 4(xy+yz+zt+tx)$$

with $x=a^2$ and similar.

69.

if $a \geq 2$, $b \geq 2$, $c \geq 2$ are reals, then prove that

$$8\left(a^3+b\right)\left(b^3+c\right)\left(c^3+a\right) \geq 125\,(a+b)\,(b+c)\,(c+a)$$

SOLUTiON:
Lets write LHS as

$$8*(a^3b^3c^3+abc+a^4b^3+b^4c^3+c^4a^3+a^4c+b^4a+c^4b)$$

From the Muirheads inequality We have that

$$a^4b^3+b^4c^3+c^4a^3 \geq \sum a^3b^2c^2 = a^2b^2c^2(a+b+c)$$

and

$$a^4c+b^4a+c^4b \geq \sum a^3bc = abc(a^2+b^2+c^2)$$

$.(**) Nowletsseethat$
$a^2b^2c^2=(ab)(bc)(ca) \geq (a+b)(b+c)(c+a) Thisiseasytoprove.$
$Fromthe (**) Wegetthat$
$LHS \geq 8(a^3b^3c^3+abc+a^2b^2c^2(a+b+c)+abc(a^2+b^2+c^2))$

$$=8a^2b^2c^2(abc+\frac{1}{abc}+a+b+c+\frac{a^2+b^2+c^2}{abc})$$

so We have to prove that :

$$abc+\frac{1}{abc}+a+b+c+\frac{a^2+b^2+c^2}{abc} \geq \frac{125}{8}$$

or

$$8a^2b^2c^2+8+8abc(a+b+c)+8(a^2+b^2+c^2)\geq 125abc$$

WLOG let $a\geq b\geq c$
Now let $abc=P$ We will make that following change

$$a\rightarrow\frac{a}{\epsilon}$$

and $b\rightarrow b\epsilon$ where $\epsilon\geq 1$ and $a\epsilon$.

The RHS doesnt change. in the LHS the first part also doesn't change.

$$a+b\geq\frac{a}{\epsilon}+b\epsilon$$

equivalent to $(\epsilon-1)(a-b\epsilon)$ which is true. Also We get that

$$a^2+b^2\geq\frac{a^2}{\epsilon^2}+b^2\epsilon^2$$

So as We get the numbers closer to each other the LHS decreases while the RHS remains

the same so it is enough to prove the inequality for the case $a=b=c$ which is equivalent to :

$$8a^6+8+24a^4+24a^2\geq 125a^3$$

Which is pretty easy.
70,
Let $a,b,c\geq 0$. Prove that:

$$\frac{1}{\sqrt{2a^2+ab+bc}}+\frac{1}{\sqrt{2b^2+bc+ca}}+\frac{1}{\sqrt{2c^2+ca+ab}}\geq\frac{9}{2(a+b+c)}.$$

Solution:
We have:

$$\sum_{cyc}\frac{1}{\sqrt{2a^2+ab+bc}}=\sum_{cyc}\frac{2a+b+c}{2\cdot\frac{2a+b+c}{2}\sqrt{2a^2+ab+bc}}\geq$$

$$\geq\sum_{cyc}\frac{2a+b+c}{\left(\frac{2a+b+c}{2}\right)^2+2a^2+ab+bc}.$$

But

$$\sum_{cyc}\frac{2a+b+c}{\left(\frac{2a+b+c}{2}\right)^2+2a^2+ab+bc}\geq\frac{9}{2(a+b+c)}\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}(100a^6+600a^5b+588a^5c+1123a^4b^2-357a^4c^2-1842a^3b^3+$$

$$+1090a^4bc-1414a^3b^2c+1330a^3c^2b-1218a^2b^2c^2)\geq 0,$$

which is easy.
71. Let a,b,c $>$ 0 .Prove that :

$$\frac{a}{a+2b}+\frac{b}{b+2c}+\frac{c}{c+2a}\leq\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}$$

Solution:

$$\Longleftrightarrow \sum\left(1-\frac{a}{a+2b}\right) \ge 3-\frac{3(a^2+b^2+c^2)}{(a+b+c)^2}=\frac{6(ab+bc+ca)}{(a+b+c)^2}$$

$$\Longleftrightarrow \frac{b}{a+2b}+\frac{c}{b+2c}+\frac{a}{c+2a} \ge \frac{3(ab+bc+ca)}{(a+b+c)^2}$$

By Cauchy-Schwarz We get

$$\sum \frac{b}{a+2b}\sum b(a+2b) \ge (a+b+c)^2$$

it suffice to show that

$$(a+b+c)^4 \ge 3(ab+bc+ca)(ab+bc+ca+2a^2+2b^2+2c^2)$$

Without loss of generosity,assume that $ab+bc+ca=3$,then it becomes

$$\left[(a+b+c)^2-9\right]^2 \ge 0$$

which is obvious.

72.

Let a,b,c,d be positive real numbers. Prove that the following inequality holds

$$\frac{a^4+b^4}{(a+b)(a^2+ab+b^2)}+\frac{b^4+c^4}{(b+c)(b^2+bc+c^2)}+\frac{c^4+d^4}{(c+d)(c^2+cd+d^2)}+\frac{d^4+a^4}{(d+a)(d^2+da+a^2)} \ge \frac{a^2+b^2+c^2+d^2}{a+b+c+d}$$

Solution:

$$\frac{a^4+b^4}{(a+b)(a^2+ab+b^2)} \ge \frac{\frac{1}{2}(a^2+b^2)^2}{(a+b)(a^2+ab+b^2)}$$

Thus , it remains to prove that

$$\sum_{cyc}\frac{(a^2+b^2)^2}{(a+b)(a^2+ab+b^2)} \ge \frac{2(a^2+b^2+c^2+d^2)}{a+b+c+d}$$

Acording to Cauchy-Shwarz inequality We have :

$$LHS \ge \frac{4(a^2+b^2+c^2+d^2)^2}{A}$$

where $A=\sum_{cyc}(a+b)(a^2+ab+b^2)=2(a^3+b^3+c^3+d^3)+2\sum_{cyc}ab(a+b)$
it suffices to show that

$$2(a^2+b^2+c^2+d^2)(a+b+c+d) \ge 2(a^3+b^3+c^3+d^3)+2\sum_{cyc}ab(a+b)$$

$$\Leftrightarrow \sum_{cyc}\left(2a^3+2a^2b+2a^2c+2a^2d-2a^3-2ab(a+b)\right)=\sum_{cyc}2a^2c \ge 0$$

73.

if a,b,c are nonnegative real numbers, then

$$\sum a\sqrt{a^2+4b^2+4c^2} \ge (a+b+c)^2.$$

Solution.

in the nontrivial case when two of a,b,c are nonzero, We take square both sides and write the inequality as

$$\sum a^2(a^2+4b^2+4c^2)+2\sum ab\sqrt{(a^2+4b^2+4c^2)(4a^2+b^2+4c^2)} \geq \geq \left(\sum a\right)^4.$$

Applying the Cauchy-Schwarz inequality in combination with the trivial inequality

$$\sqrt{(u+4v)(v+4u)} \geq 2u+2v+\frac{2uv}{u+v} \ \forall u,v \geq 0, u+v>0,$$

We get

$$\sum ab\sqrt{(a^2+4b^2+4c^2)(4a^2+b^2+4c^2)} \geq \sum ab\left[\sqrt{(a^2+4b^2)(b^2+4a^2)}+4c^2\right]$$
$$\geq \sum ab\left(2a^2+2b^2+\frac{2a^2b^2}{a^2+b^2}+4c^2\right).$$

Therefore, it suffices to prove that

$$\sum a^4+8\sum a^2b^2+4\sum ab(a^2+b^2)+8abc\sum a+4\sum \tfrac{a^3b^3}{a^2+b^2} \geq$$
$$\geq \left(\sum a\right)^4.$$

This inequality reduces to.

74.

Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{b^2+c^2}{a^2+bc}+\frac{c^2+a^2}{b^2+ca}+\frac{a^2+b^2}{c^2+ab} \geq \frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}.$$

Solution.

By the Cauchy-Schwarz inequality, We have

$$\sum \frac{b^2+c^2}{a^2+bc} \geq \frac{\left[\sum(b^2+c^2)\right]^2}{\sum(b^2+c^2)(a^2+bc)} = \frac{4(a^2+b^2+c^2)^2}{\sum ab(a^2+b^2)+2\sum a^2b^2}.$$

Therefore, it suffices to prove that

$$2(a^2+b^2+c^2)^2 \geq \sum \frac{a\left[\sum ab(a^2+b^2)+2\sum a^2b^2\right]}{b+c}.4$$

Since

$$\sum ab(a^2+b^2)+2\sum a^2b^2 =$$
$$= (b+c)[a^3+2a^2(b+c)+bc(b+c)+a(b^2-bc+c^2)]-4a^2bc,$$

this inequality can be written as

$$2\left(\sum a^2\right)^2+4abc\sum \frac{a^2}{b+c} \geq$$
$$\geq \sum a[a^3+2a^2(b+c)+bc(b+c)+a(b^2-bc+c^2)],$$

or equivalently,

$$\sum a^4+2\sum a^2b^2+4abc\sum \frac{a^2}{b+c} \geq abc\sum a+2\sum a^3(b+c).$$

Now, by Chebyshev's inequality, We have

$$\sum \frac{a^2}{b+c} \geq \frac{3(a^2+b^2+c^2)}{2(a+b+c)},$$

and thus, it suffices to show that

$$\sum a^4 + 2\sum a^2b^2 + \frac{6abc\sum a^2}{\sum a} \geq abc\sum a + 2\sum a^3(b+c).$$

After some simple computations, We can write this inequality as

$$a^3(a-b)(a-c) + b^3(b-c)(b-a) + c^3(c-a)(c-b) \geq 0,$$

which is Schur's inequality. The **Solution** is completed. Equality holds if and only if $a = b = c$, or$a = b$ and $c = 0$, or any cyclic permutation..

$$\sum a^2b^2 + 2\sum \frac{a^3b^3}{a^2+b^2} \geq 2abc\sum a,$$

or

$$\sum \frac{a^2b^2(a+b)^2}{a^2+b^2} \geq 2abc\sum a.$$

By the Cauchy-Schwarz inequality, We have

$$\sum \frac{a^2b^2(a+b)^2}{a^2+b^2} \geq \frac{[ab(a+b)+bc(b+c)+ca(c+a)]^2}{(a^2+b^2)+(b^2+c^2)+(c^2+a^2)},$$

and thus, it is enough to to check that

$$[ab(a+b)+bc(b+c)+ca(c+a)]^2 \geq 4abc(a+b+c)(a^2+b^2+c^2).$$

Without loss of generality, assume that b is betien $a$and c. From the AM-GM inequality, We have

$$4abc(a+b+c)(a^2+b^2+c^2) \leq [ac(a+b+c)+b(a^2+b^2+c^2)]^2.$$

On the other hand, one has

$$\begin{gathered} ac(a+b+c)+b(a^2+b^2+c^2)-[ab(a+b)+bc(b+c)+ca(c+a)] = \\ = -b(a-b)(b-c) \leq 0. \end{gathered}$$

Combining these two inequalities, the conclusion follows. Equality occurs if and only if a=b=c, or $a = b = 0$, or$b = c = 0, orc = a = 0$.

75. Let a, b, c be positive real number. Prove that:

$$\sum \sqrt{\frac{2(b+c)}{a}} \geq \frac{27(a+b)(b+c)(c+a)}{4(a+b+c)(ab+bc+ca)}$$

Solution.

By AM-GM inequality We have

$$\frac{1}{a\sqrt{2(a^2+bc)}} = \frac{\sqrt{b+c}}{\sqrt{2a}.\sqrt{(ab+ac)(a^2+bc)}} \geq \frac{\sqrt{2(b+c)}}{\sqrt{a}.(a+b)(a+c)}$$

it suffices to show that

$$\sum (b+c)\sqrt{\frac{2(b+c)}{a}} \geq \frac{9(a+b)(b+c)(c+a)}{2(ab+bc+ca)}$$

By Chebyselv inequality We have

$$\sum (b+c)\sqrt{\frac{2(b+c)}{a}} \geq \frac{2}{3}(a+b+c).\sum \sqrt{\frac{2(b+c)}{a}}$$

Hence, it suffices to show that

$$\sum \sqrt{\frac{2(b+c)}{a}} \geq \frac{27(a+b)(b+c)(c+a)}{4(a+b+c)(ab+bc+ca)}$$

By Cauchy-Schwarz inequality, We get

$$\left(\sum \sqrt{\frac{2(b+c)}{a}}\right)^2 \left(\sum a(b+c)^2\right) \geq 16(a+b+c)^3$$

And by AM-GM inequality,

$$27(a+b)(b+c)(c+a) \leq 8(a+b+c)^3$$

Finally, We need to show that

$$\frac{16(a+b+c)^3}{\sum a(b+c)^2} \geq \frac{27.8(a+b+c)^3(a+b)(b+c)(c+a)}{16(a+b+c)^2(ab+bc+ca)^2}$$

or

$$32(a+b+c)^2(ab+bc+ca)^2 \geq 27(a+b)(b+c)(c+a)\left((a+b)(b+c)(c+a)+4abc\right)$$

or

$$5x^2+32y^2 \geq 44xy$$

where We setting $x=(a+b)(b+c)(c+a)$, $y=abc$ and using the equality $(a+b+c)(ab+bc+ca)=x+y$

The last inequality is true because it equivalent $(x-8y)(5x-4y) \geq 0$, obviously.

76.

if a,b,c are positive real numbers, then

$$\sum \frac{1}{a\sqrt{2(a^2+bc)}} \geq \frac{9}{2(ab+bc+ca)}.$$

First **Solution**.

By Holder's inequality, We have

$$\left(\sum \frac{1}{a\sqrt{a^2+bc}}\right)^2 \left(\sum \frac{a^2+bc}{a}\right) \geq \left(\sum \frac{1}{a}\right)^3.$$

it follows that

$$\left(\sum \frac{1}{a\sqrt{a^2+bc}}\right)^2 \geq \frac{(ab+bc+ca)^3}{a^2b^2c^2\left(\sum a^2b^2+\sum a^2bc\right)},$$

and hence, it suffices to prove that

$$2(ab+bc+ca)^5 \geq 81a^2b^2c^2\left(\sum a^2b^2+\sum a^2bc\right).$$

Setting $x=bc, y=ca$ and $z=ab$, this inequality becomes

$$2(x+y+z)^5 \geq 81xyz(x^2+y^2+z^2+xy+yz+zx).$$

Using the ill-known inequality

$$xyz \leq \frac{(x+y+z)(xy+yz+zx)}{9},$$

We see that it is enough to check that

$$2(x+y+z)^4 \geq 9(xy+yz+zx)(x^2+y^2+z^2+xy+yz+zx),$$

which is equivalent to the obvious inequality

$$(x^2+y^2+z^2-xy-yz-zx)(2x^2+2y^2+2z^2+xy+yz+zx) \geq 0.$$

The **Solution** is completed. Equality holds if and only if $a=b=c$.
Second **Solution**. By the AM-GM inequality, We have

$$\frac{1}{a\sqrt{2(a^2+bc)}} = \frac{\sqrt{b+c}}{\sqrt{2a}\sqrt{(ab+ac)(a^2+bc)}} \geq \frac{\sqrt{2(b+c)}}{\sqrt{a}(a+b)(a+c)}.$$

Therefore, it suffices to prove that

$$\sum \sqrt{\frac{b+c}{2a}} \cdot \frac{1}{(a+b)(a+c)} \geq \frac{9}{4(ab+bc+ca)}.$$

Without loss of generality, assume that $a \geq b \geq c$. Since

$$\sqrt{\frac{b+c}{2a}} \leq \sqrt{\frac{c+a}{2b}} \leq \sqrt{\frac{a+b}{2c}}$$

and

$$\frac{1}{(a+b)(a+c)} \leq \frac{1}{(b+c)(b+a)} \leq \frac{1}{(c+a)(c+b)},$$

by Chebyshev's inequality, We get

$$\begin{aligned}\sum \sqrt{\frac{b+c}{2a}} \cdot \frac{1}{(a+b)(a+c)} &\geq \frac{1}{3}\left(\sum \sqrt{\frac{b+c}{2a}}\right)\left[\sum \frac{1}{(a+b)(a+c)}\right] \\ &= \frac{2(a+b+c)}{3(a+b)(b+c)(c+a)} \sum \sqrt{\frac{b+c}{2a}}.\end{aligned}$$

So, it suffices to show that

$$\sum \sqrt{\frac{b+c}{2a}} \geq \frac{27(a+b)(b+c)(c+a)}{8(a+b+c)(ab+bc+ca)}.$$

Setting

$$t = \sqrt[6]{\frac{(a+b)(b+c)(c+a)}{8abc}}$$

$t \geq 1$. By the AM-GM inequality, We have

$$\sum \sqrt{\frac{b+c}{2a}} \geq 3t.$$

Also, it is easy to verify that

$$\frac{27(a+b)(b+c)(c+a)}{8(a+b+c)(ab+bc+ca)} = \frac{27t^6}{8t^6+1}.$$

So, it is enough to check that

$$3t \geq \frac{27t^6}{8t^6+1},$$

or

$$8t^6 - 9t^5 + 1 \geq 0.$$

Since $t \geq 1$, this inequality is true and the **Solution** is completed.
76.
Give $a_1, a_2, ..., a_n \geq 0$are numbers have sum is 1. Prove that if $n > 3$ so

$$a_1a_2 + a_2a_3 + ... + a_na_1 \leq \frac{1}{4}$$

Solution:
Let $n = 2k$, where $k \in \mathbb{N}$ and $a_1 + a_3 + ... + a_{2k-1} = x$.
Hence,

$$a_1a_2 + a_2a_3 + ... + a_na_1 \leq (a_1 + a_3 + ... + a_{2k-1})(a_2 + a_4 + ... + a_{2k}) =$$
$$= x(1-x) \leq \frac{1}{4}$$

Let $n = 2k - 1$ and $a_1 = \min_i\{a_i\}$.
Hence,

$$a_1a_2 + a_2a_3 + ... + a_na_1 \leq$$
$$\leq a_1a_2 + a_2a_3 + ... + a_na_2 \leq (a_1 + a_3 + ... + a_{2k-1})(a_2 + a_4 + ... + a_{2k-2}) =$$
$$= x(1-x) \leq \frac{1}{4}$$

77.
Let a, b, c be non-negative real numbers. Prove that

$$\frac{a^3 + 2abc}{a^3 + (b+c)^3} + \frac{b^3 + 2abc}{b^3 + (c+a)^3} + \frac{c^3 + 2abc}{c^3 + (a+b)^3} \geq 1$$

Solution
We have

$$1 - \sum \frac{a^3}{a^3 + (b+c)^3} = \sum \left(\frac{a^3}{a^3 + b^3 + c^3} - \frac{a^3}{a^3 + (b+c)^3} \right) = \sum \frac{3a^3bc(b+c)}{(a^3 + b^3 + c^3)(a^3 + (b+c)^3)}$$

Hence, it suffices to show that

$$2\sum \frac{a^3 + b^3 + c^3}{a^3 + (b+c)^3} \geq \sum \frac{3a^2(b+c)}{a^3 + (b+c)^3}$$
$$\Leftrightarrow \sum \frac{2a^3 - 3a^2(b+c) + 2b^3 + 2c^3}{a^3 + (b+c)^3} \geq 0$$
$$\Leftrightarrow \sum \frac{(a-b)(a^2 - 2ab - 2b^2) + (a-c)(a^2 - 2ac - 2c^2)}{a^3 + (b+c)^3} \geq 0$$
$$\Leftrightarrow \sum \frac{(a-b)^3 - (c-a)^3 + 3c^2(c-a) - 3b^2(a-b)}{a^3 + (b+c)^3} \geq 0$$

it suffices to show that

$$\sum \frac{c^2(c-a) - b^2(a-b)}{a^3 + (b+c)^3} \geq 0$$

and

$$\sum \frac{(a-b)^3 - (c-a)^3}{a^3 + (b+c)^3} \geq 0$$

The first inequality is equivalent to

$$\Leftrightarrow \sum(a-b)\left(\frac{a^2}{b^3+(c+a)^3}-\frac{b^2}{a^3+(b+c)^3}\right)\geq 0$$

Finally, to finish the **Solution**, We will show that if $a \geq b$, then

$$\frac{a^2}{b^3+(c+a)^3}\geq\frac{b^2}{a^3+(b+c)^3}$$
$$\Leftrightarrow a^5-b^5\geq b^2(c+a)^3-a^2(b+c)^3$$
$$\Leftrightarrow a^5-b^5\geq a^2b^2(a-b)+c^3(b^2-a^2)+3c^2ab(b-a)$$

which is obviously true since $a \geq b$ and $c \geq 0$.

And the second inequality is equivalent to

$$\sum(a-b)^3\left(\frac{1}{a^3+(b+c)^3}-\frac{1}{b^3+(c+a)^3}\right)\geq 0$$
$$\Leftrightarrow \sum\frac{(a-b)^4(3c(a+b)+3c^2)}{(a^3+(b+c)^3)(b^3+(c+a)^3)}\geq 0$$

which is obviously true.

Equality holds for $a=b=c$ or $abc=0$

78.

Let a, b, c are positive real numbers, prove that

$$\sqrt{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}+2\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}\geq 5$$

Solution:

By using the ill known

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+1\geq\frac{21\left(a^2+b^2+c^2\right)}{(a+b+c)^2}$$

Setting $x=\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}\leq 1$. it suffices to show that

$$\sqrt{\frac{3\left(10-x^2\right)}{2x^2+1}}+2x\geq 5$$
$$\Leftrightarrow \frac{3\left(10-x^2\right)}{2x^2+1}\geq 4x^2-20x+25$$
$$\Leftrightarrow \frac{-8x^4+40x^3-57x^2+20x+5}{2x^2+1}\geq 0$$
$$\Leftrightarrow \frac{(x-1)\left(-8x^3+32x^2-25x-5\right)}{2x^2+1}\geq 0$$

which is clearly true.

$$\sqrt{\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3}}\left(\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3}\right)\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)\geq 1$$

And We also note that the folloid is not true

$$\left(\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3}\right)\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)\geq 1$$

79.
Let a, b, c be the side-lengths of a triangle such that $a^2+b^2+c^2=3$. Prove that

$$\frac{bc}{1+a^2}+\frac{ca}{1+b^2}+\frac{ab}{1+c^2}\geq\frac{3}{2}.$$

Solution.
Write the inequality as

$$\sum\frac{2bc}{4a^2+b^2+c^2}\geq 1.$$

Since $1=\sum\frac{b^2c^2}{a^2b^2+b^2c^2+c^2a^2}$, this inequality is equivalent to

$$\sum\left(\frac{2bc}{4a^2+b^2+c^2}-\frac{b^2c^2}{a^2b^2+b^2c^2+c^2a^2}\right)\geq 0,$$

or

$$\frac{abc}{a^2b^2+b^2c^2+c^2a^2}\sum\frac{(2a^2-bc)(b-c)^2}{a(4a^2+b^2+c^2)}\geq 0.$$

Without loss of generality, assume that $a\geq b\geq c$. Since

$$\frac{(2a^2-bc)(b-c)^2}{a(4a^2+b^2+c^2)}\geq 0,$$

it suffices to prove that

$$\frac{(2b^2-ca)(c-a)^2}{b(4b^2+c^2+a^2)}+\frac{(2c^2-ab)(a-b)^2}{c(4c^2+a^2+b^2)}\geq 0.$$

Since a, b, c are the side-lengths of a triangle and $a\geq b\geq c$, We have

$$2b^2-ca\geq c(b+c)-ca=c(b+c-a)\geq 0,$$

and

$$a-c-\frac{b}{c}(a-b)=\frac{(b-c)(b+c-a)}{c}\geq 0.$$

Therefore,

$$\frac{(2b^2-ca)(c-a)^2}{b(4b^2+c^2+a^2)}\geq\frac{b(2b^2-ca)(a-b)^2}{c^2(4b^2+c^2+a^2)}.$$

it suffices to show that

$$\frac{b(2b^2-ca)}{4b^2+c^2+a^2}+\frac{c(2c^2-ab)}{4c^2+a^2+b^2}\geq 0,$$

or

$$\frac{b(2b^2-ca)}{4b^2+c^2+a^2}\geq\frac{c(ab-2c^2)}{4c^2+a^2+b^2}.$$

Since $ab-2c^2-(2b^2-ca)=a(b+c)-2(b^2+c^2)\leq a(b+c)-(b+c)^2\leq 0$, it is enough to check that

$$\frac{b}{4b^2+c^2+a^2}\geq\frac{c}{4c^2+a^2+b^2},$$

which is true because

$$b(4c^2+a^2+b^2)-c(4b^2+c^2+a^2)=(b-c)[(b-c)^2+(a^2-bc)]\geq 0.$$

The **Solution** is completed.
80.
Let $a, b, c, d > 0$ such that $a^2 + b^2 + c^2 + d^2 = 4$,then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 2 + \frac{2}{abcd}$$

Solution
write the inequality as

$$abc + bcd + cda + dab \leq 2abcd + 2.$$

Without loss of generality, assume that $a \geq b \geq c \geq d$.

Let

$$t = \sqrt{\frac{a^2 + b^2}{2}},$$

=> $1 \leq t \leq \sqrt{2}$. Since

$$\begin{aligned} 2\left(\frac{1}{c} + \frac{1}{d}\right) \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} &\geq \frac{16}{a + b + c + d} \\ &\geq \frac{16}{\sqrt{4(a^2 + b^2 + c^2 + d^2)}} = 4, \end{aligned}$$

We have$c + d \geq 2cd$.
Therefore,

$$\begin{aligned} abc + bcd + cda + dab - 2abcd &= ab(c + d - 2cd) + cd(a + b) \\ &\leq \frac{a^2 + b^2}{2}(c + d - 2cd) + cd\sqrt{2(a^2 + b^2)} \\ &= t^2(c + d - 2cd) + 2tcd. \end{aligned}$$

it suffices to prove that

$$t^2(c + d - 2cd) + 2tcd \leq 2,$$

or

$$2tcd(1 - t) + t^2(c + d) \leq 2.$$

Using the AM-GM inequality, We get

$$c + d \leq \frac{(c + d)^2 + 4}{4} = \frac{(4 - 2t^2 + 2cd) + 4}{4} = \frac{4 - t^2 + cd}{2}.$$

So, it is enough to check that

$$4tcd(1 - t) + t^2(4 - t^2 + cd) \leq 4,$$

or

$$tcd(4 - 3t) \leq (2 - t^2)^2.$$

Since $2 - t^2 = \frac{c^2 + d^2}{2} \geq cd$, We have

$$(2 - t^2)^2 - tcd(4 - 3t) \geq cd(2 - t^2) - tcd(4 - 3t) = 2cd(t - 1)^2 \geq 0.$$

The **Solution** is completed.
81.
Let a, b, c be positive real number .Prove:

$$\sum_{cyc} \sqrt[3]{(a^2+ab+b^2)^2} \le \sqrt[3]{3\left(\sum_{cyc}(2a^2+bc)\right)^2}$$

Solution:

$$\sum_{cyc} \sqrt[3]{(a^2+ab+b^2)^2} \le \sqrt[3]{3\left(\sum_{cyc}(2a^2+bc)\right)^2} \leftrightarrow \sum_{cyc} \sqrt[3]{3(a^2+ab+b^2)^2} \le \sqrt[3]{9\left(\sum_{cyc}(2a^2+bc)\right)^2}$$

By holder's inequality:

$$\sum_{cyc} \sqrt[3]{3(a^2+ab+b^2)^2} \le \sqrt[3]{9(\sum_{cyc}(a^2+ab+b^2))^2}$$

So We must prove:

$$\sqrt[3]{9(\sum_{cyc}(a^2+ab+b^2))^2} \le \sqrt[3]{(\sum_{cyc}(2a^2+bc))}^2$$

$$<=> (\sum_{cyc}(a^2+ab+b^2))^2 \le 3\sum_{cyc}(2a^2+bc)^2$$

Using Cauchy-Schawrz's inequality,

$$3\sum_{cyc}(2a^2+bc)^2 \ge (\sum_{cyc}(2a^2+bc))^2 = \sum_{cyc}(a^2+ab+b^2)^2$$

Q.E.D
.

82.
Let $a, b, c > 0$.prove that:

$$\sum_{cyc} \frac{1}{a^2+bc} \ge \sum_{cyc} \frac{1}{a^2+2bc} + \frac{ab+bc+ca}{2(a^2b^2+b^2c^2+c^2a^2)}$$

Solution:
1...

$$(a^2b^2+b^2c^2+c^2a^2)\left(\sum_{cyc}\frac{1}{a^2+bc}\right) = \sum_{cyc}\left(bc + \frac{a^2(b^2+c^2-bc)}{a^2+bc}\right)$$

$$<=> 2\sum_{cyc}\frac{a^2(b^2-bc+c^2)}{a^2+bc} \ge ab+bc+ca$$

$$<=> 2\sum_{cyc} a^2\left(1+\frac{b^2-bc+c^2}{a^2+bc}\right) \ge 2(a^2+b^2+c^2)+ab+bc+ca$$

$$<=> \sum_{cyc}\frac{a^2}{a^2+bc} \ge 1 + \frac{ab+bc+ca}{2(a^2+b^2+c^2)}$$

By Cauchy-Schwarz,

$$\sum_{cyc} \frac{a^2}{a^2+bc} \geq \frac{(a+b+c)^2}{\sum_{cyc} a^2 + \sum_{cyc} bc} = 1 + \frac{ab+bc+ca}{\sum_{cyc} a^2 + \sum_{cyc} bc} \geq 1 + \frac{ab+bc+ca}{2(a^2+b^2+c^2)}$$

Q.E.D

2..
Ta có:

$$(a^2b^2+b^2c^2+c^2a^2)\left(\sum_{cyc} \frac{1}{a^2+bc}\right) = \sum_{cyc}\left(b^2+c^2-bc\frac{b^2+c^2-bc}{a^2+bc}\right)$$

$$<=> 2(a^2+b^2+c^2) - \sum_{cyc} bc\frac{b^2+c^2-bc}{a^2+bc} \leq \frac{3}{2}(a^2+b^2+c^2)$$

Hay l

$$2\sum_{cyc} bc\frac{b^2+c^2-bc}{a^2+bc} \geq a^2+b^2+c^2$$

By AM-GM's inequality:

$$2(b^2+c^2-bc) \geq b^2+c^2$$

And We will prove:

$$\sum_{cyc} \frac{bc(b^2+c^2)}{a^2+bc} \geq a^2+b^2+c^2$$

By Cauchy-Schwarz's inequality:

$$LHS \geq \frac{(\sum ab\sqrt{a^2+b^2})^2}{\sum bc(a^2+bc)}$$

$$(\sum bc\sqrt{b^2+c^2})^2 \geq (\sum a^2)(abc\sum a + \sum a^2b^2)$$

Using Cauchy-Schwarz,

$$\sqrt{a^2+b^2}\sqrt{a^2+c^2} \geq a^2+bc$$

$$<=> abc(a^3+b^3+c^3+3abc-\sum a^2b \geq 0)$$

it is true.
83.
if a,b,c are positive real numbers, then

$$\frac{a^2(b+c)}{b^2+c^2} + \frac{b^2(c+a)}{c^2+a^2} + \frac{c^2(a+b)}{a^2+b^2} \geq a+b+c.$$

First **Solution**.
We have

$$\sum\left[\frac{a^2(b+c)}{b^2+c^2} - a\right] = \sum \frac{ab(a-b)-ca(c-a)}{b^2+c^2}$$

$$= \sum ab(a-b)\left(\frac{1}{b^2+c^2} - \frac{1}{c^2+a^2}\right) = \sum \frac{ab(a+b)(a-b)^2}{(a^2+c^2)(b^2+c^2)} \geq 0.$$

Thus, it follows that

$$\sum \left[\frac{a^2(b+c)}{b^2+c^2} - a\right] \geq 0,$$

or

$$\frac{a^2(b+c)}{b^2+c^2} + \frac{b^2(c+a)}{c^2+a^2} + \frac{c^2(a+b)}{a^2+b^2} \geq a+b+c,$$

which is just the desired inequality. Equality holds if and only if $a = b = c$.
Second **Solution**.
Having in view of the identity

$$\frac{a^2(b+c)}{b^2+c^2} = \frac{(b+c)(a^2+b^2+c^2)}{b^2+c^2} - b - c,$$

We can write the desired inequality as

$$\frac{b+c}{b^2+c^2} + \frac{c+a}{c^2+a^2} + \frac{a+b}{a^2+b^2} \geq \frac{3(a+b+c)}{a^2+b^2+c^2}.$$

Without loss of generality, assume that $a \geq b \geq c$. Since $a^2+c^2 \geq b^2+c^2$and

$$\frac{b+c}{b^2+c^2} - \frac{a+c}{a^2+c^2} = \frac{(a-b)(ab+bc+ca-c^2)}{(a^2+c^2)(b^2+c^2)} \geq 0,$$

by Chebyshev's inequality, We have

$$[(b^2+c^2)+(a^2+c^2)]\left(\frac{b+c}{b^2+c^2} + \frac{a+c}{a^2+c^2}\right) \geq 2[(b+c)+(a+c)],$$

or

$$\frac{b+c}{b^2+c^2} + \frac{a+c}{a^2+c^2} \geq \frac{2(a+b+2c)}{a^2+b^2+2c^2}.$$

Therefore, it suffices to prove that

$$\frac{2(a+b+2c)}{a^2+b^2+2c^2} + \frac{a+b}{a^2+b^2} \geq \frac{3(a+b+c)}{a^2+b^2+c^2},$$

which is equivalent to the obvious inequality

$$\frac{c(a^2+b^2-2c^2)(a^2+b^2-ac-bc)}{(a^2+b^2)(a^2+b^2+c^2)(a^2+b^2+2c^2)} \geq 0.$$

Solution 3
Note that from Cauchy-Schwartz inequality We have

$$\sum_{cyc} \frac{a^2(b+c)}{(b^2+c^2)} \geq \sum_{cyc} \frac{\left[\sum_{cyc} a^2(b+c)\right]^2}{\sum_{cyc} a^2(b+c)(b^2+c^2)}$$

Therefore it suffices to show that

$$\left[\sum_{cyc} a^2(b+c)\right]^2 \geq (a+b+c)\sum_{cyc} a^2(b+c)(b^2+c^2)$$

After expansion and using the convention $p = a+b+c; q = ab+bc+ca; r = abc$ this is equivalent to with:

$$\Leftrightarrow r(2p^3+9r-7pq) \geq 0$$

But, since (from trivial inequality) We have$p^2-3q\geq 0$, hence it suffices to show that $p^3+9r\geq 4pq$, which follows from Schur's inequality.
Equality occurs if and only if a=b=c or when $a=b; c=0$ and its cyclic permutations.
84.
if a,b,c are positive numbers such that $a+b+c=3$ then

$$\frac{a}{3a+b^2}+\frac{b}{3b+c^2}+\frac{c}{3c+a^2}\leq\frac{3}{4}.$$

Solution:
is equivalent to

$$\sum\frac{a}{3a+b^2}\leq\frac{3}{4}$$
$$\sum\left(\frac{3a}{3a+b^2}-1\right)\leq-\frac{3}{4}$$

or

$$\sum\frac{b^2}{b^2+3a}\geq\frac{3}{4}$$

By Cauchy Schwarz inequality, We have

$$LHS\geq\frac{(a^2+b^2+c^2)^2}{\sum a^4+(a+b+c)\sum ab^2}$$

it suffices to prove

$$4(a^2+b^2+c^2)^2\geq 3\sum a^4+3\sum a^2b^2+3\sum ab^3+3\sum a^2bc$$
$$\Leftrightarrow (a^2+b^2+c^2)^2-3\sum ab^3+3(\sum a^2b^2-\sum a^2bc)\geq 0$$

By VasC's inequality, We have

$$(a^2+b^2+c^2)^2-3\sum ab^3\geq 0$$

By Am -GM inequality,

$$\sum a^2b^2-\sum a^2bc\geq 0$$

85.
if $a\geq b\geq c\geq d\geq 0$ and $a+b+c+d=2$, then

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b)\leq 1.$$

Solution:
First,let us prove a lemma:
Lemma:
For any $a+b+c+d=2$ and $a\geq b\geq c\geq d\geq 0$

$$a^2b+b^2c+c^2d+d^2a\geq ab^2+bc^2+cd^2+da^2$$

Solution of lemma:
Let

$$F(a)=(b-d)a^2+(d^2-b^2)a+b^2c+c^2d-bc^2-cd^2$$
$$F^{'}(a)=(b-d)(2a-b-d)\geq 0 <=> F(a)\geq F(b)=(c-d)b^2+(d^2-c^2)b+cd(c-d)$$

$$F^{'}(b)=(c-d)(2b-c-d)\geq 0 <=> F(b)\geq F(c)=0$$

Now,let us turn back the **Solution** of the problem. From lemma We have:

$$a^2b+b^2c+c^2d+d^2a+ab^2+bc^2+cd^2+da^2+2(abc+bcd+cda+dab)\geq 2(ab^2+bc^2+cd^2+da^2+abc+bcd+cda+dab$$

it follows that;

$$(a+b+c+d)(a+c)(b+d)\geq 2(ab(b+c)+bc(c+d)+cd(d+a)+da(a+b))$$

But, by AM-GM inequality:

$$(a+c)(b+d)=(a+c)(b+d)\leq \frac{(a+b+c+d)^2}{4}=1$$

86.

if x,y,z,p,q be nonnegative real numbers such that

$$(p+q)(yz+zx+xy)>0$$

Prove that:

$$\frac{2(p+q)}{(y+z)(py+qz)}+\frac{2(p+q)}{(z+x)(pz+qx)}+\frac{2(p+q)}{(x+y)(px+qy)}\geq \frac{9}{yz+xy+zx}$$

Solution:

$$\sum_{cyc}\frac{2(p+q)}{(y+z)(py+qz)}-\frac{9}{yz+zx+xy}\equiv \frac{F(x,y,z)}{(y+z)(z+x)(x+y)(py+qz)(pz+qx)(px+qy)(yz+zx+xy)}.$$

$$F(x,y,z)=F(x,x+s,x+t)$$
$$=16x^4\left(p^3+q^3\right)\left(s^2-st+t^2\right)$$
$$+x^3\left\{16(p+q)\left(p^2+q^2\right)(s+t)(s-t)^2+\left[(p-2q)^2(15p+7q)\right.\right.$$
$$\left.+5q^2(p+q)s^2t+\left[(q-2p)^2(15q+7p)+5p^2(q+p)\right]st^2\right.$$
$$+x^2\left\{\frac{2(s-t)^2[p^2(p+47q)s^2+51pq(p+q)st+q^2(q+47p)t^2]}{3}\right.$$
$$+\frac{2[(5p+q)(5p-12q)^2+6q^3]s^4}{75}+\frac{[(77p-145q)^2(7918p+6699q)+3003p^3+14297pq^2]s^3t}{5633859}$$
$$+\frac{34(p+q)(p-q)^2s^2t^2}{3}+\frac{[(77q-145p)^2(7918q+6699p)+3003q^3+14297qp^2]st^3}{5633859}$$
$$\left.+\frac{2[(5q+p)(5q-12p)^2+6p^3]t^4}{75}\right\}$$
$$+x\left\{[(2p+5q)s-(2q+5p)t]^2\left[\frac{4pq(p+q)s^3}{(2p+5q)^2}\right.\right.$$
$$+\frac{(4p^4+40p^3q+65p^2q^2+113pq^3+30q^4)s^2t}{(2p+5q)^3}$$
$$\left.+\frac{(4q^4+40q^3p+65q^2p^2+113qp^3+30p^4)st^2}{(2q+5p)^3}+\frac{4qp(q+p)t^3}{(2q+5p)^2}\right]$$
$$+\frac{s^3t^2}{(2p+5q)^2(2q+5p)^3}\left[(2p-3q)^2\left(1505p^6+9948p^5q+19439p^4q^2\right.\right.$$

$$+16869p^3q^3+6709p^2q^4+852pq^5+117q^6$$

$$+4960p^7q+6800p^6q^2+p^5q^3+4p^4q^4+5p^3q^5+8q^6p^2+4pq^7+7q^8\Big]$$

$$+\frac{s^2t^3}{(2p+5q)^3(2q+5p)^2}\Big[(2q-3p)^2\left(1505q^6+9948q^5p+19439q^4p^2+16869q^3p^3+6709q^2p^4+852qp^5+117p^6\right)$$

$$+4960q^7p+6800q^6p^2+q^5p^3+4q^4p^4+5q^3p^5+8p^6q^2+4qp^7+7p^8\Big]\Bigg\}$$

$$+st\Bigg\{[(13p+47q)s-(13q+47p)t]^2\Bigg[\frac{2pq(p+q)s^2}{(13p+47q)^2}$$

$$+\frac{49pq(p+q)st}{6(743p^2+6914pq+743q^2)}+\frac{2qp(q+p)t^2}{(13q+47p)^2}\Bigg]$$

$$+(p-q)^2st\Bigg[\frac{q(324773p^3+3233274p^2q+836101pq^2+419052q^3)s^2}{6(13p+47q)(743p^2+6914pq+743q^2)}$$

$$+\frac{2(p+q)(p-q)^2(832132509p^4+9284734492p^3q+9070265998p^2q^2+9284734492pq^3+832132509q^4)st}{3(13p+47q)^2(13q+47p)^2(743p^2+6914pq+743q^2)}$$

$$+\frac{p(324773q^3+3233274q^2p+836101qp^2+419052p^3)t^2}{6(13q+47p)(743p^2+6914pq+743q^2)}\Bigg]\Bigg\}\geq 0,$$

which is clearly true for $x=\min\{x,y,z\}$.

87.

Let a,b,c be positive numbers such that $ab+bc+ca=3$. Prove that

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq 1+\frac{3}{2(a+b+c)}.$$

Solution:

1.....Let $f(a,b,c)=\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-1-\frac{3}{2(a+b+c)}$ and $a=\min\{a,b,c\}$. Then

$$f(a,b,c)-f\left(a,\sqrt{(a+b)(a+c)}-a,\sqrt{(a+b)(a+c)}-a\right)=$$

$$=\left(\sqrt{a+b}-\sqrt{a+c}\right)^2\Bigg(\frac{1}{(a+b)(a+c)}$$

$$-\frac{1}{2(b+c)\left(\sqrt{(a+b)(a+c)}-a\right)}+\frac{3}{2(a+b+c)\left(2\cdot\sqrt{(a+b)(a+c)}-a\right)}\geq$$

$$\geq\left(\sqrt{a+b}-\sqrt{a+c}\right)^2\left(\frac{1}{4bc}-\frac{1}{2(b+c)\cdot\sqrt{bc}}+\frac{2}{3(b+c)^2}\right)\geq 0$$

since, $\sqrt{(a+b)(a+c)}\geq a+\sqrt{bc}$ and $2\cdot\sqrt{(a+b)(a+c)}-a\leq a+b+c\leq\frac{3(b+c)}{2}$. Thus, remain to prove that $f(a,b,b)\geq 0$, which equivalent to

$$(a-b)^2(2a^3+9a^2b+12ab^2+b^3)\geq 0.$$

2.......

The inequality is equivalent to:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{3}{\sqrt{3(ab+bc+ca)}}+\frac{3}{2(a+b+c)}$$

$$\leftrightarrow\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}-\frac{9}{2(a+b+c)}\geq\frac{3}{\sqrt{3(ab+bc+ca)}}-\frac{3}{a+b+c}$$

$$\leftrightarrow \frac{(a-b)^2}{2(a+c)(b+c)(a+b+c)}+\frac{(b-c)^2}{2(a+b)(a+c)(a+b+c)}+\frac{(c-a)^2}{2(b+c)(a+b)(a+b+c)} \geq$$

$$\frac{3[(a-b)^2+(b-c)^2+(c-a)^2]}{2(a+b+c+\sqrt{3(ab+bc+ca)})(a+b+c)(\sqrt{3(ab+bc+ca)})}$$

$$\leftrightarrow (a-b)^2.M+(b-c)^2.N+(c-a)^2.P \geq 0$$

with :

$$M=(a+b)[(a+b+c)+\sqrt{3(ab+bc+ca)}]\sqrt{3(ab+bc+ca)}-3(a+b)(b+c)(c+a)$$

$$N=(b+c)[(a+b+c)+\sqrt{3(ab+bc+ca)}]\sqrt{3(ab+bc+ca)}-3(a+b)(b+c)(c+a)$$

$$P=(c+a)[(a+b+c)+\sqrt{3(ab+bc+ca)}]\sqrt{3(ab+bc+ca)}-3(a+b)(b+c)(c+a)$$

Suppose that:$a \geq b \geq c$.
So We have:

$$M=(a+b)([(a+b+c)+\sqrt{3(ab+bc+ca)}]\sqrt{3(ab+bc+ca)}-3(b+c)(c+a)) \geq 0$$

Because $(a+b+c)\sqrt{3(ab+bc+ca)} \geq 3c^2$

$$P=(a+c)([(a+b+c)+\sqrt{3(ab+bc+ca)}]\sqrt{3(ab+bc+ca)}-3(a+b)(b+c)) \geq 0$$

Because $(a+b+c)\sqrt{3(ab+bc+ca)} \geq 3b^2$ So We must prove:

$$N+P \geq 0$$

it ‘s equivalent to:

$$X=[(a+b+c)+\sqrt{3(ab+bc+ca)}]\sqrt{3(ab+bc+ca)}(a+b+2c)-6(a+b)(b+c)(c+a) \geq 0$$

Put

$$x=a+b+c; y=\sqrt{3(ab+bc+ca)}$$

$$X \geq [(a+b+c)+\sqrt{3(ab+bc+ca)}]\sqrt{3(ab+bc+ca)}(a+b+c)-6(a+b+c)(ab+bc+ca)$$

$$\leftrightarrow x^2y \geq xy^2$$

$$\leftrightarrow x \geq y$$

(it ‘s true for all positive numbers a,b,c).

88.
Let a, b, cbe positive real number . Prove that:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{a+b+c}{2(ab+bc+ca)}+\frac{3}{a+b+c}$$

Solution: Let put $p=a+b+c, q=ab+bc+ca, r=abc$, This inequality is equivalent to:

$$\frac{p^2+q}{pq-r} \geq \frac{p}{2q}+\frac{3}{p}$$

$$\Longleftrightarrow \frac{p^2+3}{3p-r} \geq \frac{p}{6}+\frac{3}{p}$$

By expanding expression We have:

$$(p^2+3)6p-p^2(3p-r)-18(3p-r) \geq 0$$

$$\iff 3p^3 + p^2r - 36p + 18r \geq 0$$

From the ill-known inequality, the third degree Schur's inequality states:

$$p^3 - 4pq + 9r \geq 0 \iff p^3 - 12p + 9r \geq 0$$

We have:

$$\iff 3p^3 + p^2r - 36p + 18r \geq 0$$

$$\iff 3(p^3 - 12p + 9r) + r(p^2 - 9) \geq 0$$

On the other hand, We have:

$$r(p^2 - 9) \geq 0 \iff (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$$

89. if x, y, z are nonnegative real numbers such that $x + y + z = 3$, then

$$4(\sqrt{x} + \sqrt{y} + \sqrt{z}) + 15 \leq 9(\sqrt{\frac{x+y}{2}} + \sqrt{\frac{y+z}{2}} + \sqrt{\frac{z+x}{2}})$$

Solution: The inequality's true when $x = 3, y = z = 0$. if no two of x, y, z are 0, set $x = a^2$ etc. it becomes

$$8(a+b+c) + 10\sqrt{3a^2+3b^2+3c^2} \leq 9\left(\sqrt{2a^2+2b^2} + \sqrt{2b^2+2c^2} + \sqrt{2c^2+2a^2}\right)$$

$$\iff 10\left(\sqrt{3a^2+3b^2+3c^2} - (a+b+c)\right) \leq 9\sum_{\text{cyc}} \sqrt{2a^2+2b^2} - (a+b)$$

$$\iff 10\sum_{\text{cyc}} \frac{(a-b)^2}{a+b+c+\sqrt{3a^2+3b^2+3c^2}} \leq 9\sum_{\text{cyc}} \frac{(a-b)^2}{a+b+\sqrt{2a^2+2b^2}}$$

$$\iff \sum_{\text{cyc}} \left(\frac{9}{a+b+\sqrt{2a^2+2b^2}} - \frac{10}{a+b+c+\sqrt{3a^2+3b^2+3c^2}}\right)(a-b)^2 \geq 0$$

Now each term is nonnegative, for in fact,

$$9(a+b+\sqrt{3a^2+3b^2}) \geq 10(a+b+\sqrt{2a^2+2b^2})$$

because

$$\left(\frac{9\sqrt{3}}{\sqrt{2}} - 10\right)\sqrt{2a^2+2b^2} > \sqrt{2a^2+2b^2} \geq a+b$$

90.

if a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$\frac{1}{4a^2+b^2+c^2} + \frac{1}{4b^2+c^2+a^2} + \frac{1}{4c^2+a^2+b^2} \leq \frac{1}{2}$$

Solution:

$$\frac{1}{4a^2+b^2+c^2} + \frac{1}{4b^2+c^2+a^2} + \frac{1}{4c^2+a^2+b^2} \leq \frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \sum_{sym}(a^6 - 4a^5b + 13a^4b^2 - 2a^4bc - 6a^3b^3 - 12a^3b^2c + 10a^2b^2c^2) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}(a-b)^2(2c^4 + 2(a^2-4ab+b^2)c^2 + a^4 - 2a^3b + 4a^2b^2 - 2ab^3 + b^4) \geq 0,$$

which true because

$$(a^2-4ab+b^2)^2 - 2(a^4 - 2a^3b + 4a^2b^2 - 2ab^3 + b^4) =$$

$$= -(a-b)^2(a^2+6ab+b^2) \le 0.$$

91.
Let a,b,c be nonnegative real numbers such that$a+b+c=3$. Prove that

$$\frac{a^2b}{4-bc}+\frac{b^2c}{4-ca}+\frac{c^2a}{4-ab} \le 1.$$

Solution.
Since

$$\frac{4a^2b}{4-bc} = a^2b + \frac{a^2b^2c}{4-bc}$$

the inequality can be written as

$$abc\sum \frac{ab}{4-bc} \le 4 - \sum a^2b.$$

Using the ill-known inequality $a^2b+b^2c+c^2a+abc \le 4$, We get

$$4-(a^2b+b^2c+c^2a) \ge abc,$$

and hence, it suffices to prove that

$$abc\sum \frac{ab}{4-bc} \le abc,$$

or equivalently,

$$\frac{ab}{4-bc}+\frac{bc}{4-ca}+\frac{ca}{4-ab} \le 1.$$

Since

$$ab+bc+ca \le \frac{(a+b+c)^2}{3} = 3,$$

We get

$$\frac{ab}{4-bc} \le \frac{ab}{\frac{4}{3}(ab+bc+ca)-bc} = \frac{3ab}{4ab+bc+4ca}.$$

Therefore, it is enough to check that

$$\frac{x}{4x+4y+z}+\frac{y}{4y+4z+x}+\frac{z}{4z+4x+y} \le \frac{1}{3},$$

where $x=ab, y=ca$ and $z=bc$. This is a ill-known inequality. 92.
if a,b,c are nonnegative real numbers, then

$$a^3+b^3+c^3+12abc \le a^2\sqrt{a^2+24bc}+b^2\sqrt{b^2+24ca}+c^2\sqrt{c^2+24ab}$$

Solution:

$$\begin{gathered}
\left(a^2+24bc\right)\left[7\left(a^2+b^2+c^2\right)+8(bc+ca+ab)\right]^2 \\
-\left[7a^3+8a^2(b+c)+7a\left(b^2+c^2\right)+92abc+48bc(b+c)\right]^2 \\
=24bc\big[(b-c)^2\left(109a^2+77ab+77ac+49b^2+89bc+49c^2\right) \\
+(b+c-2a)^2(25bc+7ab+7ca)\big] \ge 0 \Longrightarrow \\
\sum a^2\sqrt{a^2+24bc} \\
\ge \sum \frac{a^2[7a^3+8a^2(b+c)+7a(b^2+c^2)+92abc+48bc(b+c)]}{7(a^2+b^2+c^2)+8(bc+ca+ab)}
\end{gathered}$$

$$= a^3 + b^3 + c^3 + 12abc.$$

93. where a, b, c, d are nonnegative real numbers.Prove the inequality:

$$a\sqrt{9a^2 + 7b^2} + b\sqrt{9b^2 + 7c^2} + c\sqrt{9c^2 + 7d^2} + d\sqrt{9d^2 + 7a^2} \geq (a + b + c + d)^2$$

Solution:
By CauchySchwarz, We have

$$4\sum a\sqrt{9a^2 + 7b^2} \geq \sum a(9a + 7b)$$

it suffices to prove that

$$9(a^2+b^2+c^2+d^2)+7(a+c)(b+d) \geq 4(a+b+c+d)^2 = 4(a+c)^2+4(b+d)^2+8(a+c)(b+d)$$

$$\Leftrightarrow 9(a^2 + b^2 + c^2 + d^2) \geq 4(a + c)^2 + 4(b + d)^2 + (a + c)(b + d)$$

which is true because

$$a^2 + b^2 + c^2 + d^2 \geq (a + c)(b + d)$$

$$2(a^2 + b^2 + c^2 + d^2) = 2(a^2 + c^2) + 2(b^2 + d^2) \geq (a + c)^2 + (b + d)^2$$

94. Let a, b, c be positive . Prove that.

$$\sum \sqrt{\frac{3a^2}{7a^2 + 5(b + c)^2}} \leq 1 \leq \sum \sqrt{\frac{a^2}{a^2 + 2(b + c)^2}}$$

Solution:
The right hand is trivial by the Holder inequality since

$$\left(\sum \frac{a}{\sqrt{a^2 + 2\left(b + c\right)^2}}\right)^2 \left[\sum a\left(a^2 + 2\left(b + c\right)^2\right)\right] \geq \left(\sum a\right)^3$$

And $\left(\sum a\right)^3 \geq \left[\sum a\left(a^2 + 2\left(b + c\right)^2\right)\right] \Leftrightarrow \sum ab\left(a + b\right) \geq 6abc$. For the left hand by the Cauchy Schwarz inequality We have

$$\left(\sum \frac{a}{\sqrt{7a^2 + 5\left(b + c\right)^2}}\right)^2 \leq \left(\sum a\right)\left(\sum \frac{a}{7a^2 + 5\left(b + c\right)^2}\right)$$

Assume $a + b + c = 3$, denote $ab + bc + ca = \frac{9-q^2}{3}, r = abc$ then We will prove

$$\sum \frac{a}{12a^2 - 30a + 45} \leq \frac{1}{9}$$

$$\Leftrightarrow f\left(r\right) = 48r^2 + \left(222 + 52q^2\right)r + 20q^4 + 75q^2 - 270 \geq 0$$

We have

$$r \geq \max\left(0, \frac{\left(3 + q\right)^2\left(3 - 2q\right)}{27}\right)$$

Therefor, if

$$q \geq \frac{3}{2}$$

then get $r \geq 0$ and

$$f(r) \geq f(0) = 20\left(q - \frac{3}{2}\right)\left(q + \frac{3}{2}\right)\left(q^2 + 6\right) \geq 0$$

if $q \leq \frac{3}{2}$ then get $r \geq \frac{(3+q)^2(3-2q)}{27}$ and

$$f(r) \geq f\left(\frac{(3+q)^2(3-2q)}{27}\right) = \frac{q^2(2q-3)\left(96q^3 - 396q^2 + 2322q - 5103\right)}{729} \geq 0$$

We have done. Equality holds if an only if $a = b = c$ or $a = b, c = 0$ or any cyclic permutations.

95.

if a, b, c, d, e are positive real numbers such that $a + b + c + d + e = 5$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{20}{a^2 + b^2 + c^2 + d^2 + e^2} \geq 9$$

Solution, $\sum_{sym} f(a,b)$ means $f(a,b) + f(a,c) + f(a,d) + f(a,e) + f(b,c) + f(b,d)$ $+ f(b,e) +$ $f(c,d) + f(c,e) + f(d,e)$. We will firstly rewrite the inequality as

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} - \frac{25}{a+b+c+d+e} \geq 4 - \frac{4(a+b+c+d+e)^2}{5(a^2+b^2+c^2+d^2+e^2)}.$$

Using the identities

$$(a+b+c+d+e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) - 25 = \sum_{sym} \frac{(a-b)^2}{ab}$$

and $5(a^2+b^2+c^2+d^2+e^2) - (a+b+c+d+e)^2 = \sum_{sym}(a-b)^2$ We can rewrite again the inequality as

$$\frac{1}{a+b+c+d+e} \sum_{sym} \frac{(a-b)^2}{ab} \geq \frac{4}{5} \times \frac{\sum_{sym}(a-b)^2}{a^2+b^2+c^2+d^2+e^2}$$

or $\sum_{sym} S_{ab}(a-b)^2 \geq 0$ where

$$S_{xy} = \frac{1}{xy} - \frac{4}{a^2+b^2+c^2+d^2+e^2}$$

for all $x, y \in \{a, b, c, d, e\}$. Assume that $a \geq b \geq c \geq d \geq e > 0$. We will show that $S_{bc} + S_{bd} \geq 0$ and $S_{ab} + S_{ac} + S_{ad} + S_{ae} \geq 0$. indeed, We have

$$S_{bc} + S_{bd} = \frac{1}{bc} + \frac{1}{bd} - \frac{8}{a^2+b^2+c^2+d^2+e^2} > \frac{1}{bc} + \frac{1}{bd} - \frac{8}{b^2+b^2+c^2+d^2}$$
$$\geq \frac{1}{bc} + \frac{1}{bd} - \frac{8}{2bc+2bd} \geq 0$$

and

$$S_{ab}+S_{ac}+S_{ad}+S_{ae} = \frac{1}{ab}+\frac{1}{ac}+\frac{1}{ad}+\frac{1}{ae}-\frac{16}{a^2+b^2+c^2+d^2+e^2} \geq \frac{16}{a(b+c+d+e)} - \frac{16}{a^2+\frac{1}{4}(b+c+d+e)^2} \geq 0.$$

Hence, with notice that
$S_{bd} \geq S_{bc}$ and $S_{ae} \geq S_{ad} \geq S_{ac} \geq S_{ab}$

We have $S_{bd} \geq 0$ and $S_{ae} \geq 0, S_{ae} + S_{ad} \geq 0, S_{ae} + S_{ad} + S_{ac} \geq 0$.
Thus,

$$S_{bd}(b-d)^2 + S_{bc}(b-c)^2 \geq (S_{bd} + S_{bc})(b-c)^2 \geq 0(1)$$

and

$$S_{ae}(a-e)^2+S_{ad}(a-d)^2+S_{ac}(a-c)^2+S_{ab}(a-b)^2 \geq (S_{ae}+S_{ad})(a-d)^2+S_{ac}(a-c)^2+S_{ab}(a-b)^2$$

$$\geq (S_{ae} + S_{ad} + S_{ac})(a-c)^2 + S_{ab}(a-b)^2 \geq (S_{ae} + S_{ad} + S_{ac} + S_{ab})(a-b)^2 \geq 0(2)$$

On the other hand, $S_{be} \geq S_{bd} \geq 0$ and $S_{de} \geq S_{ce} \geq S_{cd} \geq S_{bd} \geq 0(3)$.

Therefore, from (1), (2) and (3) We get $\sum_{sym} S_{ab}(a-b)^2 \geq 0$.

Equality occurs when $a = b = c = d = e$ or $a = 2b = 2c = 2d = 2e$.
96.
Let a, b, c be nonnegative real numbers. Prove that

$$1 + \frac{3abc}{a^2b + b^2c + c^2a} \geq \frac{2(ab + bc + ca)}{a^2 + b^2 + c^2}$$

Solution:
We can prove it as follow:
Rewriting the inequality as

$$\frac{3abc}{a^2b + b^2c + c^2a} \geq \frac{2(ab + bc + ca) - a^2 - b^2 - c^2}{a^2 + b^2 + c^2}$$

if $2(ab + bc + ca) \leq a^2 + b^2 + c^2$, it is trivial.
if $2(ab + bc + ca) \geq a^2 + b^2 + c^2$, applying Schur's inequality:

$$3abc \geq \frac{(a + b + c)[2(ab + bc + ca) - a^2 - b^2 - c^2]}{3}$$

it suffices to show that

$$\frac{(a + b + c)[2(ab + bc + ca) - a^2 - b^2 - c^2]}{3(a^2b + b^2c + c^2a)} \geq \frac{2(ab + bc + ca) - a^2 - b^2 - c^2}{a^2 + b^2 + c^2}$$

$$(a + b + c)(a^2 + b^2 + c^2) \geq 3(a^2b + b^2c + c^2a)$$

$$b(a-b)^2 + c(b-c)^2 + a(c-a)^2 \geq 0$$

(True)
97.
Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{6}{a + b + c} \geq 5.$$

Solution:
1....WLOG assume $a \geq b \geq c$.
Let

$$f(a, b, c) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{6}{a + b + c}$$

$$f(a, b, c) \geq f(a, \sqrt{bc}, \sqrt{bc})$$

$$<=> \frac{(\sqrt{b} - \sqrt{c})^2}{bc(a + b + c)(a + 2\sqrt{bc})}((a + b + c)(a + 2\sqrt{bc}) - 6bc) \geq 0 \Leftrightarrow$$

$$(a+b+c)(a+2\sqrt{bc}) \geq 6bc.$$

As

$$a \geq \frac{b+c}{2} \geq \sqrt{bc}$$

so

$$(a+b+c)(a+2\sqrt{bc}) \geq 9bc \geq 6bc$$

hence the above inequality is true.

$$f(\frac{1}{x^2}, x, x) \geq 5 \Leftrightarrow$$

$$(x-1)^2(2x^4+4x^3-4x^2-x+2) \geq 0.$$

As $2x^4+4x^3-4x^2-x+2 > 0$ if $x > 0$, so the above inequality is true.
Therefore

$$f(a,b,c) \geq f(a, \sqrt{bc}, \sqrt{bc}) = f(\frac{1}{bc}, \sqrt{bc}, \sqrt{bc}) \geq 5.$$

=> Q.E.D

2.........
Assume that $a \geq b, c$. Write $x = \sqrt{a}, y = \sqrt{\frac{b}{c}}$. Then $x \geq 1$ and the inequality

$$(ab+bc+ca)(a+b+c)+6 \geq 5(a+b+c)$$

becomes

$$x^3(y+y^{-1}) - 5x^2 + (y^2+9+y^{-2}) - 5x^{-1}(y+y^{-1}) + x^{-3}(y+y^{-1}) \geq 0$$

This can be seperated as

$$2x^3 - 5x^2 + 11 - 10x^{-1} + 2x^{-3} \geq 0$$

and

$$x^3(y+y^{-1}-2) + (y^2+y^{-2}-2) - 5x^{-1}(y+y^{-1}-2) + x^{-3}(y+y^{-1}-2) \geq 0$$

The first one is easy. About the second one,
Note that $x^3 + x^{-3} \geq 2 \geq 2x^{-1}$ and $(y^2+y^{-2}-2) \geq 3(y^1+y^{-1}-2) \geq 3x^{-1}(y^1+y^{-1}-2)$
since

$$y^2 - 3y + 4 - 3y^{-1} + y^{-2} = (y-1)^2(y-1+y^{-1})$$

3.........
Lema of Vaile Cirtoaje

$$(a+b)(b+c)(c+a) + 7 \geq 5(a+b+c)$$

(Can easy prove by MV)
But

$$\begin{aligned}(a+b)(b+c)(c+a) &= a^2b+a^2c+b^2c+b^2a+c^2a+c^2b+2abc \\ &= \left(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b+3abc\right) - abc\end{aligned}$$

$$= (a+b+c)(bc+ca+ab) - abc = (a+b+c)(bc+ca+ab) - 1$$

where We have used that abc = 1 in the last step of our calculation. Thus, We have

$$((a+b+c)(bc+ca+ab) - 1) + 7 \geq 5(a+b+c)$$

; in other words,

$$(a+b+c)(bc+ca+ab) + 6 \geq 5(a+b+c)$$

Upon division by a + b + c, this becomes

$$(bc+ca+ab) + \frac{6}{a+b+c} \geq 5$$

Finally, since $abc = 1$,
We have $bc = \frac{1}{a}$, $ca = \frac{1}{b}$ and $ab = \frac{1}{c}$, and thus We get

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{6}{a+b+c} \geq 5$$

98.
Let $a, b, c > 0$ and with all $k \geq -3/2$. Prove the inequality:

$$\sum \frac{a^3 + (k+1)abc}{b^2 + kbc + c^2} \geq a + b + c$$

Solution:
Our inequality is equivalent to

$$\frac{a(a^2+bc-b^2-c^2)}{b^2+kbc+c^2} + \frac{b(b^2+ca-c^2-a^2)}{c^2+kca+a^2} + \frac{c(c^2+ab-a^2-b^2)}{a^2+kab+b^2} \geq 0,$$

$$(a^2-b^2)\left(\frac{a}{b^2+kbc+c^2} - \frac{b}{a^2+kac+c^2}\right) + c\left[\frac{a(b-c)}{b^2+kbc+c^2} + \frac{b(a-c)}{a^2+kac+c^2} + \frac{c^2+ab-a^2-b^2}{a^2+kab+b^2}\right] \geq 0.$$

From now, We see that

$$\frac{a}{b^2+kbc+c^2} - \frac{b}{a^2+kac+c^2} = \frac{(a-b)(a^2+b^2+c^2+ab+kac+kbc)}{(b^2+kbc+c^2)(a^2+kac+c^2)},$$

$$\frac{c^2+ab-a^2-b^2}{a^2+kab+b^2} = \frac{(c-a)(c-b) - a(b-c) - b(a-c) - (a-b)^2}{a^2+kab+b^2},$$

$$\frac{a(b-c)}{b^2+kbc+c^2} - \frac{a(b-c)}{a^2+kab+b^2} = \frac{a(a-c)(b-c)(a+c+kb)}{(a^2+kab+b^2)(b^2+kbc+c^2)},$$

$$\frac{b(a-c)}{a^2+kac+c^2} - \frac{b(a-c)}{a^2+kab+b^2} = \frac{a(a-c)(b-c)(b+c+ka)}{(a^2+kab+b^2)(a^2+kac+c^2)}.$$

Therefore, the inequality can be rewritten as

$$A(a-b)^2 + \frac{c(a-c)(b-c)}{a^2+kab+b^2} B \geq 0,$$

where

$$A = \frac{(a+b)(a^2+b^2+c^2+ab+kac+kbc)}{(a^2+kac+c^2)(b^2+kbc+c^2)} - \frac{c}{a^2+kab+b^2},$$

and

$$B = \frac{a(a+c+kb)}{b^2+kbc+c^2} + \frac{b(b+c+ka)}{a^2+kac+c^2} + 1.$$

Now, using the symmetry, We can assume that $a \geq b \geq c$, then We will prove $A \geq 0$ and $B \geq 0$ to finish our **Solution**.
Solution of $A \geq 0$. We have the following estimations

$$(a^2+b^2+c^2+ab+kac+kbc)-(a^2+kac+c^2)=b(a+b+kc)\geq 0,$$

$$a+b\geq 2c,$$

$$2(a^2+kab+b^2)-(b^2+kbc+c^2)=b(2a-c)k+2a^2+b^2-c^2)\geq -\frac{3}{2}b(2a-c)+2a^2+b^2-c^2=(2a-b)(a-b)+c\left(\frac{3}{2}b-c\right)\geq 0$$

From these inequalities, We can easily see that $A \geq 0$.
Solution of $B \geq 0$. To prove this, We will consider 2 case:
+ if $b^2 \geq ac$, consider the function

$$f(k)=\frac{a(a+c+kb)}{b^2+kbc+c^2}+\frac{b(b+c+ka)}{a^2+kac+c^2}+1,$$

We have

$$f'(k)=\frac{ab(a^2-bc)}{(a^2+kac+c^2)}+\frac{ab(b^2-ac)}{(b^2+kbc+c^2)^2}\geq 0,$$

therefore $f(k)$ is increasing and it suffices to us to show that $f\left(-\frac{3}{2}\right)\geq 0$, i.e.

$$\frac{2a(2a+2c-3b)}{2b^2-3bc+2c^2}+\frac{2b(2b+2c-3a)}{2a^2-3ac+2c^2}+2\geq 0.$$

We have

$$\frac{2a(2a+2c-3b)}{2b^2-3bc+2c^2}+1=\frac{(a-b)(a+b+3c)+3(a-b)^2+ac+2c^2}{2b^2-3bc+2c^2}\geq\frac{(a-b)(a+b+3c)}{2b^2-3bc+2c^2},$$

Similarly,

$$\frac{2b(2b+2c-3a)}{2a^2-3ac+2c^2}+1\geq\frac{(b-a)(a+b+3c)}{2a^2-3ac+2c^2}.$$

it follows that

$$LHS\geq (a-b)(a+b+3c)\left(\frac{1}{2b^2-3bc+2c^2}-\frac{1}{2a^2-3ac+2c^2}\right)\geq 0.$$

+ if $ac \geq b^2$, then rewrite our inequality as

$$a(a+c+kb)\cdot\frac{a^2+kac+c^2}{b^2+kbc+c^2}+b(b+c+ka)+a^2+c^2+kac\geq 0.$$

Since $a \geq b, ac \geq b^2$ and $k \geq -\frac{3}{2}$, We have $a+c+kb \geq 0$ and $a^2+kac+c^2 \geq b^2+kbc+c^2$, therefore

$$LHS\geq a(a+c+kb)+b(b+c+ka)+a^2+c^2+kac=a(2b+c)k+2a^2+b^2+ac+bc+c^2\geq$$
$$\geq -\frac{3}{2}a(2b+c)+2a^2+b^2+ac+bc+c^2=(a-b)(2a-b-c)+\frac{1}{2}ac+c^2\geq 0.$$

This ends our **Solution**. Equality holds if and only if $a=b=c$ or $a=b, c=0$ and its permutations.
99. if a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{a(4b+4c-a)}{b^2+c^2}+\frac{b(4c+4a-b)}{c^2+a^2}+\frac{c(4a+4b-c)}{a^2+b^2}\leq\frac{21}{2}$$

Solution:
The inequality still holds for a,b,c are real numbers, but it is just a trivial corollary of the

case nonnegative real numbers as

$$\frac{a(4b+4c-a)}{b^2+c^2}=\frac{4ab+4ac-a^2}{b^2+c^2}\le\frac{4|ab|+4|ac|-a^2}{b^2+c^2}.$$

As

$$\sum\frac{a(b+c)}{b^2+c^2}-3=\sum\frac{b(a-b)-c(c-a)}{b^2+c^2}=\sum(a-b)\left(\frac{b}{b^2+c^2}-\frac{a}{a^2+c^2}\right)=\sum\frac{(a-b)^2(ab-c^2)}{(a^2+c^2)(b^2+c^2)}$$

and

$$\sum\frac{a^2}{b^2+c^2}-\frac{3}{2}=\frac{1}{2}\sum\frac{(a^2-b^2)^2}{(a^2+c^2)(b^2+c^2)}$$

We can rewrite our inequality as

$$\sum(a-b)^2(a^2+b^2)(a^2+b^2+8c^2-6ab)\ge 0$$

Assume that $a\ge b\ge c$, then it is easy to verify that

$$(b-c)^2(b^2+c^2+8a^2-6bc)\ge 0$$

and We can reduce the inequality into

$$(a-c)^2(a^2+c^2)(a^2+c^2+8b^2-6ca)+(a-b)^2(a^2+b^2)(a^2+b^2+8c^2-6ab)\ge 0$$

which is true as

$$a-c\ge a-b\ge 0,\quad (a-c)(a^2+c^2)\ge(a-b)(a^2+b^2)\ge 0$$

and

$$(a^2+c^2+8b^2-6ca)+(a^2+b^2+8c^2-6ab)=(a-3b)^2+(a-3c)^2\ge 0.$$

This completes our **Solution**.
Equality holds when $a=b=c$ or $a=3b=3c$ and its cyclic permutations.
100.
if a,b,c are nonnegative real numbers, then

$$(d)\qquad \left(\frac{a}{b-c}\right)^2+\left(\frac{b}{c-a}\right)^2+\left(\frac{c}{a-b}\right)^2+\frac{a^2+b^2+c^2}{ab+bc+ca}\ge 4$$

Solution:
Without loss of generality, We may assume that c is the smallest number among a,b,c. We will show that

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(a-c)^2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\ge\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{a^2+b^2}{ab}.$$

indeed, this inequality is equivalent to

$$\frac{a^2}{(b-c)^2}-\frac{a^2}{b^2}+\frac{b^2}{(a-c)^2}-\frac{b^2}{a^2}\ge\frac{a^2+b^2}{ab}-\frac{a^2+b^2+c^2}{ab+bc+ca},$$

or

$$\frac{ca^2(2b-c)}{b^2(b-c)^2}+\frac{cb^2(2a-c)}{a^2(a-c)^2}\ge\frac{c[(a+b)(a^2+b^2)-abc]}{ab(ab+bc+ca)}.$$

Since

$$\frac{2b-c}{(b-c)^2}\ge\frac{2b-2c}{(b-c)^2}=\frac{2}{b-c}\ge\frac{2}{b},\quad \frac{2a-c}{(a-c)^2}\ge\frac{2}{a},$$

and
$$\frac{(a+b)(a^2+b^2)-abc}{ab(ab+bc+ca)} \le \frac{(a+b)(a^2+b^2)}{a^2b^2},$$
it suffices to prove that
$$\frac{2ca^2}{b^3}+\frac{2cb^2}{a^3} \ge \frac{c(a+b)(a^2+b^2)}{a^2b^2},$$
which is true because
$$\frac{2a^2}{b^3}+\frac{2b^2}{a^3}=\frac{2a^4}{a^2b^3}+\frac{2b^4}{a^3b^2} \ge \frac{2(a^2+b^2)^2}{a^2b^2(a+b)} \ge \frac{(a+b)(a^2+b^2)}{a^2b^2}.$$
Now, according to the above inequality, one can reduce the problem into proving that
$$\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{a^2+b^2}{ab} \ge 4,$$
which is true.

101.

For $a,b,c>0$ and $-1 \le k \le 2$, Prove:
$$\left(\frac{a}{b-c}\right)^2+\left(\frac{b}{c-a}\right)^2+\left(\frac{c}{a-b}\right)^2 \ge k+\frac{(4-2k)(ab+bc+ca)}{a^2+b^2+c^2}$$
Solution:

using Cauchy Schwarz inequality as follows:

Let $x=a^2+b^2+c^2$ and $y=ab+bc+ca$. Applying the Cauchy Schwarz inequality, We have
$$\sum \frac{a^2}{(b-c)^2} \ge \frac{(a+b+c)^2}{\sum(b-c)^2}=\frac{x+2y}{2(x-y)}.$$
Notice also that $\frac{2-k}{3} \ge 0$, so from the above inequality, We get
$$\frac{2-k}{3}\sum \frac{a^2}{(b-c)^2} \ge \frac{(2-k)(x+2y)}{6(x-y)}. \tag{1}$$
On the other hand, since
$$\sum \frac{a}{b-c}\cdot\frac{b}{c-a}=-1,$$
We have
$$\sum \frac{a^2}{(b-c)^2}-2=\sum \frac{a^2}{(b-c)^2}+2\sum \frac{a}{b-c}\cdot\frac{b}{c-a}=\left(\sum \frac{a}{b-c}\right)^2 \ge 0,$$
and since $\frac{k+1}{3} \ge 0$, this inequality gives us that
$$\frac{k+1}{3}\sum \frac{a^2}{(b-c)^2} \ge \frac{2(k+1)}{3}. \tag{2}$$
From (1) and (2), We obtain the following inequality
$$\sum \frac{a^2}{(b-c)^2}=\frac{2-k}{3}\sum \frac{a^2}{(b-c)^2}+\frac{k+1}{3}\sum \frac{a^2}{(b-c)^2} \ge \frac{(2-k)(x+2y)}{6(x-y)}+\frac{2(k+1)}{3}.$$
Thus, it is enough to show that
$$\frac{(2-k)(x+2y)}{6(x-y)}+\frac{2(k+1)}{3} \ge k+\frac{(4-2k)y}{x}.$$

This inequality can be written as

$$(2-k)\left[\frac{x+2y}{6(x-y)}+\frac{1}{3}-\frac{2y}{x}\right]\geq 0,$$

or

$$\frac{(2-k)(x-2y)^2}{2x(x-y)}\geq 0.$$

The last one is obviously true, so our **Solution** is completed.
102.
if a, b, c are nonnegative real numbers such that $a+b+c=2$, then

$$\frac{bc}{a^2+1}+\frac{ca}{b^2+1}+\frac{ab}{c^2+1}\leq 1.$$

First **Solution**.
in the nontrivial case when two of a,b,c are nonzero, We claim that the following inequality holds

$$\frac{bc}{a^2+1}\leq\frac{bc(b+c)}{ab(a+b)+bc(b+c)+ca(c+a)},$$

or

$$(b+c)(a^2+1)\geq bc(b+c)+a(b^2+c^2)+a^2(b+c).$$

Since $a(b^2+c^2)=a(b+c)^2-2abc$ this inequality can be written as

$$(b+c)(a^2+1)-a(b+c)^2-a^2(b+c)\geq bc(b+c)-2abc,$$

or

$$(b+c)(1-ab-ac)\geq bc(b+c-2a).$$

By the AM-GM inequality, We have

$$1-ab-ac=1-a(b+c)\geq 1-\frac{(a+b+c)^2}{4}=0.$$

Thus, We can easily see that the above inequality is clearly true for $b+c\leq 2a$.
Let us assume now that $b+c\geq 2a$. in this case, using the AM-GM inequality, We have

$$(b+c)(1-ab-ac)-bc(b+c-2a)\geq(b+c)(1-ab-ac)-\frac{(b+c)^2}{4}(b+c-2a)=\frac{b+c}{4(4-2a(b+c)-(b+c)^2)}$$

$$=\frac{b+c}{4}[(a+b+c)^2-2a(b+c)-(b+c)^2]=\frac{a^2(b+c)}{4}\geq 0.$$

This completes the **Solution** of the claim and by using it, We get

$$\sum\frac{bc}{a^2+1}\leq\sum\frac{bc(b+c)}{ab(a+b)+bc(b+c)+ca(c+a)}=1.$$

This is what We want to prove.
Equality holds if and only if $a=b=1$ and $c=0$, or any cyclic permutation.
Second **Solution**.
Since $\frac{bc}{a^2+1}=bc-\frac{a^2bc}{a^2+1}$, the inequality can be written as

$$abc\left(\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\right)+1-(ab+bc+ca)\geq 0.$$

Notice that for any nonnegative real number x, We have

$$\frac{1}{x^2+1} = \frac{2-x}{2} + \frac{x(x-1)^2}{2(x^2+1)} \geq \frac{2-x}{2}.$$

Using this inequality, it suffices to prove that

$$abc\left(a \cdot \frac{2-a}{2} + b \cdot \frac{2-b}{2} + c \cdot \frac{2-c}{2}\right) + 1 - (ab+bc+ca) \geq 0,$$

or

$$abc(ab+bc+ca) + 1 - (ab+bc+ca) \geq 0.$$

Setting $q = ab+bc+ca$, $0 < q \leq \frac{4}{3}$. From the fourth degree Schur's inequality

$$a^4+b^4+c^4+abc(a+b+c) \geq ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)$$

and the given hypothesis, We get

$$abc \geq \frac{(q-1)(4-q)}{3}.$$

it follows that

$$\begin{aligned} abc(ab+bc+ca)+1-(ab+bc+ca) &\geq \frac{(q-1)(4-q)}{3} \cdot q + 1 - q \\ &= \frac{(3-q)(q-1)^2}{3} \geq 0. \end{aligned}$$

The **Solution** is completed.

Third **Solution**. Write the inequality as

$$abc\left(\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1}\right) \geq ab+bc+ca-1.$$

Since the case $ab+bc+ca \leq 1$ is trivial, let us assume that $ab+bc+ca \geq 1$. Setting

$$q = ab+bc+ca, 1 \leq q \leq \frac{4}{3}.$$

From the fourth degree Schur's inequality

$$a^4+b^4+c^4+abc(a+b+c) \geq ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2),$$

We get

$$q-1 \leq \frac{3abc}{4-q}.$$

Using this result, We see that it suffices to prove that

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \geq \frac{3}{4-q}.$$

This inequality is equivalent to

$$\left(\frac{a}{a^2+1} + \frac{1}{2}\right) + \left(\frac{b}{b^2+1} + \frac{1}{2}\right) + \left(\frac{c}{c^2+1} + \frac{1}{2}\right) \geq \frac{3}{4-q} + \frac{3}{2},$$

or
$$\frac{(a+1)^2}{a^2+1}+\frac{(b+1)^2}{b^2+1}+\frac{(c+1)^2}{c^2+1}\geq\frac{3(6-q)}{4-q}.$$
By the Cauchy-Schwarz inequality, We have
$$\frac{(a+1)^2}{a^2+1}+\frac{(b+1)^2}{b^2+1}+\frac{(c+1)^2}{c^2+1}\geq\frac{(a+1+b+1+c+1)^2}{a^2+1+b^2+1+c^2+1}=\frac{25}{7-2q}.$$
Therefore, it suffices to prove that
$$\frac{25}{7-2q}\geq\frac{3(6-q)}{4-q},$$
which is equivalent to the obvious inequality
$$(q-1)(13-3q)\geq 0.$$
The **Solution** is completed.

103.

Let $a>0,b>0,c>0,\ ab+ac+bc=3$. Prove that
$$\frac{a+b+c}{3}\geq\sqrt[36]{\frac{a^4+b^4+c^4}{3}}(*)$$
Solution:

As you work,put $a+b+c=3u;abc=3$ then
$$a^4+b^4+c^4=(9u^2-6)^2-18+12uv\leq(9u^2-6)^2-18+12=81u^4-108u^2+30$$
So that:
$$a^4+b^4+c^4\leq\frac{81u^4-108u^2+30}{3}=27u^4-36u^2+10$$
We need to prove that
$$u^{36}\geq 27u^4-36u^2+10$$
Let $f(x)=x^{36}-27x^4+36x^2-10$.We will prove that $f(x)\geq 0$ for all $x\geq 1$
$$f'(x)=36x^{35}-108x^3+72x=x(36x^{34}-108x^2+72)$$
Apply AM-GM's inequality We have
$$36x^{34}+72\geq 108\sqrt[108]{x^{34.36}}\geq 108x^2$$
(because $x\geq 1$)

So that $f'(x)\geq 0$ for all $x\geq 1$

Hence $f(x)\geq f(1)=0$ for all $x\geq 1$

Therefore the problem is proved :

Remark:

a) $a,b,c,d>0$ satisfy $ab+ac+ad+bc+bd+cd=6$.Prove that:
$$\frac{a+b+c+d}{4}\geq\sqrt[27]{\frac{a^3+b^3+c^3+d^3}{4}}$$
b) $a,b,c,d>0$ satisfy $ab+ac+ad+bc+bd+cd=6$.Prove that:
$$\frac{a+b+c+d}{4}\geq\sqrt[64]{\frac{a^4+b^4+c^4+d^4}{4}}$$

104.
Let a, b and c are nonnegative numbers such that $bc + ca + ab = 3$. Prove that:

$$\frac{a+b+c}{3} \geq \sqrt[16]{\frac{a^3+b^3+c^3}{3}}.$$

Solution:
To prove this inequality, We may write it as

$$\left(\frac{a^3+b^3+c^3}{3}\right)^2\left(\frac{ab+bc+ca}{3}\right)^{13} \leq \left(\frac{a+b+c}{3}\right)^{32}.$$

This is a homogeneous ineuality of a, b, c, so We may forget the condition $ab + bc + ca = 3$ To normalize for $a + b + c = 3$. Now, applying the AM-GM inequality, We have

$$LHS = \left[\left(\frac{a^3+b^3+c^3}{3}\right)\left(\frac{ab+bc+ca}{3}\right)\right]^2\left(\frac{ab+bc+ca}{3}\right)^{11} \leq$$

$$\frac{1}{13^{13}}\left[2\left(\frac{a^3+b^3+c^3}{3}\right)\left(\frac{ab+bc+ca}{3}\right)+11\left(\frac{ab+bc+ca}{3}\right)\right]^{13}.$$

it is thus sufficient to prove that

$$2\left(\frac{a^3+b^3+c^3}{3}\right)\left(\frac{ab+bc+ca}{3}\right)+11\left(\frac{ab+bc+ca}{3}\right) \leq 13,$$

which is equivalent to

$$\frac{13(a+b+c)^2}{ab+bc+ca} \geq \frac{54(a^3+b^3+c^3)}{(a+b+c)^3}+33.$$

105.Let a, b, c be positive real numbers such that $ab + bc + ca + abc \geq 4$. Prove that

$$\frac{1}{(a+1)^2(b+c)}+\frac{1}{(b+1)^2(c+a)}+\frac{1}{(c+1)^2(a+b)} \leq \frac{3}{8}.$$

Solution.
Letting $a = tx, b = ty$ and $c = tz$, where $t > 0$ and $x, y, z > 0$ such that $xy+yz+zx+xyz = 4$. The condition $ab + bc + ca + abc \geq 4, t \geq 1$, and the inequality becomes

$$\frac{1}{t(tx+1)^2(y+z)}+\frac{1}{t(ty+1)^2(z+x)}+\frac{1}{t(tz+1)^2(x+y)} \leq \frac{3}{8}.$$

We see that it suffices to prove this inequality for $t = 1$.
in this case, We may write the inequality in the form

$$\frac{1}{(x+1)^2(y+z)}+\frac{1}{(y+1)^2(z+x)}+\frac{1}{(z+1)^2(x+y)} \leq \frac{3}{8}.$$

Now, since $x, y, z > 0$ and $xy + yz + zx + xyz = 4$, there exist some positive real numbers u, v, w such that

$$x = \frac{2u}{v+w}, y = \frac{2v}{w+u} and z = \frac{2w}{u+v}.$$

The above inequality becomes

$$\sum \frac{(u+v)(u+w)(v+w)^2}{(2u+v+w)^2[v(u+v)+w(u+w)]} \leq \frac{3}{4}.$$

Using the AM-GM inequality and the Cauchy-Schwarz inequality, We get

$$\frac{(u+v)(u+w)(v+w)^2}{(2u+v+w)^2[v(u+v)+w(u+w)]} \le \frac{(v+w)^2}{4[v(u+v)+w(u+w)]} \le \frac{1}{4}\left(\frac{v}{u+v}+\frac{w}{u+w}\right).$$

Therefore,

$$\sum \frac{(u+v)(u+w)(v+w)^2}{(2u+v+w)^2[v(u+v)+w(u+w)]} \le \frac{1}{4}\sum\left(\frac{v}{u+v}+\frac{w}{u+w}\right) = \frac{3}{4}.$$

The **Solution** is completed. Equality holds if and only if $a=b=c=1$. Remark.
The **Solution** of this problem gives us the fourth **Solution** of the previous problem, because the condition $abc \ge 1, ab+bc+ca+abc \ge 4$.
106.
For $a,b,c>0$, such that $abc=1$, prove that the following inequality holds

$$a\cdot\frac{a^4-b}{a^4+2b}+b\cdot\frac{b^4-c}{b^4+2c}+c\cdot\frac{c^4-a}{c^4+2a} \ge 0$$

Solution:
Note that the inequality is equivalent to

$$\sum \frac{a^5}{a^4+2b^2ac} \ge \frac{a+b+c}{3}$$

due to Holder inequality We have

$$\left(\sum \frac{a^5}{a^4+2b^2ac}\right)\left(\sum a^4+2b^2ac\right)\sum a \ge \left(\sum a^2\right)^3$$

We have to prove that

$$\left(\sum a^2\right)^3 \ge \frac{(a+b+c)^2}{3}\left(\sum a^4+2b^2ac\right)$$

which is true because

$$\left(\sum a^4+2b^2ac\right) \le \sum a^4+2a^2b^2 = \left(\sum a^2\right)^2$$

The **Solution** is completed , equality occurs when $a=b=c$
107.
Let a,b,c be positive real number. Prove that:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \le 4+\frac{(ab+bc+cd+da+ac+bd)^2}{3abcd}$$

Solution:
Let $A=b+c+d$, $B=bc+cd+db$ and $C=bcd$. The inequality can be written as

$$\frac{(aA+B)^2}{3aC}+4 \ge (a+A)\left(\frac{1}{a}+\frac{B}{C}\right),$$

which is equivalent to

$$\frac{(aA+B)^2}{a}+12C \ge 3(a+A)\left(\frac{C}{a}+B\right),$$

$$A^2a+2AB+\frac{B^2}{a}+12C \ge 3C+3aB+\frac{3AC}{a}+3AB,$$

$$(A^2-3B)a+\frac{B^2-3AC}{a}\ge AB-9C.$$

Since $A^2-3B\ge 0$ and $B^2-3AC\ge 0$, using the AM-GM inequality, We have

$$(A^2-3B)a+\frac{B^2-3AC}{a}\ge 2\sqrt{(A^2-3B)(B^2-3AC)}.$$

Therefore, it suffices to prove that

$$4(A^2-3B)(B^2-3AC)\ge (AB-9C)^2,$$

or

$$4(b^2+c^2+d^2-bc-cd-db)[b^2c^2+c^2d^2+d^2b^2-bcd(b+c+d)]\ge [b(c^2+d^2)+c(d^2+b^2)+d(b^2+c^2)-6bcd]^2,$$

or

$$[(b-c)^2+(c-d)^2+(d-b)^2][d^2(b-c)^2+b^2(c-d)^2+c^2(d-b)^2]\ge [d(b-c)^2+b(c-d)^2+c(d-b)^2]^2,$$

which is true according to the Cauchy-Schwarz inequality.

108:

Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\ge\left(\frac{a+c}{b+c}\right)^2+\left(\frac{b+a}{c+a}\right)^2+\left(\frac{c+b}{a+b}\right)^2.$$

Solution.

The inequality is equivalent to

$$\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{(b+c)^2-(a+c)^2}{(b+c)^2}\right)\ge 3$$

$$\iff \sum_{cyc}\frac{a^2(b+c)^2+b^2(b+c)^2-b^2(a+c)^2}{b^2(b+c)^2}\ge 3$$

$$\iff \sum_{cyc}\frac{2a^2bc+a^2c^2+b^4+2b^3c-2ab^2c}{b^2(b+c)^2}\ge 3$$

Now, from AM-GM, We have $a^2c^2+b^4\ge 2ab^2c$.

it's suffice to show that

$$\sum_{cyc}\frac{2a^2bc+2b^3c}{b^2(b+c)^2}\ge 3$$

And it's true since

$$\sum_{cyc}\frac{2a^2bc+2b^3c}{b^2(b+c)^2}=\sum_{cyc}\frac{bc(2a^2+2b^2)}{b^2(b+c)^2}\ge\sum_{cyc}\frac{bc(a+b)^2}{b^2(b+c)^2}\ge 3$$

by AM-GM.

109.

if a, b, c, x are positive real numbers, then

$$\frac{a^x}{b^x}+\frac{b^x}{c^x}+\frac{c^x}{a^x}\ge\left(\frac{a+c}{b+c}\right)^x+\left(\frac{b+a}{c+a}\right)^x+\left(\frac{c+b}{a+b}\right)^x.$$

Solution:
First, We shall prove the following lemma.
Lemma. if x and y are positive real numbers such that $(x-y)(y-1) \geq 0$, then

$$x^2 + \frac{2}{x} \geq y^2 + \frac{2}{y}.$$

Solution.
After factorizing, We can write this inequality as

$$\frac{(x-y)[xy(x+y)-2]}{xy} \geq 0.$$

Now, if $x \geq y$, then from the given hypothesis, We have $x \geq y \geq 1$, and thus it is clear that $(x-y)[xy(x+y)-2] \geq 0$.
if $x \leq y$, then from $(x-y)(y-1) \geq 0$, We have $x \leq y \leq 1$, which gives $xy(x+y) \leq 2$ and so $(x-y)[xy(x+y)-2] \geq$
0.
Turning back to our problem. Squaring both sides, We can write the inequality as

$$\sum \left(\frac{a^{2x}}{b^{2x}} + 2\frac{b^x}{a^x} \right) \geq \sum \left[\frac{(a+c)^{2x}}{(b+c)^{2x}} + 2\frac{(b+c)^x}{(a+c)^x} \right].$$

This inequality follows from adding the inequality

$$\frac{a^{2x}}{b^{2x}} + 2\frac{b^x}{a^x} \geq \frac{(a+c)^{2x}}{(b+c)^{2x}} + 2\frac{(b+c)^x}{(a+c)^x}$$

and its analogous inequalities.
This inequality is true because

$$\left[\frac{a^x}{b^x} - \frac{(a+c)^x}{(b+c)^x} \right] \left[\frac{(a+c)^x}{(b+c)^x} - 1 \right] \geq 0$$

(this is a trivial inequality).
110.
For $a, b, c > 0$, such that $abc = 1$, prove that the following inequality holds

$$\frac{a}{a+b^4+c^4} + \frac{b}{b+c^4+a^4} + \frac{c}{c+a^4+b^4} \leq 1$$

Give a generalization to this inequality.
Solution:
By Cauchy-Schwarz ineq, We have:

$$\begin{aligned} \frac{a}{a+b^4+c^4} &= \frac{a(a^3+2)}{(a+b^4+c^4)(a^3+1+1)} \\ &\leq \frac{a(a^3+2)}{(a^2+b^2+c^2)^2} \end{aligned}$$

Similarly, We get:

$$LHS \leq \frac{a^4+b^4+c^4+2(a+b+c)}{(a^2+b^2+c^2)^2}$$

And We need to prove that:

$$(a^2+b^2+c^2)^2 \geq a^4+b^4+c^4+2(a+b+c)$$

$$\Leftrightarrow a^2b^2 + b^2c^2 + c^2a^2 \geq a + b + c$$

which is true because:

$$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c) = a + b + c$$

Equality holds when $a = b = c = 1$
REMARK:

$$1.\frac{a}{a + b^n + c^n} + \frac{b}{b + c^n + a^n} + \frac{c}{c + a^n + b^n} \leq 1$$

for $a, b, c > 0$ satisfying $abc = 1$ and $n \geq 3$
2..if $a_1, a_2, ..., a_n$ are positive real numbers satisfying $a_1a_1...a_n = 1$, then

$$\sum \frac{a_1}{a_1 + a_2^k + ... + a_n^k} \leq 1$$

for any $k \geq 1$.
Solution:
let $S = \sum_{p=1}^{n} a_p^k$ The inequality is equivalent to

$$\sum \frac{S - a_p^k}{a_p + S - a_p^k} \geq n - 1$$

According to Cauchy-Shwarz inequality

$$\sum \frac{S - a_p^k}{a_p + S - a_p^k} \geq \frac{\left(\sqrt{S - a_1^k} + \sqrt{S - a_2^k} + ... + \sqrt{S - a_n^k}\right)^2}{(a_1 + a_2... + a_n) + (n - 1)S}$$

it suffices to prove that

$$\left(\sqrt{S - a_1^k} + \sqrt{S - a_2^k} + ... + \sqrt{S - a_n^k}\right)^2 \geq (n - 1)(a_1 + a_2 + ... + a_n) + (n - 1)^2 S$$

but

$$\left(\sqrt{S - a_1^k} + \sqrt{S - a_2^k} + ... + \sqrt{S - a_n^k}\right)^2$$

$$= \sum (S - a_p^k) + \sum_{i \neq j} \sqrt{(S - a_i^k)(S - a_j^k)}$$

$$= (n - 1)S + \sum_{i \neq j} \sqrt{(S - a_i^k)(S - a_j^k)}$$

but from Cauchy Shwarz inequality

$$\sqrt{(S - a_i^k)(S - a_j^k)} \geq S - a_i^k - a_j^k + a_i^{\frac{k}{2}} a_j^{\frac{k}{2}}$$

$$\Rightarrow \sum_{i \neq j} \sqrt{(S - a_i^k)(S - a_j^k)} \geq (n - 1)^2 S - (n - 1)S + \sum_{i \neq j} a_i^{\frac{k}{2}} a_j^{\frac{k}{2}}$$

Hence We need prove that

$$\sum_{i \neq j} a_i^{\frac{k}{2}} a_j^{\frac{k}{2}} \geq (n - 1) \sum a_i$$

or

$$\sum_{i\neq j} a_i^{\frac{k}{2}} a_j^{\frac{k}{2}} \geq (n-1)\sum (a_1a_2...a_n)^{\frac{k-1}{n}} a_1$$

which is just Muirhead inequality

The **Solution** is completed,equality occurs when $a_1 = a_2 = ... = a_n = 1$.

111.

Let $x_1, x_2, \ldots, x_n$ be positive real numbers with sum 1. Find the integer part of:

$$E = x_1 + \frac{x_2}{\sqrt{1-x_1^2}} + \frac{x_3}{\sqrt{1-(x_1+x_2)^2}} + \cdots + \frac{x_n}{\sqrt{1-(x_1+x_2+\cdots+x_{n-1})^2}}$$

Solution:

Because $\sqrt{1-(x_1+x_2+\cdots+x_i)} \leq 1$ holds for every i,

We have that $E \geq x_1 + x_2 + \cdots + x_n = 1$.

Now take $a_1 = 0$ and $a_i = \arccos(x_1 + x_2 + x_3 + \cdots + x_{i-1})$.

its equivalent to $\cos(a_i) = x_1 + x_2 + x_3 + \cdots + x_{i-1}$.

Which implies $x_i = \cos(a_{i+1}) - \cos(a_i)$ So the expression transforms in:

$$E = \frac{\cos(a_2)-\cos(a_1)}{\sin(a_1)} + \frac{\cos(a_3)-\cos(a_2)}{\sin(a_2)} + \ldots + \frac{\cos(a_{n+1})-\cos(a_n)}{\sin(a_n)}$$

We have:

$$\frac{\cos(a_{i+1})-\cos(a_i)}{\sin(a_i)} = \frac{2\sin(\frac{a_{i+1}+a_i}{2})\cdot\sin(\frac{a_i-a_{i+1}}{2})}{\sin(a_i)}$$

Because $a_{i+1} < a_i$ We have

$$\frac{2\sin(\frac{a_{i+1}+a_i}{2})\cdot\sin(\frac{a_i-a_{i+1}}{2})}{\sin(a_i)} < 2\sin(\frac{a_i-a_{i+1}}{2})$$

Applying and adding this relation for $i = 1, 2, \ldots n$ We have:

$$E < \sin(a_1-a_2)+\sin(a_2-a_3)+\cdots+\sin(a_n-a_{n+1}) < a_1-a_2+a_2-a_3+\cdots-a_{n+1} = a_1-a_{n+1} = \frac{\pi}{2} < 2$$

The last equality is right because $\sin(x) < x$.

Because $1 < E < 2$ We have $[E] = 1$.

112.For $a, b, c > 0$, such that $abc = 1$, prove that the following inequalities hold

$$\frac{a^2}{b(a^2+2c)} + \frac{b^2}{c(b^2+2a)} + \frac{c^2}{a(c^2+2b)} \geq 1.$$

Solution:

We replace a, b, c by

$$\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$$

respectively yields that The inequality is equivalent to

$$\sum \frac{a^3c}{a^3b+2b^3c} \geq 1$$

but due to Cauchy-Shwarz We have

$$\sum \frac{a^3c}{a^3b+2b^2c} \sum c^3a(a^3b+2b^3c) \geq \left(\sum a^2b^2\right)^2$$

We have to prove that

$$\left(\sum a^2b^2\right)^2 \geq 3\sum a^4c^3b = 3\sum (ab)^3(bc)$$

which is just Vasc inequality

The **Solution** is completed , equality occurs when $a=b=c=1$

113.let a,b,c. prove that

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{1}{8}.\frac{(a+b)(b+c)(c+a)}{a^3+b^3+c^3} \geq \frac{4}{3}$$

We rewrite this into

$$\frac{(a+b+c)^3}{(a+b+c)(ab+bc+ca)}+\frac{(a+b)(b+c)(c+a)}{8(a^3+b^3+c^3)} \geq \frac{10}{3}.$$

Note that We also have, from AM-GM that

$$(a+b+c)(ab+bc+ca) \leq \frac{9}{8}(a+b)(b+c)(c+a).$$

So We have to prove that

$$\frac{8(a+b+c)^3}{9(a+b)(b+c)(c+a)}+\frac{(a+b)(b+c)(c+a)}{8(a^3+b^3+c^3)} \geq \frac{10}{3};$$

Which rewrites into

$$\frac{8(a^3+b^3+c^3)}{9(a+b)(b+c)(c+a)}\frac{(a+b)(b+c)(c+a)}{8(a^3+b^3+c^3)} \geq \frac{2}{3};$$

Which is just a direct application of the AM-GM inequality and thus perfectly true.

114.

Let a,b,cbe positive real numbers satisfying

$$\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}=3.$$

Prove that

$$\sqrt[5]{a^5+5}+\sqrt[5]{b^5+5}+\sqrt[5]{c^5+5} \leq \sqrt[5]{5a^5+1}+\sqrt[5]{5b^5+1}+\sqrt[5]{5c^5+1}.$$

Solution:

Using Holder's inequality, We have

$$(\sum \sqrt[5]{a^5+5})^5 \leq \sum a^4(\sum \frac{a^5+5}{a^4}) = (\sum a)^4(\sum a+5\sum \frac{1}{a}) \leq (\sum a)^4(\sum a+15)$$

because

$$\sum \frac{1}{a} \leq 3\sqrt[5]{\frac{\sum \frac{1}{a^5}}{3}}=3.$$

115.

Let a, b, c, d be nonnegative real numbers, no three of which are zero. Prove that

$$\sqrt{\frac{a}{b+2c+d}}+\sqrt{\frac{b}{c+2d+a}}+\sqrt{\frac{c}{d+2a+b}}+\sqrt{\frac{d}{a+2b+c}} \geq \sqrt{2}.$$

Solution:
1... We have, using the AM-GM inequality that

$$\sum_{cyc}\sqrt{\frac{a}{b+2c+d}}=\sum_{cyc}\frac{2\sqrt{2}a}{2\sqrt{2a(b+2c+d)}}\geq\sum_{cyc}\frac{2\sqrt{2}a}{2a+b+2c+d}.$$

Now, from the Cauchy-Schwarz inequality We obtain

$$\sqrt{2}\cdot\sum_{cyc}\frac{2a}{2a+b+2c+d}\geq\sqrt{2}\cdot\frac{2(a+b+c+d)^2}{2\sum a^2+2\sum_{sym}ab}$$

Therefore it suffices to show that

$$2(a+b+c+d)^2\geq\sum_{cyc}a^2+2\sum_{sym}ab;$$

Which leads to $\sum_{sym}ab\geq 0$; which is obvious.

Hence We are done. Equality holds iff $a=b; c=d=0$ and its cyclic permutations. $\square$ 2...

$$\begin{aligned}\sum\sqrt{\frac{a}{b+2c+d}}&\geq\sum\sqrt{\frac{a}{2b+2c+2d}}\\&=\sum\frac{\sqrt{2}a}{\sqrt{2a(2b+2c+2d)}}\\&\geq\sum\frac{2\sqrt{2}a}{2a+2b+2c+2d}\\&=\sum\frac{\sqrt{2}a}{a+b+c+d}=\sqrt{2}\end{aligned}$$

We complete the **Solution**
116.
Let x, y, z be non negative real numbers. Prove that

$$(x+yz)(y+zx)(z+xy)\left(xyz+(x+y+z)^2\right)\geq(x+y)^2(y+z)^2(z+x)^2.$$

Solution:
1..Notice that $(x+yz)(y+zx)=z(x+y)^2+xy(1-z)^2$ and

$$(z+xy)[xyz+(x+y+z)^2]=z(x+y+z+xy)^2+xy(x+y)^2.$$

Thus, using the Cauchy-Schwarz inequality, We get

$$(x+yz)(y+zx)(z+xy)[xyz+(x+y+z)^2]=[z(x+y)^2+xy(1-z)^2][z(x+y+z+xy)^2+xy(x+y)^2]\geq$$
$$[z(x+y)(x+y+z+xy)+xy(1-z)(x+y)]^2=(x+y)^2(y+z)^2(z+x)^2.$$

The **Solution** is completed.
2...
Notice that

$$(x+yz)(y+zx)(z+xy)=xyz(x+y+z-1)^2+(xy+yz+zx-xyz)^2.$$

By Cauchy Schwarz inequality, We have

$$\begin{aligned}
&(x+yz)(y+zx)(z+xy)\left(xyz+(x+y+z)^2\right)\\
=&\left(xyz(x+y+z-1)^2+(xy+yz+zx-xyz)^2\right)\left(xyz+(x+y+z)^2\right)\\
\geq&\left(xyz(x+y+z-1)+(xy+yz+zx-xyz)(x+y+z)\right)^2\\
=&(x+y)^2(y+z)^2(z+x)^2.
\end{aligned}$$

117.

Solution:

This inequality is equivalent to

$$\left(\sum\sqrt{a^2+bc}\right)^2\geq\left(\sum a+\left(\sqrt{2}-1\right)\sum\sqrt{ab}\right)^2,$$

or

$$\begin{aligned}
&\sum\left(a^2+bc\right)+2\sum\sqrt{\left(a^2+bc\right)\left(b^2+ca\right)}\geq(a+b+c)^2\\
&+\left(2\sqrt{2}-2\right)(a+b+c)\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+\left(3-2\sqrt{2}\right)\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)^2,
\end{aligned}$$

or

$$\begin{aligned}
&2\sum\sqrt{\left(a^2+bc\right)\left(b^2+ca\right)}\geq\left(2\sqrt{2}-2\right)\sum\left(\sqrt{a^3b}+\sqrt{ab^3}\right)\\
&+\left(4-2\sqrt{2}\right)\sum ab+\left(4-2\sqrt{2}\right)\sum\sqrt{a^2bc}.
\end{aligned}$$

By Cauchy Schwarz inequality, We have

$$\begin{aligned}
&\sum\sqrt{\left(a^2+bc\right)\left(b^2+ca\right)}\\
=&\sum\sqrt{\left(a^2+bc\right)\left(ca+b^2\right)}\\
\geq&\sum\left(\sqrt{ca^3}+\sqrt{b^3c}\right),
\end{aligned}$$

it suffices to prove that

$$2\sum\left(\sqrt{ca^3}+\sqrt{b^3c}\right)\geq\left(2\sqrt{2}-2\right)\sum\left(\sqrt{a^3b}+\sqrt{ab^3}\right)+\left(4-2\sqrt{2}\right)\sum ab+\left(4-2\sqrt{2}\right)\sum\sqrt{a^2bc},$$

or

$$\left(4-2\sqrt{2}\right)\sum\left(\sqrt{a^3b}+\sqrt{ab^3}\right)\geq\left(4-2\sqrt{2}\right)\sum ab+\left(4-2\sqrt{2}\right)\sum\sqrt{a^2bc},$$

which is true because

$$\sum\left(\sqrt{a^3b}+\sqrt{ab^3}\right)\geq2\sum ab\geq\sum2\sqrt{a^2bc}.$$

118.

Let $a,b,c\geq0$ such that $a^2+b^2+c^2=1$,prove that

$$\frac{1-ab}{7-3ac}+\frac{1-bc}{7-3ba}+\frac{1-ca}{7-3cb}\geq\frac{1}{3}$$

Solution.

First, We will show that

$$\frac{1}{7-3ab}+\frac{1}{7-3bc}+\frac{1}{7-3ca}\leq\frac{1}{2}.$$

Using the Cauchy-Schwarz inequality, We have

$$\frac{1}{7-3ab}=\frac{1}{3(1-ab)+4}\leq\frac{1}{9}\left[\frac{1}{3(1-ab)}+1\right].$$

it follows that

$$\sum\frac{1}{7-3ab}\leq\frac{1}{27}\sum\frac{1}{1-ab}+\frac{1}{3},$$

and thus, it is enough to show that

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\leq\frac{9}{2},$$

which is Vasc's inequality.
Now, We write the original inequality as

$$\frac{3-3ab}{7-3ac}+\frac{3-3bc}{7-3ba}+\frac{3-3ca}{7-3cb}\geq 1,$$

or

$$\frac{7-3ab}{7-3ac}+\frac{7-3bc}{7-3ba}+\frac{7-3ca}{7-3cb}\geq 1+4\left(\frac{1}{7-3ab}+\frac{1}{7-3bc}+\frac{1}{7-3ca}\right).$$

Since

$$4\left(\frac{1}{7-3ab}+\frac{1}{7-3bc}+\frac{1}{7-3ca}\right)\leq 2,$$

it is enough to show that

$$\frac{7-3ab}{7-3ac}+\frac{7-3bc}{7-3ba}+\frac{7-3ca}{7-3cb}\geq 3,$$

which is true according to the AM-GM inequality.
119.
Let $a,b,c\geq 0$, such that $a+b+c>0$ and $b+c\geq 2a$.
For $x,y,z>0$, such that $xyz=1$, prove that the following inequality holds

$$\frac{1}{a+x^2(by+cz)}+\frac{1}{a+y^2(bz+cx)}+\frac{1}{a+z^2(bx+cy)}\geq\frac{3}{a+b+c}$$

Solution:
Setting $u=\frac{1}{x}$, $v=\frac{1}{y}$ and $w=\frac{1}{z}$ and using the condition $uvw=1$, the inequality can be rewritten as

$$\sum\frac{u}{au+cv+bw}=\sum\frac{u^2}{au^2+cuv+bwu}\geqslant\frac{3}{a+b+c}.$$

Applying Cauchy, it suffices to prove

$$\frac{(u+v+w)^2}{a\sum u^2+(b+c)\sum uv}\geqslant\frac{3}{a+b+c}$$
$$\Longleftrightarrow\frac{1}{2}\cdot(b+c-2a)\left(\sum(x-y)^2\right)\geqslant 0,$$

which is obvious due to the condition for a,b,c.
120.
Let

$$a_1\geq a_2\geq\ldots\geq a_n\geq 0, b_1\geq b_2\geq\ldots\geq b_n\geq 0$$

$$\sum_{i=1}^{n} a_i = 1 = \sum_{i=1}^{n} b_i$$

Find the maxmium of $\sum_{i=1}^{n}(a_i - b_i)^2$

Solution:

Without loss of generality, assume that $a_1 \geq b_1$.

Notice that for $a \geq x \geq 0$, b, $y \geq 0$, We have

$$(a-x)^2 + (b-y)^2 - (a+b-x)^2 - y^2 = -2b(a-x+y) \leq 0.$$

According to this inequality, We have

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 \leq (a_1 + a_2 - b_1)^2 + b_2^2,$$
$$(a_1 + a_2 - b_1)^2 + (a_3 - b_3)^2 \leq (a_1 + a_2 + a_3 - b_1)^2 + b_3^2,$$
$$(a_1 + a_2 + \cdots + a_{n-1} - b_1)^2 + (a_n - b_n)^2 \leq (a_1 + a_2 + \cdots + a_n - b_1)^2 + b_n^2.$$

Adding these inequalities, We get

$$\sum_{i=1}^{n}(a_i - b_i)^2 \leq (1-b_1)^2 + b_2^2 + b_3^2 + \cdots + b_n^2$$
$$\leq (1-b_1)^2 + b_1(b_2 + b_3 + \cdots + b_n) =$$
$$(1-b_1)^2 + b_1(1-b_1) = 1 - b_1 \leq 1 - \frac{1}{n}.$$

Equality holds for example when $a_1 = 1, a_2 = a_3 = \cdots = a_n = 0$ and $b_1 = b_2 = \cdots = b_n = \frac{1}{n}$.

121. Let a, b, c be positive real numbers such that

$$a^2 + b^2 + c^2 = \frac{1}{3}.$$

Prove that

$$\frac{1}{a^2 - bc + 1} + \frac{1}{b^2 - ca + 1} + \frac{1}{c^2 - ab + 1} \leq 3.$$

Solution:

This inequality is equivalent to

$$\sum \frac{a^2 - bc}{a^2 - bc + 1} \geq 0,$$

WLOG: $a \geq b \geq c$, We ha

$$\left(a^2 - bc + 1\right)(b+c) - \left(b^2 - ca + 1\right)(c+a)$$
$$= (a-b)\left(ab + 2c(a+b) + c^2 - 1\right)$$
$$= (a-b)\left(ab + 2ca + 2cb + c^2 - 3a^2 - 3b^2 - 3c^2\right)$$
$$= -(a-b)\left(\left(2a^2 + 2b^2 - ab\right) + (c-a)^2 + (c-b)^2\right) \leq 0,$$

similarly, We can **Solution** that

$$\left(b^2 - ca + 1\right)(c+a) \leq \left(c^2 - ab + 1\right)(a+b)$$
$$\Rightarrow \frac{1}{\left(a^2 - bc + 1\right)(b+c)} \geq \frac{1}{\left(b^2 - ca + 1\right)(c+a)} \geq \frac{1}{\left(c^2 - ab + 1\right)(a+b)}$$

And consider

$$
\begin{aligned}
&\left(a^2-bc\right)(b+c)-\left(b^2-ca\right)(c+a)\\
&=(a-b)\left(ab+2c(a+b)+c^2\right)\geq 0,
\end{aligned}
$$

similarly, We have

$$\left(b^2-ca\right)(c+a)\geq\left(c^2-ab\right)(a+b).$$

By Chebyshev's inequality, We have

$$
\begin{aligned}
&\sum\frac{a^2-bc}{a^2-bc+1}\\
&=\sum\frac{\left(a^2-bc\right)(b+c)}{\left(a^2-bc+1\right)(b+c)}\\
&\geq\frac{1}{3}\left(\sum\left(a^2-bc\right)(b+c)\right)\left(\sum\frac{1}{\left(a^2-bc+1\right)(b+c)}\right)=0
\end{aligned}
$$

122.
Let a,b,c be non negative real numbers. Prove that

$$\sqrt{5a^2+4bc}+\sqrt{5b^2+4ca}+\sqrt{5c^+4ab}\geq\sqrt{3(a^2+b^2+c^2)}+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})$$

Solution:
Case: $ab+bc+ca=0$ is trivial.
Now consider for the case $ab+bc+ca>0$.

$$
\begin{aligned}
&\sqrt{5a^2+4bc}-2\sqrt{bc}=\frac{5a^2}{\sqrt{5a^2+4bc}+2\sqrt{bc}}\\
&\geq\frac{5a^2}{\sqrt{3+2}\sqrt{\frac{5a^2+4bc}{3}+2bc}}=\frac{a^2}{\sqrt{\frac{a^2+2bc}{3}}}\\
&\geq\frac{a^2}{\sqrt{\frac{a^2+b^2+c^2}{3}}}\Rightarrow\sum\left(\sqrt{5a^2+4bc}-2\sqrt{bc}\right)\\
&\geq\sum\frac{a^2}{\sqrt{\frac{a^2+b^2+c^2}{3}}}=\sqrt{3\left(a^2+b^2+c^2\right)}.
\end{aligned}
$$

123. Let a,b,c be non negative real numbers. Prove that

$$\sqrt{5a^2+4bc}+\sqrt{5b^2+4ca}+\sqrt{5c^2+4ab}\geq\sqrt{3\left(a^2+b^2+c^2\right)}+2\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right).$$

Solution:

$$
\begin{aligned}
&<=>\sum(\sqrt{5a^2+4bc}-2\sqrt{bc})\geq\sqrt{3(a^2+b^2+c^2)}\\
&<=>\frac{5a^2}{\sqrt{5a^2+4bc}+2\sqrt{bc}}\geq\sqrt{3(a^2+b^2+c^2)}
\end{aligned}
$$

By cauchy-scharzt,We have

$$\sum\frac{5a^2}{\sqrt{5a^2+4bc}+2\sqrt{bc}}\geq\frac{5(a^2+b^2+c^2)^2}{\sum(a^2\sqrt{5a^2+4bc}+2\sqrt{bc})}$$

We have,by cauchy-scharzt:

$$\sum a^2\sqrt{5a^2+4bc} \le \sqrt{(a^2+b^2+c^2)[5(a^4+b^4+c^4)+4abc(a+b+c)]}$$

and

$$\sqrt{3abc}(\sqrt{a^3}+\sqrt{b^3}+\sqrt{c^3}) \le \sqrt{3abc}\sqrt{(a^2+b^2+c^2)(a+b+c)} \le \sqrt{a^2+b^2+c^2}(ab+bc+ca)$$

Finally,We only need to prove that:

$$5(a^2+b^2+c^2) \ge \sqrt{15(a^4+b^4+c^4)+12abc(a+b+c)}+2(ab+bc+ca)$$
$$<=> [5(a^2+b^2+c^2)-2(ab+bc+ca)]^2 \ge 15(a^4+b^4+c^4)+12abc(a+b+c)$$
$$<=> 10(a^4+b^4+c^4)+54(a^2b^2+b^2c^2+c^2a^2)-24abc(a+b+c)-10\sum ab(a^2+b^2) \ge 0$$

From

$$24(a^2b^2+b^2c^2+c^2a^2) \ge 24abc(a+b+c)$$

Finally,We only need to prove that

$$10(a^4+b^4+c^4)+30(a^2b^2+b^2c^2+c^2a^2) \ge 10\sum ab(a^2+b^2)$$
$$<=> a^4+b^4+c^4+3(a^2b^2+b^2c^2+c^2a^2) \ge 2\sum ab(a^2+b^2)$$

$<=>$ $(a-b)^2+(b-c)^2+(c-a)^2 \ge 0$,which is obivious true. Our **Solution** are completed,equality occur if and if only $a=b=c$. 124. Let a,b,c be nonnegative real numbers satisfying $a+b+c=\sqrt{5}$.
Prove that:

$$(a^2-b^2)(b^2-c^2)(c^2-a^2) \le \sqrt{5}$$

Solution:
For this one, We can assume WLOG that $c \ge b \ge a$; so that We have

$$P=(a^2-b^2)(b^2-c^2)(c^2-a^2)=(c^2-b^2)(c^2-a^2)(b^2-a^2) \le b^2c^2(c^2-b^2).$$

Also note that $\sqrt{5}=a+b+c \ge b+c$ since $a \ge 0$.
Now, using the AM-GM inequality We have

$$(c+b)\cdot\left(\left(\frac{\sqrt{5}}{2}-1\right)\cdot c\right)^2\cdot\left(\left(\frac{\sqrt{5}}{2}+1\right)b\right)^2\cdot(c-b)$$
$$\le (c+b)\left\{\frac{\sqrt{5}(b+c)}{5}\right\}^5 \le \sqrt{5};$$

So that We get

$$P \le \sqrt{5}.$$

And hence We are done.

Equality holds if and only if $(a,b,c)=\left(\frac{\sqrt{5}}{2}+1;\frac{\sqrt{5}}{2}-1;0\right)$ and all its cyclic permutations. $\square$
125.

Let x, y, z be positive real numbers such that $xyz = 1$. Prove that

$$\frac{x^3+1}{\sqrt{x^4+y+z}} + \frac{y^3+1}{\sqrt{y^4+z+x}} + \frac{z^3+1}{\sqrt{z^4+x+y}} \geq 2\sqrt{xy+yz+zx}.$$

Solution:
Using the AM-GM inequality, We have

$$2\sqrt{(x^4+y+z)(xy+yz+zx)} = 2\sqrt{[x^4+xyz(y+z)](xy+yz+zx)} = 2\sqrt{(x^3+y^2z+yz^2)(x^2y+x^2z+xyz)}$$
$$\leq (x^3+y^2z+yz^2)+(x^2y+x^2z+xyz) = (x+y+z)(x^2+yz) = \frac{(x+y+z)(x^3+1)}{x}.$$

it follows that

$$\frac{x^3+1}{\sqrt{x^4+y+z}} \geq \frac{2x\sqrt{xy+yz+zx}}{x+y+z}.$$

Adding this and it analogous inequalities, the result follows.
126.
Let a, b, c be positive real numbers such that

$$16(a+b+c) \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$\sum \frac{1}{\left[a+b+\sqrt{2(a+c)}\right]^3} \leq \frac{8}{9}.$$

Solution:
Using the AM-GM inequality, We have:

$$a+b+\sqrt{2(c+a)} = a+b+\sqrt{\frac{c+a}{2}}+\sqrt{\frac{c+a}{2}} \geq 3\sqrt[3]{\frac{(a+b)(c+a)}{2}}.$$

So that:

$$\sum \frac{1}{\left[a+b+\sqrt{2(c+a)}\right]^3} \leq \sum \frac{2}{27(a+b)(c+a)}.$$

Thus, it's enough to check that:

$$\sum \frac{1}{3(a+b)(c+a)} \leq 4 \iff 6(a+b)(b+c)(c+a) \geq a+b+c,$$

which is true since $9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca)$ and:

$$16abc(a+b+c) \geq ab+bc+ca \Rightarrow \frac{16(ab+bc+ca)^2}{3} \geq ab+bc+ca \iff ab+bc+ca \geq \frac{3}{16}.$$

The **Solution** is completed. Equality holds if and only if $a=b=c=\frac{1}{4}$.
127.
if a_1, a_2, $\dots$, a_n are positive real numbers such that $a_1+a_2+\cdots+a_n = n$, then

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}+\frac{2n\sqrt{n-1}}{a_1^2+a_2^2+\cdots+a_n^2} \geq n+2\sqrt{n-1}.*$$

Solution:
With $n=2$ We have a beautiful ineq :$a_1+a_2=2$

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{4}{a_1^2+a_2^2} \geq 4$$

$$<=> \frac{(a_1-a_2)^4}{a_1a_2(a_1+a_2)(a_1^2+a_2^2)} \geq 0$$

We have

$$* <=> \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-\frac{n^2}{a_1+a_2+\cdots+a_n} \geq 2\sqrt{n-1}(1-\frac{n}{a_1^2+a_2^2+\cdots+a_n^2})$$

$$<=> \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-\frac{n^2}{a_1+a_2+\cdots+a_n} \geq \frac{2\sqrt{n-1}}{n}(\frac{n(a_1^2+a_2^2+\cdots+a_n^2)-(a_1+a_2+\cdots+a_n)^2}{a_1^2+a_2^2+\cdots+a_n^2})$$

$$<=> \sum \frac{(a_We-a_j)^2}{a_ia_j(a_1+a_2+\cdots+a_n} \geq \frac{2\sqrt{n-1}}{n}(\frac{\sum(a_We-a_j)^2}{a_1^2+a_2^2+\cdots+a_n^2})$$

$$<=> \sum[\frac{1}{a_ia_j(a_1+a_2+\cdots+a_n)}-\frac{2\sqrt{n-1}}{n(a_1^2+a_2^2+\cdots+a_n^2)}](a_We-a_j)^2 \geq 0$$

$$<=> \sum[\frac{1}{a_ia_j}-\frac{2\sqrt{n-1}}{(a_1^2+a_2^2+\cdots+a_n^2)}]\frac{(a_We-a_j)^2}{n} \geq 0$$

Q.E.D

128.
Let $a, b, c > 0$.Prove that:

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} \geq \frac{1}{a^2+b^2+c^2}+\frac{2}{ab+bc+ca}$$

Solution:
This inequality is equivalent to

$$\sum \frac{a^2+b^2+c^2}{a^2+2bc} \geq \frac{2\left(a^2+b^2+c^2\right)}{ab+bc+ca}+1,$$

or

$$\sum \frac{(b-c)^2}{a^2+2bc} \geq \frac{\sum(b-c)^2}{ab+bc+ca}.$$

Case $a=b=c$ is trivial. Now, Consider for the case

$$(a-b)^2+(b-c)^2+(c-a)^2>0.$$

By Cauchy Schwarz inequality, We have

$$\left(\sum\left(a^2+2bc\right)(b-c)^2\right)\left(\sum \frac{(b-c)^2}{a^2+2bc}\right) \geq \left(\sum(b-c)^2\right)^2.$$

And because

$$\sum\left(a^2+2bc\right)(b-c)^2=(ab+bc+ca)\left(\sum(b-c)^2\right).$$

Therefore

$$\sum \frac{(b-c)^2}{a^2+2bc} \geq \frac{\left(\sum(b-c)^2\right)^2}{(ab+bc+ca)\left(\sum(b-c)^2\right)}=\frac{\sum(b-c)^2}{ab+bc+ca}$$

as require

129.
Let a, b, c be non negative real numbers such that $ab+bc+ca>0$. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}+\frac{1}{ab+bc+ca} \geq \frac{12}{(a+b+c)^2}.$$

Solution:

$$<=> \sum \frac{(a+b)(a+c)}{2a^2+bc} + \sum \frac{a^2+bc}{2a^2+bc} - 2 \geq \frac{12(ab+bc+ca)}{(a+b+c)^2}$$

From $\sum \frac{2a^2+2bc}{2a^2+bc} - 3 = \frac{bc}{2a^2+bc} \geq 1$,We get

$$\sum \frac{a^2+bc}{2a^2+bc} - 2 \geq 0$$

Now,We will prove the stronger

$$\sum \frac{(a+b)(a+c)}{2a^2+bc} \geq \frac{12(ab+bc+ca)}{(a+b+c)^2}$$

From cauchy-scharzt,We have

$$\sum \frac{(a+b)(a+c)}{2a^2+bc} = (a+b)(b+c)(c+a)(\sum \frac{1}{(2a^2+bc)(b+c)} \geq \frac{3(a+b)(b+c)(c+a)}{ab(a+b)+bc(b+c)+ca(c+a)}$$

Finally,We only need to prove that

$$\frac{(a+b)(b+c)(c+a)}{ab(a+b)+bc(b+c)+ca(c+a)} \geq \frac{4(ab+bc+ca)}{(a+b+c)^2}$$

$$<=> \frac{(a+b+c)^2}{ab+bc+ca} \geq \frac{4[ab(a+b)+bc(b+c)+ca(c+a)}{(a+b)(b+c)(c+a)} = 4 - \frac{8abc}{(a+b)(b+c)(c+a)}$$

$$<=> \frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{8abc}{(a+b)(b+c)(c+a)} \geq 2$$

which is old problem. Our,**Solution** are completed,equality occur if and if only a=b=c,a=b,c=0 or any cyclic permution.

130.

Let$x, y, z > 0$and $x + y + z = 1$. Prove that:

$$\frac{x^2y^2}{z+xy} + \frac{y^2z^2}{x+yz} + \frac{z^2x^2}{y+zx} \geq \frac{xy+yz+zx}{4}$$

Solution:

The inequality can be written:

$$\frac{x^2y^2}{(x+z)(y+z)} + \frac{y^2z^2}{(y+x)(z+x)} + \frac{z^2x^2}{(z+y)(x+y)} \geq \frac{xy+yz+zx}{4}.$$

Since

$$\frac{x^2y^2}{(x+z)(y+z)} = xy - \frac{xyz(x+y+z)}{(x+z)(y+z)},$$

the above inequality is equivalent to

$$\frac{3}{4}(xy+yz+zx) \geq xyz(x+y+z)\sum \frac{1}{(x+z)(y+z)},$$

or

$$\frac{3(xy+yz+zx)}{xyz(x+y+z)} \geq \frac{8(x+y+z)}{(x+y)(y+z)(z+x)},$$

which is true because

$$\frac{3(xy+yz+zx)}{xyz(x+y+z)} \geq \frac{9}{xy+yz+zx},$$

and

$$\frac{9}{xy+yz+zx} \geq \frac{8(x+y+z)}{(x+y)(y+z)(z+x)}.$$

131. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$12\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 4(a^3+b^3+c^3)+21.$$

Solution:
WLOG $a \leq b \leq c$
Denote

$$f(a,b,c) = 12\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - 4(a^3+b^3+c^3)+21$$

$$f(\frac{a+b}{2},\frac{a+b}{2},c) = 12\left(\frac{4}{a+b}+\frac{1}{c}\right) - 4c^3+(a+b)^3+21$$

Then $f(a,b,c)-f(\frac{a+b}{2},\frac{a+b}{2},c)$

$$=12\left(\frac{1}{a}+\frac{1}{b}-\frac{4}{a+b}+\frac{1}{c}\right)+(a+b)^3-4(a^3+b^3)=3(a-b)^2(\frac{4}{ab(a+b)}-(a+b))$$

By AM-GM ineq, notice that:

$$ab(a+b)^2 \leq \frac{(a+b)^4}{4} \leq 4$$

So $\frac{4}{ab(a+b)}-(a+b) \geq 0$ and

$$f(a,b,c) \geq f(\frac{a+b}{2},\frac{a+b}{2},c)$$

Now We prove:

$$f(\frac{a+b}{2},\frac{a+b}{2},c) \geq 0 \Leftrightarrow 12\left(\frac{4}{3-c}+\frac{1}{c}\right)-4c^3+(3-c)^3+21 \geq 0$$

$$\Leftrightarrow \frac{36(c+1)}{c(3-c)} \geq 3c^3+9c^2-27c+48$$

$$\Leftrightarrow 12(c+1) \geq c(3-c)(c^3+3c^2-9c+16)$$

$$\Leftrightarrow c^5-18c^3+43c^2-36c+12 \geq 0$$

$$\Leftrightarrow (c-2)^2(c-1)(c^2+3c-3) \geq 0$$

which is true because $c \geq 1$ We complete the **Solution**, equality hold when $(a,b,c) = (2,\frac{1}{2},\frac{1}{2})$
132.
Let a,b,c,d be positive real numbers such three of them are side-lengths of a triangle. Prove that

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}} \leq 4.$$

Solution:
Without loss of generality, assume that $d=\min\{a,\ b,\ c\}$. Using the known result

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}} \leq 3-\sqrt{\frac{2c}{c+a}},(1)$$

We see that it suffices to prove that

$$\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}} \leq 1+\sqrt{\frac{2c}{c+a}}.(2)$$

This inequality can be written as follows

$$\frac{2c}{c+d}+\frac{2d}{d+a}+2\sqrt{\frac{4cd}{(c+d)(d+a)}}\le 1+\frac{2c}{c+a}+2\sqrt{\frac{2c}{c+a}},(3)$$

$$2\sqrt{2c}\left[\frac{1}{\sqrt{a+c}}-\sqrt{\frac{2d}{(c+d)(d+a)}}\right]\ge\frac{2c}{c+d}+\frac{2d}{d+a}-\frac{2c}{c+a}-1,(4)$$

$$2\sqrt{2c}\left[\frac{1}{a+c}-\frac{2d}{(c+d)(d+a)}\right]\ge\frac{(a-d)(c-d)(a-c)}{(c+d)(d+a)(a+c)}\left[\frac{1}{\sqrt{a+c}}+\sqrt{\frac{2d}{(c+d)(d+a)}}\right],(5)$$

$$2\sqrt{2c}\ge(a-c)\left[\frac{1}{\sqrt{a+c}}+\sqrt{\frac{2d}{(c+d)(d+a)}}\right].(6)$$

if $a\le c$, then (6) is clearly true. Let us consider now the case $a\ge c$. Since

$$\sqrt{\frac{2d}{(c+d)(d+a)}}\le\frac{1}{\sqrt{a+c}},(7)$$

it suffices to prove that

$$\sqrt{2c(a+c)}\ge a-c.(8)$$

From the given hypothesis, We have $2c\ge c+d\ge a$. Therefore,

$$\sqrt{2c(a+c)}\ge\sqrt{a(a+c)}>a>a-c.(9)$$

The **Solution** is completed.

133.

Let a, b, c be positive real numbers. Prove that

$$A+B\ge 2C,$$

where

$$A=\frac{a}{b}+\frac{b}{c}+\frac{c}{a},B=\frac{a+c}{b+c}+\frac{b+a}{c+a}+\frac{c+b}{a+b},C=\frac{b+c}{a+c}+\frac{c+a}{b+a}+\frac{a+b}{c+b}.$$

Solution:

Let's denote

$$x=\frac{a}{b},y=\frac{b}{c},z=\frac{c}{a}$$

Then $A=x+y+z$

$$\frac{a+c}{b+c}=\frac{a}{b+c}+\frac{c}{b+c}=\frac{1}{\frac{b}{a}+\frac{c}{a}}+\frac{1}{\frac{b}{c}+1}=\frac{x}{xz+1}+\frac{1}{y+1}$$

Acting analogously, We will obtain

$$B=\frac{x}{xz+1}+\frac{y}{yx+1}+\frac{z}{zy+1}+\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}$$

$$C=\frac{x}{xz+1}+\frac{y}{yx+1}+\frac{z}{zy+1}+\frac{x}{x+1}+\frac{y}{y+1}+\frac{z}{z+1}$$

Thus, We have to prove that for $xyz=1$

$$x+y+z+\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\ge\frac{x}{xz+1}+\frac{y}{yx+1}+\frac{z}{zy+1}+\frac{2x}{x+1}+\frac{2y}{y+1}+\frac{2z}{z+1}$$

$$x+y+z \geq \frac{x}{xz+1}+\frac{y}{yx+1}+\frac{z}{zy+1}+\frac{2x-1}{x+1}+\frac{2y-1}{y+1}+\frac{2z-1}{z+1}$$

Note that $\frac{x}{xz+1}=\frac{xy}{y+1}$, therefore it remains:

$$x+y+z \geq \frac{2x-1+xz}{x+1}+\frac{2y-1+xy}{y+1}+\frac{2z-1+yz}{z+1}$$

Now,

$$z=\frac{zx+z}{x+1}$$

so We have:

$$0 \geq \frac{2x-1-z}{x+1}+\frac{2y-1-x}{y+1}+\frac{2z-1-y}{z+1}$$

or

$$\frac{z-2x+1}{x+1}+\frac{x-2y+1}{y+1}+\frac{y-2z+1}{z+1} \geq 0$$

This can be rewritten as

$$\frac{z+3}{x+1}+\frac{x+3}{y+1}+\frac{y+3}{z+1} \geq 6$$

From the principle of arranged sets,

$$\frac{z+3}{x+1}+\frac{x+3}{y+1}+\frac{y+3}{z+1} \geq \frac{x+3}{x+1}+\frac{y+3}{y+1}+\frac{z+3}{z+1}$$

So, for $xyz=1$ it suffices to prove that

$$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1} \geq 1,5$$

After returning back to a,b,c it turns into a ill-known inequality

$$\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a} \geq 1,5$$

that completes the **Solution** .

134.

1)Let a,b,c be sides of triangle.Prove that:

$$\frac{a}{2a^2+bc}+\frac{b}{2b^2+ca}+\frac{c}{2c^2+ab} \geq \frac{a+b+c}{a^2+b^2+c^2}$$

Solution:

From:

$$\frac{a}{2a^2+bc}-\frac{1}{a+b+c}=\frac{-(a-b)(a-c)}{(2a^2+bc)(a+b+c)}$$

and

$$\frac{a+b+c}{a^2+b^2+c^2}-\frac{3}{a+b+c}=\frac{-2\sum(a-b)(a-c)}{(a^2+b^2+c^2)(a+b+c)}$$

We can write this inequality in the form

$$X(a-b)(a-c)+Y(b-a)(b-c)+Z(c-a)(c-b) \geq 0$$

with

$$X=\frac{2}{a^2+b^2+c^2}-\frac{1}{2a^2+bc}$$

and smilar with Y,Z.
Bacause a,b,c be sides of triagle,We have $3a^2 \geq a^2 \geq (b-c)^2$

$$X = \frac{3a^2-(b-c)^2}{(a^2+b^2+c^2)(2a^2+bc} \geq 0$$

So $X, Y, Z \geq 0$
assume that $a \geq b \geq c$
We have

$$X - Y = \frac{(a-b)(2a+2b-c)}{(2a^2+bc)(2b^2+ac)} \geq 0$$

$$X(a-b)(a-c)+Y(b-a)(b-c)+z(c-a)(c-b) \geq (a-b)[X(a-c)-Y(b-c)] \geq X(a-b)^2 \geq 0$$

Our **Solution** are completed,equality occur if and if only a=b=c,a=b,c=0 or any cyclic permution.
2) Let a, b, c, d be real number. Prove that:

$$|\frac{a-b}{a+b} + \frac{c-d}{c+d} + \frac{ad+bc}{ac-bd}| \geq \sqrt{3}$$

Proof:

$$|\frac{a-b}{a+b} + \frac{c-d}{c+d} + \frac{ad+bc}{ac-bd}| \geq \sqrt{3} \Leftrightarrow (\frac{a-b}{a+b} + \frac{c-d}{c+d} + \frac{ad+bc}{ac-bd})^2 \geq 3$$

. We have

$$(x+y+z)^2 \geq 3|xy+yz+zx|$$

Hence

$$(\frac{a-b}{a+b} + \frac{c-d}{c+d} + \frac{ad+bc}{ac-bd})^2 \geq 3|\frac{a-b}{a+b}.\frac{c-d}{c+d} + \frac{c-d}{c+d}.\frac{ad+bc}{ac-bd} + \frac{ad+bc}{ac-bd}.\frac{a-b}{a+b}$$

$$= 3|\frac{(a-b)(c-d)(ac-bd)+(ad+bc)[(a+b)(c-d)+(a-b)(c+d)]}{(a+b)(c+d)(ac-cd)}|$$

$$= 3|\frac{(a-b)(c-d)(ac-bd)+2(ad+bc)(ac-bd)}{(a+b)(c+d)(ac-cd)}| = 3|\frac{(a+b)(c+d)(ac-bd)}{(a+b)(c+d)(ac-cd)}| = 3$$

. Q.E.D
.
Enquality holds when

$$\begin{cases} ad = bc \\ a^2c+b^2d = 3ab(c+d) \\ ac^2+bd^2 = 3cd(a+b) \end{cases} \Leftrightarrow \begin{cases} ad = bc \\ a+b = c+d \end{cases}$$

135.
Let a, b, c be real variables, such that $a+b+c=3$. Prove that the following inequality holds

$$\frac{a^2}{b^2-2b+3} + \frac{b^2}{c^2-2c+3} + \frac{c^2}{a^2-2a+3} \geq \frac{3}{2}$$

Solution:
Using the Cauchy-Schwarz inequality, We have

$$(b-1)^2 = [(a-1)+(c-1)]^2 \leq 2[(a-1)^2+(c-1)^2].(1)$$

it follows that

$$(b-1)^2 \le \frac{2}{3}[(a-1)^2+(b-1)^2+(c-1)^2] = \frac{2}{3}(a^2+b^2+c^2-3), (2)$$

which implies

$$b^2-2b+3 = (b-1)^2+2 \le \frac{2}{3}(a^2+b^2+c^2).(3)$$

From (3), We obtain

$$\frac{a^2}{b^2-2b+3} \ge \frac{3}{2}\cdot\frac{a^2}{a^2+b^2+c^2}, (4)$$

and by adding this to its analogous inequalities, We get the desired result.

136.Let a, b, c be nonnegative real numbers such that $ab+bc+ca>0$. Prove that

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \ge \frac{2}{ab+bc+ca}.$$

Solution:

The inequality is equivalent to

$$\sum \frac{ab+bc+ca}{2a^2+bc} \ge 2, (1)$$

or

$$\sum \frac{a(b+c)}{2a^2+bc} + \sum \frac{bc}{bc+2a^2} \ge 2.(2)$$

Using the Cauchy-Schwarz inequality, We have

$$\sum \frac{bc}{bc+2a^2} \ge \frac{\left(\sum bc\right)^2}{\sum bc(bc+2a^2)} = 1.(3)$$

Therefore, it suffices to prove that

$$\sum \frac{a(b+c)}{2a^2+bc} \ge 1.(4)$$

Since

$$\frac{a(b+c)}{2a^2+bc} \ge \frac{a(b+c)}{2(a^2+bc)},$$

it is enough to check that

$$\sum \frac{a(b+c)}{a^2+bc} \ge 2, (5)$$

which is a known result.

137.

Let $a, b, c > 0$.prove that:

$$\sum \sqrt[3]{\frac{a^2+bc}{b^2+c^2}} \ge 9.\frac{\sqrt[3]{abc}}{(a+b+c)}$$

Solution :

This ineq is equivalent to:

$$\sum \frac{a^2+bc}{\sqrt[3]{abc(a^2+bc)^2(b^2+c^2)}} \ge \frac{9}{(a+b+c)^3}$$

By AM-GM ineq , We have:

$$\frac{a^2+bc}{\sqrt[3]{abc(a^2+bc)^2(b^2+c^2)}} = \frac{a^2+bc}{\sqrt[3]{(a^2+bc)c(a^2+bc)b(b^2+c^2)a}} \ge \frac{3(a^2+bc)}{\sum\limits_{sym} a^2b}$$

Similarly, this ineq is true if We prove that:

$$\frac{3(a^2+b^2+c^2+ab+bc+ca)}{\sum\limits_{sym} a^2b} \geq \frac{9}{(a+b+c)^3}$$

$$\Leftrightarrow a^3+b^3+c^3+3abc \geq \sum_{sym} a^2b$$

(which is true by Schur ineq)
equality holds when $a=b=c$.
138.
Let a,b,c be the positive real numbers. Prove that:

$$1+\frac{ab^2+bc^2+ca^2}{(ab+bc+ca)(a+b+c)} \geq \frac{4.\sqrt[3]{(a^2+ab+bc)(b^2+bc+ca)(c^2+ca+ab)}}{(a+b+c)^2}$$

Solution:
And this is our **Solution** for it:
Multiplying both sides of the above inequality with $(a+b+c)^2$ it's equivalent to prove that

$$(a+b+c)^2+\frac{(a+b+c)(ab^2+bc^2+ca^2)}{ab+bc+ca}$$
$$\geq 4.\sqrt[3]{(a^2+ab+bc)(b^2+bc+ca)(c^2+ca+ab)}$$

We have

$$(a+b+c)^2+\frac{(a+b+c)(ab^2+bc^2+ca^2)}{ab+bc+ca}=\sum\frac{(a^2+ab+bc)(c+a)(c+b)}{ab+bc+ca}$$

By using AM-GM inequality We get

$$\sum\frac{(a^2+ab+bc)(c+a)(c+b)}{ab+bc+ca} \geq 3.\frac{\sqrt[3]{(a^2+ab+bc)(b^2+bc+ca)(c^2+ca+ab)[(a+b)(b+c)(c+a)]^2}}{ab+bc+ca}$$

Since it's suffices to show that

$$\sqrt{3}.\sqrt[3]{(a+b)(b+c)(c+a)} \geq 2.\sqrt{ab+bc+ca}$$

which is clearly true by AM-GM inequality again.
The **Solution** is completed. Equality holds for $a=b=c$ 139. For any $a,b,c>0$. Prove that:

$$3(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1) \geq \frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}+\frac{b+c}{a+b}+\frac{c+a}{c+a}+\frac{a+b}{c+a}$$

Solution: The ineq is equivalent to :

$$3(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}) \geq 6+2\sum\frac{a}{b+c}$$

But :

$$3(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}) \geq 3\frac{(a+b+c)^2}{ab+ac+bc}=6+3\frac{a^2+b^2+c^2}{ab+ac+bc}$$

it's enough to prove that :

$$\frac{a^2+b^2+c^2}{ab+ac+bc} \geq \frac{2}{3}\sum\frac{a}{b+c} \Leftrightarrow a^2+b^2+c^2 \geq \frac{2}{3}(a^2+b^2+c^2+abc(\sum\frac{1}{a+b}))$$

$$\Leftrightarrow a^2+b^2+c^2 \geq 2(a.\frac{bc}{b+c}+b.\frac{ac}{a+c}+c.\frac{ab}{a+b})$$

But :

$$2(a.\frac{bc}{b+c}+b.\frac{ac}{a+c}+c.\frac{ab}{a+b})\leq 2.\sum(a.\frac{b+c}{4})=ab+ac+bc\leq a^2+b^2+c^2$$

Q.E.D

140. Let a,b,c be positive real numbers. Prove that:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\sqrt{3(a^2+b^2+c^2)}$$

Solution:
The sharper inequalities hold:

$$(a)\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca};$$

This ineq is equivalent to :

$$(ab+bc+ca)(\sum_{cyc}\frac{a^2}{b})\geq(a+b+c)(a^2+b^2+c^2)$$

$$\Leftrightarrow a^3+b^3+c^3+a^2c+c^2b+b^2a+\sum_{cyc}\frac{a^3c}{b}\geq a^3+b^3+c^3+\sum_{sym}a^2b$$

$$\Leftrightarrow\frac{a^3c}{b}+\frac{b^3a}{c}+\frac{c^3b}{a}\geq ac^2+cb^2+ba^2$$

By AM-Gm ineq, We have:

$$\frac{a^3c}{b}+\frac{b^3a}{c}\geq 2a^2b$$

$$\frac{b^3a}{c}+\frac{c^3b}{a}\geq 2b^2c$$

$$\frac{a^3c}{b}+\frac{c^3b}{a}\geq 2c^2a$$

Adding up these ineqs, We have done, equality hold when $a=b=c$

140. Let a,b,c be the side-lengths of a triangle inequality. Prove that

$$\frac{a}{\sqrt{a+2b+2c}}+\frac{b}{\sqrt{b+2c+2a}}+\frac{c}{\sqrt{c+2a+2b}}\leq\sqrt{\frac{2}{3}(a+b+c)}.$$

Solution:
Using the Cauchy-Schwarz inequality We have

$$\left(\sum_{cyc}\frac{a}{\sqrt{a+2b+2c}}\right)^2\leq(a+b+c)\left(\sum_{cyc}\frac{a}{a+2b+2c}\right);$$

So that it suffices to check that

$$\sum_{cyc}\frac{a}{a+2b+2c}\leq\frac{2}{3};$$

Which, on the substitution $a=x+y;b=y+z;c=z+x$ is equivalent to with

$$\sum_{cyc}\frac{3(x+y)}{3(x+y)+4z}\leq 2;$$

Which, again equivalents

$$\sum_{cyc} \frac{4z}{4z+3(x+y)} \geq 1.$$

Note that We have; using the Cauchy-Schwarz inequality that

$$\sum_{cyc} \frac{4z}{4z+3(x+y)} \geq \frac{4(x+y+z)^2}{4(x^2+y^2+z^2)+6(xy+yz+zx)};$$

So that it is enough to check that

$$4(x+y+z)^2 \geq 4(x^2+y^2+z^2)+6(xy+yz+zx);$$

Which, on turn, can be rewritten into the following obvious inequality:

$$2(xy+yz+zx) \geq 0.$$

Hence proved. Equality occurs if and only if $x=y=0$ i.e. $a=b; c=0$; and its cyclics.□

141.Let a,b,c are positive real numbers.prove that:

$$\sqrt[110]{4ab(a+b)} + \sqrt[110]{4cb(b+c)} + \sqrt[110]{4ca(c+a)} \leq 3\sqrt[110]{(a+b)(b+c)(c+a)}$$

Solution:

This ineq is equivalent to :

$$\sqrt[110]{\frac{4ab}{(c+a)(c+b)}} + \sqrt[110]{\frac{4cb}{(a+c)(a+b)}} + \sqrt[110]{\frac{4ca}{(b+c)(b+a)}} \leq 3$$

By AM-GM ineq , We have:

$$\sqrt[110]{\frac{4ab}{(c+a)(c+b)}} \leq \frac{108+\frac{2a}{a+c}+\frac{2b}{b+c}}{110}$$

Similarly, addding up these ineqs, We have: $LHS \leq 3$

Equality holds when $a=b=c$

144. Let a,b,c be positive real numbers such that $3(a^2+b^2+c^2)+ab+bc+ca=12$. Prove that

$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \leq \frac{3}{\sqrt{2}}.$$

Solution:

Let

$$A=a^2+b^2+c^2, B=ab+bc+ca$$

$$\Rightarrow 2A+B=2\sum a^2+\sum ab \leq \frac{3}{4}\left(3\sum a^2+\sum ab\right)=9.$$

By Cauchy Schwarz inequality, We have

$$\sum \frac{a}{\sqrt{a+b}}$$

$$=\sum \sqrt{a}\sqrt{\frac{a}{a+b}}$$

$$\leq \sqrt{a+b+c}\sqrt{\sum \frac{a}{a+b}}.$$

By Cauchy Schwarz inequality again, We have

$$\sum \frac{b}{a+b} = \sum \frac{b^2}{b(a+b)}$$

$$\geq \frac{(a+b+c)^2}{\sum b(a+b)} = \frac{A+2B}{A+B}$$

$$\Rightarrow \sum \frac{a}{a+b} = 3 - \sum \frac{b}{a+b} \leq 3 - \frac{A+2B}{A+B} = \frac{2A+B}{A+B}$$

hence, it suffices to prove that

$$(a+b+c)\cdot \frac{2A+B}{A+B} \leq \frac{9}{2}$$

Consider

$$(a+b+c)\sqrt{2A+B}$$

$$= \sqrt{(A+2B)(2A+B)} \leq \frac{(A+2B)+(2A+B)}{2}$$

$$= \frac{3}{2}(A+B) \Rightarrow (a+b+c)\cdot \frac{2A+B}{A+B} \leq \frac{3}{2}\sqrt{2A+B} \leq \frac{9}{2}$$

as require

145.

Let a,b,c be positive real numbers such that $3(a^2+b^2+c^2)+ab+bc+ca=12$. Prove that

$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \leq \frac{3}{\sqrt{2}}.$$

Solution:

By AM-GM ineq , easy to see that: $3 \leq a^2+b^2+c^2 \leq 4$

By Cauchy-Schwarz ineq, We have:

$$LHS^2 = (\sum \frac{a\sqrt{a+c}}{\sqrt{(a+b)(a+c)}}) \leq (a^2+b^2+c^2+ab+bc+ca)(\sum \frac{a}{(a+b)(a+c)})$$

Using the familiar ineq : $9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca)$, We have:

$$\sum \frac{a}{(a+b)(a+c)} = \frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)} \leq \frac{9}{4(a+b+c)}$$

And We need to prove that:

$$\frac{9(a^2+b^2+c^2+ab+bc+ca)}{4(a+b+c)} \leq \frac{9}{2} \Leftrightarrow \frac{6-(a^2+b^2+c^2)}{\sqrt{24-5(a^2+b^2+c^2)}} \leq 1$$

$$\Leftrightarrow (6-(a^2+b^2+c^2))^2 \leq 24-5(a^2+b^2+c^2)$$

$$\Leftrightarrow (3-(a^2+b^2+c^2))(4-(a^2+b^2+c^2)) \leq 0$$

which is true

We are done , equality holds when $a=b=c=1$

147.

Let $a,b,c>0$ such that $abc \geq 1$. Prove that

$$\frac{1}{a^4+b^3+c^2} + \frac{1}{b^4+c^3+a^2} + \frac{1}{c^4+a^3+b^2} \leq 1$$

Solution:
1..By Cauchy Schwarz inequality, We have

$$\sum \frac{1}{a^4+b^3+c^2} = \sum \frac{1+b+c^2}{(a^4+b^3+c^2)(1+b+c^2)}$$
$$\leq \sum \frac{1+b+c^2}{(a^2+b^2+c^2)^2} = \frac{3+(a+b+c)+\left(a^2+b^2+c^2\right)}{(a^2+b^2+c^2)^2}$$

By AM-GM inequality, We have

$$a+b+c \geq 3\sqrt[3]{abc} \geq 3,$$
$$a^2+b^2+c^2 \geq 3\sqrt[3]{a^2b^2c^2} \geq 3.$$
$$\Rightarrow \frac{3+(a+b+c)+\left(a^2+b^2+c^2\right)}{(a^2+b^2+c^2)^2}$$
$$\leq \frac{\frac{1}{3}(a+b+c)^2+\frac{1}{3}(a+b+c)^2+\left(a^2+b^2+c^2\right)}{(a^2+b^2+c^2)^2}$$
$$\leq \frac{3}{a^2+b^2+c^2} \leq 1.$$

2..
So if $abc > 1$ We can let a replaced by a_1 which is smaller than a and the LHS will be bigger so We just need to prove the situation which $abc = 1$
Thus according to the CS inequality

$$(a^4+b^3+c^2)(1+b+c^2) \geqslant (\sum a^2)$$
$$\frac{(1+b+c^2)}{\sum a^2} \geqslant \frac{1}{a^4+b^3+c^2}$$

and the other two are simillar so We have

$$LHS \leqslant \frac{\sum a^2+\sum a+3}{(\sum a^2)^2}$$

then We just need to prove

$$\sum a^4+2\sum a^2b^2 \geqslant \sum(a^2+a+1)$$

for

$$a^4+1 \geqslant 2a^2$$

We just need to prove $\sum a^2+2\sum a^2b^2 \geqslant a+b+c+6$ then $a^2+1 \geqslant 2a$
so what We want to show is

$$\sum a+2\sum a^2b^2 \geqslant 9$$

which is a straight applied of AM-GM and $abc = 1$
148.
Let $a, b, c \geq 0$ such that $a+b+c = a^2+b^2+c^2$. Prove that

$$\left(\frac{a+b+c}{2}\right)^4+1 \geq \sqrt{a^2b+b^2c+c^2a}+\sqrt{ab^2+bc^2+ca^2}$$

Solution:
Using AM-Gm and CS We have:

$$\left(\frac{a+b+c}{2}\right)^4+1\geq 2\left(\frac{a+b+c}{2}\right)^2$$

$$\left(\sqrt{a^2b+b^2c+c^2a}+\sqrt{ab^2+bc^2+ca^2}\right)^2\leq 2(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2)\leq 2(a+b+c)(ab+bc+ca)$$

We need to prove that:

$$4\left(\frac{a+b+c}{2}\right)^4\geq 2(a+b+c)(ab+bc+ca)$$

Setting

$$a+b+c=a^2+b^2+c^2=t\geq 0\Rightarrow ab+bc+ca=\frac{t^2-t}{2}$$

our inequality equivalent to:

$$t^3\geq 4(t^2-t)\Leftrightarrow t(t-2)^2\geq 0$$

149.Let a;b;c>0.Prove that:

$$\sqrt{\frac{(a+b)}{c}}+\sqrt{\frac{(b+c)}{a}}+\sqrt{\frac{(a+c)}{b}}\geq 2(\sqrt{\frac{c}{a+b}}+\sqrt{\frac{a}{c+b}}+\sqrt{\frac{b}{a+c}})$$

Solution:

$$\sum_{cyc}\sqrt{\frac{y+z}{x}}\geq 2\sum_{cyc}\sqrt{\frac{x}{y+z}}\Leftrightarrow\sum_{cyc}\frac{y+z-2x}{\sqrt{x(y+z)}}\geq 0\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{z-x}{\sqrt{x(y+z)}}-\frac{x-y}{\sqrt{x(y+z)}}\right)\geq 0\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{x-y}{\sqrt{y(x+z)}}-\frac{x-y}{\sqrt{x(y+z)}}\right)\geq 0\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\frac{z(x-y)^2}{(\sqrt{x(y+z)}+\sqrt{y(x+z)})\sqrt{xy(x+z)(y+z)}}\geq 0.$$

150.For positive reals such that $a^2b^2+b^2c^2+c^2a^2=3$, prove that

$$\sqrt{\frac{a+bc^2}{2}}+\sqrt{\frac{b+ca^2}{2}}+\sqrt{\frac{c+ab^2}{2}}\leq\frac{3}{abc}.$$

Solution:
1....

$$\left(\sum_{cyc}\sqrt{\frac{a+bc^2}{2}}\right)^2\leq\frac{3}{2}\cdot\sum_{cyc}(a+a^2c).$$

Let $a=\frac{1}{x}$,$b=\frac{1}{y}$ and $c=\frac{1}{z}$.
Hence,

$$a+b+c\leq\frac{3}{a^2b^2c^2}\Leftrightarrow(a+b+c)^4\leq\frac{(a^2b^2+a^2c^2+b^2c^2)^7}{27a^8b^8c^8}\Leftrightarrow$$

$$\Leftrightarrow(x^2+y^2+z^2)^7\geq 27(xy+xz+yz)^4x^2y^2z^2,$$

which true and

$$a^2c+b^2a+c^2b \le \frac{3}{a^2b^2c^2} \Leftrightarrow (a^2c+b^2a+c^2b)^4 \le \frac{(a^2b^2+a^2c^2+b^2c^2)^9}{243a^8b^8c^8} \Leftrightarrow$$

$$\Leftrightarrow (x^2+y^2+z^2)^9 \ge 243(x^2y+y^2z+z^2x)^4x^2y^2z^2,$$

which true since,

$$(x^2+y^2+z^2)^3 \ge 3(x^2y+y^2z+z^2x)^2$$

2.......

$$\sum\sqrt{\frac{a+bc^2}{2}} = \sum\sqrt{\frac{ab+b^2c^2}{2b}} \le \sum\frac{1}{2}(\frac{1}{b}+\frac{ab+b^2c^2}{2})$$

$$= \frac{1}{4}(\sum\frac{2}{a}+\sum ab+\sum a^2b^2)$$

Since

$$ab+bc+ca \le \sqrt{3(a^2b^2+b^2c^2+c^2a^2)} = 3, (LHS) \le \frac{1}{2}(3+\sum\frac{1}{a}$$

$$= \frac{3}{2}+\frac{ab+bc+ca}{2abc} \le \frac{3}{2}+\frac{3}{2abc} \le \frac{3}{abc}$$

since

$$3 = a^2b^2+b^2c^2+c^2a^2 \ge 3(abc)^{\frac{4}{3}}$$

Q.E.D

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151.

if a,b,c are nonnegative real numbers such that $ab+bc+ca>0$, then

$$\frac{b+c}{b^2+bc+c^2}+\frac{c+a}{c^2+ca+a^2}+\frac{a+b}{a^2+ab+b^2} \ge \frac{4(a+b+c)}{a^2+b^2+c^2+ab+bc+ca}.$$

Solution:

$$\sum\frac{(a+b)(a+b+c)}{a^2+ab+b^2} = 3+\sum\frac{ab+bc+ca}{a^2+ab+b^2}$$

Or it suffices to prove that

$$\sum\frac{ab+bc+ca}{a^2+ab+b^2} \ge 1+\frac{4(ab+bc+ca)}{a^2+b^2+c^2+ab+bc+ca}$$

or

$$\frac{4(a^2+b^2+c^2)}{a^2+b^2+c^2+ab+bc+ca}-2 \ge \sum\left(1-\frac{ab+bc+ca}{a^2+ab+b^2}\right)$$

$$\sum\frac{(a-b)^2}{a^2+b^2+c^2+ab+bc+ca} \ge \sum\frac{a(a-c)+b(b-c)}{a^2+ab+b^2}$$

$$\sum\frac{(a-b)^2}{a^2+b^2+c^2+ab+bc+ca} \ge \sum\frac{a(a-c)}{a^2+ab+b^2}-\frac{c(a-c)}{b^2+bc+c^2}$$

$$\sum\frac{(a-c)^2}{a^2+b^2+c^2+ab+bc+ca} \ge \sum\frac{(a-c)^2(b^2-ac)}{(a^2+ab+b^2)(b^2+bc+c^2)}$$

but it follows from the inequality

$$(a^2+ab+b^2)(b^2+bc+c^2) \ge (b^2-ac)(a^2+b^2+c^2+ab+bc+ca)$$

which is obviously true after expanding.

The **Solution** is completed , equality occurs when $a=b=c$

152.
Let a,b,c be positive real numbers, prove that:

$$\frac{(a^2+bc)(b+c)}{b^2+bc+c^2}+\frac{(b^2+ca)(c+a)}{c^2+ca+a^2}+\frac{(c^2+ab)(a+b)}{a^2+ab+b^2}\geq a+b+c+\frac{9abc}{(a+b+c)^2}$$

Solution:
Firstly, We will show that

$$\frac{bc(a^2+bc)}{b+c}+\frac{ca(b^2+ca)}{c+a}+\frac{ab(c^2+ab)}{a+b}\geq 3abc.(1)$$

Since

$$\frac{bc(a^2+bc)}{b+c}=\frac{bc(a+b)(a+c)}{b+c}-abc,$$

this inequality can be written as

$$\sum\frac{bc(a+b)(a+c)}{b+c}\geq 6abc.$$

Now, using the AM-GM inequality, We get

$$\sum\frac{bc(a+b)(a+c)}{b+c}\geq 3\sqrt[3]{a^2b^2c^2(a+b)(b+c)(c+a)}\geq 6abc.$$

Turning back to our problem. Using the AM-GM inequality, We have

$$\begin{aligned}\frac{1}{b^2+bc+c^2}&=\frac{4(ab+bc+ca)}{4(ab+bc+ca)(b^2+bc+c^2)}\\&\geq\frac{4(ab+bc+ca)}{[(ab+bc+ca)+(b^2+bc+c^2)]^2}\\&=\frac{4(ab+bc+ca)}{(b+c)^2(a+b+c)^2}.\end{aligned}$$

Thus, it suffices to prove that

$$4\sum\frac{(a^2+bc)(ab+bc+ca)}{b+c}\geq(a+b+c)^3+9abc,$$

or

$$4\sum a(a^2+bc)+4\sum\frac{bc(a^2+bc)}{b+c}\geq(a+b+c)^3+9abc.$$

Now, using (1), We see that it suffices to show that

$$4(a^3+b^3+c^3+3abc)+12abc\geq(a+b+c)^3+9abc,$$

or

$$4(a^3+b^3+c^3)+15abc\geq(a+b+c)^3,$$

which is Schur's inequality
153.
Let a,b,cbe positive real number. Prove that:

$$\frac{(a^2+bc)(b+c)}{b^2+bc+c^2}+\frac{(b^2+ca)(c+a)}{c^2+ca+a^2}+\frac{(c^2+ab)(a+b)}{a^2+ab+b^2}\geq\frac{4}{3}(a+b+c).$$

Solution:
Notice that this inequality can be written as

$$\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}+\frac{1}{a^2+ab+b^2}\geq$$
$$\geq\frac{7}{3}\cdot\frac{a+b+c}{a^2(b+c)+b^2(c+a)+c^2(a+b)+abc},$$

which is stronger than the known result of Vasc

$$\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}+\frac{1}{a^2+ab+b^2}\geq\frac{9}{(a+b+c)^2},$$

because

$$7(a+b+c)^3\geq 27[a^2(b+c)+b^2(c+a)+c^2(a+b)+abc].$$

it is true by Schur's inequality.
154.
Prove that:

$$\sum\frac{1}{\sqrt{1+(n^2-1)a_i}}\geq 1$$

if $a_1a_2...a_n=1$and $a_i>0$,
Solution:
Let $a_i=e^{x_i}$. Hence, $\sum\limits_{i=1}^{n}x_i=0$ and We need to prove that $\sum\limits_{i=1}^{n}f(x_i)\geq 0$,
where

$$f(x)=\frac{1}{\sqrt{1+(n^2-1)e^x}}$$

But

$$f''(x)=\frac{(n^2-1)e^x((n^2-1)e^x-2)}{4\left(1+(n^2-1)e^x\right)^{2.5}}$$

Thus, f is a convex function on

$$\left[\ln\frac{2}{n^2-1},+\infty\right)$$

and f is a concave function on $\left(-\infty,\ln\frac{2}{n^2-1}\right]$.
if at least two numbers from $\{x_i\}$ are smaller than $\ln\frac{2}{n^2-1}$ so

$$\sum_{i=1}^{n}f(x_i)>\frac{2}{\sqrt{1+(n^2-1)e^{\ln\frac{2}{n^2-1}}}}=\frac{2}{\sqrt{3}}>1$$

if all

$$x_i\in\left[\ln\frac{2}{n^2-1},+\infty\right)$$

so by Jensen

$$\sum_{i=1}^{n}f(x_i)\geq nf\left(\frac{\sum\limits_{i=1}^{n}x_i}{n}\right)=nf(0)=1$$

The last case:
Exactly one number from $\{x_i\}$ smaller than $\ln\frac{2}{n^2-1}$. Lets mark him as t.
Hence, a sum of the remain numbers equal $-t$.

Hence, by Jensen again for f on $\left[\ln\frac{2}{n^2-1},+\infty\right)$ enough to prove that:

$$\frac{1}{\sqrt{1+(n^2-1)e^t}}+\frac{n-1}{\sqrt{1+(n^2-1)e^{-\frac{t}{n-1}}}}\geq 1$$

Let $e^t=x^{n-1}$, where

$$0<x<\left(\frac{2}{n^2-1}\right)^{\frac{1}{n-1}}<1$$

and

$$g(x)=\frac{1}{\sqrt{1+(n^2-1)x^{n-1}}}+\frac{n-1}{\sqrt{1+\frac{n^2-1}{x}}}$$

We obtain:

$$g'(x)=\frac{(n-1)^2(n+1)(x^n-1)\left((n^2-1)^3x^{2n-3}-x^n-3(n^2-1)x^{n-1}-1\right)}{2x^2\left((1+(n^2-1)x^{n-1})(1+\frac{n^2-1}{x})\right)^{1.5}\left(x^n\left(1+\frac{n^2-1}{x}\right)^{1.5}+(1+(n^2-1)x^{n-1})^{1.5}\right)}$$

We see that g' has only two positive roots: 1 and x_1, where $0<x_1<1$ (all this for $n\geq 3$, but for $n=2$ the inequality is obvious).
id est, $x_{max}=x_1$, $g(1)=1$ and since $\lim\limits_{x\to 0^+}g(x)=1$,
155.
Suppose that a,b,c be three positive real numbers such that $a+b+c=3$. Prove that :

$$\frac{1}{2+a^2+b^2}+\frac{1}{2+b^2+c^2}+\frac{1}{2+c^2+a^2}\leq\frac{3}{4}$$

Solution:
1)Write the inequality as

$$\sum\frac{a^2+b^2}{a^2+b^2+2}\geq\frac{3}{2},$$

or

$$\sum\frac{(a+b)^2}{(a+b)^2+\frac{2(a+b)^2}{a^2+b^2}}\geq\frac{3}{2}.$$

Applying the Cauchy Schwarz inequality, We have

$$LHS\geq\frac{4(a+b+c)^2}{\sum(a+b)^2+2\sum\frac{(a+b)^2}{a^2+b^2}}=\frac{36}{\sum(a+b)^2+2\sum\frac{(a+b)^2}{a^2+b^2}}.$$

it suffices to prove that

$$\sum(a+b)^2+2\sum\frac{(a+b)^2}{a^2+b^2}\leq 24.$$

Because

$$12-\sum(a+b)^2=\frac{4}{3}(a+b+c)^2-\sum(a+b)^2=-\frac{1}{3}\sum(a-b)^2,$$

and

$$12-2\sum\frac{(a+b)^2}{a^2+b^2}=2\sum\frac{(a-b)^2}{a^2+b^2},$$

this inequality is equivalent to

$$\sum(a-b)^2\left(\frac{6}{a^2+b^2}-1\right)\geq 0.$$

Under the assumption that $a\geq b\geq c$, We see that this inequality is obviously true if $a^2+b^2\leq 6$.

Let us consider now the case $a^2+b^2 \geq 6$, in this case We have

$$\frac{1}{a^2+b^2+2} \leq \frac{1}{8},$$

and

$$\frac{1}{a^2+c^2+2}+\frac{1}{b^2+c^2+2} \leq \frac{1}{a^2+2}+\frac{1}{b^2+2} \leq \frac{1}{8-b^2}+\frac{1}{b^2+2} \leq \frac{1}{2}+\frac{1}{2},$$

(because $0 \leq b^2 \leq 6$)

Hence

$$\sum \frac{1}{a^2+b^2+2} \leq \frac{1}{8}+\frac{1}{8}+\frac{1}{2}=\frac{3}{4}.$$

2)Suppose that a,b,c be three positive real numbers such that $a+b+c=3$. Prove that :

$$\frac{1}{2+a^2+b^2}+\frac{1}{2+b^2+c^2}+\frac{1}{2+c^2+a^2} \leq \frac{3}{4}$$

Solution:(Messigem)

Lemma :With $x,y,z>0$,We have:

$$(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2 \geq 2\sqrt{3(xy+yz+zx)}+4(x+y+z)$$

Put $m=\sqrt{b+c}+\sqrt{c+a}-\sqrt{a+b}; n=\sqrt{c+a}+\sqrt{a+b}-\sqrt{b+c}$

$$p=\sqrt{a+b}+\sqrt{b+c}-\sqrt{c+a}$$

then $m,n,p>0$.This inequality become:

$$mn+np+pm \geq \sqrt{3mnp(m+n+p)}$$

Which is obvious true. Equality holds if and only if $x=y=z$

-Now,com back the problem.

it is equivalent to:

$$\frac{a^2+b^2}{2+a^2+b^2}+\frac{b^2+c^2}{2+b^2+c^2}+\frac{c^2+a^2}{2+c^2+a^2} \geq \frac{3}{2}$$

By Cauchy-Schwarz inequality and Lemma,We have;

$$LHS \geq \frac{(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2})^2}{6+2(a^2+b^2+c^2)}$$

$$=\frac{2\sum a^2+2\sum \sqrt{(a^2+b^2)(a^2+c^2)}}{6+2(a^2+b^2+c^2)} \geq \frac{2\sum a^2+2(\sum a^2+\sum ab)}{6+2(a^2+b^2+c^2)}=\frac{3}{2}$$

Then,We have Q.E.D

Equality holds if and only if $a=b=c=1$

3)Let

$$f(a,b,c)=\frac{1}{2+a^2+b^2}+\frac{1}{2+b^2+c^2}+\frac{1}{2+c^2+a^2}$$

$$=> f(a,t,t)=\frac{2}{2+a^2+t^2}+\frac{1}{2+2t^2}$$

with $t = \frac{b+c}{2}$
We have

$$f(a,t,t)-f(a,b,c) = (b^2+c^2-\frac{(b+c)^2}{2})(\frac{1}{(b^2+c^2+2)(2+\frac{(b+c)^2}{2})}-\frac{1}{(4+2a^2+b^2+c^2)(4+2a^2+\frac{(b+c)^2}{2})}) \geq 0$$

it's true because $2(b^2+c^2) \geq (b+c)^2$ and

$$4+2a^2+b^2+c^2 \geq b^2+c^2+2, 4+2a^2+\frac{(b+c)^2}{2} \geq 2+\frac{(b+c)^2}{2}$$

And then,We prove that

$$f(a,t,t) \leq \frac{3}{4} \Leftrightarrow (a-1)^2(15a^2-78a+111) \geq 0$$

(which is obvious true)(Q.E.D
) Equality holds if and only if $a = b = c = 1$
156.
Let a, b, c be positive real numbers such that $a+b+c = 3$.
Prove that:

$$1+8abc \geq 9min(a,b,c)$$

Solution:
WLOG let $a = min\{a,b,c\} => a \leq 1$,$b = a+p, c = a+q$,where $3a+p+q = 3$ and $a,p,q \geq 0$, then

$$1+8abc \geq 9min(a,b,c) \iff 1+8a(a+p)(a+q) \geq 9a$$

$$\iff 1+8a^3+8(p+q)a^2+8pqa \geq 9a \iff 1+8a^3+8(3-3a)a^2+8pqa \geq 9a$$

$$\iff 0 \geq 16a^3-24a^2+(9-8pq)a-1$$

since $p, q \geq 0$ and

$$a = \frac{3-p-q}{3} \leq 1$$

We have

$$16a^3-24a^2+(9-8pq)a-1 \leq 16a^3-24a^2+9a-1 = (a-1)(4a-1)^2 \leq 0$$

it is True.
157.Let a, b, c be real positive numbers such that $abc = 1$, prove or disprove that

$$\frac{b+c}{a^3+bc}+\frac{c+a}{b^3+ca}+\frac{a+b}{c^3+ab} \leq \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Solution:
Note that, since $abc = 1$, We have $\sqrt[3]{abc} = 1$.
On the other hand, by the GM-HM inequality, We have

$$\sqrt[3]{abc} \geq \frac{3}{1/a+1/b+1/c}$$

and thus

$$1 \geq \frac{3}{1/a+1/b+1/c}$$

This yields

$$\frac{b+c}{a^3+bc} \leq \frac{b+c}{a^3+bc \cdot \frac{3}{1/a+1/b+1/c}}$$

Similarly,

$$\frac{c+a}{b^3+ca} \le \frac{c+a}{b^3+ca\cdot\frac{3}{1/a+1/b+1/c}}$$

and

$$\frac{a+b}{c^3+ab} \le \frac{a+b}{c^3+ab\cdot\frac{3}{1/a+1/b+1/c}}$$

Now We will show:

Lemma 1. For any three positive numbers a, b, c, We have

$$\frac{b+c}{a^3+bc\cdot\frac{3}{1/a+1/b+1/c}} + \frac{c+a}{b^3+ca\cdot\frac{3}{1/a+1/b+1/c}} + \frac{a+b}{c^3+ab\cdot\frac{3}{1/a+1/b+1/c}}$$
$$\le \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Of course, this Lemma 1 will then yield the problem, since We will have

$$\frac{b+c}{a^3+bc}+\frac{c+a}{b^3+ca}+\frac{a+b}{c^3+ab}$$
$$\le \frac{b+c}{a^3+bc\cdot\frac{3}{1/a+1/b+1/c}} + \frac{c+a}{b^3+ca\cdot\frac{3}{1/a+1/b+1/c}} + \frac{a+b}{c^3+ab\cdot\frac{3}{1/a+1/b+1/c}}$$
$$\le \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Hence, in order to solve the problem, it is enough to prove Lemma 1.

Solution of Lemma 1 .

Denoting

$$x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$$

We have

$$a=\frac{1}{x}, b=\frac{1}{y}, c=\frac{1}{z}$$

Thus, after some algebra,

$$\frac{b+c}{a^3+bc\cdot\frac{3}{1/a+1/b+1/c}} = \frac{x^3\left(y+z\right)\left(x+y+z\right)}{2x^3+\left(x^3+yz^2+zy^2+xyz\right)}$$

By the AM-GM inequality,

$$x^3+yz^2+zy^2+xyz \ge 4\sqrt[4]{x^3\cdot yz^2\cdot zy^2\cdot xyz} = 4\sqrt[4]{x^4y^4z^4} = 4xyz$$

and thus,

$$\frac{b+c}{a^3+bc\cdot\frac{3}{1/a+1/b+1/c}} = \frac{x^3\left(y+z\right)\left(x+y+z\right)}{2x^3+\left(x^3+yz^2+zy^2+xyz\right)}$$
$$\le \frac{x^3\left(y+z\right)\left(x+y+z\right)}{2x^3+4xyz} = \frac{x+y+z}{2}\cdot\frac{x^2\left(y+z\right)}{x^2+2yz}$$

Similarly,

$$\frac{c+a}{b^3+ca\cdot\frac{3}{1/a+1/b+1/c}} = \frac{x+y+z}{2}\cdot\frac{y^2\left(z+x\right)}{y^2+2zx}$$

and

$$\frac{a+b}{c^3+ab\cdot\frac{3}{1/a+1/b+1/c}} = \frac{x+y+z}{2}\cdot\frac{z^2\left(x+y\right)}{z^2+2xy}$$

Now We come to another lemma: Lemma 2.
For any three positive reals x, y, z, the inequality

$$\frac{x^2(y+z)}{x^2+2yz}+\frac{y^2(z+x)}{y^2+2zx}+\frac{z^2(x+y)}{z^2+2xy}\le\frac{2\left(x^2+y^2+z^2\right)}{x+y+z}$$

holds.
From this lemma, We will be able to conclude

$$\frac{x+y+z}{2}\cdot\left(\frac{x^2(y+z)}{x^2+2yz}+\frac{y^2(z+x)}{y^2+2zx}+\frac{z^2(x+y)}{z^2+2xy}\right)\le x^2+y^2+z^2$$

thus, using what We have found before,

$$\begin{aligned}&\frac{b+c}{a^3+bc\cdot\frac{3}{1/a+1/b+1/c}}+\frac{c+a}{b^3+ca\cdot\frac{3}{1/a+1/b+1/c}}+\frac{a+b}{c^3+ab\cdot\frac{3}{1/a+1/b+1/c}}\\ \le&\frac{x+y+z}{2}\cdot\frac{x^2(y+z)}{x^2+2yz}+\frac{x+y+z}{2}\cdot\frac{y^2(z+x)}{y^2+2zx}+\frac{x+y+z}{2}\cdot\frac{z^2(x+y)}{z^2+2xy}\\ &=\frac{x+y+z}{2}\cdot\left(\frac{x^2(y+z)}{x^2+2yz}+\frac{y^2(z+x)}{y^2+2zx}+\frac{z^2(x+y)}{z^2+2xy}\right)\\ &\le x^2+y^2+z^2=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\end{aligned}$$

since

$$x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}$$

thus, Lemma 1 will be proven. Hence, it only remains to prove Lemma 2. **Solution** of Lemma 2.
We note that

$$\frac{x^2(y+z)}{x^2+2yz}=(y+z)-\frac{2(y+z)yz}{x^2+2yz}$$

Similarly,

$$\frac{y^2(z+x)}{y^2+2zx}=(z+x)-\frac{2(z+x)zx}{y^2+2zx}$$

$$\frac{z^2(x+y)}{z^2+2xy}=(x+y)-\frac{2(x+y)xy}{z^2+2xy}$$

Hence,

$$\begin{aligned}&\frac{x^2(y+z)}{x^2+2yz}+\frac{y^2(z+x)}{y^2+2zx}+\frac{z^2(x+y)}{z^2+2xy}\\ =&\left((y+z)-\frac{2(y+z)yz}{x^2+2yz}\right)+\left((z+x)-\frac{2(z+x)zx}{y^2+2zx}\right)+\left((x+y)-\frac{2(x+y)xy}{z^2+2xy}\right)\\ &=2\left((x+y+z)-\left(\frac{(y+z)yz}{x^2+2yz}+\frac{(z+x)zx}{y^2+2zx}+\frac{(x+y)xy}{z^2+2xy}\right)\right)\end{aligned}$$

Therefore, the inequality that We have to prove,

$$\frac{x^2(y+z)}{x^2+2yz}+\frac{y^2(z+x)}{y^2+2zx}+\frac{z^2(x+y)}{z^2+2xy}\le\frac{2\left(x^2+y^2+z^2\right)}{x+y+z}$$

is equivalent to

$$2\left((x+y+z)-\left(\frac{(y+z)yz}{x^2+2yz}+\frac{(z+x)zx}{y^2+2zx}+\frac{(x+y)xy}{z^2+2xy}\right)\right)\le\frac{2\left(x^2+y^2+z^2\right)}{x+y+z}$$

$$=> (x+y+z) - \left(\frac{(y+z)\,yz}{x^2+2yz} + \frac{(z+x)\,zx}{y^2+2zx} + \frac{(x+y)\,xy}{z^2+2xy} \right) \le \frac{x^2+y^2+z^2}{x+y+z}$$

Subtracting both sides from $x+y+z$, We get

$$\frac{(y+z)\,yz}{x^2+2yz} + \frac{(z+x)\,zx}{y^2+2zx} + \frac{(x+y)\,xy}{z^2+2xy} \ge (x+y+z) - \frac{x^2+y^2+z^2}{x+y+z}$$

what simplifies to

$$\frac{(y+z)\,yz}{x^2+2yz} + \frac{(z+x)\,zx}{y^2+2zx} + \frac{(x+y)\,xy}{z^2+2xy} \ge \frac{2\,(yz+zx+xy)}{x+y+z}$$

Now, a rather strange way to rewrite this is to make the fractions on the left hand side more complicated:

$$\frac{((y+z)\,yz)^2}{(x^2+2yz)\,(y+z)\,yz} + \frac{((z+x)\,zx)^2}{(y^2+2zx)\,(z+x)\,zx} + \frac{((x+y)\,xy)^2}{(z^2+2xy)\,(x+y)\,xy} \ge \frac{2\,(yz+zx+xy)}{x+y+z}$$

Now, applying the Cauchy-Schwarz inequality in the Engel form , We have

$$\frac{((y+z)\,yz)^2}{(x^2+2yz)\,(y+z)\,yz} + \frac{((z+x)\,zx)^2}{(y^2+2zx)\,(z+x)\,zx} + \frac{((x+y)\,xy)^2}{(z^2+2xy)\,(x+y)\,xy}$$

$$\ge \frac{((y+z)\,yz + (z+x)\,zx + (x+y)\,xy)^2}{(x^2+2yz)\,(y+z)\,yz + (y^2+2zx)\,(z+x)\,zx + (z^2+2xy)\,(x+y)\,xy}$$

$$= \frac{((y+z)\,yz + (z+x)\,zx + (x+y)\,xy)^2}{2\,(x+y+z)\,(y^2z^2+z^2x^2+x^2y^2)}$$

Now, the last important lemma in our **Solution** will be:
Lemma 3. For any three positive reals x, y, z, We have

$$((y+z)\,yz + (z+x)\,zx + (x+y)\,xy)^2 \ge 4\,(yz+zx+xy)\left(y^2z^2+z^2x^2+x^2y^2\right)$$

in fact, once this Lemma 3 will be shown, We will have

$$\frac{((y+z)\,yz + (z+x)\,zx + (x+y)\,xy)^2}{2\,(x+y+z)\,(y^2z^2+z^2x^2+x^2y^2)} \ge \frac{4\,(yz+zx+xy)\left(y^2z^2+z^2x^2+x^2y^2\right)}{2\,(x+y+z)\,(y^2z^2+z^2x^2+x^2y^2)}$$

$$= \frac{2\,(yz+zx+xy)}{x+y+z}$$

and thus We will have

$$\frac{((y+z)\,yz)^2}{(x^2+2yz)\,(y+z)\,yz} + \frac{((z+x)\,zx)^2}{(y^2+2zx)\,(z+x)\,zx} + \frac{((x+y)\,xy)^2}{(z^2+2xy)\,(x+y)\,xy}$$

$$\ge \frac{((y+z)\,yz + (z+x)\,zx + (x+y)\,xy)^2}{2\,(x+y+z)\,(y^2z^2+z^2x^2+x^2y^2)} \ge \frac{2\,(yz+zx+xy)}{x+y+z}$$

what will finish the **Solution** of Lemma 2. Thus, all We need is to show Lemma 3.
First **Solution** of Lemma 3. We have

$$((y+z)\,yz + (z+x)\,zx + (x+y)\,xy)^2 - 4\,(yz+zx+xy)\left(y^2z^2+z^2x^2+x^2y^2\right)$$

$$= y^2z^2\,(y-z)^2 + z^2x^2\,(z-x)^2 + x^2y^2\,(x-y)^2$$

$$+2xyz\left(x^3+y^3+z^3+3xyz-yz^2-zy^2-zx^2-xz^2-xy^2-yx^2\right)$$

what is ≥ 0 since squares are nonnegative and since

$$x^3+y^3+z^3+3xyz-yz^2-zy^2-zx^2-xz^2-xy^2-yx^2 \geq 0$$

by Schur. Hence,

$$((y+z)\,yz+(z+x)\,zx+(x+y)\,xy)^2 \geq 4\,(yz+zx+xy)\left(y^2z^2+z^2x^2+x^2y^2\right)$$

and Lemma 3 is proven.
Second **Solution** of Lemma 3. We have

$$\begin{aligned}((y+z)\,yz+(z+x)\,zx+(x+y)\,xy)^2 &= ((x\cdot zx+y\cdot xy+z\cdot yz)+(x\cdot xy+y\cdot yz+z\cdot zx))^2\\ &\geq 4\,(x\cdot zx+y\cdot xy+z\cdot yz)\,(x\cdot xy+y\cdot yz+z\cdot zx)\end{aligned}$$

(by the inequality $(u+v)^2 \geq 4uv$ which holds for any two reals u and v). Hence, it remains to prove that

$$(x\cdot zx+y\cdot xy+z\cdot yz)\,(x\cdot xy+y\cdot yz+z\cdot zx) \geq (yz+zx+xy)\left(y^2z^2+z^2x^2+x^2y^2\right)$$

Since this inequality is symmetric in its variables x, y, z, We can WLOG assume that $x \geq y \geq z$.
Denote $A=x$, $B=y$, $C=z$. Then, $A \geq B \geq C$.
Also, denote $X=yz$, $Y=zx$, $Z=xy$. Then, $y \leq x$ yields $yz \leq zx$, so that $X \leq Y$.
Hence, applying Theorem 9 a) from Vornicu-Schur inequality and its variations to the three reals A, B, C and the three nonnegative reals X, Y, Z,
We get

$$(AY+BZ+CX)\,(AZ+BX+CY) \geq (X+Y+Z)\,(XBC+YCA+ZAB)$$

This rewrites as

$$(x\cdot zx+y\cdot xy+z\cdot yz)\,(x\cdot xy+y\cdot yz+z\cdot zx) \geq (yz+zx+xy)\,(yz\cdot yz+zx\cdot zx+xy\cdot xy)$$

Equivalently,

$$(x\cdot zx+y\cdot xy+z\cdot yz)\,(x\cdot xy+y\cdot yz+z\cdot zx) \geq (yz+zx+xy)\left(y^2z^2+z^2x^2+x^2y^2\right)$$

and Lemma 3 is proven.
158.
Find the largest positive real number p such that

$$\frac{a}{(a+1)(b+p)}+\frac{b}{(b+1)(c+p)}+\frac{c}{(c+1)(a+p)} \geq \frac{3}{2(1+p)},$$

for any positive real numbers a,b,c such that $abc=1$.
Solution.
Setting

$$a=t^2 and b=c=\frac{1}{t}, t>0,$$

the inequality becomes

$$\frac{t^3}{(t^2+1)(pt+1)}+\frac{t}{(t+1)(pt+1)}+\frac{1}{(t+1)(t^2+p)} \geq \frac{3}{2(1+p)}.$$

$$Letting t \to \infty, We get \frac{1}{p} \geq \frac{3}{2(1+p)}, or p \leq 2$$

We will show that 2 is that largest value of p such that the desired inequality holds; that is, to prove that

$$\frac{a}{(a+1)(b+2)} + \frac{b}{(b+1)(c+2)} + \frac{c}{(c+1)(a+2)} \geq \frac{1}{2}.$$

Since $a, b, c > 0$ and $abc = 1$, there exist some positive real numbers x,y,z such that

$$a = \frac{x}{y}, b = \frac{z}{x} and c = \frac{y}{z}$$

After making this substitution, the above inequality becomes

$$\frac{x^2}{(x+y)(2x+z)} + \frac{y^2}{(y+z)(2y+x)} + \frac{z^2}{(z+x)(2z+y)} \geq \frac{1}{2}.$$

By the Cauchy-Schwarz inequality, We have

$$\sum \frac{x^2}{(x+y)(2x+z)} \geq \frac{(x+y+z)^2}{\sum (x+y)(2x+z)},$$

and hence, it suffices to show that

$$2(x+y+z)^2 \geq (x+y)(2x+z) + (y+z)(2y+x) + (z+x)(2z+y),$$

which is just an identity.

159.

if $a \geq b \geq c \geq 0 and a+b+c > 0$, then

$$\frac{(a-c)^2}{2(a+c)} \leq a+b+c-3\sqrt[3]{abc} \leq \frac{2(a-c)^2}{a+c}.$$

Solution. (a) We prove that

$$\frac{(a-c)^2}{2(a+c)} \leq a+b+c-3\sqrt[3]{abc}.$$

By the Cauchy-Schwarz inequality, We have

$$\frac{(a-c)^2}{2(a+c)} \leq \frac{(a-c)^2}{\left(\sqrt{a}+\sqrt{c}\right)^2} = \left(\sqrt{a}-\sqrt{c}\right)^2 = a+c-2\sqrt{ac}.$$

Therefore, it suffices to prove that

$$b + 2\sqrt{ac} \geq 3\sqrt[3]{abc},$$

which is true according to the AM-GM inequality. Equality holds if and only if $a = b = c$.

(b) We prove that

$$a+b+c-3\sqrt[3]{abc} \leq \frac{2(a-c)^2}{a+c}.$$

Setting $a = x^3, b = y^3$ and $c = z^3$, this inequality becomes

$$x^3 + y^3 + z^3 - 3xyz \leq \frac{2(x^3-z^3)^2}{x^3+z^3},$$

or equivalently,

$$\frac{1}{2}(x+y+z)[(x-y)^2+(y-z)^2+(z-x)^2] \leq \frac{2(x^3-z^3)^2}{x^3+z^3}.$$

Since $x+y+z \leq 2x+zand(x-y)^2+(y-z)^2 \leq (x-z)^2$ it suffices to show that

$$(2x+z)(x-z)^2 \leq \frac{2(x^3-z^3)^2}{x^3+z^3},$$

or

$$(2x+z)(x^3+z^3) \leq 2(x^2+xz+z^2)^2.$$

By the AM-GM inequality, We have

$$(2x+z)(x^3+z^3) \leq (2x+4z)(x^3+2xz^2) = 2(x^2+2xz)(x^2+2z^2) \leq \tfrac{1}{2}[(x^2+2xz)+(x^2+2z^2)]^2 = 2(x^2+xz+z^2)$$

Therefore the last inequality is true and our **Solution** is completed.
Equality holds if and only if$a=b=c, ora=b$ and $c=0$.
Another **Solution** for (b). in the nontrivial case ab+bc+ca>0, by the AM-GM inequality,
We get

$$\sqrt[3]{abc} = \frac{(ab+bc+ca)\sqrt[3]{abc}}{ab+bc+ca} \geq \frac{3abc}{ab+bc+ca}.$$

Therefore, it suffices to prove that

$$a+b+c-\frac{9abc}{ab+bc+ca} \leq \frac{2(a-c)^2}{a+c},$$

or

$$(a+b+c)(ab+bc+ca)-9abc \leq \frac{2(ab+bc+ca)(a-c)^2}{a+c}.$$

Since

$$\frac{2(ab+bc+ca)(a-c)^2}{a+c} = 2b(a-c)^2+\frac{2ac(a-c)^2}{a+c} \geq 2b(a-c)^2$$

it is enough to check that

$$(a+b+c)(ab+bc+ca)-9abc \leq 2b(a-c)^2.$$

This inequality is equivalent to

$$(a+c)(a-b)(b-c) \geq 0,$$

which is obviously true.
160.
a)Let a, b, c, d be positive real numbers. Prove that

$$\sqrt{\frac{ab+ac+ad+bc+bd+cd}{6}} \geq \sqrt[3]{\frac{abc+bcd+cda+dab}{4}}.$$

Solution.
Due to symmetry and homogeneity, We may assume that

$$a \geq b \geq c \geq d and ab+ac+ad+bc+bd+cd=6.$$

The inequality becomes

$$abc+bcd+cda+dab \leq 4.$$

By the Cauchy-Schwarz inequality, We get

$$\begin{aligned}(abc+bcd+cda+dab)^2 &\le (ab+bc+cd+da)(abc^2+bcd^2+cda^2+dab^2)\\ &= (ab+bc+cd+da)[ac(bc+da)+bd(ab+cd)].\end{aligned}$$

Now, since $a \ge b \ge c \ge d$, We have $ac \ge bd$ and

$$bc+da-(ab+cd) = -(a-c)(b-d) \le 0.$$

Thus, by Chebyshev's inequality,

$$ac(bc+da)+bd(ab+cd) \le \frac{1}{2}(ac+bd)(bc+da+ab+cd).$$

Combining this with the above inequality, We get

$$\begin{aligned}(abc+bcd+cda+dab)^2 \le \frac{1}{2}(ac+bd)(ab+bc+cd+da)^2 &= \frac{1}{4}\cdot 2(ac+bd)\cdot(ab+bc+cd+da)^2\\ &\le \frac{1}{4}\left[\frac{2(ac+bd)+2(ab+bc+cd+da)}{3}\right]^3 = 4.\end{aligned}$$

The **Solution** is completed. Equality holds if and only if $a=b=c=d$.
Q.E.D

b)Let ABC be a triangle such that: $A \ge B \ge \frac{\pi}{3} \ge C$.Prove that:

$$\sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}} \ge \sqrt{9+\frac{R-2r}{R}}$$

Where r, R be the radius and circumradius of a triangle ABC
Solution:
Lemma 1. Let ABC be a triangle so that its angles satisfy a relation of type :

$$A \ge B \ge 60^\circ \ge C$$

Prove that :

$$s \ \ge \ \sqrt{3}\cdot(R+r)$$

Proof. Since the angles of $\triangle\ ABC$ satisfy a relation of type

$$A \ge B \ge 60^\circ \ge C \implies \left\{\begin{array}{l} \tan\frac{A}{2}-\frac{\sqrt{3}}{3} \ \ge\ 0\\ \\ \tan\frac{B}{2}-\frac{\sqrt{3}}{3} \ \ge\ 0\\ \\ \tan\frac{C}{2}-\frac{\sqrt{3}}{3} \ \le\ 0\end{array}\right| \implies \prod\left(\tan\frac{A}{2}-\frac{\sqrt{3}}{3}\right) \ \le\ 0 \iff \prod\tan\frac{A}{2}$$

$$-\frac{\sqrt{3}}{3}\cdot\sum\tan\frac{B}{2}\tan\frac{C}{2}+\frac{1}{3}\cdot\sum\tan\frac{A}{2}-\frac{\sqrt{3}}{9} \ \le\ 0 \iff$$

$$\iff \frac{r}{s}-\frac{\sqrt{3}}{3}+\frac{1}{3}\cdot\frac{4R+r}{s}$$

$$-\frac{\sqrt{3}}{9} \ \le\ 0 \iff 9r+3(4R+r)-4s\sqrt{3} \ \le\ 0 \iff s \ \ge \sqrt{3}\cdot(R+r)$$

Lemma 2.

In any triangle ABC the following inequality holds :

$$\sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}} \ \geq \ \frac{s+3r\cdot(2-\sqrt{3})}{\sqrt{2Rr}}$$

Proof. Apply the Jensen's inequality for the convex function

$$f(x)=\tan\frac{x}{4}$$

on the interval

$$\left(0\ ,\ \frac{\pi}{4}\right)$$

$$3{\cdot}\tan\left(\frac{\frac{A}{4}+\frac{B}{4}+\frac{C}{4}}{3}\right) \ \leq \ \tan\frac{A}{4}{+}\tan\frac{B}{4}{+}\tan\frac{C}{4} \iff 3{\cdot}(2{-}\sqrt{3}) \ \leq \ \tan\frac{A}{4}{+}\tan\frac{B}{4}{+}\tan\frac{C}{4}(*)$$

On the other hand, we may write :

$$\sum\tan\ \frac{A}{4}=\sum\frac{1-\cos\frac{A}{2}}{\sin\frac{A}{2}}=$$

$$\sum\frac{1-\frac{s-a}{AI}}{\frac{r}{AI}}=\sum\frac{AI-s+a}{r}=\frac{AI+BI+CI-s}{r}$$

Thus, the inequality $(*)$ is equivalent to :

$$AI+BI+CI \ \geq \ s+3r\cdot(2-\sqrt{3})\ (**)$$

Since

$$AI=\sqrt{2Rr}\cdot\sqrt{\frac{b+c-a}{a}}$$

and so on, the inequality $(**)$ finally becomes :

$$\sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}} \ \geq \ \frac{s+3r\cdot(2-\sqrt{3})}{\sqrt{2Rr}}$$

Problem.Prove that in any triangle ABC that satisfies a relation of type

$$A\geq B\geq 60^\circ\geq C$$

the following inequality holds :

$$\sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}}\geq\frac{6r+\sqrt{3}\cdot(R-2r)}{\sqrt{2Rr}}\geq\sqrt{9+\frac{R-2r}{R}}$$

Proof.

► Applylemma 1 lemma 2

$$\Longrightarrow\sum\sqrt{\frac{b+c-a}{a}}\ \overset{(2)}{\geq}\ \frac{s+3r\cdot(2-\sqrt{3})}{\sqrt{2Rr}}\ \overset{(1)}{\geq}\ \frac{6r+\sqrt{3}\cdot(R-2r)}{\sqrt{2Rr}}$$

► The inequality

$$\frac{6r+\sqrt{3}\cdot(R-2r)}{\sqrt{2Rr}}\geq\sqrt{9+\frac{R-2r}{R}}$$

becomes :

$$6r + \sqrt{3} \cdot (R - 2r) \geq \sqrt{2r(10R - 2r)}$$

$$\iff 3(R - 2r)^2 + 12\sqrt{3} \cdot r(R - 2r) + 36r^2 \ \geq \ 20Rr - 4r^2$$

$$\iff (R - 2r) \cdot \left[3R + (12\sqrt{3} - 26) \cdot r\right] \geq 0$$

c)Let ABC is triagle,Prove that:

$$\frac{R}{2r} \geq \frac{m_a}{h_a}.$$

Solution:
The Inequalities can rewrite:
a, b, c are the side-lengths of a triangle, then

$$b^2c^2(a + b + c) \geq (2b^2 + 2c^2 - a^2)(b + c - a)(c + a - b)(a + b - c).$$

Setting $x = b+c-a$, $y = c+a-b$ and $z = a+b-c$. Clearly, x, y, $z > 0$. The inequality becomes

$$(x + y)^2(x + z)^2(x + y + z) \geq 4[4x(x + y + z) + (y - z)^2]xyz,$$

which is equivalent to

$$(x + y + z)[(x + y)^2(x + z)^2 - 16x^2yz] \geq 4xyz(y - z)^2.$$

Since

$$\begin{aligned}(x + y)^2(x + z)^2 - 16x^2yz &= (x - y)^2(x + z)^2 + 4xy(x - z)^2\\ &\geq 4xz(x - y)^2 + 4xy(x - z)^2,\end{aligned}$$

it suffices to prove that

$$(x + y + z)[z(x - y)^2 + y(x - z)^2] \geq yz(y - z)^2.$$

Now, we see that $x + y + z > y + z$. So, it is enough to check the following inequality

$$(y + z)[z(x - y)^2 + y(x - z)^2] \geq yz(y - z)^2,$$

which is true according to the Cauchy-Schwarz Inequality

$$(y + z)[z(x - y)^2 + y(x - z)^2] \geq \left[\sqrt{y}\sqrt{z}(x - y) + \sqrt{z}\sqrt{y}(z - x)\right]^2 = yz(y - z)^2$$

The proof is completed. 161.
Given an integer $n \geq 2$, find the maximal constant $\lambda(n)$ having the following property:
if a sequence of real numbers $a_0, a_1, \ldots, a_n satisfies 0 = a_0 \leq a_1 \leq \cdots \leq a_n$ and $2a_i \geq a_{i-1} + a_{i+1} for i = 1, 2, \ldots, n - 1$, then

$$\left(\sum_{i=1}^{n} ia_i\right)^2 \geq \lambda(n) \sum_{i=1}^{n} a_i^2.$$

Solution.
We choose $(a_0, a_1, a_2, \ldots, a_n) = (0, 1, \ldots, 1)$.

This sequence satisfies the given hypothesis and after substituting into the desired inequality, We get

$$\lambda(n) \leq \frac{n(n+1)^2}{4}.$$

We will show that the maximal constant$\lambda(n) is \frac{n(n+1)^2}{4}$; that is, to show that the following inequality holds

$$\left(\sum_{i=1}^{n} ia_i\right)^2 \geq \frac{n(n+1)^2}{4} \sum_{i=1}^{n} a_i^2.$$

We will prove this inequality by induction on n. For $n = 2$, We have

$$(a_1 + 2a_2)^2 - \frac{9}{2}(a_1^2 + a_2^2) = \frac{1}{2}(a_2 - a_1)(7a_1 - a_2) \geq 0,$$

because $2a_1 \geq a_0 + a_2 = a_2$. So, the inequality is true for $n = 2$.
Now, assume that it holds for $n \geq 2$, We will show that it also holds for $n + 1$
Since $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$and $2a_i \geq a_{i-1} + a_{i+1}$ for all $i = 1, 2, \ldots, n-1$, by the inductive hypothesis, We have

$$\sum_{i=1}^{n} a_i^2 \leq \frac{4}{n(n+1)^2} \left(\sum_{i=1}^{n} ia_i\right)^2.$$

Therefore, it suffices to prove that

$$\left[\sum_{i=1}^{n} ia_i + (n+1)a_{n+1}\right]^2 \geq \frac{(n+1)(n+2)^2}{4} \left[\frac{4}{n(n+1)^2} \left(\sum_{i=1}^{n} ia_i\right)^2 + a_{n+1}^2\right].$$

Setting $\sum_{i=1}^{n} ia_i = \frac{n(n+1)}{2}A, A \leq a_{n+1}$, the above inequality becomes

$$(n+1)\left(\frac{n}{2}A + a_{n+1}\right)^2 \geq \frac{(n+2)^2}{4}(nA^2 + a_{n+1}^2),$$

or

$$\frac{n}{4}(a_{n+1} - A)[(3n+4)A - na_{n+1}] \geq 0.$$

From this, We see that the inequality for $n + 1$ holds if We have

$$A \geq \frac{n}{3n+4} a_{n+1},$$

or

$$\sum_{i=1}^{n} ia_i \geq \frac{n^2(n+1)}{2(3n+4)} a_{n+1}.$$

Since

$$2a_i \geq a_{i-1} + a_{i+1}$$

for all $i = 1, 2, \ldots, n$, We can easily deduce that $a_{i-1} \geq \frac{i-1}{i} a_i$ for all $i = 2, 3, \ldots, n+1$, and hence We get

$$a_{i-1} \geq \frac{i-1}{i} a_i \geq \frac{i-1}{i} \cdot \frac{i}{i+1} a_{i+1} \geq \cdots \geq \frac{i-1}{i} \cdots \frac{n}{n+1} a_{n+1} = \frac{i-1}{n+1} a_{n+1}.$$

it follows that

$$\sum_{i=1}^{n} ia_i \geq \frac{a_{n+1}}{n+1} \sum_{i=1}^{n} i(i-1) = \frac{a_{n+1}}{n+1} \cdot \frac{n(n^2-1)}{3} = \frac{n(n-1)}{3} a_{n+1}.$$

Since $\dfrac{n(n-1)}{3} > \dfrac{n^2(n+1)}{2(3n+4)}$ for $n \geq 2$, We obtain

$$\sum_{i=1}^{n} ia_i > \frac{n^2(n+1)}{2(3n+4)}a_{n+1},$$

and the **Solution** is completed.

162.

Let a, b, c, d be nonnegative real numbers. Prove that

$$\sum a^4 + 8abcd \geq \sum abc(a+b+c).$$

Solution.

Without loss of generality, We may assume that $d = \min\{a, b, c, d\}$

Write the inequality as

$$\sum_{a,b,c} a^4 - abc\sum_{a,b,c} a + d^4 + 8abcd - d\sum_{a,b,c} ab(a+b) - d^2\sum_{a,b,c} ab \geq 0.$$

By the fourth degree Schur's inequality, We have

$$\sum_{a,b,c} a^4 - abc\sum_{a,b,c} a \geq \sum_{a,b,c} a(b+c)(b-c)^2 \geq d\sum_{a,b,c} a(b-c)^2,$$

and hence, it suffices to show that

$$\sum_{a,b,c} a(b-c)^2 + d^3 + 8abc - \sum_{a,b,c} ab(a+b) - d\sum_{a,b,c} ab \geq 0,$$

or

$$a(2bc - bd - cd) + d^3 - bcd \geq 0.$$

Since

$$2bc - bd - cd \geq 0$$

and

$$a \geq d,$$

We get

$$a(2bc - bd - cd) + d^3 - bcd \geq d(2bc - bd - cd) + d^3 - bcd = d(b-d)(c-d) \geq 0.$$

The **Solution** is completed. On the assumption $d = \min\{a, b, c, d\}$, equality holds for $a = b = c = d$,and again for$a = b = c and d = 0$.

163.

Let a,b,c be nonnegative real numbers, not all are zero. Prove that

$$a^2 + b^2 + c^2 + \sqrt{3} \cdot \frac{\sqrt[3]{abc}(ab+bc+ca)}{\sqrt{a^2+b^2+c^2}} \geq 2(ab+bc+ca).$$

First **Solution**. By the AM-GM inequality and the Cauchy-Schwarz inequality, We have

$$\frac{\sqrt{3}\sqrt[3]{abc}(ab+bc+ca)}{\sqrt{a^2+b^2+c^2}} \geq \frac{3\sqrt{3}abc}{\sqrt{a^2+b^2+c^2}} = \frac{3abc\sqrt{3(a^2+b^2+c^2)}}{a^2+b^2+c^2} \geq \frac{3abc(a+b+c)}{a^2+b^2+c^2}.$$

Therefore, it suffices to prove that

$$a^2 + b^2 + c^2 + \frac{3abc(a+b+c)}{a^2+b^2+c^2} \geq 2(ab+bc+ca).$$

Using some simple computations, We can write this inequality as

$$\sum a^4 + abc\sum a \geq \sum ab(a^2+b^2),$$

which is true because it is the fourth degree Schur's inequality.
Equality holds if and only if $a=b=c$, or $a=b$ and $c=0$, or any cyclic permutation.
Second **Solution**.
We consider two cases.
The first case is when

$$\sqrt{3}\cdot\frac{\sqrt[3]{abc}(ab+bc+ca)}{\sqrt{a^2+b^2+c^2}} \geq \frac{9abc}{a+b+c}.$$

in this case, We can see immediately that the inequality can be deduced from the third degree Schur's inequality

$$a^2+b^2+c^2+\frac{9abc}{a+b+c} \geq 2(ab+bc+ca).$$

in the second case

$$\frac{9abc}{a+b+c} > \sqrt{3}\cdot\frac{\sqrt[3]{abc}(ab+bc+ca)}{\sqrt{a^2+b^2+c^2}},$$

We have

$$\sqrt{3}\sqrt[3]{abc} > \sqrt[4]{\frac{(a+b+c)^2(ab+bc+ca)^2}{3(a^2+b^2+c^2)}} \geq \sqrt[4]{\frac{(ab+bc+ca)^3}{a^2+b^2+c^2}}.$$

Therefore, it suffices to prove that

$$a^2+b^2+c^2+\frac{ab+bc+ca}{\sqrt{a^2+b^2+c^2}}\sqrt[4]{\frac{(ab+bc+ca)^3}{a^2+b^2+c^2}} \geq 2(ab+bc+ca),$$

which is equivalent to

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^{\frac{3}{4}} \geq 2.$$

But this is clearly true because

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^{\frac{3}{4}} \geq \frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{ab+bc+ca}{a^2+b^2+c^2} \geq 2.$$

The **Solution** is completed.
164.
if

$$a \geq b \geq c \geq 0$$

and

$$a+b+c > 0,$$

then

$$\frac{(a-c)^2}{4(a+b+c)} \leq \frac{a+b+c}{3} - \sqrt[3]{abc} \leq \frac{4(a-c)^2}{3(a+b+c)}.$$

Solution. (a) We prove that

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \geq \frac{(a-c)^2}{4(a+b+c)}.$$

There are two cases to consider.
Case 1. $2b \geq a+c$. By the AM-GM inequality, We have

$$\sqrt[3]{abc} \leq \frac{b+\sqrt{ac}+\sqrt{ac}}{3}.$$

Therefore, it suffices to prove that

$$\frac{a+c-2\sqrt{ac}}{3} \geq \frac{(a-c)^2}{4(a+b+c)},$$

or

$$4(a+b+c) \geq 3\left(\sqrt{a}+\sqrt{c}\right)^2.$$

We have

$$displaystyle 4(a+b+c) \geq 6(a+c) \geq 3\left(\sqrt{a}+\sqrt{c}\right)^2,$$

so this inequality is true.
Case 2.

$$a+c \geq 2b.$$

Setting

$$x=\frac{a}{a+c}, y=\frac{b}{a+c}, z=\frac{c}{a+c} and t=xz.$$

We have

$$x+z=1, y \leq \frac{1}{2} and 0 \leq t \leq \frac{1}{4}.$$

The inequality can be written as

$$\frac{1+y}{3}-\sqrt[3]{yt} \geq \frac{1-4t}{4(1+y)},$$

or

$$\frac{t}{1+y}-\sqrt[3]{yt}+\frac{1+y}{3}-\frac{1}{4(y+1)} \geq 0.$$

By the AM-GM inequality, We have

$$\frac{t}{1+y}+\sqrt{\frac{y(1+y)}{27}}+\sqrt{\frac{y(1+y)}{27}} \geq \sqrt[3]{yt}.$$

Thus, it suffices to show that

$$\frac{1+y}{3}-\frac{1}{4(y+1)}-2\sqrt{\frac{y(1+y)}{27}} \geq 0,$$

or

$$\sqrt{3}(4y^2+8y+1) \geq 8(y+1)\sqrt{y(1+y)}.$$

This inequality is true because

$$3(4y^2+8y+1)^2-64y(1+y)^3=(1-2y)^3(3+2y) \geq 0.$$

The **Solution** is completed. Equality holds if and only if

$$a=b=c$$

(b) We prove that

$$\frac{a+b+c}{3}-\sqrt[3]{abc} \leq \frac{4(a-c)^2}{3(a+b+c)}.$$

By the AM-GM inequality, We have

$$\sqrt[3]{abc} = \frac{\sqrt[3]{a^2b^2c^2}}{\sqrt[3]{abc}} \geq \frac{3\sqrt[3]{a^2b^2c^2}}{a+b+c}.$$

Therefore, it suffices to prove that

$$(a+b+c)^2 - 9\sqrt[3]{a^2b^2c^2} \leq 4(a-c)^2.$$

Since

$$(a-c)^2 \geq (a-b)^2 + (b-c)^2,$$

We have

$$4(a-c)^2 \geq 2(a-b)^2 + 2(b-c)^2 + 2(c-a)^2,$$

and thus, We see that the above inequality is true if

$$(a+b+c)^2 - 9\sqrt[3]{a^2b^2c^2} \leq 2(a-b)^2 + 2(b-c)^2 + 2(c-a)^2,$$

or equivalently,

$$a^2 + b^2 + c^2 + 3\sqrt[3]{a^2b^2c^2} \geq 2(ab+bc+ca),$$

which is true according to the third degree Schur's inequality and the AM-GM inequality

$$\sum a^2 + 3\sqrt[3]{a^2b^2c^2} \geq \sum \sqrt[3]{a^2b^2}\left(\sqrt[3]{a^2} + \sqrt[3]{b^2}\right) \geq 2\sum ab.$$

Equality holds if and only if $a = b = c, or a = b$ and $c = 0$. 165.
Let a,b,c be positive real numbers. Prove that

$$\sum \sqrt{\frac{b+c}{a}} \geq 2\sqrt{1 + \frac{(a+b)(b+c)(c+a) - 8abc}{4\sum a(a+b)(a+c)}\sum \sqrt{\frac{a}{b+c}}}.$$

Solution
. By the Cauchy-Schwarz inequality and the AM-GM inequality, We have

$$\left(\sum \sqrt{\frac{b+c}{a}}\right)^2 = \sum \frac{b+c}{a} + 2\sum \sqrt{\frac{(a+b)(a+c)}{bc}}$$

$$\geq \sum \frac{b+c}{a} + 2\sum \frac{a+\sqrt{bc}}{\sqrt{bc}}$$

$$= \sum \frac{b+c}{a} + 2\sum \frac{a}{\sqrt{bc}} + 6 \geq \sum \frac{b+c}{a} + 4\sum \frac{a}{b+c} + 6$$

and

$$\left(\sum \sqrt{\frac{a}{b+c}}\right)^2 \leq \left(\sum a\right)\left(\sum \frac{1}{b+c}\right) = 3 + \sum \frac{a}{b+c}.$$

Therefore, it suffices to prove that

$$\sum \frac{b+c}{a} + 4\sum \frac{a}{b+c} + 6 \geq$$

$$\geq \left[4 + \frac{(a+b)(b+c)(c+a) - 8abc}{\sum a(a+b)(a+c)}\right]\left(3 + \sum \frac{a}{b+c}\right),$$

which is in succession equivalent to

$$\sum \frac{b+c}{a} - 6 \geq \frac{(a+b)(b+c)(c+a) - 8abc}{\sum a(a+b)(a+c)} \left(3 + \sum \frac{a}{b+c}\right),$$

$$\frac{(a+b)(b+c)(c+a) - 8abc}{abc} \geq \frac{(a+b)(b+c)(c+a) - 8abc}{\sum a(a+b)(a+c)} \left(3 + \sum \frac{a}{b+c}\right),$$

$$\frac{\sum a(a+b)(a+c)}{abc} \geq 3 + \sum \frac{a}{b+c}.$$

The last inequality is true because

$$\frac{\sum a(a+b)(a+c)}{abc} > \frac{\sum \left[a^2(b+c) + abc\right]}{abc} = \sum \frac{a(b+c)}{bc} + 3$$
$$\geq 4 \sum \frac{a(b+c)}{(b+c)^2} + 3 > 3 + \sum \frac{a}{b+c}.$$

The **Solution** is completed. Equality holds if and only if $a = b = c$. 166.
Let ABC be a given triangle. Prove that for any positive real numbers x,y,z, the inequality holds

$$\sum \sqrt{\frac{x}{y+z}} \sin A \leq \sqrt{\frac{(x+y+z)^3}{(x+y)(y+z)(z+x)}}.$$

Solution.
Setting

$$u = \sqrt{\frac{x}{y+z}}, v = \sqrt{\frac{y}{z+x}}, w = \sqrt{\frac{z}{x+y}}.$$

Since

$$\sin C = \sin(A+B) = \sin A \cos B + \sin B \cos A,$$

the inequality can be written as

$$(u + w\cos B)\sin A + w \sin B \cos A + v \sin B \leq \sqrt{\frac{(x+y+z)^3}{(x+y)(y+z)(z+x)}}.$$

By the Cauchy-Schwarz inequality, We have

$$(u + w\cos B)\sin A + w \sin B \cos A + v \sin B \leq$$
$$\leq \sqrt{(u + w\cos B)^2 + w^2 \sin^2 B} + v \sin B$$
$$\leq \sqrt{(1+v^2)[(u + w\cos B)^2 + w^2 \sin^2 B + \sin^2 B]}.$$

Therefore, it suffices to prove that

$$(1+v^2)[(u + w\cos B)^2 + w^2 \sin^2 B + \sin^2 B] \leq \frac{(x+y+z)^3}{(x+y)(y+z)(z+x)}.$$

Since

$$1 + v^2 = \frac{x+y+z}{z+x}$$

and

$$(u + w\cos B)^2 + w^2 \sin^2 B + \sin^2 B = u^2 + w^2 + 2uw \cos B + \sin^2 B,$$

this inequality is equivalent to

$$u^2 + w^2 + 2uw\cos B + \sin^2 B \le \frac{(x+y+z)^2}{(x+y)(y+z)}.$$

By the AM-GM inequality, We have

$$2uw\cos B \le u^2w^2 + \cos^2 B = u^2w^2 + 1 - \sin^2 B,$$

and thus, We obtain

$$u^2 + w^2 + 2uw\cos B + \sin^2 B \le u^2 + w^2 + u^2w^2 + 1 = \frac{(x+y+z)^2}{(x+y)(y+z)}.$$

The **Solution** is completed.
Remark.
From the **Solution** above, We can see that the following more general statement holds:
if x,y,z are real numbers and A,B,C are three angles of a triangle, then

$$x\sin A + y\sin B + z\sin C \le \sqrt{(1+x^2)(1+y^2)(1+z^2)}.$$

167.
Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{a^2}{4a^2+5bc}} + \sqrt{\frac{b^2}{4b^2+5ca}} + \sqrt{\frac{c^2}{4c^2+5ab}} \le 1.$$

Solution.
For the sake of contradiction, assume that there exist positive real numbers

$$a, b, c$$

such that

$$\sqrt{\frac{a^2}{4a^2+5bc}} + \sqrt{\frac{b^2}{4b^2+5ca}} + \sqrt{\frac{c^2}{4c^2+5ab}} > 1.$$

Setting

$$x = \sqrt{\frac{a^2}{4a^2+5bc}}, \quad y = \sqrt{\frac{b^2}{4b^2+5ca}}, \quad z = \sqrt{\frac{c^2}{4c^2+5ab}}.$$

it is easy to see that $x, y, z < \frac{1}{2}$ and

$$\frac{bc}{a^2} = \frac{1-4x^2}{5x^2}, \quad \frac{ca}{b^2} = \frac{1-4y^2}{5y^2}, \quad \frac{ab}{c^2} = \frac{1-4z^2}{5z^2}.$$

Therefore,

$$(1-4x^2)(1-4y^2)(1-4z^2) = 5^3x^2y^2z^2.$$

Since

$$x, y, z < \frac{1}{2}$$

and

$$x + y + z > 1,$$

We have

$$\prod(1-4x^2) < \prod[(x+y+z)^2-4x^2] = \prod(y+z-x)\cdot\prod(3x+y+z) \le \prod(y+z-x)\cdot\frac{5^3(x+y+z)^3}{27}$$

$$\leq \prod(y+z-x)\cdot\frac{5^3(x+y+z)(x^2+y^2+z^2)}{9} = \frac{5^3}{9}[2(x^2y^2+y^2z^2+z^2x^2)-(x^4+y^4+z^4)](x^2+y^2+z^2).$$

it follows that

$$[2(x^2y^2+y^2z^2+z^2x^2)-(x^4+y^4+z^4)](x^2+y^2+z^2) > 9x^2y^2z^2,$$

or

$$2(x^2y^2+y^2z^2+z^2x^2)-(x^4+y^4+z^4) > \frac{9x^2y^2z^2}{x^2+y^2+z^2}.$$

But it is clear from the third degree Schur's inequality that

$$2(x^2y^2+y^2z^2+z^2x^2)-(x^4+y^4+z^4) \leq \frac{9x^2y^2z^2}{x^2+y^2+z^2}.$$

So, what We have assumed is false. Or in the other words, for any positive real numbers a, b, c, We must have

$$\sqrt{\frac{a^2}{4a^2+5bc}}+\sqrt{\frac{b^2}{4b^2+5ca}}+\sqrt{\frac{c^2}{4c^2+5ab}} \leq 1.$$

The **Solution** is completed. Equality holds if and only if

$$a = b = c.$$

168.
if

$$a, b, c, x, y, z$$

are positive real numbers such that

$$y^2 \leq zxandz^2 \leq xy,$$

then

$$\frac{a}{ax+by+cz}+\frac{b}{bx+cy+az}+\frac{c}{cx+ay+bz} \leq \frac{3}{x+y+z}.$$

Solution. Write the inequality as

$$\sum\left(\frac{1}{x}-\frac{a}{ax+by+cz}\right) \geq \frac{3}{x}-\frac{3}{x+y+z},$$

or

$$\sum\frac{by+cz}{ax+by+cz} \geq \frac{3(y+z)}{x+y+z}.$$

By the Cauchy-Schwarz inequality, We have

$$\sum\frac{by+cz}{ax+by+cz} \geq \frac{\left[\sum(by+cz)\right]^2}{\sum(by+cz)(ax+by+cz)}$$

$$= \frac{(y+z)^2\left(\sum a\right)^2}{(y^2+z^2)\sum a^2+(xy+xz+2yz)\sum ab}.$$

Therefore, it suffices to prove that

$$(y+z)(x+y+z)\left(\sum a^2+2\sum ab\right) \geq$$

$$\geq 3(y^2+z^2)\sum a^2+3(xy+xz+2yz)\sum ab,$$

which is equivalent to

$$(xy + xz + 2yz - 2y^2 - 2z^2)\left(\sum a^2 - \sum ab\right) \geq 0.$$

Since

$$\sum a^2 \geq \sum ab,$$

it suffices to show that

$$xy + xz + 2yz - 2y^2 - 2z^2 \geq 0.$$

Without loss of generality, assume that

$$y \geq z.$$

From

$$xz \geq y^2,$$

We have

$$x \geq \frac{y^2}{z},$$

and hence

$$\begin{aligned} xy + xz + 2yz - 2y^2 - 2z^2 &\geq \frac{y^3}{z} + y^2 + 2yz - 2y^2 - 2z^2 \\ &= \frac{(y-z)(y^2+2z^2)}{z} \geq 0. \end{aligned}$$

The **Solution** is completed. Equality holds if and only if

$$x = y = z$$

or

$$a = b = c.$$

169.
Let a,b,c be positive real numbers. Prove that

$$\frac{ab^2}{a^2+2b^2+c^2} + \frac{bc^2}{b^2+2c^2+a^2} + \frac{ca^2}{c^2+2a^2+b^2} \leq \frac{a+b+c}{4}.$$

Solution. By the Cauchy-Schwarz inequality, We have

$$\frac{9}{a^2+b^2+c^2} + \frac{1}{b^2} \geq \frac{16}{a^2+2b^2+c^2}.$$

According to this inequality, We get

$$\frac{ab^2}{a^2+2b^2+c^2} \leq \frac{1}{16}\left(a + \frac{9ab^2}{a^2+b^2+c^2}\right).$$

Therefore,

$$\sum \frac{ab^2}{a^2+2b^2+c^2} \leq \frac{a+b+c}{16} + \frac{9(ab^2+bc^2+ca^2)}{16(a^2+b^2+c^2)}.$$

From this, We see that the desired inequality is true if

$$3(ab^2+bc^2+ca^2) \leq (a+b+c)(a^2+b^2+c^2).$$

But this inequality is true, since it is equivalent to the obvious one

$$b(a-b)^2+c(b-c)^2+a(c-a)^2 \geq 0.$$

The **Solution** is completed. Equality holds if and only if

$$a=b=c.$$

Remark. in the same manner, one can also prove that

$$\frac{ab^3}{a^3+2b^3+c^3}+\frac{bc^3}{b^3+2c^3+a^3}+\frac{ca^3}{c^3+2a^3+b^3} \leq \frac{a+b+c}{4}.$$

170.
Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\frac{a^2b}{4-bc}+\frac{b^2c}{4-ca}+\frac{c^2a}{4-ab} \leq 1.$$

Solution.
Since

$$\frac{4a^2b}{4-bc}=a^2b+\frac{a^2b^2c}{4-bc},$$

the inequality can be written as

$$abc\sum\frac{ab}{4-bc} \leq 4-\sum a^2b.$$

Using the ill-known inequality $a^2b+b^2c+c^2a+abc \leq 4$, We get

$$4-(a^2b+b^2c+c^2a) \geq abc,$$

and hence, it suffices to prove that

$$abc\sum\frac{ab}{4-bc} \leq abc,$$

or equivalently,

$$\frac{ab}{4-bc}+\frac{bc}{4-ca}+\frac{ca}{4-ab} \leq 1.$$

Since

$$ab+bc+ca \leq \frac{(a+b+c)^2}{3}=3,$$

We get

$$\frac{ab}{4-bc} \leq \frac{ab}{\frac{4}{3}(ab+bc+ca)-bc}=\frac{3ab}{4ab+bc+4ca}.$$

Therefore, it is enough to check that

$$\frac{x}{4x+4y+z}+\frac{y}{4y+4z+x}+\frac{z}{4z+4x+y} \leq \frac{1}{3},$$

where $x=ab, y=ca$ and $z=bc$. This is a ill-known inequality.
171.
if a,b,c are positive real numbers such that $a+b+c=1$, then

$$(a+b)^2(1+2c)(2a+3c)(2b+3c) \geq 54abc.$$

Solution.
Write the inequality as

$$(a+b)^2(1+2c)\left(2+\frac{3c}{a}\right)\left(2+\frac{3c}{b}\right) \geq 54c.$$

By the Cauchy-Schwarz inequality and the AM-GM inequality, We have

$$\left(2+\frac{3c}{a}\right)\left(2+\frac{3c}{b}\right) \geq \left(2+\frac{3c}{\sqrt{ab}}\right)^2 \geq \left(2+\frac{6c}{a+b}\right)^2 = \frac{4(1+2c)^2}{(a+b)^2}.$$

Therefore, it suffices to prove that

$$(1+2c)^3 \geq \frac{27}{2}c,$$

which is true according to the AM-GM inequality

$$(1+2c)^3 = \left(\frac{1}{2}+\frac{1}{2}+2c\right)^3 \geq 27 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 2c = \frac{27}{2}c.$$

Equality holds if and only if

$$a = b = \frac{3}{8}$$

and $c = \dfrac{1}{4}$. 172.
fa, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$\sqrt{\frac{a^2+ab+b^2}{ab+2c^2+3}} + \sqrt{\frac{b^2+bc+c^2}{bc+2a^2+3}} + \sqrt{\frac{c^2+ca+a^2}{ca+2b^2+3}} \geq \frac{3}{\sqrt{2}}.$$

Solution.
After using the AM-GM inequality, We see that it suffices to prove that the stronger inequality holds

$$8\prod(a^2+ab+b^2) \geq \prod(ab+2c^2+3).$$

This is equivalent to

$$\prod[a^2+b^2+(a+b)^2] \geq \prod[ab+c^2+(a+c)(b+c)].$$

By the Cauchy-Schwarz inequality, We have

$$[a^2+b^2+(a+b)^2][a^2+c^2+(a+c)^2] \geq [a^2+bc+(a+b)(a+c)]^2.$$

Multiplying this and its analogous inequalities, We get the desired inequality. Equality holds if and only if $a = b = c$.
173.
Let a,b,c be positive real numbers. Prove that

$$\sqrt{\frac{a^2+2b^2}{a^2+ab+bc}} + \sqrt{\frac{b^2+2c^2}{b^2+bc+ca}} + \sqrt{\frac{c^2+2a^2}{c^2+ca+ab}} \geq 3.$$

Solution.
After using the AM-GM inequality, We see that it suffices to prove that the stronger inequality holds

$$\prod(a^2+2b^2) \geq \prod(a^2+ab+bc).$$

By the Cauchy-Schwarz inequality, We have

$$(b^2+b^2+a^2)(b^2+c^2+c^2) \geq (b^2+bc+ca)^2.$$

Multiplying this and the two analogous inequalities, We get the desired result. Equality holds if and only if $a=b=c$. 174.
if a,b,c are positive real numbers such that $a^3+b^3+c^3+abc=12$then

$$\frac{19(a^2+b^2+c^2)+6(ab+bc+ca)}{a+b+c} \geq 36.$$

Solution.
By the AM-GM inequality, We have

$$\begin{aligned}19\sum a^2+6\sum ab &= 8\sum a^2+8\sum a^2+3\left(\sum a\right)^2 \\ &\geq 12\sqrt[3]{3\left(\sum a^2\right)^2\left(\sum a\right)^2}.\end{aligned}$$

it follows that

$$\frac{19(a^2+b^2+c^2)+6(ab+bc+ca)}{a+b+c} \geq 12\sqrt[3]{\frac{3(a^2+b^2+c^2)^2}{a+b+c}}.$$

Therefore, it is enough to prove that

$$(a^2+b^2+c^2)^2 \geq 9(a+b+c).$$

After homogenizing, this inequality becomes

$$4(a^2+b^2+c^2)^2 \geq 3(a+b+c)(a^3+b^3+c^3+abc),$$

which is equivalent to the obvious inequality

$$\sum(b-c)^2(b+c-3a)^2 \geq 0.$$

Equality holds if and only if $a=2b=2c$, or any cyclic permutation.
175.
if a,b,c are positive real numbers, then

$$\sqrt{a+\sqrt[3]{b+\sqrt[4]{c}}} > \sqrt[32]{abc}.$$

Solution.
Setting$A=\sqrt{a+\sqrt[3]{b+\sqrt[4]{c}}}$,then it is clear that

$$A^2>a, A^6>bandA^{24}>c.$$

Multiplying these inequalities, We get

$$A^{32}>abc,$$

from which the conclusion follows.
176.
Let a,b,c,d be nonnegative real numbers, no three of which are zero. Prove that

$$\sum\frac{a}{b^2+c^2+d^2} \geq \frac{4}{a+b+c+d}.$$

Solution. By the AM-GM inequality, We have

$$(a^2+b^2+c^2+d^2)^2 \geq 4a^2(b^2+c^2+d^2).$$

it follows that

$$a(a^2+b^2+c^2+d^2)^2 \geq 4a^3(b^2+c^2+d^2),$$

and hence

$$\frac{a}{b^2+c^2+d^2} \geq \frac{4a^3}{(a^2+b^2+c^2+d^2)^2}.$$

it suffices to prove that

$$\sum \frac{a^3}{(a^2+b^2+c^2+d^2)^2} \geq \frac{1}{a+b+c+d},$$

or equivalently,

$$(a^3+b^3+c^3+d^3)(a+b+c+d) \geq (a^2+b^2+c^2+d^2)^2,$$

which is obviously true according to the Cauchy-Schwarz inequality. Equality holds if and only if$a=b$ and $c=d=0$, or any cyclic permutation.

177.

if a,b,c are positive real numbers, then

$$\frac{a^2}{b(a^2+ab+b^2)}+\frac{b^2}{c(b^2+bc+c^2)}+\frac{c^2}{a(c^2+ca+a^2)} \geq \frac{3}{a+b+c}.$$

Solution.

1)By the Cauchy-Schwarz inequality and the AM-GM inequality, We have

$$\left[\sum \frac{a^2}{b(a^2+ab+b^2)}\right]\left(\sum \frac{a^2+ab+b^2}{b}\right) \geq \left(\sum \frac{a}{b}\right)^2 \geq 3\sum \frac{a}{b}$$

Therefore, it suffices to prove that

$$\left(\sum a\right)\left(\sum \frac{a}{b}\right) \geq \sum \frac{a^2+ab+b^2}{b},$$

or equivalently,

$$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \geq a+b+c.$$

This is obviously true according to the AM-GM inequality, so our **Solution** is completed. Equality holds if and only if $a=b=c$.

2)

The inequality is equivalent with:

$$\sum_{cyc}\frac{a^2}{a^2+ab+b^2}+\sum_{cyc}\frac{a^2(a+c)}{b(a^2+ab+b^2)} \geq 3$$

i'll split the inequality into 2 parts. First We show that:

$$\sum_{cyc}\frac{a^2}{a^2+ab+b^2} \geq 1$$

Denote

$$A=a^2+ab+b^2, B=b^2+bc+c^2 and C=c^2+ca+a^2.$$

We want to prove that:

$$\frac{a^2}{A}+\frac{b^2}{B}+\frac{c^2}{C}-1\geq 0$$
$$\iff \left(\frac{a^2}{A}+\frac{b^2}{B}+\frac{c^2}{C}-1\right)\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)\geq 0$$
$$\iff \sum\frac{a^2}{A^2}+\sum\left(\frac{a^2+b^2}{AB}-\frac{1}{B}\right)\geq 0$$
$$\iff \sum\frac{a^2}{A^2}-\sum\frac{ab}{AB}\geq 0 \iff \sum\left(\frac{a}{A}-\frac{b}{B}\right)^2\geq 0$$

which is obvious.
We are left to show that:

$$\sum_{cyc}\frac{a^2(a+c)}{b(a^2+ab+b^2)}\geq 2$$

From Cauchy-Schwartz inequality We get that:

$$LHS=\sum_{cyc}\frac{a^2(a+c)^2}{b(a+c)(a^2+ab+b^2)}$$
$$\geq\frac{\left(\sum_{cyc}a^2+\sum_{cyc}ab\right)^2}{\sum_{cyc}ab(a^2+ab+b^2)+2abc\sum_{cyc}a+\sum_{cyc}a^3b}$$

So it's enough to prove that:

$$a^4+b^4+c^4+a^2b^2+b^2c^2+c^2a^2\geq 2(a^3b+b^3c+c^3a)$$
$$\iff a^2(a-b)^2+b^2(b-c)^2+c^2(c-a)^2\geq 0.$$

The **Solution** is complete.
178.
Let a,b,c be nonnegative real numbers. Prove that

$$(a)\sum\sqrt{a^2+ab+b^2}\geq\sqrt{4(a^2+b^2+c^2)+5(ab+bc+ca)};$$
$$(b)\sum\sqrt{a^2+ab+b^2}\leq 2\sqrt{a^2+b^2+c^2}+\sqrt{ab+bc+ca}.$$

Solution.
(a) Without of generality, We may assume that b is betien a and c. Now, with notice that

$$\sqrt{a^2+ab+b^2}=\sqrt{\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}},\sqrt{a^2+ac+c^2}=\sqrt{\left(a+\frac{c}{2}\right)^2+\frac{3c^2}{4}},$$

and

$$\sqrt{b^2+bc+c^2}=\sqrt{\left(b+\frac{c}{2}\right)^2+\frac{3c^2}{4}},$$

We may apply Minkowski's inequality to get

$$\sum\sqrt{a^2+ab+b^2}\geq\sqrt{\left(2a+\frac{3b}{2}+c\right)^2+\frac{3}{4}(b+2c)^2}.$$

it suffices to prove that

$$\left(2a+\frac{3b}{2}+c\right)^2+\frac{3}{4}(b+2c)^2 \geq 4(a^2+b^2+c^2)+5(ab+bc+ca).$$

This reduces to $(a-b)(b-c) \geq 0$, which is obviously true.
The **Solution** is completed.
Equality if and only if $a=b=c$, or $b=c=0$, or $c=a=0$, or $a=b=0$. (b) in the nontrivial case when two of a,b,c are nonzero, We assume that $a=\max\{a,b,c\}$. Setting

$$A=\sqrt{2a^2+b^2+c^2+ab+ac}, B=\sqrt{b^2+bc+c^2},$$

$$C=\sqrt{a^2+b^2+c^2}, D=\sqrt{ab+bc+ca}.$$

By the Cauchy-Schwarz inequality, We have

$$\sqrt{a^2+ab+b^2}+\sqrt{a^2+ac+c^2} \leq \sqrt{2}A.$$

Therefore, it suffices to prove that

$$\sqrt{2}A+B \leq 2C+D,$$

or equivalently,

$$\frac{\sqrt{2}(A^2-2C^2)}{A+\sqrt{2}C} \leq \frac{D^2-B^2}{D+B}.$$

Since $A^2-2C^2=D^2-B^2=ab+ac-b^2-c^2 \geq 0$, the last inequality is equivalent to

$$\frac{\sqrt{2}}{A+\sqrt{2}C} \leq \frac{1}{D+B},$$

or

$$D+B \leq \frac{A}{\sqrt{2}}+C.$$

This is true because

$$A^2-2B^2=(2a^2-b^2-c^2)+(ab+ac-2bc) \geq 0$$

and

$$C^2-D^2=a^2+b^2+c^2-ab-bc-ca \geq 0.$$

The **Solution** is completed. Equality holds if and only if $a=b=c$, or $b=c=0$, or $c=a=0$, or $a=b=0$.
Another **Solution** of (a). After squaring, the inequality can be written as

$$\sum\sqrt{(a^2+ab+b^2)(a^2+ac+c^2)} \geq (a+b+c)^2.$$

Since

$$a^2+ab+b^2=\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4} and a^2+ac+c^2=\left(a+\frac{c}{2}\right)^2+\frac{3c^2}{4}$$

by the Cauchy-Schwarz inequality, We get

$$\sqrt{(a^2+ab+b^2)(a^2+ac+c^2)} \geq \left(a+\frac{b}{2}\right)\left(a+\frac{c}{2}\right)+\frac{3bc}{4}.$$

Adding this and its analogous inequalities, We get the desired result.
179.

Let a,b,c be positive real numbers such that $a^2+b^2+c^2=1$. Prove that

$$\frac{a}{\sqrt{1+bc}}+\frac{b}{\sqrt{1+ca}}+\frac{c}{\sqrt{1+ab}}\le\frac{3}{2}.$$

Solution.
By the Cauchy-Schwarz inequality, We have

$$\left(\sum\frac{a}{\sqrt{1+bc}}\right)^2\le\left(\sum a\right)\left(\sum\frac{a}{1+bc}\right).$$

Therefore, it suffices to prove that

$$\left(\sum a\right)\left(\sum\frac{a}{1+bc}\right)\le\frac{9}{4}.$$

Now, by using the ill-known inequality

$$(a+b+c)(ab+bc+ca)\le\frac{9}{8}(a+b)(b+c)(c+a),$$

We get

$$a+b+c\le\frac{9(a+b)(b+c)(c+a)}{8(ab+bc+ca)}.$$

From this, We see that the above inequality is true if We have

$$\sum\frac{a}{1+bc}\le\frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)},$$

or equivalently,

$$\sum\frac{a}{1+bc}\le\sum\frac{a}{(a+b)(a+c)}.$$

Since

$$\frac{a}{(a+b)(a+c)}-\frac{a}{1+bc}=\frac{a(b^2+c^2-ab-ac)}{(a+b)(a+c)(1+bc)}=\frac{ca(c-a)-ab(a-b)}{(a+b)(a+c)(1+bc)},$$

this inequality can be written as follows

$$\sum\frac{ca(c-a)}{(a+b)(a+c)(1+bc)}-\sum\frac{ab(a-b)}{(a+b)(a+c)(1+bc)}\ge 0,$$

$$\sum\frac{ab(a-b)}{(b+c)(b+a)(1+ca)}-\sum\frac{ab(a-b)}{(a+b)(a+c)(1+bc)}\ge 0,$$

$$\sum\frac{ab(a-b)^2(1-c^2)}{(a+b)(b+c)(c+a)(1+bc)(1+ca)}\ge 0.$$

Since $1-c^2=a^2+b^2>0$, the last inequality is obviously true, and the **Solution** is completed.
Equality holds if and only if $a=b=c=\frac{1}{\sqrt{3}}$..
180.
if a,b,c,x,y,z are nonnegative real numbers such that
$a+b+c=x+y+z=1$, then

$$ax+by+cz+16abc\le 1.$$

Solution.
Without loss of generality, We may assume that $a=\max\{a,b,c\}$. Then, We have

$$ax+by+cz\le a(x+y+z)=a=1-b-c\le 1-2\sqrt{bc}.$$

it suffices to prove that

$$\sqrt{bc} \geq 8abc,$$

or

$$8a\sqrt{bc} \leq 1.$$

This is true, since by the AM-GM inequality, We have

$$8a\sqrt{bc} \leq 4a(b+c) \leq [a+(b+c)]^2 = 1.$$

The **Solution** is completed. On the assumption $a = \max\{a, b, c\}$, equality holds for$a = \frac{1}{2}, b = c = \frac{1}{4}, x =$1 and $y = z = 0$, and again for $a = x = 1$ and$b = c = y = z = 0$. 181.
Let x,y,z be real numbers such that $0 \leq x < y \leq z \leq 1$ and $3x + 2y + z \leq 4$. Find the maximum value of the expression

$$S = 3x^2 + 2y^2 + z^2.$$

Solution.
We will show that $S \leq \frac{10}{3}$ with equality if $x = \frac{1}{3}$ and $y = z = 1$. Let us consider two cases
Case 1. $0 < x < \frac{1}{3}. Since 0 < y \leq z \leq 1$, We have

$$S \leq 3 \cdot \left(\frac{1}{3}\right)^2 + 2 \cdot 1^2 + 1 \cdot 1^2 = \frac{10}{3}.$$

Case 2. $x \geq \frac{1}{3}$. Since $0 < x \leq y \leq z$, We have $4 \geq 3x + 2y + z > 6x$, and hence

$$\frac{1}{3} \leq x < \frac{2}{3}.$$

According to this result, We have

$$(3x-1)(3x-2) \leq 0,$$

or

$$3x^2 \leq 3x - \frac{2}{3}.$$

Combining this with the obvious inequalities $y^2 \leq y, z^2 \leq z$, We get

$$3x^2 + 2y^2 + z^2 \leq 3x + 2y + z - \frac{2}{3} \leq 4 - \frac{2}{3} = \frac{10}{3}.$$

The **Solution** is completed.
183.
Let a,b,c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{2ab^2+1} + \frac{1}{2bc^2+1} + \frac{1}{2ca^2+1} \geq 1.$$

First **Solution**.
By the Cauchy-Schwarz inequailty, We have

$$\sum \frac{1}{2ab^2+1} \geq \frac{(a+b+c)^2}{\sum c^2(2ab^2+1)} = \frac{(a+b+c)^2}{\sum a^2 + 2abc\sum ab}.$$

Then, it suffices to prove that

$$(a+b+c)^2 \geq a^2 + b^2 + c^2 + 2abc(ab+bc+ca),$$

or

$$2(ab+bc+ca)(1-abc) \geq 0.$$

This is true because by the AM-GM inequality, . Equality holds if and only if $a=b=c=1$.
Second **Solution**.
Since

$$\frac{1}{2ab^2+1} = 1 - \frac{2ab^2}{2ab^2+1},$$

the inequality can be written as

$$\frac{ab^2}{2ab^2+1} + \frac{bc^2}{2bc^2+1} + \frac{ca^2}{2ca^2+1} \leq 2.$$

Now, by the AM-GM inequality, We have

$$\frac{ab^2}{2ab^2+1} \leq \frac{ab^2}{3\sqrt[3]{a^2b^4}} = \frac{\sqrt[3]{ab^2}}{3} \leq \frac{a+2b}{9}.$$

Adding this and its analogous inequalities, We get the desired result.
Remark.
Actually, the more general statement holds: Let a,b,c be nonnegative real numbers such that $a+b+c=3$. if $0 \leq k \leq 8$, then

$$\frac{1}{ab^2+k} + \frac{1}{bc^2+k} + \frac{1}{ca^2+k} \geq \frac{3}{1+k}.$$

185.
Let a,b,c be the side-lengths of a triangle. Prove that

$$\frac{\prod(b+c-a)^2}{8a^2b^2c^2} \leq \frac{\prod[2a^2-(b-c)^2]}{(b+c)^2(c+a)^2(a+b)^2}.$$

Solution.
Let $a=y+z, b=z+x$ and $c=x+y$, where x,y,z are positive real numbers. The inequality becomes

$$\frac{(y^2+6yz+z^2)(z^2+6zx+x^2)(x^2+6xy+y^2)}{(2x+y+z)^2(2y+z+x)^2(2z+x+y)^2} \geq$$
$$\geq \frac{8x^2y^2z^2}{(x+y)^2(y+z)^2(z+x)^2},$$

or equivalently,

$$\frac{(y^2+6yz+z^2)(z^2+6zx+x^2)(x^2+6xy+y^2)}{x^2y^2z^2} \geq$$
$$\geq \frac{8(2x+y+z)^2(2y+z+x)^2(2z+x+y)^2}{(x+y)^2(y+z)^2(z+x)^2}.$$

Let

$$A = \frac{(y-z)^2}{yz}, \quad B = \frac{(z-x)^2}{zx}, \quad C = \frac{(x-y)^2}{xy}$$

and

$$X = \frac{(y-z)^2}{(x+y)(x+z)}, \quad Y = \frac{(z-x)^2}{(y+z)(z+x)}, \quad Z = \frac{(x-y)^2}{(z+x)(z+y)}.$$

The last inequality can be written as

$$(8+A)(8+B)(8+C) \geq 8(4+X)(4+Y)(4+Z),$$

or

$$64\left(\sum A-2\sum X\right)+8\left(\sum AB-4\sum XY\right)+(ABC-8XYZ)\geq 0.$$

Since $4x^2yz\leq(x+y)(x+z)(y+z)^2$, We have $BC\geq 4YZ$and hence

$$AB+BC+CA-4XY-4YZ-4ZX\geq 0.$$

Also, since $(x+y)^2(y+z)^2(z+x)^2>8x^2y^2z^2$, We have $ABC-8XYZ\geq 0$.
Now, from these two inequalities, We see that the above inequality is true if We have

$$A+B+C\geq 2X+2Y+2Z.$$

This inequality is equivalent to

$$\sum\frac{(y-z)^2}{yz}\geq 2\sum\frac{(y-z)^2}{(x+y)(x+z)},$$

or

$$\sum\frac{y+z}{x}-6\geq 4\sum\frac{x}{y+z}-6,$$

which is true because

$$4\sum\frac{x}{y+z}\leq\sum x\left(\frac{1}{y}+\frac{1}{z}\right)=\sum\frac{y+z}{x}.$$

The **Solution** is completed.
186.
if a,b,c are positive real numbers such that $a+b+c=abc$, then

$$\frac{a^2}{\sqrt{a^2+1}}+\frac{b^2}{\sqrt{b^2+1}}+\frac{c^2}{\sqrt{c^2+1}}\geq\frac{\sqrt{3}}{2}(a+b+c).$$

Solution.
By Holder's inequality, We have

$$\left(\sum\frac{a^2}{\sqrt{a^2+1}}\right)^2\left(\sum\frac{a^2+1}{a}\right)\geq(a+b+c)^3.$$

Therefore, it suffices to prove that

$$4(a+b+c)\geq 3\sum\frac{a^2+1}{a}.$$

This inequality is equivalent to

$$a+b+c\geq 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

or

$$abc(a+b+c)\geq 3(ab+bc+ca).$$

By using the given hypothesis, this inequality can be written as

$$(a+b+c)^2\geq 3(ab+bc+ca),$$

which is obviously true. Equaity holds if and only if $a=b=c=\sqrt{3}$.
187.
Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{b+c}{\sqrt{a^2+bc}}+\frac{c+a}{\sqrt{b^2+ca}}+\frac{a+b}{\sqrt{c^2+ab}}\geq 4.$$

Solution.
By Holder's inequality, We have

$$\left(\sum \frac{b+c}{\sqrt{a^2+bc}}\right)^2 \left[\sum (b+c)(a^2+bc)\right] \ge 8(a+b+c)^3.$$

Therefore, it suffices to prove that

$$(a+b+c)^3 \ge 2\sum (b+c)(a^2+bc).$$

This inequality is equivalent to

$$\sum a^3 + 6abc \ge \sum ab(a+b),$$

which is true according to Schur's inequality.
Equality holds if and only if $a=b$ and $c=0$, or any cyclic permutation.
188.
Let $a,b,c \in \left[\frac{1}{2},1\right]$. Prove that:

$$9\sqrt[3]{abc} \ge (a+b+c)^2.$$

First **Solution**.
The inequality written:

$$(a+b+c)^2 \cdot \frac{1}{\sqrt[3]{abc}} \le 9.$$

Using AM-GM's inequality We have:

$$(a+b+c)^2 \cdot \frac{1}{\sqrt[3]{abc}} \le \frac{1}{81}\left[2(a+b+c)+\frac{3}{\sqrt[3]{abc}}\right]^3, (1)$$

and

$$\frac{3}{\sqrt[3]{abc}} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}. (2)$$

From (1) and (2) We will prove that:

$$2(a+b+c)+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le 9.$$

$$<=> \sum \left(2a+\frac{1}{a}-3\right) \le 0,$$

$$<=> \sum \frac{(2a-1)(a-1)}{a} \le 0.$$

Do

$$\frac{1}{2} \le a,b,c \le 1,$$

so the inequality is true.
Equality holds when $a=b=c=1$.
Second **Solution**2.
WLOG

$$a \ge b \ge c. => 2b \ge a \ge bvc \ge a \ge c.$$

We will prove that:

$$a^2b \ge \frac{(a+b)^3}{8}.$$

$$<=> (a-b)(4ab+b^2-a^2) \geq 0,$$

it is true by $2b \geq a$. We have:

$$a^2c \geq \frac{(a+c)^3}{8}.$$

And $1 \geq a > 0$, We have

$$\sqrt[3]{abc} \geq a\sqrt[3]{abc} = \sqrt[3]{(a^2b)(a^2c)} \geq \frac{(a+b)(a+c)}{4}.$$

We must prove:

$$9(a+b)(a+c) \geq 4(a+b+c)^2.$$

$$<=> 5a^2+ab+ac+bc-4b^2-4c^2 \geq 0.$$

Because $a \geq b \geq c$,We have:

$$\begin{aligned} 5a^2+ab+ac+bc-4b^2-4c^2 &\geq 5b^2+b^2+bc+bc-4b^2-4c^2 \\ &= 2(b-c)(b+2c) \geq 0. \end{aligned}$$

Q.E.D

189.
if a,b,c are nonnegative real numbers, then

$$\frac{abc+1}{(a+1)(b+1)(c+1)} \geq \frac{(abc-1)^2}{(ab+a+1)(bc+b+1)(ca+c+1)}.$$

Solution. Let us consider two cases:
The first case is when $abc \leq 3$. in this case, We have

$$(ab+a+1)(bc+b+1)(ca+c+1) \geq (a+1)(b+1)(c+1)$$

and

$$abc+1-(abc-1)^2 = abc(3-abc) \geq 0,$$

so the inequality holds.
For the second case, We have abc $>$ 3. Then, by using the estimation

$$\begin{aligned} (ab+a+1)(bc+b+1)(ca+c+1) &\geq (ab+a)(bc+b)(ca+c) \\ &= abc(a+1)(b+1)(c+1), \end{aligned}$$

it suffices to prove that

$$abc(abc+1) \geq (abc-1)^2,$$

which is true because

$$abc(abc+1)-(abc-1)^2 = 3abc-1 > 0.$$

The **Solution** is completed. Equality holds if and only if $a=b=c=0$.
190.
Let a,b,c be positive real numbers. Prove that

$$a+b+c \leq \frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}+\frac{1}{2}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right).$$

Solution.
Setting $a = yz, b = zx$ and $c = xy$, where x,y,z are some positive real numbers. The inequality becomes

$$\frac{1}{2}(x^2+y^2+z^2)+xyz\left(\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\right)\geq xy+yz+zx.$$

By the Cauchy-Schwarz inequality, We have

$$\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}\geq\frac{9}{2(x+y+z)},$$

and hence, it suffices to prove that

$$x^2+y^2+z^2+\frac{9xyz}{x+y+z}\geq 2(xy+yz+zx),$$

which is Schur's inequality, and the **Solution** is completed. Equality holds if and only if $a = b = c$.
191.
if a,b,c are nonnegative real numbers such that $ab + bc + ca > 0$ and $a^2 + b^2 + c^2 = 3$, then

$$\frac{b+c}{a^2+bc}+\frac{c+a}{b^2+ca}+\frac{a+b}{c^2+ab}\geq 3.$$

Solution.
By the Cauchy-Schwarz inequality, We have

$$\sum\frac{b+c}{a^2+bc}\geq\frac{4(a+b+c)^2}{\sum(b+c)(a^2+bc)}=\frac{2(a+b+c)^2}{\sum ab(a+b)}.$$

it suffices to prove that

$$2(a+b+c)^2\sqrt{3(a^2+b^2+c^2)}\geq 9\sum ab(a+b),$$

or equivalently,

$$2(a+b+c)^2\sqrt{3(a^2+b^2+c^2)}+27abc\geq 9(a+b+c)(ab+bc+ca).$$

By Schur's inequality, We see that

$$27abc\geq 3(a+b+c)[2(ab+bc+ca)-(a^2+b^2+c^2)],$$

and thus, it is enough to check the following inequality

$$2\left(\sum a\right)\sqrt{3\sum a^2}+3\left(2\sum ab-\sum a^2\right)\geq 9\sum ab,$$

or

$$2(a+b+c)\sqrt{3(a^2+b^2+c^2)}\geq 3(a^2+b^2+c^2+ab+bc+ca).$$

This is true, because

$$\begin{aligned}&4(a+b+c)^2(a^2+b^2+c^2)-3(a^2+b^2+c^2+ab+bc+ca)^2=\\&\quad=(a^2+b^2+c^2-ab-bc-ca)(a^2+b^2+c^2+3ab+3bc+3ca)\geq 0.\end{aligned}$$

The **Solution** is completed. Equality holds if and only if $a = b = c = 1$. 192.
if a,b,c are positive real numbers, then

$$\frac{a^4}{1+a^2b}+\frac{b^4}{1+b^2c}+\frac{c^4}{1+c^2a}\geq\frac{abc(a+b+c)}{abc+1}.$$

Solution. By the Cauchy-Schwarz inequality, We have

$$\sum \frac{a^4}{1+a^2b} \geq \frac{\left(\sum a^2\sqrt{c}\right)^2}{\sum c(1+a^2b)} = \frac{\left(\sum a^2\sqrt{c}\right)^2}{(a+b+c)(abc+1)}.$$

Therefore, it is enough to prove that

$$\left(\sum a^2\sqrt{c}\right)^2 \geq abc(a+b+c)^2.$$

This is equivalent to

$$\sum a^2\sqrt{c} \geq \sqrt{abc}(a+b+c),$$

or

$$\sqrt{\frac{a^3}{b}} + \sqrt{\frac{b^3}{c}} + \sqrt{\frac{c^3}{a}} \geq a+b+c.$$

By the AM-GM inequality, We have

$$\sqrt{\frac{a^3}{b}} + \sqrt{\frac{a^3}{b}} + b \geq 3a.$$

Adding this and its analogous inequalities, the conclusion follows.
Equality holds if and only if $a = b = c$.
193.
Let a, b, c be positive ral number such that abc=1. Prove that"

$$\frac{b+c+6}{a} + \frac{c+a+6}{b} + \frac{a+b+6}{c} \geq 8(a+b+c).$$

Solution:
Notice that

$$\sum \frac{6}{c} = 6\sum ab = \sum c(a+b) + 4\sum ab$$

and

$$8\sum a = 2\sum(a+b) + 4\sum a.$$

Hence the inequality written :

$$\sum \frac{(a+b)(c-1)^2}{c} \geq 4\sum a - 4\sum ab.$$

Because

$$\begin{aligned}4\sum a - 4\sum ab &= 4\sum a - 4\sum ab + 4(abc-1)\\ &= 4(a-1)(b-1)(c-1),\end{aligned}$$

The inequality become:

$$\sum \frac{(a+b)(c-1)^2}{c} \geq 4(a-1)(b-1)(c-1).$$

it is true with $(a-1)(b-1)(c-1) \leq 0$.
Case $(a-1)(b-1)(c-1) > 0$, Using AM-GM's inequality, We will prove the inequality:

$$3\sqrt[3]{(a+b)(b+c)(c+a)(a-1)^2(b-1)^2(c-1)^2} \geq 4(a-1)(b-1)(c-1),$$
$$<=> (a+b)(b+c)(c+a) \geq \frac{64}{27}(a-1)(b-1)(c-1).$$

Do $\frac{64}{27} < 3$ Hence We will prove:

$$(a+b)(b+c)(c+a) \geq 3(a-1)(b-1)(c-1).$$

Notice:

$$(a+b)(b+c)(c+a) = \sum \frac{a}{b} + \sum \frac{b}{a} + 2,$$

$$<=> \sum \frac{a}{b} + \sum \frac{b}{a} + 3\sum ab + 2 \geq 3\sum a.$$

it is treu because:

$$\sum \frac{b}{a} + 3\sum ab \geq 2\sqrt{3}\sum b > 3\sum b.$$

Q.E.D

. The enquality holds when $a = b = c = 1$. 194.

Let a,b,c be nonnegative real numbers satisfying $a + b + c = 3$. Prove that

$$\frac{ab}{\sqrt{b+c}} + \frac{bc}{\sqrt{c+a}} + \frac{ca}{\sqrt{a+b}} + \frac{1}{\sqrt{2}} \geq \frac{(a+b)(b+c)(c+a)}{2\sqrt{2}}.$$

Solution.

Let $x = ab + bc + ca$ and $y = abc$. By Holder's inequality, We have

$$\left(\sum \frac{ab}{\sqrt{b+c}}\right)^2 \left[\sum ab(b+c)\right] \geq (ab + bc + ca)^3 = x^3.$$

Also,

$$\sum ab(b+c) = (ab^2 + bc^2 + ca^2 + abc) + 2abc \leq 4 + 2abc = 4 + 2y.$$

From these two inequalities, We deduce that

$$\sum \frac{ab}{\sqrt{b+c}} \geq \sqrt{\frac{x^3}{2y+4}} = \frac{x^2}{\sqrt{2}\sqrt{x(y+2)}} \geq \frac{2x^2}{\sqrt{2}(x+y+2)}.$$

Thus, it is sufficient to show that

$$\frac{4x^2}{x+y+2} + 2 \geq (a+b)(b+c)(c+a).$$

Since $(a+b)(b+c)(c+a) = 3x - y$, this is equivalent to

$$\frac{4x^2}{x+y+2} + 2 \geq 3x - y,$$

or

$$\frac{4x^2}{x+y+2} + (x+y+2) \geq 4x,$$

which is obviously true according to the AM-GM inequality.

Equality holds if and only if $a = b = c = 1$, or $a = 1, b = 2$ and $c = 0$, or any cyclic permutation thereof

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Let a,b,c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^a(b+c)} + \frac{1}{b^b(c+a)} + \frac{1}{c^c(a+b)} \leq \frac{3}{2}.$$

Solution. Without loss of generality, assume that $c=\min\{a,b,c\}$. From $abc=1$, We get $c\le 1$. Thus, by Bernoulli's inequality, We have

$$\frac{1}{c^c}=\left(1+\frac{1}{c}-1\right)^c\le 1+c\left(\frac{1}{c}-1\right)=2-c.$$

On the other hand, it is known that $x^x\ge x$ for any positive real number x. According these two inequalities, We see that it suffices to show that

$$\frac{1}{a(b+c)}+\frac{1}{b(a+c)}+\frac{2-c}{a+b}\le\frac{3}{2}.$$

Since

$$\frac{1}{a(b+c)}+\frac{1}{b(a+c)}=\frac{bc}{b+c}+\frac{ac}{a+c}=2c-c^2\left(\frac{1}{a+c}+\frac{1}{b+c}\right)$$

and

$$\frac{1}{a+c}+\frac{1}{b+c}-\frac{2}{\sqrt{ab}+c}=\frac{\left(\sqrt{a}-\sqrt{b}\right)^2\left(\sqrt{ab}-c\right)}{(a+c)(b+c)\left(\sqrt{ab}+c\right)}\ge 0,$$

We get

$$\frac{1}{a(b+c)}+\frac{1}{b(a+c)}\le 2c-\frac{2c^2}{\sqrt{ab}+c}.$$

Besides, it is clear from the AM-GM inequality that

$$\frac{2-c}{a+b}\le\frac{2-c}{2\sqrt{ab}}.$$

Therefore, it suffices to prove that

$$2c-\frac{2c^2}{\sqrt{ab}+c}+\frac{2-c}{2\sqrt{ab}}\le\frac{3}{2}.$$

Setting $t=\sqrt{ab}$, We get

$$c=\frac{1}{t^2}$$

and the inequality becom

$$\frac{2}{t^2}-\frac{\frac{2}{t^4}}{t+\frac{1}{t^2}}+\frac{2-\frac{1}{t^2}}{2t}\le\frac{3}{2},$$

which is equivalent to the obvious inequality

$$\frac{(3t^4+4t^3+t^2+2t+1)(t-1)^2}{2t^3(t^3+1)}\ge 0.$$

The **Solution** is completed. Equality holds if and only if $a=b=c=1$.
197.
Let a, b, c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{(a+1)^2(b+c)}+\frac{1}{(b+1)^2(c+a)}+\frac{1}{(c+1)^2(a+b)}\le\frac{3}{8}.$$

First **Solution**.
Setting $a=x^2, b=y^2 and c=z^2$, where x,y,z are positive real numbers. The inequality becomes

$$\frac{1}{(x^2+1)^2(y^2+z^2)}+\frac{1}{(y^2+1)^2(z^2+x^2)}+\frac{1}{(z^2+1)^2(x^2+y^2)}\le\frac{3}{8}.$$

By the Cauchy-Schwarz inequality, We have

$$\sqrt{(x^2+1)(y^2+z^2)} \geq xy + z = \frac{1}{z} + z = \frac{z^2+1}{z},$$

and

$$\sqrt{(1+x^2)(y^2+z^2)} \geq y + xz = y + \frac{1}{y} = \frac{y^2+1}{y}.$$

Multiplying these two inequalities, We get

$$(x^2+1)(y^2+z^2) \geq \frac{(y^2+1)(z^2+1)}{yz}.$$

it follows that

$$\sum \frac{1}{(x^2+1)^2(y^2+z^2)} \leq \sum \frac{yz}{(x^2+1)(y^2+1)(z^2+1)}.$$

Therefore, it suffices to prove that

$$(x^2+1)(y^2+1)(z^2+1) \geq \frac{8}{3}(xy+yz+zx).$$

Using again the Cauchy-Schwarz inequality, We have

$$\sqrt{(x^2+1)(1+y^2)} \geq x+y, \quad \sqrt{(y^2+1)(1+z^2)} \geq y+z,$$
$$\sqrt{(z^2+1)(1+x^2)} \geq z+x.$$

Multiplying these three inequalities and then using the known inequality

$$(x+y)(y+z)(z+x) \geq \frac{8}{9}(x+y+z)(xy+yz+zx),$$

We get

$$\begin{aligned}(x^2+1)(y^2+1)(z^2+1) &\geq (x+y)(y+z)(z+x)\\ &\geq \frac{8}{9}(x+y+z)(xy+yz+zx).\end{aligned}$$

Therefore, it suffices to show that

$$x+y+z \geq 3,$$

which is true according to the AM-GM inequality. Equality holds if and only if $a=b=c=1$.
Second **Solution**. Setting $a=x^3, b=y^3$ and $c=z^3$, We have

$$\frac{1}{(a+1)^2(b+c)} = \frac{x^3y^3z^3}{(x^3+xyz)^2(y^3+z^3)} = \frac{xy^3z^3}{(x^2+yz)^2(y^3+z^3)}.$$

By the AM-GM inequality, We get

$$(x^2+yz)(y+z) = y(x^2+z^2) + z(x^2+y^2) \geq 2\sqrt{yz(x^2+y^2)(x^2+z^2)}.$$

This yields

$$(x^2+yz)^2(y+z) \geq \frac{4yz(x^2+y^2)(x^2+z^2)}{y+z}.$$

Using this in combination with the obvious inequality $2(y^2-yz+z^2) \geq y^2+z^2$, We get

$$\begin{aligned}\frac{1}{(a+1)^2(b+c)} &\leq \frac{xy^2z^2(y+z)}{4(x^2+y^2)(x^2+z^2)(y^2-yz+z^2)}\\ &\leq \frac{xy^2z^2(y+z)}{2(x^2+y^2)(x^2+z^2)(y^2+z^2)}.\end{aligned}$$

Therefore, it suffices to prove that

$$4\sum xy^2z^2(y+z) \le 3(x^2+y^2)(y^2+z^2)(z^2+x^2).$$

Dividing each side of this inequality by $x^2y^2z^2$, it becomes

$$3\left(\frac{x}{y}+\frac{y}{x}\right)\left(\frac{y}{z}+\frac{z}{y}\right)\left(\frac{z}{x}+\frac{x}{z}\right) \ge 4\sum\left(\frac{x}{y}+\frac{y}{x}\right),$$

or

$$3\left[2+\frac{(x-y)^2}{xy}\right]\left[2+\frac{(y-z)^2}{yz}\right]\left[2+\frac{(z-x)^2}{zx}\right] \ge 24+4\sum\frac{(x-y)^2}{xy}.$$

Now, using the trivial inequality

$$(2+u)(2+v)(2+w) \ge 8+4(u+v+w) \quad \forall u,v,w \ge 0,$$

We get

$$\begin{aligned}3\left[2+\frac{(x-y)^2}{xy}\right]\left[2+\frac{(y-z)^2}{yz}\right]\left[2+\frac{(z-x)^2}{zx}\right] &\ge 24+12\sum\frac{(x-y)^2}{xy}\\ &\ge 24+4\sum\frac{(x-y)^2}{xy},\end{aligned}$$

which completes the **Solution**. Third **Solution**.
Since a, b, c>0 and abc=1, there exist some positive real numbers x, y, z such that

$$a=\frac{y}{x}, b=\frac{x}{z} and c=\frac{z}{y}$$

After making this substitution, the inequality becomes

$$\frac{x^2yz}{(x+y)^2(xy+z^2)}+\frac{y^2zx}{(y+z)^2(yz+x^2)}+\frac{z^2xy}{(z+x)^2(zx+y^2)} \le \frac{3}{8}.$$

By the AM-GM inequality, We have

$$xy+z^2 \ge 2z\sqrt{xy},$$

and

$$(x+y)^2 \ge 2\sqrt{2xy(x^2+y^2)}.$$

Therefore,

$$\frac{x^2yz}{(x+y)^2(xy+z^2)} \le \frac{x}{4\sqrt{2(x^2+y^2)}}.$$

it suffices to show that

$$\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{y^2+z^2}}+\frac{z}{\sqrt{z^2+x^2}} \le \frac{3}{\sqrt{2}},$$

which is just a known result.
fourd **Solution**:
Case1. if $ab+bc+ca \ge a+b+c$
Using AM-GM's inequality, We have:

$$(1+a)(b+c) \ge 2\sqrt{a}.2\sqrt{bc}=4$$

$$(1+b)(c+a) \ge 4; (1+c)(a+b) \ge 4.$$

Hence We must prove:

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\le\frac{3}{2}\Leftrightarrow ab+bc+ca\ge a+b+c$$

it is true.
Case2. if $ab+bc+ca\le a+b+c$ Using Am-GM's niequality:

$$(1+a)^2\ge 4a=\frac{4}{bc},nn\frac{1}{(1+a)^2(b+c)}\le\frac{bc}{4(b+c)}.Tngt\frac{1}{(1+b)^2(c+a)}$$

$$\le\frac{ca}{4(c+a)};\frac{1}{(1+c)^2(a+b)}\le\frac{ab}{4(a+b)}.$$

So We must prove :

$$\frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a}\le\frac{3}{2}\Leftrightarrow(ab+bc+ca)^2+abc(a+b+c)\le$$

$$\frac{3}{2}(a+b+c)(ab+bc+ca)-\frac{3}{2}abc\Leftrightarrow\frac{3}{2}(a+b+c)(ab+bc+ca)\ge(ab+bc+ca)^2+a+b+c+\frac{3}{2}$$

Because$ab+bc+ca\le a+b+c$So

$$(a+b+c)(ab+bc+ca)\ge(ab+bc+ca)^2;(1)$$

$$\frac{1}{3}(a+b+c)(ab+bc+ca\ge(a+b+c).\sqrt[3]{a^2b^2c^2}=a+b+c;(2)$$

$$\frac{1}{6}(a+b+c)(ab+bc+ca)\ge\frac{1}{6}.9abc=\frac{3}{2}(3)$$

From (1) ,92) and (3) We have Q.E.D

.

Remark.

The **Solution**s of this problem gives us various **Solution**s of the previous problem, because We have $4x^x\ge(x+1)^2$ for any $x>0$. 198.

Let a,b,c be positive real numbers. Prove that

$$\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\sqrt{3}\sum\frac{a+2b}{\sqrt{a^2+2b^2}}\ge 13.$$

Solution. Denote

$$P=\frac{a+2b}{\sqrt{a^2+2b^2}}+\frac{b+2c}{\sqrt{b^2+2c^2}}+\frac{c+2a}{\sqrt{c^2+2a^2}}.$$

By the AM-GM inequality and the Cauchy-Schwarz inequality, We have

$$P=\sqrt{3}\sum\frac{(a+2b)^2}{(a+2b)\sqrt{3(a^2+2b^2)}}\ge 2\sqrt{3}\sum\frac{(a+2b)^2}{(a+2b)^2+3(a^2+2b^2)}$$

$$\ge\frac{2\sqrt{3}\left[\sum(a+2b)\right]^2}{\sum[(a+2b)^2+3(a^2+2b^2)]}=\frac{9\sqrt{3}\left(\sum a\right)^2}{7\sum a^2+2\sum ab}.$$

Therefore, it suffices to prove that

$$\frac{4(a^2+b^2+c^2)}{ab+bc+ca}+\frac{27(a+b+c)^2}{7(a^2+b^2+c^2)+2(ab+bc+ca)}\ge 13,$$

which is equivalent to the obvious inequality

$$\frac{28(a^2+b^2+c^2-ab-bc-ca)^2}{(ab+bc+ca)[7(a^2+b^2+c^2)+2(ab+bc+ca)]}\ge 0.$$

Equality holds if and only if $a=b=c$.
199.
if a,b,c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{9(a^2+b^2+c^2)}{(a+b+c)^2}.$$

Solution.
By applying the known inequality

$$(x+y+z)^3\geq\frac{27}{4}(x^2y+y^2z+z^2x+xyz)\quad\forall x,\ y,\ z\geq 0$$

for$x=\frac{a}{b},y=\frac{b}{c}andz=\frac{c}{a}$, We get

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^3\geq\frac{27}{4}\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}+1\right)=\frac{27}{4}\left(\frac{a^3+b^3+c^3}{abc}+1\right).$$

Therefore, it suffices to prove that

$$\frac{a^3+b^3+c^3}{abc}+1\geq\frac{108(a^2+b^2+c^2)^3}{(a+b+c)^6},$$

or

$$\frac{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)}{abc}+4\geq\frac{108(a^2+b^2+c^2)^3}{(a+b+c)^6}.$$

Using now the obvious inequality $3abc(a+b+c)\leq(ab+bc+ca)^2$, We have

$$\frac{a+b+c}{abc}=\frac{3(a+b+c)^2}{3abc(a+b+c)}\geq\frac{3(a+b+c)^2}{(ab+bc+ca)^2},$$

and hence, it is enough to check that

$$\frac{3(a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)}{(ab+bc+ca)^2}+4\geq\frac{108(a^2+b^2+c^2)^3}{(a+b+c)^6}.$$

Setting $t=\frac{3(a^2+b^2+c^2)}{(a+b+c)^2},1\leq t<3$. The above inequality is equivalent to

$$\frac{54(t-1)}{(3-t)^2}+4\geq 4t^3,$$

or

$$(t-1)(9-6t-8t^2+10t^3-2t^4)\geq 0.$$

But this is true because

$$9-6t-8t^2+10t^3-2t^4=2(3+3t-t^2)(t-1)^2+3>0.$$

The **Solution** is completed. Equality holds if and only if $a=b=c$. 200.
Let a,b,c be non-negative real numbers, prove that:

$$(a+b)^2(b+c)^2(c+a)^2\geq\left(a^2+b^2+c^2+ab+bc+ca\right)(ab+bc+ca)^2+10a^2b^2c^2$$

Solution Setting $\sum ab\,(a+b)=S,\ abc=T$, so $S\geq 6T$ and:

$$(a+b+c)\,(ab+bc+ca)=S+3T$$
$$(a+b)\,(b+c)\,(c+a)=S+2T$$

The inequality is equilvalent to:

$$(S+3T)^2-(S+2T)^2+10T^2 \le$$
$$\le (a+b+c)^2(ab+bc+ca)^2-\left(a^2+b^2+c^2+ab+bc+ca\right)(ab+bc+ca)^2$$
$$\Leftrightarrow T\left(2S+5T\right)+10T^2 \le (ab+bc+ca)^3$$

From a ill-known inequality

$$(ab+bc+ca)^2 \ge 3abc\left(a+b+c\right)$$

,We have:

$$(ab+bc+ca)^3 \ge 3T\left(a+b+c\right)\left(ab+bc+ca\right) = 3T\left(S+3T\right)$$

Therefore, it suffices to prove that:

$$T\left(2S+5T\right)+10T^2 \le 3T\left(S+3T\right) \Leftrightarrow 6T \le S$$

which is true.
The **Solution** is completed. Equlity holds for $a=b=c$, or$a=b=0, c \ge 0$
201.
Let a,b,c be non-negative real numbers, prove that:

$$(a+b)^2(b+c)^2(c+a)^2 \ge 4\left(a^2+bc\right)\left(b^2+ca\right)\left(c^2+ac\right)+32a^2b^2c^2$$

Solution:
Without loss of generality, We may assume that a is betien b and c, or$(a-c)(a-b) \le 0$.
From AM-GM inequality, We have

$$(b+c)^2(a+b)^2(c+a)^2 = (b+c)^2\left(a^2+bc+ab+ac\right)^2 \ge 4(b+c)^2\left(a^2+bc\right)a\left(b+c\right)$$

Therefore, it suffices to prove that:

$$a(b+c)^3\left(a^2+bc\right) \ge \left(a^2+bc\right)\left(b^2+ca\right)\left(c^2+ac\right)+8a^2b^2c^2$$

Which is equilvalent to:

$$\left(a^2+bc\right)\left[a(b+c)^3-\left(b^2+ca\right)\left(c^2+ac\right)\right] \ge 8a^2b^2c^2$$
$$\Leftrightarrow \left(a^2+bc\right)\left[3abc\left(b+c\right)-b^2c^2-a^2bc\right] \ge 8a^2b^2c^2$$
$$\Leftrightarrow \left(a^2+bc\right)\left(b-a\right)\left(a-c\right)+2\left(a^2+bc\right)a\left(b+c\right) \ge 8a^2bc$$

We have, from AM-GM inequality and the assumption:

$$\left(a^2+bc\right)\left(b-a\right)\left(a-c\right) \ge 02\left(a^2+bc\right)a\left(b+c\right) \ge 8\sqrt{a^2bc}.a.\sqrt{bc} = 8a^2bc$$

The **Solution** is completed. Equality holds for $a=b=c$, or $c=0, a=b$.
202.
Let a,b,c be positive real number such that

$$16(a+b+c) \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

. Prove that:

$$\frac{1}{\left(a+b+\sqrt{2(a+c)}\right)^3}+\frac{1}{\left(b+c+\sqrt{2(b+a)}\right)^3}+\frac{1}{\left(c+a+\sqrt{2(c+b)}\right)^3}\le\frac{8}{9}$$

Solution:
Using AM-GM's inequality

$$a+b+\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a+c}{2}}\ge 3.\sqrt[3]{\frac{(a+b)(a+c)}{2}}$$

$$=>\frac{1}{\left(a+b+\sqrt{2(a+c)}\right)^3}\le\frac{2}{27(a+b)(a+c)}$$

and

$$\sum\frac{1}{\left(a+b+\sqrt{2(a+c)}\right)^3}\le\frac{4(a+b+c)}{27(a+b)(b+c)(c+a)}$$

and We have

$$(a+b)(b+c)(c+a)\ge\frac{8}{9}(a+b+c)(ab+bc+ca)$$

so

$$\sum\frac{1}{\left(a+b+\sqrt{2(a+c)}\right)^3}\le\frac{1}{6(ab+bc+ca)}$$

using that inequality easy: $(ab+bc+ca)^2\ge 3abc(a+b+c)$,We have

$$16(a+b+c)\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{3(a+b+c)}{ab+bc+ca}$$

$$=>ab+bc+ca\ge\frac{3}{16}$$

$$=>\sum\frac{1}{\left(a+b+\sqrt{2(a+c)}\right)^3}\le\frac{8}{9}$$

Enquality hold $a=b=c=1/4$
203.
Let a,b,c be positive real number . prove that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}\ge\frac{(a+b+c)^2}{a^2+b^2+c^2}$$

Solution:
We will prove that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}$$

Using Cauchy-Schwarz's inequality

$$\left(\sqrt{\frac{2a}{b+c}}\right)^2\le(a+b+c)\left(\sum\frac{2a}{b+c}\right)=\sum\frac{2a}{b+c}+6$$

So We will prove

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2\ge\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}+6$$

Using inequalities

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{b^2}{c^2}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$
$$\frac{1}{a}+\frac{1}{b}\geq\frac{4}{a+b}$$
$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\geq 3$$

Hence We have

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2=\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+2\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)$$
$$\geq a\left(\frac{1}{b}+\frac{1}{c}\right)+b\left(\frac{1}{c}+\frac{1}{a}\right)+c\left(\frac{1}{a}+\frac{1}{b}\right)+3$$
$$\geq\sum\frac{4a}{b+c}+3\geq\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}+6$$

And

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}\geq\frac{(a+b+c)^2}{a^2+b^2+c^2}$$

By Am-GM's inequality

$$\sqrt{\frac{2a}{b+c}}=\frac{2a}{\sqrt{2a}\sqrt{b+c}}\geq\frac{4a}{2a+b+c}$$

So

$$\sum\sqrt{\frac{2a}{b+c}}\geq\left(\frac{4a}{2a+b+c}+\frac{4b}{2b+c+a}+\frac{4c}{2c+a+c}\right)$$

and by Cauchy-Schwarz's inequality:

$$a^2+b^2+c^2\geq ab+bc+ca$$

We have

$$\sum\frac{a}{2a+b+c}=\sum\frac{a^2}{2a^2+ab+ac}\geq\frac{(a+b+c)^2}{2(a^2+b^2+c^2+ab+bc+ca)}\geq\frac{(a+b+c)^2}{4(a^2+b^2+c^2)}$$

So

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}\geq\frac{(a+b+c)^2}{a^2+b^2+c^2}$$

Q.E.D
Enquality holds $a=b=c$
204.
if a,b,c are angles of an acute triangle, prove that

$$\pi^\pi a^b b^c c^a\leq(a^2+b^2+c^2)^\pi.$$

Solution:
The function$f(x)=\ln x$ is strictly concave, so from the general iighted Jensen inequality with iights the
a,b,c We have that $\dfrac{b\ln a+c\ln b+a\ln c}{a+b+c}\leq\ln\left(\dfrac{ab+bc+ca}{a+b+c}\right)$,or$\dfrac{b\ln a+c\ln b+a\ln c}{\pi}\leq$

$\ln\left(\frac{ab+bc+ca}{\pi}\right)$. But, the last relation can be rewritten as :

$$\frac{1}{\pi}\cdot\ln(a^b b^c c^a)\leq\ln\left(\frac{ab+bc+ca}{\pi}\right), thatis \ln\left(a^b b^c c^a\right)^{1/\pi}\leq\ln\left(\frac{ab+bc+ca}{\pi}\right).$$

Removing the logarithm We get

$$\left(a^b b^c c^a\right)^{1/\pi}\leq\frac{ab+bc+ca}{\pi}\Longrightarrow\pi^\pi a^b b^c c^a\leq(ab+bc+ca)^\pi\leq(a^2+b^2+c^2)^\pi, Q.E.D.$$

205.

if it holds that

$$(\sin x+\cos x)^k\leq 8(\sin^n x+\cos^n x),$$

then find the value of:

$$(n+k)_{\max}.$$

1st **Solution**:

From the Poir-Mean inequality We have that

$$\sqrt[n]{\frac{\sin^n x+\cos^n x}{2}}\geq\sqrt{\frac{\sin^2 x+\cos^2 x}{2}}\Longrightarrow\sin^n x+\cos^n x\geq 2\cdot\left(\frac{\sqrt{2}}{2}\right)^n.$$

So, multiplying by

$$8 We have that 8\left(\sin^n x+\cos^n x\right)\geq 16\cdot\left(\frac{\sqrt{2}}{2}\right)^n.$$

Taking now in hand the left hand side, We know that

$$\sin x+\cos x\leq\sqrt{2}\Longrightarrow(\sin x+\cos x)^k\leq\left(\sqrt{2}\right)^k.$$

So, We know that

$$16\cdot\left(\frac{\sqrt{2}}{2}\right)^n\geq\left(\sqrt{2}\right)^k.$$

Doing the manipulations on both sides We get that

$$n+k\leq 8, hence (n+k)_{\max}=8, Q.E.D.$$

2nd **Solution** (An idea by Vo Quoc Ba Can):

The inequality is symmetric on $\sin x,\cos x$. So, We only need to find the maximum of those two constants for the values of which $\sin x=\cos x$,that is

$$x=\frac{\pi}{4}.$$

So, plugging on the above inequality the value

$$x=\frac{\pi}{4}$$

We get the desired maximum result, Q.E.D

. 206.

if $x,y,z>0$ prove that

$$\sqrt{x^2+xy+y^2}+\sqrt{y^2+yz+z^2}+\sqrt{z^2+zx+x^2}\geq 3\sqrt{xy+yz+zx}.$$

Solution:
From the ill-known lemma

$$4(a^2+ab+b^2)\ge 3(a+b)^2 We deduce that: 2\sqrt{a^2+ab+b^2}\ge \sqrt{3}(a+b).$$

Doing that cyclic for x, y, z and adding up the 3 relations We get that

$$2\sum_{cyc}\sqrt{x^2+xy+y^2}\ge \sqrt{3}\cdot 2\sum_{cyc}x \Longrightarrow \sum\sqrt{x^2+xy+y^2}\ge \sqrt{3}\sum_{cyc}x.$$

So, it is enough to prove that $\sqrt{3}\sum_{cyc}x\ge 3\sqrt{xy+yz+zx}$.
Squaring both sides We come to the conclusion

$$\left(\sum_{cyc}x\right)^2\ge 3\sum_{cyc}xy,$$

Q.E.D
. 207.
Leta, b,c be positive real numbers such that $a+b+c=3$. Prove that

$$b\cdot\frac{(2a+b+c)^2}{2a^2+(b+c)^2}+c\cdot\frac{(a+2b+c)^2}{2b^2+(c+a)^2}+a\cdot\frac{(a+b+2c)^2}{2c^2+(a+b)^2}\le 8.$$

Solution:
From the hypothesis, the inequality is of the form

$$b\cdot\frac{(a+3)^2}{2a^2+(3-a)^2}+c\cdot\frac{(b+3)^2}{2b^2+(3-b)^2}+a\cdot\frac{(c+3)^2}{2c^2+(3-c)^2}\le 8.$$

if We expand the nominators and the denominators, then We get that

$$b\cdot\frac{a^2+6a+9}{3a^2-6a+9}+c\cdot\frac{b^2+6b+9}{3b^2-6b+9}+a\cdot\frac{c^2+6c+9}{3c^2-6c+9}\le 8.$$

Now let us use once again the Cauchy-Reverse technique.

$$\frac{a^2+6a+9}{3a^2-6a+9}=\frac{1}{3}\cdot\frac{3a^2+18a+27}{3a^2-6a+9}=\frac{1}{3}\cdot\frac{(3a^2-6a+9)+(24a+18)}{3a^2-6a+9}.$$

Thus, We have

$$\frac{1}{3}\cdot\left(1+\frac{24a+18}{3a^2-6a+9}\right).$$

Moreover,

$$3a^2-6a+9=3(a-1)^2+6\ge 6\Longrightarrow\frac{1}{3a^2-6a+9}\le\frac{1}{6}.$$

So,

$$\frac{1}{3}\cdot\left(1+\frac{24a+18}{3a^2-6a+9}\right)\le\frac{1}{3}\cdot\left(1+\frac{24a+18}{6}\right).$$

Multiplying by

$$b We acquire b\cdot\frac{1}{3}\cdot\left(1+\frac{24a+18}{3a^2-6a+9}\right)\le b\cdot\frac{1}{3}\cdot\left(1+\frac{24a+18}{6}\right)=b\cdot\left(\frac{1}{3}+\frac{8a+6}{6}\right).$$

Now, if We sum up the 3 inequalities it remains to prove that

$$LHS\le\sum_{cyc}b\cdot\left(\frac{1}{3}+\frac{8a+6}{6}\right)=\frac{1}{3}\sum_{cyc}b+\frac{8}{6}\sum_{cyc}ab+\sum_{cyc}b\le 8,$$

which reduces to the obvious inequality

$$\sum_{cyc} ab \leq 3,$$

Q.E.D

.

208.

if a, b, c are positive real numbers such that $abc = 1$, then prove that

$$\frac{c\sqrt{a^3+b^3}}{a^2+b^2} + \frac{a\sqrt{b^3+c^3}}{b^2+c^2} + \frac{b\sqrt{c^3+a^3}}{c^2+a^2} \geq \frac{3}{\sqrt{2}}.$$

Solution:

From the Cauchy-Schwarz inequality We deduce that

$$(a^3+b^3)(a+b) \geq (a^2+b^2)^2$$

Removing the square We get that

$$\sqrt{a^3+b^3} \cdot \sqrt{a+b} \geq (a^2+b^2). Letusnowdivideby a^2+b^2.$$

Then We have

$$\frac{\sqrt{a^3+b^3}}{a^2+b^2} \geq \frac{1}{\sqrt{a+b}}.$$

Moreover, multiply by c. i, thus, acquire

$$\frac{c\sqrt{a^3+b^3}}{a^2+b^2} \geq \frac{c}{\sqrt{a+b}}.$$

So,We have proved that

$$\frac{c\sqrt{a^3+b^3}}{a^2+b^2} + \frac{a\sqrt{b^3+c^3}}{b^2+c^2} + \frac{b\sqrt{c^3+a^3}}{c^2+a^2} \geq \frac{c}{\sqrt{a+b}} + \frac{b}{\sqrt{c+a}} + \frac{a}{\sqrt{b+c}}.$$

We will now apply Holder's inequality, that is

$$\left(\frac{c}{\sqrt{a+b}} + \frac{b}{\sqrt{c+a}} + \frac{a}{\sqrt{b+c}}\right)^2 \cdot [c(a+b)+b(c+a)+a(b+c)] \geq (a+b+c)^3, or\ (LHS)^2 \geq \frac{(a+b+c)^3}{2(ab+bc+ca)}.$$

Rewrite the sum

$$(a+b+c)^3 as (a+b+c)^2 \cdot (a+b+c).$$

Then We get that:

$$\frac{(a+b+c)^3}{2(ab+bc+ca)} = \frac{(a+b+c)^2 \cdot (a+b+c)}{2(ab+bc+ca)} \geq \frac{3(ab+bc+ca)(a+b+c)}{2(ab+bc+ca)} = \frac{3(a+b+c)}{2}.$$

And finally, from the AM-GM inequality We have

$$\frac{3(a+b+c)}{2} \geq \frac{3 \cdot 3\sqrt[3]{abc}}{2} = \frac{9}{2}$$

So, We have proved that

$$(LHS)^2 \geq \frac{9}{2} \Longrightarrow LHS \geq \frac{3}{\sqrt{2}}$$

$$\Longrightarrow \frac{c\sqrt{a^3+b^3}}{a^2+b^2} + \frac{a\sqrt{b^3+c^3}}{b^2+c^2} + \frac{b\sqrt{c^3+a^3}}{c^2+a^2} \geq \frac{3}{\sqrt{2}},$$

Q.E.D

.

209.

if a, b, c are positive real numbers satisfying the equality $abc = 1$, then prove that

$$\frac{a^{\frac{2}{3}}}{\sqrt{b+c}} + \frac{b^{\frac{2}{3}}}{\sqrt{c+a}} + \frac{c^{\frac{2}{3}}}{\sqrt{a+b}} \geq \frac{3}{\sqrt{2}}.$$

Solution:

Without loss of generality, assume that $a \geq b \geq c$.

Since

$$a^{\frac{2}{3}} \geq b^{\frac{2}{3}} \geq c^{\frac{2}{3}} and \frac{1}{\sqrt{b+c}} \geq \frac{1}{\sqrt{c+a}} \geq \frac{1}{\sqrt{a+b}},$$

using Chebyshev's inequality We get:

$$\frac{a^{\frac{2}{3}}}{\sqrt{b+c}} + \frac{b^{\frac{2}{3}}}{\sqrt{c+a}} + \frac{c^{\frac{2}{3}}}{\sqrt{a+b}} \geq \frac{1}{3}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}}\right) \cdot \left(\frac{1}{\sqrt{b+c}} + \frac{1}{\sqrt{c+a}} + \frac{1}{\sqrt{a+b}}\right).$$

Moreover, From the AM-GM inequality We have that

$$\frac{1}{3}\left(a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}}\right) \cdot \left(\frac{1}{\sqrt{b+c}} + \frac{1}{\sqrt{c+a}} + \frac{1}{\sqrt{a+b}}\right) \geq \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}}}{\sqrt[6]{(a+b)(b+c)(c+a)}}.$$

Therefore, it suffices to prove that

$$2\left(a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}}\right)^2 \geq 9\sqrt[3]{(a+b)(b+c)(c+a)}.$$

Set $a = x^3, b = y^3, c = z^3$.

The above inequality can be written now as

$$2(x^2+y^2+z^2)^2 \geq 9\sqrt[3]{(x^3+y^3)(y^3+z^3)(z^3+x^3)}, or 2(x^2+y^2+z^2)^2 \geq 9\sqrt[3]{xyz(x^3+y^3)(y^3+z^3)(z^3+x^3)}.$$

From the AM-GM inequality once again, We acquire that

$$\sqrt[3]{xyz(x^3+y^3)(y^3+z^3)(z^3+x^3)} \leq \frac{x(y^3+z^3) + y(z^3+x^3) + z(x^3+y^3)}{3}.$$

And thus, it is enough to check that

$$2(x^2+y^2+z^2)^2 \geq 3\left[x(y^3+z^3) + y(z^3+x^3) + z(x^3+y^3)\right],$$

which is equivalent to the obvious inequality

$$(x^2-xy+y^2)(x-y)^2 + (y^2-yz+z^2)((y-z)^2 + (z^2-zx+x^2)(z-x)^2 \geq 0,$$

Q.E.D

.

210.

Let a, b, c be positive real numbers. Prove that

$$1 + \frac{8abc}{(a+b)(b+c)(c+a)} \geq \frac{2(ab+bc+ca)}{a^2+b^2+c^2}.$$

Solution (An idea by Silouanos Brazitikos):

From the above inequality is it enough to show that

$$\frac{8abc}{(a+b)(b+c)(c+a)} \geq \frac{2(ab+bc+ca) - a^2 + b^2 + c^2}{a^2+b^2+c^2}.$$

But from Schur's inequality We know that

$2(ab+bc+ca)-a^2-b^2-c^2 \leq \frac{9abc}{a+b+c}. So it is enough to check that 8(a^2+b^2+c^2)(a+b+c) \geq 9(a+b)(b+c)(c+a).$

From Cauchy-Schwarz inequality We know that

$$3(a^2+b^2+c^2) \geq (a+b+c)^2.$$

Therefore We only need to prove that

$$\left(\frac{a+b+b+c+c+a}{3}\right)^3 \geq (a+b)(b+c)(c+a),$$

which is obviously true from AM-GM inequality,
Q.E.D
.
211.
if x_i for $i = 1, 2, ..., n$ are positive real numbers then prove that:

$$\sum_{i=1}^{n}\left(5\sqrt[5]{x_i^3} - 3\sqrt[3]{\left(\frac{3x_i+2}{5}\right)^5}\right) \leq 2n$$

Solution(An idea by Vo Quoc Ba Can):
We only need to prove that

$$5\sqrt[5]{a^3} - 3\sqrt[3]{\left(\frac{3a+2}{5}\right)^5} \leq 2$$

for all $a > 0$. So, using the AM-GM inequality We have that

$$a + a + a + 1 + 1 \geq 5\sqrt[5]{a^3}.$$

it follows that

$$\sqrt[3]{\left(\frac{3a+2}{5}\right)^5} \geq \sqrt[3]{\left(\sqrt[5]{a^3}\right)^5} = a.$$

Therefore it suffices to prove that $5\sqrt[5]{a^3} - 2 \leq 3a$, which is obviously true from the AM-GM inequality,
Q.E.D
.
212.
Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$.
Prove that

$$(1-a)(1-b)(1-c)(1-d) \geq abcd$$

1st **Solution**:
We divide the inequality with a, b, c, d.
Then We get that

$$\frac{1-a}{a} \cdot \frac{1-b}{b} \cdot \frac{1-c}{c} \cdot \frac{1-d}{d} \geq 1.$$

Let

$$x = \frac{1-a}{a}, y = \frac{1-b}{b}, z = \frac{1-c}{c}, w = \frac{1-d}{d}.$$

We need to prove that $xyzw \geq 1$. But from the hypothesis We get that

$$1 = \sum_{cyc} \frac{1}{(1+x)^2}.$$

From Jensen's inequality We get that

$$1 = \sum_{cyc} \frac{1}{(1+x)^2} \geq \frac{1}{1+xy} + \frac{1}{1+zw}.$$

After some calculations, We get the desired result, that is $xyzw \geq 1$. 2nd **Solution**:
From AM-GM inequality We get that

$$c^2 + d^2 \geq 2cd \Longrightarrow 1 - a^2 - b^2 \geq 2cd.$$

And hence

$$2(1-a)(1-b) - 2cd \geq 2(1-a)(1-b) - 1 + a^2 + b^2 = (1-a-b)^2 \geq 0$$

Similarly We can prove that $(1-c)(1-d) \geq ab$.
So We prooved that $(1-a)(1-b) \geq cd$.
Similarly We can prove that

$$(1-c)(1-d) \geq ab.$$

Multiplying these inequalities the desired inequality follows,
Q.E.D

.
213.
Let a, b, c be non-negative numbers, no two of them are zero. Prove that

$$\frac{a^2}{a^2+ab+b^2} + \frac{b^2}{b^2+bc+c^2} + \frac{c^2}{c^2+ca+a^2} \geq 1.$$

1st **Solution**:
Let

$$A = a^2 + ab + b^2, B = b^2 + bc + c^2, C = c^2 + ca + a^2.$$

We have

$$\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right)\left(\frac{a^2}{A} + \frac{b^2}{B} + \frac{c^2}{C} - 1\right) = \sum_{cyc} \frac{a^2}{A^2} + \sum_{cyc} \frac{b^2+c^2}{BC} - \sum_{cyc} \frac{1}{A}$$

$$= \frac{1}{2} \sum_{cyc} \left(\frac{b}{B} - \frac{c}{C}\right)^2 \geq 0$$

from which the desired inequality follows.
Equality occurs if and only if $a = b = c$. 2nd **Solution**:
Divide each fraction with a^2, b^2, c^2 respectively. Then We get that

$$\sum_{cyc} \frac{1}{1 + \frac{b}{a} + \left(\frac{b}{a}\right)^2} \geq 1.$$

Let us denote

$$\frac{b}{a} = x, \frac{c}{b} = y, \frac{a}{c} = z.$$

Then the inequality transforms to

$$\sum_{cyc} \frac{1}{x^2+x+1} \geq 1.$$

Let us now use the transformation

$$x = \frac{uv}{w^2}, y = \frac{vw}{u^2}, z = \frac{wu}{v^2}$$

which makes the inequality to

$$\sum_{cyc} \frac{u^4}{u^4+v^2w^2+u^2vw} \geq 1.$$

From Cauchy's inequality now, We get that

$$\sum_{cyc} \frac{u^4}{u^4+v^2w^2+u^2vw} \geq \frac{\left(\sum_{cyc} u^2\right)^2}{\sum_{cyc}(u^4+u^2v^2)+uvw\sum_{cyc} u}.$$

So, We only need to prove that

$$\frac{\left(\sum_{cyc} u^2\right)^2}{\sum_{cyc}(u^4+u^2v^2)+uvw\sum_{cyc} u} \geq 1 \Longleftrightarrow \sum_{cyc} u^4 \geq uvw\sum_{cyc} u,$$

which is obviously true. Equality holds only for $a=b=c$,
Q.E.D
.
214.
if a,b,c are non-negative numbers, prove that

$$3(a^2-a+1)(b^2-b+1)(c^2-c+1) \geq 1+abc+a^2b^2c^2.$$

Solution:
From the identity

$$2(a^2-a+1)(b^2-b+1) = 1+a^2b^2+(a-b)^2+(1-a)^2(1-b)^2$$

follows the inequality $2(a^2-a+1)(b^2-b+1) \geq 1+a^2b^2$.
Thus, We only need to prove that

$$3(1+a^2b^2)(c^2-c+1) \geq 2(1+abc+a^2b^2c^2)$$

which is equivalent to the quadratic in c equation

$$(3+a^2b^2)c^2-(3+2ab+3a^2b^2)c+1+3a^2b^2 \geq 0,$$

which is true since the discriminant D is equal to

$$D = -3(1-ab)^4 \leq 0.$$

Equality occurs for $a=b=c=1$,
Q.E.D

.

215. Prove that for any real numbers $a_1, a_2, ..., a_n$ the following inequality holds:

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i,j=1}^{n} \frac{ij}{i+j-1} a_i a_j.$$

Solution:

Observe that

$$\sum_{i,j=1}^{n} \frac{ij}{i+j-1} a_i a_j = \sum_{i,j=1}^{n} ia_i \cdot ja_j \int_0^1 t^{i+j-2} dt.$$

But

$$\sum_{i,j=1}^{n} \frac{ij}{i+j-1} a_i a_j$$

can be considered as a constant. So,

$$\sum_{i,j=1}^{n} ia_i \cdot ja_j \int_0^1 t^{i+j-2} dt = \int_0^1 \left(\sum_{i,j=1}^{n} ia_i \cdot ja_j \cdot t^{i-1+j-1}\right) dt.$$

Notice now that

$$\int_0^1 \left(\sum_{i,j=1}^{n} ia_i \cdot ja_j \cdot t^{i-1+j-1}\right) dt = \int_0^1 \left(\sum_{i=1}^{n} ia_i \cdot t^{i-1}\right)^2 dt.$$

So, the inequality reduces to

$$\int_0^1 \left(\sum_{i=1}^{n} ia_i \cdot t^{i-1}\right)^2 dt \geq \left(\sum_{i=1}^{n} a_i\right)^2.$$

Now, using Cauchy-Schwartz inequality for integrals, We get that

$$\int_0^1 \left(\sum_{i=1}^{n} ia_i \cdot t^{i-1}\right)^2 dt \geq \left(\int_0^1 \left(\sum_{i=1}^{n} ia_i \cdot t^{i-1}\right) dt\right)^2.$$

But, We must now observe that

$$\left(\int_0^1 \left(\sum_{i=1}^{n} ia_i \cdot t^{i-1}\right) dt\right)^2 = \left(\sum_{i=1}^{n} a_i\right)^2,$$

which comes to the conclusion,

Q.E.D

.

216.

Let x, y, z, t be positive real number such that $max(x, y, z, t) \leq min\sqrt{5}min(x, y, z, t)$. Prove that:

$$\frac{xy}{5x^2 - y^2} + \frac{yz}{5y^2 - z^2} + \frac{5z^2 - t^2}{+} \frac{zt}{5t^2 - x^2} \geq 1$$

Solution:

From $max(x, y, z, t) \leq min\sqrt{5}min(x, y, z, t)$ We have $5x^2 - y^2, 5y^2 - z^2, 5z^2 - t^2, 5t^2 - x^2 \geq 0$.

Setting

$$a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{t}, d = \frac{t}{x}$$

We have abcd=1.
The inequality can rewrite:

$$\frac{a}{5a^2-1}+\frac{b}{5b^2-1}+\frac{c}{5c^2-1}+\frac{d}{5d^2-1}\geq 1.$$

We have:

$$\frac{a}{5a^2-1}\geq\frac{1}{a^3+a^2+a+1} <=> \frac{(a-1)^2(a^2+3a+1)}{(5a^2-1)(a^3+a^2+a+1)}\geq 0(true)$$

We will prove that:

$$\frac{1}{(1+a)(1+a^2)}+\frac{1}{(1+b)(1+b^2)}+\frac{1}{(1+c)(1+c^2)}+\frac{1}{(1+d)(1+d^2)}\geq 1.$$

Without loss of generality assume that $a\geq b\geq c\geq d$.
Then from Chebyshev's inequality We have that

$$\frac{1}{(1+a)(1+a^2)}+\frac{1}{(1+b)(1+b^2)}+\frac{1}{(1+c)(1+c^2)}\geq\frac{1}{3}\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right)\left(\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}\right).$$

217.
Positive real numbers $a,b,x_1,x_2,...,x_n$ satisfy the condition $x_1+x_2+...+x_n=1$.
Prove that

$$\frac{x_1^3}{ax_1+bx_2}+\frac{x_2^3}{ax_2+bx_3}+...+\frac{x_n^3}{ax_n+bx_1}\geq\frac{1}{n(a+b)}.$$

Solution:
From Holder's inequality We have that:

$$(1+1+...+1)\left(\sum_{i=1}^{n}\frac{x_1^3}{ax_i+bx_{i+1}}\right)\left[\sum_{i=1}^{n}(ax_i+bx_{i+1})\right]\geq\left(\sum_{i=1}^{n}x_i\right)^3=1.$$

So, it remains to prove

$$\sum_{i=1}^{n}\frac{x_i^3}{ax_i+bx_{i+1}}\geq\frac{1}{n\sum_{i=1}^{n}ax_i+bx_{i+1}}\geq\frac{1}{n(a+b)}.$$

But

$$\sum_{i=1}^{n}ax_i+bx_{i+1}=a+b,$$

Q.E.D
.
218.
if a_1,a_2,a_3 are the positive real roots of the equation $4x^3-kx^2+mx-9=0$ prove that

$$k\geq 4\sqrt[3]{\sum_{cyc}\left(a_1\sqrt{a_2+a_3}\right)+3\prod_{cyc}(a_1+a_2)}.$$

Solution:
Let us divide both sides by 4 and then cube them.
We acquire

$$\left(\frac{k}{4}\right)^3\geq\sum_{cyc}a_1\sqrt{a_2+a_3}+3\prod_{cyc}(a_1+a_2).$$

But from Viete's relations We have that

$$\frac{k}{4} = a_1 + a_2 + a_3 and a_1 a_2 a_3 = \frac{9}{4}.$$

So our inequality transforms into

$$(a_1 + a_2 + a_3)^3 \geq \sum_{cyc} a_1\sqrt{a_2 + a_3} + 3\prod_{cyc}(a_1 + a_2),$$

or

$$\sum_{cyc} a_1^3 + 3\prod_{cyc}(a_1 + a_2) \geq \sum_{cyc} a_1\sqrt{a_2 + a_3} + 3\prod_{cyc}(a_1 + a_2).$$

So, it suffices to prove that

$$\sum_{cyc} a_1^3 \geq \sum_{cyc} a_1\sqrt{a_2 + a_3}.$$

But the last inequality holds because

$$a_1^3 + a_2^3 + a_3^3 \geq a_1 a_2(a_1 + a_2) + a_3^3 \geq 2\sqrt{a_1 a_2 a_3 \cdot a_3^2(a_1 + a_2)} = 3a_3\sqrt{a_1 + a_2}.$$

Adding up the 3 cyclic relations We come to the desired inequality,
Q.E.D

.

219.

if a, b, c are non-negative numbers prove that

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq (ab + bc + ca)^3.$$

Solution:
Lemma: $4(a^2 + ab + b^2) \geq 3(a + b)^2$.
Back to the inequality now, multiply both sides by 64.
Then We have that $4^3\prod_{cyc}(a^2 + ab + b^2) \geq 4^3(ab + bc + ca)^3$.
But from the lemma We reduce the current inequality to

$$27\prod_{cyc}(a + b)^2 \geq 64(ab + bc + ca)^3.$$

it also holds

$$(a + b + c)^2 \geq 3(ab + bc + ca).$$

Multiplying the last inequality with

$$\frac{64}{3}(ab + bc + ca)^2$$

We get that

$$\frac{64}{3}(ab + bc + ca)^2(a + b + c)^2 \geq 64(ab + bc + ca)^3.$$

So, it suffices to prove that

$$27\prod_{cyc}(a + b)^2 \geq \frac{64}{3}(a + b + c)^2(ab + bc + ca)^2$$

or

$$displaystyle 9(a + b)(b + c)(c + a) \geq 8(a + b + c)(ab + bc + ca),$$

which reduces to the obvious inequality

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0.$$

Equality occurs for $(a,b,c) = (1,1,1)$ and also for $(a,b,c) = (1,0,0)$ or any cyclic permutation,
Q.E.D
.
220.
Let x, y, z be non-negative numbers. if $0 \leq r \leq \sqrt{2}$,prove that

$$\sqrt{x^4+y^4+z^4} + r\sqrt{x^2y^2+y^2z^2+z^2x^2} \geq (1+r)\sqrt{x^3y+y^3z+z^3x}.$$

Solution:
if We square both sides We get

$$\sum_{cyc} x^4 + r^2 \sum_{cyc} x^2y^2 + 2r\sqrt{\sum_{cyc} x^4 \sum_{cyc} x^2y^2} \geq r^2 \sum_{cyc} x^3y + \sum_{cyc} x^3y + 2r\sum_{cyc} x^3y.$$

Now from Cauchy-Schwartz inequality We know that

$$2r\sqrt{\sum_{cyc} x^4 \sum_{cyc} x^2y^2} \geq 2r\sum_{cyc} x^3y.$$

So, it suffices to prove that

$$\sum_{cyc} x^4 + r^2 \sum_{cyc} x^2y^2 \geq (1+r^2)\sum_{cyc} x^3y.$$

For $r=0$ the inequality is true.
So, We only need to prove it for $0 < r \leq \sqrt{2}$. Rewrite the inequality in the form

$$\sum_{cyc} x^4 - \sum_{cyc} x^3y \geq r^2\left(\sum_{cyc} x^3y - \sum_{cyc} x^2y^2\right).$$

We know that $\sum_{cyc} x^4 - \sum_{cyc} x^3y \geq 0$ so, it is enough to prove it for $r = \sqrt{2}$.
For $r = \sqrt{2}$ We have that

$$\sum_{cyc} x^4 - \sum_{cyc} x^3y \geq 2\left(\sum_{cyc} x^3y - \sum_{cyc} x^2y^2\right),$$

which reduces to

$$\left(\sum_{cyc} x^2\right)^2 \geq 3\sum_{cyc} x^3y,$$

which is a ill known inequality of Vasile Cirtoaje.
if a, b, c are real numbers prove that $(a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a)$.
Solution of it.
We are going to use the following ill-known inequality: $(x+y+z)^2 \geq 3(xy+yz+zx)$.
So, if We transform the x, y, z to

$$a^2+bc-ab, b^2+ca-bc, c^2+ab-ca$$

respectively We have that

$$\left(a^2+b^2+c^2\right)^2 \geq 3\sum_{cyc}(a^2+bc-ab)(b^2+ca-bc) = 3\sum_{cyc}a^3b = 3(a^3b+b^3c+c^3a),$$

Q.E.D

.

221.

Let a, b, c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{1}{(a+kb)^3}+\frac{1}{(b+kc)^3}+\frac{1}{(c+ka)^3} \geq \frac{3}{(k+1)^3},$$

where k is a non-negative real number.

Solution:

From Holder's inequality We get that

$$\left[\sum_{cyc}\frac{1}{(a+kb)^3}\right]\left[\sum_{cyc}a(b+kc)\right]^3 \geq \left(\sum_{cyc}a^{\frac{3}{4}}\right)^4,$$

or

$$\sum_{cyc}\frac{1}{(a+kb)^3} \geq \frac{\left(a^{\frac{3}{4}}+b^{\frac{3}{4}}+c^{\frac{3}{4}}\right)^4}{(k+1)^3(ab+bc+ca)^3}.$$

So, it suffices to prove that

$$\frac{\left(a^{\frac{3}{4}}+b^{\frac{3}{4}}+c^{\frac{3}{4}}\right)^4}{(k+1)^3(ab+bc+ca)^3} \geq \frac{3}{(k+1)^3}.$$

But the last relation is equivalent to

$$\left(\sum_{cyc}a^{\frac{3}{4}}\right)^4 \geq 81 = 9\sqrt{3}(ab+bc+ca)^{3/2}.$$

Let us denote by

$$x, y, z the a^{3/4}, b^{3/4}, c^{3/4}$$

respectively. Then

$$ab=(xy)^{4/3}, bc=(yz)^{4/3}, ca=(zx)^{4/3}.$$

So, our inequality takes the form

$$(x+y+z)^4 \geq 9\sqrt{3}\sum_{cyc}(xy)^{4/3}.$$

This inequality is homogeneous so, We consider the sum$x+y+z$ equal to 3.
Doing some manipulations in left and right hand side We only need to prove that $\sum_{cyc}(xy)^{4/3} \leq 3$.

Now from the AM-GM inequality We have that

$$\sum_{cyc}xy\cdot(xy)^{1/3} \leq \sum_{cyc}xy\frac{x+y+1}{3} = \sum_{cyc}xy\frac{4-z}{3}.$$

The last one is equal to

$$\sum_{cyc}(xy)^{4/3} \leq \frac{4}{3}\sum_{cyc}xy - xyz.$$

After that, We only need to prove

$$\frac{4}{3}\sum_{cyc} xy - xyz \le 3 \Longrightarrow 4\sum_{cyc} x \sum_{cyc} xy \le 27 + 9xyz = \left(\sum_{cyc} x\right)^3 + 9xyz,$$

which is Schur's 3rd degree inequality,
Q.E.D
.
222.
Let x, y, z be non-negative real numbers. Prove that

$$\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \ge \frac{9}{4(xy+yz+zx)}.$$

1st **Solution**:
The inequality is homogeneous, so, if We normalize it We get that $xy + yz + zx = 1$.
Doing some manipulation in the left hand side We acquire:

$$4\sum_{cyc}(x+y)^2(x+z)^2 \ge 9(x+y)^2(y+z)^2(z+x)^2 \Longrightarrow 4\sum_{cyc}(x^2+1)^2 \ge 9\left(x+y+z-xyz\right)^2.$$

Let us denote by $sthex + y + z$.
Then We have that:

$$4\sum_{cyc}(x^2+1)^2 \ge 9\left(x+y+z-xyz\right)^2 = 4\sum_{cyc} x^4 + 8\sum_{cyc} x^2 + 12 \ge 9(x+y+z-xyz)^2.$$

But $s = x + y + z$, so:

$$*\sum_{cyc} x^2 = s^2 - 2$$

$$*\sum_{cyc} x^4 = s^4 - 4s^2 + 2 + 4xyzs.$$

Thus the previous inequality can be rewritten as

$$4(s^4 - 2s^2 + 1 + 4xyzs) \ge 9(s - xyz)^2.$$

Now, from Schur's inequality We know that

$$\sum_{cyc} x^4 + xyz\sum_{cyc} x \ge \sum_{cyc} xy(x^2+y^2) \Longrightarrow 6xyzs \ge (4-s^2)(s^2-1).$$

So, We come to the conclusion:

$$4(s^4-2s^2+1+4xyzs)-9(s-xyz)^2 = (s^2-4)(4s^2-1)+34xyz-9x^2y^2z^2 \ge (s^2-4)(4s^2-1)+33xyz \ge (s^2-4)(4s^2-1)+$$

2nd **Solution**:
Doing all the manipulations in the left and in the right hand side We only need to prove that

$$4\sum_{sym} x^5y + \sum_{sym} x^4yz + 3\sum_{sym} x^2y^2z^2 - 3\sum_{sym} x^3y^3 - 2\sum_{sym} x^3y^2z - \sum_{sym} x^4y^2 \ge 0.$$

But the last one holds because it is equivalent to:

$$3\left(\sum_{sym} x^5y - \sum_{sym} x^3y^3\right) + \left(\sum_{sym} x^5y - \sum_{sym} x^4y^2\right) + 2xyz\sum_{cyc} x(x-y)(x-z) \ge 0,$$

whose 2 first terms hold from Muirhead's inequality and the last one from Schur's inequality.
3nd **Solution**:
Let $a = x + y, b = y + z, c = z + x$. Then the inequality takes the following form:

$$\left(-a^2 - b^2 - c^2 + 2ab + 2bc + 2ca\right)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq 9.$$

Doing the manipulations in the left hand side We get that

$$\left(\sum_{cyc} \frac{2a}{b} + \frac{2b}{a} - 4\right) - \sum_{cyc}\left(\frac{a^2}{c^2} + \frac{b^2}{c^2} - \frac{2ab}{c^2}\right) \geq 0.$$

Thus We obtain that

$$\sum_{cyc} \frac{2}{ab}(a-b)^2 - \frac{1}{c^2}(a-b)^2 \geq 0.$$

From here We obtain that

$$\sum_{cyc}\left(\frac{2}{ab} - \frac{1}{c^2}\right)(b-c)^2 \geq 0.$$

Let us denote by S_a the

$$\frac{2}{bc} - \frac{1}{a^2}$$

and the S_b, S_c similarly.
Without loss of generality assume that $a \geq b \geq c$.
From here We have that

$$S_a \geq 0 S_a \geq S_b \geq S_c.$$

Fro the end of our **Solution** We only need to show that

$$b^2 S_b + c^2 S_c \geq 0,$$

which reduces to

$$b^3 + c^3 \geq abc \Longrightarrow b + c \geq a,$$

Q.E.D

.

223.
Let x, y, z be positive real numbers such that $xyz = 8$. Prove that

$$\frac{x^2}{\sqrt{(x^3+1)(y^3+1)}} + \frac{y^2}{\sqrt{(y^3+1)(z^3+1)}} + \frac{z^2}{\sqrt{(z^3+1)(x^3+1)}} \geq \frac{4}{3}.$$

Solution:
From the AM-GM ineuqality We know that

$$\frac{1}{\sqrt{x^3+1}} = \frac{1}{\sqrt{(x+1)(x^2-x+1)}} \geq \frac{2}{(x+1)+(x^2-x+1)} = \frac{2}{x^2+2}.$$

Doing that cyclic over the 3 fractions We get that

$$\sum_{cyc} \frac{x^2}{\sqrt{(x^3+1)(y^3+1)}} \geq \frac{4x^2}{(x^2+2)(y^2+2)}.$$

So, it suffices to prove that

$$\frac{4x^2}{(x^2+2)(y^2+2)} \geq \frac{4}{3},$$

or

$$3\sum_{cyc} x^2(z^2+2) \geq \prod_{cyc}(x^2+2).$$

After expanding, the inequality is equivalent to

$$2\sum_{cyc} x^2 + \sum_{cyc} x^2y^2 \geq 72.$$

But the last relation holds due to the AM-GM inequality and from the hypothesis, since

$$\sum_{cyc} x^2y^2 \geq 3\sqrt[3]{8^4} = 482\sum_{cyc} x^2 \geq 6\sqrt[3]{8^2} = 24.$$

Adding up these 2 relations We get the desired result,
Q.E.D
.
224.
For all non-negative real numbers a, b, c with sum 2, prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \leq 3.$$

Solution:
Assume without loss of generality that $a \geq b \geq c$. Moreover, denote by t, u the

$$\frac{a+b}{2}, \frac{a-b}{2}.$$

Then We get that $a = t+u, b = t-u$.
From the hypothesis We deduce also that $t \leq 1$.
Let us now transform the 3 factors of the inequality in terms of t, u. Thus We have that:

$$a^2+ab+b^2 = (t+u)^2 + (t+u)(t-u) + (t-u)^2 = 3t^2+u^2$$

and

$$(b^2+bc+c^2)(c^2+ca+a^2) = (t^2+tc+c^2)^2 - u^2(2tc-c^2-u^2+2t^2).$$

Define by $f\left(a,b,c\right)$ the Left Hand Side of the inequality, that is

$$f\left(a,b,c\right) = (a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2).$$

We will now prove that $f(t,t,c) - f(a,b,c) \geq 0$.
Denote the Left Hand Side by X. We must prove that $X \geq 0$.
We know that

$$X = u^2(5t^4+4t^3c-6t^2c^2-2tc^2-c^4-u^4-u^2(t-c)^2).$$

We claim that the second factor of $X$is greater than zero.
indeed, this is true as since $t = \max\{c, u\}$ We get that:

$$2t \geq t+c \Longrightarrow 4t^2 \geq (t+c)^2 t \geq c.$$

Multiplying these 2 inequalities We have that $4t^3c \geq c^2(t+c)^2$ or

$$4t^3c \geq t^2c^2+2tc^2+c^4.$$

Adding up the $5t^2c^2$ We get that

$$4t^3c + 5t^2c^2 \geq 6t^2c^2 + 2tc^3 + c^4.$$

Thus We have prove that

$$5t^4 + 4t^3c - 6t^2c^2 - 2tc^2 - c^4 \geq 5t^4 - 5t^2c^2 = 5t^2(t-c)(t+c).$$

But from the last inequality We deduce that

$$5t^2(t-c)(t+c) \geq 5(t-c)t^3 \geq 2(t-c)^4 \geq u^2(t-c)^2 + u^4$$

due to the maximized value of t.
This completes the first scale of the **Solution**.
Now We only need to prove that if $2t + c = 2$ then

$$3t^2(t^2 + tc + c^2) \leq 3$$

which is obviously true since it is of the form $(1-t)(3t^2 - 3t + 1) \geq 0$, Q.E.D
.
225.
Let ΔABC be a acute triangle, prove that:

$$\frac{cosA.cosB}{sin2C} + \frac{cosB.cosC}{sin2A} + \frac{cosC.cosA}{sin2B} \geq \frac{\sqrt{3}}{2}$$

Solution:
The inequality can be written in the algebraic form:
if a, b, c are positive real numbers, then

$$\frac{\sqrt{a(a+b)(a+c)}}{b+c} + \frac{\sqrt{b(b+c)(b+a)}}{c+a} + \frac{\sqrt{c(c+a)(c+b)}}{a+b} \geq \sqrt{3(a+b+c)}.$$

Using the known inequality

$$(x+y+z)^2 \geq 3(xy+yz+zx),$$

We see that it suffices to prove that

$$\sum \frac{(a+b)\sqrt{ab(a+c)(b+c)}}{(a+c)(b+c)} \geq a+b+c.$$

Using the Cauchy-Schwarz inequality along with the AM-GM inequality, We get

$$(a+b)\sqrt{ab(a+c)(b+c)} \geq (a+b)\left(ab + c\sqrt{ab}\right) = ab(a+b) + c\sqrt{ab}(a+b)$$
$$\geq ab(a+b) + 2abc.$$

it follows that

$$\sum \frac{(a+b)\sqrt{ab(a+c)(b+c)}}{(a+c)(b+c)} \geq \sum \frac{ab(a+b+2c)}{(a+c)(b+c)}$$
$$= \sum \left(\frac{ab}{b+c} + \frac{ab}{c+a}\right) = \sum a.$$

226.
Let a,b,c be positive real numbers . Prove that:

$$\frac{\sqrt[3]{b+c}}{a} + \frac{\sqrt[3]{c+a}}{b} + \frac{\sqrt[3]{a+b}}{c} \geq \sqrt[3]{\frac{54(a+b+c)^2}{(ab+bc+ca)(a^2+b^2+c^2)}}$$

Solution:

$$\left(\frac{\sqrt[3]{a+b}}{c}+\frac{\sqrt[3]{a+c}}{b}+\frac{\sqrt[3]{b+c}}{a}\right)^3 \geq \frac{54(a+b+c)^2}{(ab+ac+bc)\left(a^2+b^2+c^2\right)}$$

From Chebyshev (WLOG $0 \leq a \leq b \leq c$)

$$\left(\frac{\sqrt[3]{a+b}}{c}+\frac{\sqrt[3]{a+c}}{b}+\frac{\sqrt[3]{b+c}}{a}\right)^3 \geq \frac{1}{27}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3\left(\sqrt[3]{a+b}+\sqrt[3]{a+c}+\sqrt[3]{b+c}\right)^3$$

From Poir Mean

$$\left(\sqrt[3]{a+b}+\sqrt[3]{a+c}+\sqrt[3]{b+c}\right)^3 \geq 27\sqrt[3]{(a+b)(a+c)(b+c)}$$

From AM-GM

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^3 \geq \frac{27}{abc}$$

So

$$\left(\frac{\sqrt[3]{a+b}}{c}+\frac{\sqrt[3]{a+c}}{b}+\frac{\sqrt[3]{b+c}}{a}\right)^3 \geq \frac{27\sqrt[3]{(a+b)(a+c)(b+c)}}{abc}$$

So We just need to prove

$$\sqrt[3]{(a+b)(a+c)(b+c)} \geq \frac{2abc(a+b+c)^2}{(ab+ac+bc)\left(a^2+b^2+c^2\right)}$$

From GMHM

$$\sqrt[3]{(a+b)(a+c)(b+c)} \geq \frac{3}{\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}}=\frac{3(a+b)(a+c)(b+c)}{(a+b+c)^2+ab+ac+bc}$$

Expanding

$$3(a+b)(a+c)(b+c) \geq \frac{2abc(a+b+c)^4}{(ab+ac+bc)\left(a^2+b^2+c^2\right)}+\frac{2abc(a+b+c)^2}{a^2+b^2+c^2}$$

227.

Let $a,b,c>0$. Prove that following inequality holds

$$\frac{a+b}{a+b+2c}+\frac{b+c}{b+c+2a}+\frac{c+a}{c+a+2b}+\frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)} \leq \frac{13}{6}$$

Solution:

Let $a^2+b^2+c^2=t(ab+ac+bc)$. Hence,$t \geq 1$ and

$$\sum_{cyc}\frac{a+b}{a+b+2c}+\frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)} \leq \frac{13}{6} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\left(1-\frac{a+b}{a+b+2c}\right) \geq \frac{5}{6}+\frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)} \Leftrightarrow \sum_{cyc}\frac{2c}{a+b+2c} \geq \frac{5}{6}+\frac{2}{3t}.$$

But

$$\sum_{cyc}\frac{2c}{a+b+2c}=\sum_{cyc}\frac{2c^2}{ac+bc+2c^2} \geq \frac{2(a+b+c)^2}{\sum(2a^2+2ab)}=\frac{t+2}{t+1}.$$

id est, it remains to prove that

$$\frac{t+2}{t+1} \geq \frac{5}{6} + \frac{2}{3t} <=> \frac{(t-1)(t+4)}{(t+1)18t} \geq 0.$$

which is true for $t \geq 1$.
228.
Let a,b,c be three positive numbers. Prove that :

$$(\frac{a+b}{b})^2 + (\frac{b+c}{c})^2 + (\frac{c+a}{a})^2 \geq \frac{8(a^2+b^2+c^2)}{ab+bc+ca} + 4$$

Solution:
Write the inequality as

$$\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + 3\right) + 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq \frac{8(a^2+b^2+c^2)}{ab+bc+ca} + 4.$$

Applying inequality

$$x^2 + y^2 + z^2 + 2xyz + 1 \geq 2(xy + yz + zx)$$

for

$$x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a},$$

We get

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + 3 \geq 2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right).$$

Therefore, it suffices to prove that

$$\left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) \geq \frac{4(a^2+b^2+c^2)}{ab+bc+ca} + 2,$$

which is equivalent to

$$\left(\frac{a}{b} + \frac{b}{a} + 2\right) + \left(\frac{b}{c} + \frac{c}{b} + 2\right) + \left(\frac{c}{a} + \frac{a}{c} + 2\right) \geq \frac{4(a^2+b^2+c^2)}{ab+bc+ca} + 8,$$

or

$$\frac{(a+b)^2}{ab} + \frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca} \geq \frac{4(a+b+c)^2}{ab+bc+ca},$$

which is true according to the Cauchy-Schwarz inequality

$$\sum \frac{(a+b)^2}{ab} \geq \frac{\left[\sum(a+b)\right]^2}{\sum ab} = \frac{4(a+b+c)^2}{ab+bc+ca}.$$

229.
Let a, b, c be positive real number . Prove that:

$$\sum \frac{bc}{b+c-a} \leq \frac{(5R-2r)(4R+r)^2}{4s(2R-r)}$$

Solution:
We have :

$$ab + bc + ca = 4Rr + r^2$$

$$\Leftrightarrow \frac{(5R-2r)(4R+r)^2}{4s(2R-r)} = \frac{5R-2r}{2R-r}\frac{(4Rr+r^2)^2}{4sr^2} =$$

$$= \frac{5R-2r}{2R-r} \frac{(ab+bc+ca)^2}{8(a+b-c)(b+c-a)(c+a-b)}$$

$$= (2+\frac{R}{2R-r}) \frac{(ab+bc+ca)^2}{8(a+b-c)(b+c-a)(c+a-b)}$$

And

$$\frac{r}{R} = \cos A + \cos B + \cos C - 1 \Leftrightarrow \frac{R}{2R-r} = \frac{1}{2-\frac{r}{R}}$$

$$= \frac{2abc}{a^3+b^3+c^3+6abc-\sum\limits_{sym} a^2b} \geq \frac{2abc}{a^3+b^3+c^3}$$

The ineq will be true if We prove that :

$$\sum \frac{bc}{b+c-a} \leq (2+\frac{2abc}{a^3+b^3+c^3}) \frac{(ab+bc+ca)^2}{8(a+b-c)(b+c-a)(c+a-b)}$$

$$\Leftrightarrow 8\sum bc(a+b-c)(a+c-b) \leq (2+\frac{2abc}{a^3+b^3+c^3})(ab+bc+ca)^2$$

$$\Leftrightarrow 8abc(a+b+c)+16\sum b^2c^2 - 8\sum bc(b^2+c^2) \leq 4abc(a+b+c)+2\sum b^2c^2 + \frac{2abc}{a^3+b^3+c^3}(ab+bc+ca)^2$$

$$\Leftrightarrow 8\sum bc(b-c)^2 + \sum a^2(b-c)^2 \geq 2abc((a+b+c) - \frac{(ab+bc+ca)^2}{a^3+b^3+c^3})$$

$$\Leftrightarrow 8\sum bc(b-c)^2 + \sum a^2(b-c)^2 \geq abc(\frac{\sum (a+b)^2(a-b)^2 + 2\sum (a-b)^2 c(a+b)}{2(a^3+b^3+c^3)})$$

$$\Leftrightarrow \sum (a-b)^2 \left(8ab + c^2 - \frac{abc(a+b)(a+b+2c)}{2(a^3+b^3+c^3)}\right)$$

We put the inequality into the form of SOS technique:

$$S_c = 8ab + c^2 - \tfrac{abc(a+b)(a+b+2c)}{2(a^3+b^3+c^3)}$$
$$S_b = 8ac + b^2 - \tfrac{abc(a+c)(a+c+2b)}{2(a^3+b^3+c^3)}$$
$$S_a = 8bc + a^2 - \tfrac{abc(b+c)(b+c+2a)}{2(a^3+b^3+c^3)}$$

WROG $a \geq b \geq c \geq 0$
Easy to see that : $S_b, S_c, S_b + S_a \geq 0$, and We have Q.E.D
.

230. Let $a, b, c > 0$
prove that :

$$\sqrt[3]{\frac{a}{a^2b+c}} + \sqrt[3]{\frac{b}{b^2c+a}} + \sqrt[3]{\frac{c}{c^2a+b}} \geq \sqrt[3]{\frac{3(ab+bc+ca)^2}{2}}$$

Solution:
by Cauchy-Schwarz We get

$$\sum \sqrt[3]{\frac{a}{a^2b+c}} \geq \frac{(\sum a)^2}{\sum a^2 \sqrt[3]{b^2+\frac{c}{a}}}$$

applying iighted Jensen for $f(x) = \sqrt[3]{x}$ We have:

$$\sum a^2 \sqrt[3]{b^2+\frac{c}{a}} \leq 3\sqrt[3]{\frac{\sum a^2b^2 + \sum ac}{3}}$$

hence it's enough to prove that

$$(\sum a)^2 \geq 3\sqrt[3]{\frac{(\sum ab)^2(\sum a^2b^2+\sum ac)}{2}}$$

or equivalently

$$\sqrt[3]{\frac{\sum a^2}{3}}(\sum a)^2 \geq 3\sqrt[3]{\frac{(\sum ab)^2(\sum a^2b^2+\sum ac\frac{\sum a^2}{3})}{2}}$$

using $\sum a^2 \geq \sum ac$
and

$$\frac{1}{3}(\sum a^2)^2 \geq \sum a^2b^2$$

We need to prove inequality:

$$\frac{(\sum a)^2}{3} \geq \sqrt[3]{(\sum a^2)(\sum ab)^2}$$

which is true by Am-Gm.
231.
Let a,b,c be positive integer such that $abc=1$, prove that :

$$\sum \frac{1}{\sqrt{(a^2+ab+b^2)(b^2+bc+c^2)}} \geq \frac{9}{(a+b+c)(ab+bc+ca)}$$

Solution:
From Am-Gm inequality We have

$$LHS = \sum \frac{\sqrt{b^2+bc+c^2}}{\sqrt{a^2+ab+b^2}(b^2+bc+c^2)}$$
$$\geq 3\sqrt[3]{\frac{1}{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)}}$$

and Thus it suffices to show that

$$(a+b+c)(ab+bc+ca) \geq 3\sqrt[3]{\prod(a^2+ab+b^2)}$$

and We have

$$(a+b+c)(ab+bc+ca) = \sum(a^2c+b^2c+abc) \geq 3\sqrt[3]{\prod(a^2c+abc+b^2c)} = 3\sqrt[3]{\prod(a^2+ab+b^2)}$$

The **Solution** is completed equality holds if and only if $a=b=c$
232.
Give a,b,c>0 prove that:

$$\frac{ab}{c(c+a)}+\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)} \geq \frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}$$

Solution: This ineq is equivalent to

$$\frac{\sum a^2b^2(b+a)(b+c)}{abc} \geq \sum a(c+a)(c+b)$$

$$\Leftrightarrow \sum a^2b^2(b^2+ba+bc+ac) \geq \sum a^2bc(c^2+ca+cb+ab)$$

$$\Leftrightarrow (a^2b^4+b^2c^4+c^2a^4)+(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2) \geq abc(ab+bc+ca)(a+b+c)+abc(ac^2+cb^2+ba^2)$$

By AM-GM ineq, We have :

$$a^2b^2+b^2c^2+c^2a^2 \geq abc(a+b+c) \Rightarrow (ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2)abc(ab+bc+ca)(a+b+c)$$

$$a^2b^4+b^2c^4+c^2a^4 = \frac{a^2b^4+b^2c^4}{2}+\frac{b^2c^4+c^2a^4}{2}+\frac{c^2a^4+a^2b^4}{2} \geq abc(ac^2+cb^2+ba^2)$$

Adding up these ineqs , We have $LHS \geq RHS$ ang We are done. equality holds when a= $b=c$.

233.

Prove that for a,b,c positive reals

$$a\sqrt{a^2+2bc}+b\sqrt{b^2+2ac}+c\sqrt{c^2+2ba} \geq \sqrt{3}(ab+bc+ca).$$

Solution:

1)

Using Holder's inequality, We have

$$\left(\sum a\sqrt{a^2+2bc}\right)^2\left(\sum \frac{a}{a^2+2bc}\right) \geq \left(\sum a\right)^3.$$

Thus, it suffices to prove that

$$\frac{a}{a^2+2bc}+\frac{b}{b^2+2ca}+\frac{c}{c^2+2ab} \leq \frac{(a+b+c)^3}{3(ab+bc+ca)}.$$

Using now the known inequalities

$$\frac{a}{a^2+2bc}+\frac{b}{b^2+2ca}+\frac{c}{c^2+2ab} \leq \frac{a+b+c}{ab+bc+ca},$$

We see that it is enough to check that

$$\frac{a+b+c}{ab+bc+ca} \leq \frac{(a+b+c)^3}{3(ab+bc+ca)},$$

which is equivalent to the obvious inequality

$$3(ab+bc+ca) \leq (a+b+c)^2.$$

2)

$$\Longleftrightarrow (a\sqrt{a^2+2bc}+b\sqrt{b^2+2ac}+c\sqrt{c^2+2ba})^2-3(ab+bc+ca)^2 \geq 0$$

$$\Longleftrightarrow \sum_{cyc} a^2(a^2+2bc)+2\sum_{cyc} ab\sqrt{(a^2+2bc)(b^2+2ac)}-3(ab+bc+ca)^2 \geq 0$$

Note

$$\sum_{cyc} a^2(a^2+2bc)+2\sum_{cyc} ab\sqrt{(a^2+2bc)(b^2+2ac)} \geq \sum_{cyc} a^2(a^2+2bc)+2\sum_{cyc} ab(ab+2c\sqrt{ab})$$

We need only prove

$$\sum_{cyc} a^2(a^2+2bc)+2\sum_{cyc} ab(ab+2c\sqrt{ab})-3(ab+bc+ca)^2 \geq 0..........(*)$$

substitution $a=x^2, b=y^2, c=z^2,(x,y,z\geq 0)$

$$(*) \iff x^8-4x^4y^2z^2+y^8-4y^4z^2x^2+z^8-4z^4y^2x^2-y^4x^4+4y^3x^3z^2-y^4z^4+4y^3z^3x^2-z^4x^4+4z^3x^3y^2 \geq 0$$

$$=\frac{1}{2}\sum_{cyc}(x^3+x^2y+2zxy+xy^2+y^3)(x^3+x^2y-2zxy+xy^2+y^3)(x-y)^2 \geq 0$$

assume $x\geq y\geq z$
let

$$S_y=(z^3+z^2x+2zxy+x^2z+x^3)(z^3+z^2x-2zxy+x^2z+x^3)$$

$$S_x=(y^3+y^2z+2zxy+yz^2+z^3)(y^3+y^2z-2zxy+yz^2+z^3)$$

$$S_z=(x^3+x^2y+2zxy+xy^2+y^3)(x^3+x^2y-2zxy+xy^2+y^3)$$

easy prove that

$$S_y=(z^3+z^2x+2zxy+x^2z+x^3)(z^3+z^2x-2zxy+x^2z+x^3)\geq 0$$

$$S_y+S_x=(z^3+z^2x+2zxy+x^2z+x^3)(z^3+z^2x-2zxy+x^2z+x^3)$$

$$+(y^3+y^2z+2zxy+yz^2+z^3)(y^3+y^2z-2zxy+yz^2+z^3)\geq 0$$

$$S_z=(x^3+x^2y+2zxy+xy^2+y^3)(x^3+x^2y-2zxy+xy^2+y^3)\geq 0$$

Q.E.D

234. Let a,b,c be positive real number such that $a+b+c=1$ prove:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq 3(a^2+b^2+c^2)$$

Solution:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq 3(a^2+b^2+c^2)$$

$$\iff (\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a})(a+b+c)\geq 3(a^2+b^2+c^2)$$

$$\iff \frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+\frac{a^2c}{b}+\frac{b^2a}{c}+\frac{c^2b}{a}+a^2+b^2+c^2\geq 3(a^2+b^2+c^2)$$

$$\iff \frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+\frac{a^2c}{b}+\frac{b^2a}{c}+\frac{c^2b}{a}\geq 2(a^2+b^2+c^2)$$

$$(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+\frac{a^2c}{b}+\frac{b^2a}{c}+\frac{c^2b}{a})(ab+bc+ca+bc+ca+ab)\geq (a^2+b^2+c^2+ac+ba+ac)^2$$

by Cauchy-Schwarts

$$\Longrightarrow \frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+\frac{a^2c}{b}+\frac{b^2a}{c}+\frac{c^2b}{a}\geq \frac{(a^2+b^2+c^2+ac+ba+ac)^2}{2(ab+bc+ca)}$$

$$a^2+b^2+c^2+ac+ba+ac\geq 2\sqrt{(a^2+b^2+c^2)(ac+ba+ac)}$$

by AM-GM

$$\Longrightarrow (a^2+b^2+c^2+ac+ba+ac)^2 \geq 4(a^2+b^2+c^2)(ac+ba+ac)$$

$$\Longrightarrow \frac{(a^2+b^2+c^2+ac+ba+ac)^2}{2(ab+bc+ca)} \geq 2(a^2+b^2+c^2)$$

Then We are done.

234.

Prove that

$$\sum \sqrt{1-\sin A \sin B} \geq \frac{3}{2}$$

with ABC is a triangle.

Solutuon: We can rewrite this into (using the Sine rule) :

$$3R \leq \sum_{cyc} \sqrt{4R^2-ab};$$

Which is equivalent to with

$$\sum_{cyc} \left(\sqrt{4R^2-ab}-R\right) \geq 0;$$

Or,

$$\sum_{cyc} \frac{3R^2-ab}{R+\sqrt{4R^2-ab}} \geq 0.$$

The famous inequality $9R^2 \geq a^2+b^2+c^2$ gives us $3(3R^2-ab) \geq c^2-ab$ and its similar inequalities, so that We have to show that

$$\sum_{cyc} \frac{c^2}{R+\sqrt{4R^2-ab}} \geq \sum_{cyc} \frac{ab}{R+\sqrt{4R^2-ab}}.$$

Note that the sequences $\{a,b,c\}$ and $\left\{\frac{1}{R+\sqrt{4R^2-bc}};\frac{1}{R+\sqrt{4R^2-ca}};\frac{1}{R+\sqrt{4R^2-ab}}\right\}$ are similarly sorted,

so that from the Rearrangement inequality We have

$$\begin{aligned}\sum_{cyc} \frac{2c^2}{R+\sqrt{4R^2-ab}} &\geq \sum_{cyc} \frac{a^2}{R+\sqrt{4R^2-ab}} + \sum_{cyc} \frac{b^2}{R+\sqrt{4R^2-ab}} \\ &\geq \sum_{cyc} \frac{2ab}{R+\sqrt{4R^2-ab}};\end{aligned}$$

And hence We are done.

235.

Suppose $A \subset \{(a_1,a_2,\ldots,a_n) \mid a_i \in \mathbb{R}, i=1,2\ldots,n\}$. For any $\alpha=(a_1,a_2,\ldots,a_n) \in A$ and $\beta=(b_1,b_2,\ldots,b_n) \in A$, We define

$$\gamma(\alpha,\beta)=(|a_1-b_1|,|a_2-b_2|,\ldots,|a_n-b_n|),$$

$$D(A)=\{\gamma(\alpha,\beta) \mid \alpha,\beta \in A\}.$$

Show that $|D(A)| \geq |A|$

Solution: induct on n. For the base case $n=1$ let the elements of A be $z_1,z_2,\cdots,z_{|A|}$ and, without loss of generality, $z_1<z_2<z_3<\cdots<z_{|A|}$.

Thus, for all $1 \leq j \leq |A|$, $(z_We - z_1)$ is an element of $D(A)$. Since $z_We - z_1 \geq 0$ and $z_i \neq z_j$ when $i \neq j$, $D(A)$ has at least $|A|$ elements

. Now suppose the inequality is true when A consists of at $n-1$-tuples and We will prove the inequality if A consists of n-tuples.

Let A consist of the elements $e_We = \{a_{i,1}, a_{i,2}, a_{i,3}, \cdots, a_{i,n}\}$ for all $1 \leq i \leq a = |A|$.

Consider the $n-1$-tuples $e'_i = \{a_{i,2}, a_{i,3}, a_{i,4}, \cdots, a_{i,n}\}$ for all $1 \leq i \leq a$, and let $f_1, f_2, f_3, \cdots, f_t$ be the set of distinct elements e'_i,

ordered such that if n_j is the number of e'_i so that $e'_i = f_j$ for each $1 \leq j \leq t$, then $n_1 \leq n_2 \leq n_3 \leq \cdots \leq n_t$. Set A_j,

for all $1 \leq j \leq t$, to be the set of e_i so that $e'_i = f_j$. By definition, $\sum_{i=1}^{t} n_i = a$.

Define the function $f(f_i, f_j) = \{|a_{u,2} - a_{v,2}|, |a_{u,3} - a_{v,3}|, \cdots, |a_{u,n} - a_{b,n}|\}$ if $e'_u = f_i$ and $e'_v = f_j$.

Correspond these sets to vertices of a graph; let vertex v_i correspond to the set A_i in our graph.

Now define the following process. Start at v_t, draw an edge to itself, and record the $n-1$-tuple $f(f_t, f_t) = \{0, 0, 0, \cdots, 0\}$. Then, draw an edge betien v_t and v_{t-1}

and record the difference $f(f_{t-1}, f_t)$. Now, if $f(f_t, f_{t-2})$ has not yet been recorded, draw an edge betien v_t and v_{t-2} and record $f(f_t, f_{t-2})$.

Otherwise, do nothing and proceed. if $f(f_{t-1}, f_{t-2})$ has not been recorded, draw an edge betien v_{t-1} and v_{t-2} and record $f(f_{t-1}, f_{t-2})$.

Otherwise, do nothing and proceed. Do the same for the pairs (v_t, v_{t-3}), (v_{t-1}, v_{t-3}), and (v_{t-2}, v_{t-3}) in that order. Keep doing this for $t-4, t-5, \cdots, 1$.

Say that once We have determined whether to draw an edge betien v_{i-1} and v_i, We have "completed the ith set."

By Lemma 1 below, there are at least $\left\lceil \frac{n_x + n_y - 1}{2} \right\rceil$ differences betien elements of A_x and A_y (upon taking differences betien the first element of A_x and the first of A_y)

. Moreover, by the base case of the original problem, there are at least n_t differences betien elements of A_t. Let $n(A_x, A_y)$ be the number of distinct n-tuples in the set of $f(s,t)$

, where s ranges over all elements of A_x and t ranges over all elements of A_y. Summing this over all sets (x, y) so that v_x and v_y are connected gives that

$$|D(A)| \geq \sum_{v_i, v_j connected} n(A_i, A_j) + n_t \geq \sum_{v_i, v_j connected} \left\lceil \frac{n_We + n_j - 1}{2} \right\rceil + n_t$$

because differences betien A_i and A_j will form a "new difference" in the last $n-1$ elements of the n-tuple if v_i and v_j are connected. By the inductive hypothesis, after completing the jth set, there are at least j edges for all $1 \leq j \leq n$. Hence (by induction, for instance),

$$\sum_{v_i, v_j connected} \left\lceil \frac{n_We + n_j - 1}{2} \right\rceil \geq \sum_{i=2}^{t} \left\lceil \frac{n_We + n_{i-1} - 1}{2} \right\rceil + n_t$$

which is greater than or equal to $\sum_{i=1}^{t} n_We = a = |A|$ by Lemma 2 below. Hence, $|D(A)| \geq |A|$, as desired.

Lemma 1: Given two sets of reals $X = \{x_1, x_2, x_3, \cdots, x_k\}$ and $Y = \{y_1, y_2, y_3, \cdots, y_l\}$, the set containing the distinct values of $|x_We - y_j|$, where i ranges from 1 to k inclusive and j ranges from 1 to l inclusive, contains at least $\lceil \frac{k+l-1}{2} \rceil$ elements.

Solution: it suffices to show that there are at most $k + l - 1$ elements in the set consisting of distinct values of $x_We - y_j$

. Proceed by induction on $k+l$ to prove this. if $k+l=2$, this is clear. Suppose it is true for all k, l so that $k+l \leq r-1$, and We will show that it holds for k, l so that $k+l=r$.
Assume that $x_1 < x_2 < x_3 < \cdots < x_k$ and $y_1 < y_2 < y_3 < \cdots < y_l$. if $\min\{k, l\} = 1$, suppose $k = 1$, so the elements $(y_We - x_1)$, for all $1 \leq i \leq l$ are pairwise distinct and lie in our set of differences, thereby yielding l differences, as claimed.
Now, suppose $\min\{k, l\} > 1$. if $X' = X - x_k$, then there are at least $k+l-2$ distinct values among differences betien elements of X' and Y due to the inductive hypothesis. Now, $x_k - y_1 \geq x_k - y_j \geq x_We - y_j$ for all $1 \leq i \leq k$ and $1 \leq j \leq l$, where equality only occurs when $We = k$ and $j = 1$. Thus, $x_k - y_1$ is a new difference, so our set has at least $k+l-1$ elements, as claimed.
Lemma 2:Given integers $t \geq 2$ and $1 \leq n_1 \leq n_2 \leq n_3 \leq \cdots \leq n_t$,

$$\sum_{i=2}^{t} \left\lceil \frac{n_We + n_{i-1} - 1}{2} \right\rceil + n_t \geq \sum_{i=1}^{t} n_i$$

Solution: Proceed by induction on t. Consider the base case, when $t = 2$. if $n_1 = n_2$, then $\lceil \frac{n_1+n_2-1}{2} \rceil = n_1$, so

$$\left\lceil \frac{n_1 + n_2 - 1}{2} \right\rceil + n_2 = n_1 + n_2$$

as claimed. if $n_1 > n_2$, then

$$\left\lceil \frac{n_1 + n_2 - 1}{2} \right\rceil + n_2 \geq \frac{n_1 + n_1 + 1 - 1}{2} + n_2 = n_1 + n_2$$

The base case is thus proven. Now suppose that result holds for $t = r-1$, and We shall prove the result for $t = r$. if $n_{r-1} = n_r$, then

$$\sum_{i=2}^{r} \left\lceil \frac{n_We + n_{i-1} - 1}{2} \right\rceil + n_r = \sum_{i=2}^{r-1} \left\lceil \frac{n_We + n_{i-1} - 1}{2} \right\rceil + n_{r-1} + \left\lceil \frac{n_{r-1} + n_r}{2} \right\rceil$$

$$\geq \sum_{i=1}^{r-1} n_We + n_r = \sum_{i=1}^{r} n_i$$

if $n_r > n_{r-1}$, then

$$\sum_{i=2}^{r} \left\lceil \frac{n_We + n_{i-1} - 1}{2} \right\rceil + n_r = \sum_{i=2}^{r-1} \left\lceil \frac{n_We + n_{i-1} - 1}{2} \right\rceil + n_r + \left\lceil \frac{n_{r-1} + n_r - 1}{2} \right\rceil$$

$$\geq \sum_{i=2}^{r-1} \left\lceil \frac{n_We + n_{i-1} - 1}{2} \right\rceil + \frac{n_{r-1} + n_{r-1} + 1 - 1}{2} + n_r \geq \sum_{i=1}^{r} n_We$$

by the inductive hypothesis. Hence, the lemma is proven.
236.
Let a, b, c be positive reals such that $abc = 1$. Show that

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}.$$

Solution: Set

$$a = \frac{1}{x}, b = \frac{1}{y}, c = \frac{1}{z}$$

then: $x, y, z > 0, xyz = 1$

$$\frac{1}{a^5(b+2c)^2} + \frac{1}{b^5(c+2a)^2} + \frac{1}{c^5(a+2b)^2} \geq \frac{1}{3}$$

$$\Leftrightarrow \sum \frac{x^3}{(2y+z)^2} \geq \frac{1}{3}$$

By AG-GM:

$$\sum(\frac{x^3}{(2y+z)^2} + \frac{2y+z}{27} + \frac{2y+z}{27}) \geq \sum(\frac{x}{3})$$

$$\Leftrightarrow \sum \frac{x^3}{(2y+z)^2} \geq \frac{1}{9}(x+y+z) \geq \frac{1}{3}$$

equatWe if only if $a=b=c=1$

237.

Let a,b,c>0 such that $a+b+c=1$.Prove that

$$\frac{b^2}{a+b^2} + \frac{c^2}{b+c^2} + \frac{a^2}{c+a^2} \geq \frac{3}{4}$$

Solution: We have

$$\frac{b^2}{a+b^2} + \frac{c^2}{b+c^2} + \frac{a^2}{c+a^2} \geq \frac{\left(a^2+b^2+c^2\right)^2}{(a^4+b^4+c^4)+(ab^2+bc^2+ca^2)}$$

Hence it suffices to prove that

$$\frac{(a^2+b^2+c^2)^2}{(a^4+b^4+c^4)+(ab^2+bc^2+ca^2)} \geq \frac{3}{4}$$

$$\Leftrightarrow 4(\sum a^2)^2 \geq 3(\sum ab^2)(\sum a) + 3\sum a^4$$

$$\Leftrightarrow 4\sum a^4 + 8\sum a^2b^2 \geq 3\sum a^4 + 3\sum(a^2b^2 + abc^2 + a^3c)$$

$$\Leftrightarrow \sum a^4 + 5\sum a^2b^2 \geq 3abc(\sum a^4) + 3\sum a^2c$$

Since We always have

$$3(a^3c+b^3a+c^3) \leq (a^2+b^2+c^2)^2 = \left(a^4+b^4+c^4\right) + 2\left(a^2b^2+b^2c^2+c^2a^2\right)$$

Therefor it suffices to prove that

$$3\left(a^2b^2+b^2c^2+c^2a^2\right) \geq 3abc\,(a+b+c)$$

which obviously true.

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Let a , b ,c be positive real numbers. Prove that

$$\frac{a^3b}{a^4+a^2b^2+b^4} + \frac{b^3c}{b^4+b^2c^2+c^4} + \frac{c^3a}{c^4+c^2a^2+a^4} \leq 1.$$

Solution: setting

$$x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$$

We get

$$\frac{1}{\frac{a}{b}+\frac{b}{a}+\frac{b^3}{a^3}} + \frac{1}{y+\frac{1}{y}+\frac{1}{y3}} + \frac{1}{z+\frac{1}{z}+\frac{1}{z^3}} \leq 1$$

Now We have

$$x + \frac{1}{x} + \frac{1}{x^3} \geq 2 + \frac{1}{x^3}$$

$$y + \frac{1}{y} + \frac{1}{y^3} \geq 2 + \frac{1}{y^3}$$

$$z+\frac{1}{z}+\frac{1}{z^3}\geq 2+\frac{1}{z^3}$$

so the left-hand side of our equation is loir or equal to

$$\frac{1}{2+\frac{1}{x^3}}+\frac{1}{2+\frac{1}{y^3}}+\frac{1}{2+\frac{1}{z^3}}$$

the last sum is loir or equal to 1 if

$$-\frac{4\,x^3z^3y^3-x^3-y^3-z^3-1}{(2\,x^3+1)\,(2\,y^3+1)\,(2\,z^3+1)}\geq 1$$

this is true since $-4+x^3+y^3+z^3+1\geq 0$ or $\frac{x^3+y^3+z^3}{3}\geq 1$ and this is AM-GM, note that $xyz=1$ holds.

239.

Let $x,y,z>0$ satisfying $xy+yz+zx+xyz=4$. Prove the following inequality:

$$\sqrt{\frac{x+2}{3}}+\sqrt{\frac{y+2}{3}}+\sqrt{\frac{z+2}{3}}\geq 3$$

Solution(Le Viet Thai)

Setting $x+2=a,y+2=b,z+2=c$

The condition is equilvalent to:

$$(a-2)\,(b-2)+(b-2)\,(c-2)+(c-2)\,(a-2)+(a-2)\,(b-2)\,(c-2)=4$$

$$\Leftrightarrow abc=ab+bc+ca$$

$$\Leftrightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$$

From Holder's inequality, We get the desired result:

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 3^3$$

$\Rightarrow \sqrt{a}+\sqrt{b}+\sqrt{c}\geq 3\sqrt{3}$ 239

Let$a,b,c\geq 0$prove:

$$\sqrt{1+\frac{48a}{b+c}}+\sqrt{1+\frac{48b}{c+a}}+\sqrt{1+\frac{48c}{a+b}}\geq 15$$

Solution:

1.Let

$1+\frac{48a}{b+c}=(1+4x)^2$, $1+\frac{48b}{a+c}=(1+4y)^2$, $1+\frac{48c}{a+b}=(1+4z)^2$,

where

x, y and z are non-negative numbers.

Then

$$\frac{a}{b+c}=\frac{2x^2+x}{6},$$

$$\frac{b}{a+c}=\frac{2y^2+y}{6},\frac{c}{a+b}=\frac{2z^2+z}{6}.$$

Since,

$$\frac{2abc}{(a+b)(a+c)(b+c)}+\sum_{cyc}\frac{ab}{(b+c)(a+c)}=1$$

the following equality holds:

$$\frac{2(2x^2+x)(2y^2+y)(2z^2+z)}{216}+\sum_{cyc}\frac{(2x^2+x)(2y^2+y)}{36}=1\Leftrightarrow$$

$$\Leftrightarrow 108=\sum_{cyc}\left(3xy+\frac{xyz}{3}+6x^2y+6x^2z+2x^2yz+12x^2y^2+4x^2y^2z+\frac{8x^2y^2z^2}{3}\right).$$

Remain to prove that $x+y+z\geq 3$.
Let $x+y+z<3$ for some non-negative x, y and z such that
$1+\frac{48a}{b+c}=(1+4x)^2$, $1+\frac{48b}{a+c}=(1+4y)^2$ and $1+\frac{48c}{a+b}=(1+4z)^2$.
Let $x=ku$, $y=kv$, $z=kw$, where u, v and w are non-negative and $u+v+w=3$.
Hence, $0<k<1$ and

$$108=\sum_{cyc}\left(3xy+\frac{xyz}{3}+6x^2y+6x^2z+2x^2yz+12x^2y^2+4x^2y^2z+\frac{8x^2y^2z^2}{3}\right)=$$

$$=\sum_{cyc}\left(3k^2uv+\frac{k^3uvw}{3}+6k^3u^2v+6k^3u^2w+2k^4u^2vw+12k^4u^2v^2+4k^5u^2v^2w+\frac{8k^6u^2v^2w^2}{3}\right)<$$

$$<\sum_{cyc}\left(3uv+\frac{uvw}{3}+6u^2v+6u^2w+2u^2vw+12u^2v^2+4u^2v^2w+\frac{8u^2v^2w^2}{3}\right).$$

Thus,

$$108<\sum_{cyc}\left(3uv+\frac{uvw}{3}+6u^2v+6u^2w+2u^2vw+12u^2v^2+4u^2v^2w+\frac{8u^2v^2w^2}{3}\right).$$

But it's contradiction since, for all non-negative u, v and w such that $u+v+w=3$ holds:

$$108\geq\sum_{cyc}\left(3uv+\frac{uvw}{3}+6u^2v+6u^2w+2u^2vw+12u^2v^2+4u^2v^2w+\frac{8u^2v^2w^2}{3}\right)\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}(4u^6+17u^5v+17u^5w-4u^4v^2-4u^4w^2+68u^4vw-34u^3v^3+$$

$$+11u^3v^2w+11u^3w^2v-86u^2v^2w^2)\geq 0\Leftrightarrow$$

$$\Leftrightarrow 4\cdot\sum_{cyc}(u^6-u^4v^2-u^4w^2+u^2v^2w^2)+17\cdot\sum_{cyc}uv(u^2-v^2)^2+$$

$$+\sum_{cyc}(68u^4vw+11u^3v^2w+11u^3w^2v-90u^2v^2w^2)\geq 0,$$

which obviously true.
2. Without loss of generality, We may assume that $c=\min\{a,b,c\}$. Then, We notice that

$$\left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}\right)^2[a^2(b+c)+b^2(c+a)]\geq(a+b)^3,$$

from the Holder's inequality, and

$$a^2(b+c)+b^2(c+a)=c(a+b)^2+ab(a+b2c)\leq c(a+b)^2+\frac{(a+b)^2}{4}(a+b2c)=\frac{(a+b)^2(a+b+2c)}{4}.$$

Combining these two estimations, We find that

$$\left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}\right)^2 \geq \frac{4(a+b)}{a+b+2c}.$$

Now, using the Minkowsky's inequality (in combinination with this), We get

$$\sqrt{1+\frac{48a}{b+c}}+\sqrt{1+\frac{48b}{c+a}} \geq \sqrt{(1+1)^2+48\left(\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}\right)^2}$$

$$\geq \sqrt{4+48.\frac{4(a+b)}{a+b+2c}} = 2\sqrt{1+\frac{48(a+b)}{a+b+2c}},$$

and so We are left to prove that

$$2\sqrt{1+\frac{48(a+b)}{a+b+2c}}+\sqrt{1+\frac{48c}{a+b}} \geq 15,$$

which is easy to check.

3. 3. Once again We can use the uvw-theorem

Let $x=\sqrt{1+\frac{48a}{b+c}}1, y=\sqrt{1+\frac{48b}{c+a}}1, z=\sqrt{1+\frac{48c}{b+a}}1$, so $x,y,z \geq 0$.

Then We have

$$\frac{a}{b+c}=\frac{x^2+2x}{48}$$

and so on. Using the illknown:

$$\sum_{cyc}\frac{a}{b+c}\frac{b}{a+c}+2\prod_{cyc}\frac{a}{b+c}=1$$

We obtain:

$$48\sum_{cyc}(x^2+2x)(y^2+2y)+2(x^2+2x)(y^2+2y)(z^2+2z)=48^3$$

We should prove that $x+y+z \geq 12$ when

$$48\sum_{cyc}(x^2+2x)(y^2+2y)+2(x^2+2x)(y^2+2y)(z^2+2z)=48^3$$

Assume that $x+y+z<12$ when

$$48\sum_{cyc}(x^2+2x)(y^2+2y)+2(x^2+2x)(y^2+2y)(z^2+2z)=48^3$$

Then by increasing x,y,z 'till $x+y+z=12$ We will get a situation where:
$x+y+z=12$ and $48\sum_{cyc}(x^2+2x)(y^2+2y)+2(x^2+2x)(y^2+2y)(z^2+2z)>48^3$.
So it is enough to prove that:

$$48\sum_{cyc}(x^2+2x)(y^2+2y)+2(x^2+2x)(y^2+2y)(z^2+2z) \leq 48^3$$

when

$$x+y+z=12, x,y,z \geq 0$$

Let $3u=x+y+z, 3v^2=xy+yz+zx, w^3=xyz$.
Writing it in terms of u, v^2, w^3 it clearly becomes on the form $2w^6+A(u,v^2)w^3+B(u,v^2) \geq 0$

where A, B are functions in u and v^2.
So according to the uvw-theorem We only have to prove it when $xyz = 0$ and when $(xy)(y-z)(zx) = 0$.
$xyz = 0$: wlog $x = 0 \Rightarrow a = 0$.
$(xy)(yz)(zx) = 0$: wlog $y = z$. Then $\frac{b}{c+a} = \frac{c}{a+b} \iff (bc)(a+b+c) = 0 \iff b = c$.
So We only have to prove it in to cases:
$a = 0$, $b = c = 1$.
$a = 0$: $\sqrt{1+48\frac{b}{c}} + \sqrt{1+48\frac{c}{b}} \geq 14$. Squaring: $2 + 48(\frac{c}{b}+\frac{b}{c}) + 2\sqrt{1+48\frac{b}{c}}\sqrt{1+48\frac{c}{b}} \geq 14^2$, follows from $\frac{c}{b}+\frac{b}{c} \geq 2$ (AM-GM) and $\sqrt{1+48\frac{b}{c}}\sqrt{1+48\frac{c}{b}} \geq (1+48) = 49$ (Cauchy-Schwartz)
$b = c = 1$: $f(a) = \sqrt{1+24a} + 2\sqrt{1+\frac{48}{1+a}} \geq 15$
Then $f'(a) = \frac{12}{\sqrt{1+24a}\sqrt{49+a}\sqrt{(1+a)^3}(\sqrt{49+a}\sqrt{(1+a)^3}+2\sqrt{16(1+24a)})}$ $((49+a)(1+a)^3 16(1+24a))$
So $f'(a) \geq 0 \iff$
$(a1)(a^3 + 54a^2 + 204a33) \geq 0$
$a^3 + 54a^2 + 20433$ has exactly one positive root, and this root is less than 1.
Let it be α. Then $f'(a) \geq 0 \iff (a1)(a\alpha) \geq 0$. Hence f is increasing in $[0;\alpha] \cap [1;+\infty]$ and decreasing in $[\alpha;1)$. So $f(a) \geq \min\{f(0); f(1)\}$. And since $f(0) = f(1) = 15$, We see that $f(a) \geq 15$
, and We are done. Equality when $a = b = c$ or when $a = b, c = 0$ and permutations
340.
Letx, y, z are positive numbers and$x + y + z = 3$.
prove:

$$\frac{x^3}{y^3+8} + \frac{y^3}{z^3+8} + \frac{z^3}{x^3+8} \geq \frac{1}{9} + \frac{2}{27}(xy+yz+zx)$$

Solution:
By AG-GM, We have:

$$\sum[\frac{x^3}{(y+2)(y^2-2y+4)} + \frac{x(y+2)}{9(y^2-2y+4)} + \frac{2}{3}.\frac{y^2-2y+4}{9y^2}] \geq$$

$$\geq \sum[\frac{2}{3}.\frac{x^2}{y^2-2y+4} + \frac{2}{3}.\frac{y^2-2y+4}{9y^2}] \geq \frac{4}{9}\sum[\frac{x}{y}] \geq \frac{4}{3}$$

$$\Rightarrow \sum \frac{x^3}{(y+2)(y^2-2y+4)} \geq \frac{4}{3} - \sum \frac{x(y+2)}{9(y^2-2y+4)} - \sum \frac{2}{3}.\frac{y^2-2y+4}{9y^2}$$

On the other hand:

$$\sum \frac{x(y+2)}{9(y^2-2y+4)} = \sum \frac{x(y+2)}{9(y-1)^2+27} \leq \leq \sum \frac{xy+2x}{27} = \frac{xy+yz+zx}{27} + \frac{2(x+y+z)}{27} = \frac{xy+yz+zx}{27} + \frac{2}{9}$$

And:

$$\frac{2}{3}\sum \frac{y^2-2y+4}{9y^2} \leq \frac{2}{3}\sum[\frac{2}{3}y - \frac{1}{3}]$$

(*) Really:

$$(*) \Leftrightarrow \sum \frac{(y-1)^2(3y+1)}{9y^2} \geq 0$$

So:

$$\frac{4}{3} - \sum \frac{x(y+2)}{9(y^2-2y+4)} - \sum \frac{2}{3}.\frac{y^2-2y+4}{9y^2} \geq$$

$$\geq \frac{4}{3} - \frac{xy+yz+zx}{27} - \frac{2}{9} - \frac{2}{3}\sum[\frac{2}{3}y - \frac{1}{3}] = \frac{4}{9} - \frac{xy+yz+zx}{27}$$

So:

$$\sum \frac{x^3}{y^3+8} \geq \frac{4}{9} - \frac{xy+yz+zx}{27}$$

and:

$$\frac{4}{9} - \frac{xy+yz+zx}{27} \geq \frac{1}{9} + \frac{2}{27}(xy+yz+zx) \Leftrightarrow xy+yz+zx \leq 3$$

True.

341.

Let a, b, c, p be real numbers. We denote

$$F(a,b,c) = \sum \frac{p(3-p)a^2 + 2(1-p)bc}{pa^2+b^2+c^2} - \frac{3(1+p)(2-p)}{2+p}$$

Prove that $(p-1).F(a,b,c) \geq 0$ for all real numbers a, b, c and all positive real number p with equality if and only if $p = 1$ or $(a,b,c) = (1,1,1)$ or $(a,b,c) = (1,1,\frac{2}{p})$ and their permutations.

Solution.

Because

$$(2+p)[p(3-p)a^2+2(1-p)bc]-(1+p)(2-p)(pa^2+b^2+c^2) = 2p(2a^2-b^2-c^2)+(p-1)(p+2)(b-c)^2$$

, We have

$$F(a,b,c) = \sum \left(\frac{p(3-p)a^2 + 2(1-p)bc}{pa^2+b^2+c^2} - \frac{(1+p)(2-p)}{2+p} \right)$$
$$= \frac{2p}{2+p} \sum \frac{2a^2-b^2-c^2}{pa^2+b^2+c^2} + (p-1)\sum \frac{(b-c)^2}{pa^2+b^2+c^2}.$$

Notice that

$$\sum \frac{2a^2-b^2-c^2}{pa^2+b^2+c^2} = \sum (a^2-b^2)\left(\frac{1}{pa^2+b^2+c^2} - \frac{1}{pb^2+c^2+a^2}\right)$$
$$= (1-p)\sum \frac{(a^2-b^2)^2}{(pa^2+b^2+c^2)(pb^2+c^2+a^2)}$$

, We have

$$F(a,b,c) = \frac{2p(1-p)}{2+p}\sum \frac{(a^2-b^2)^2}{(pa^2+b^2+c^2)(pb^2+c^2+a^2)} + \; + (p-1)\sum \frac{(b-c)^2}{pa^2+b^2+c^2}$$
$$= (p-1)\left(\sum \frac{(a-b)^2}{pc^2+a^2+b^2} - \frac{2p}{2+p}\sum \frac{(a^2-b^2)^2}{(pa^2+b^2+c^2)(pb^2+c^2+a^2)}\right).$$

Hence the original inequality is equivalent to

$$\sum \frac{(a-b)^2}{pc^2+a^2+b^2} \geq \frac{2p}{2+p}\sum \frac{(a^2-b^2)^2}{(pa^2+b^2+c^2)(pb^2+c^2+a^2)}$$

for all reals a, b, c and positive real p. From the inequality $(x-y)^2 \geq (|x|-|y|)^2 \; \forall x, y$, We see that it suffits to prove the above inequality for $a, b, c \geq 0$ and $p > 0$. This inequality is equivalent to

$$\sum (a-b)^2 \left(\frac{2+p}{pc^2+a^2+b^2} - \frac{2p(a+b)^2}{(pa^2+b^2+c^2)(pb^2+c^2+a^2)}\right) \geq 0.$$

, which can be rewritten as

$$S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 0$$

with

$$S_c = (p+2)(pa^2+b^2+c^2)(pb^2+c^2+a^2) - 2p(a+b)^2(pc^2+a^2+b^2)$$

and S_a, S_b are determined similarly. We can rewritten S_c as two forms

$$S_c = (pc^2+a^2+b^2)(pb-2a)^2 + (p+2)(c^2-a^2)[pc^2+a^2+b^2+(1-p)(pb^2+a^2+c^2)]$$

$$S_c = (pc^2+a^2+b^2)(pa-2b)^2 + (p+2)(c^2-b^2)[pc^2+a^2+b^2+(1-p)(pa^2+b^2+c^2)]$$

and similarly for S_a and S_b.

it is very useful We rewrite in both two above forms. Now, We assume $a \geq b \geq c \geq 0$ and consider three cases of p:

First case : $0 < p \leq 1$. Clearly that $S_a \geq 0, S_b \geq 0$, it suffits to prove $S_b + S_c \geq 0$ (because $a-c \geq a-b \geq 0$). We have

$$S_b + S_c = (pb^2+c^2+a^2)(pa-2c)^2 + (pc^2+a^2+b^2)(pa-2b)^2 - (p+2)(1-p)(b^2-c^2)^2.$$

Using the inequality $2(x^2+y^2) \geq (x+y)^2 \ \forall x, y$ We have

$$(pb^2+c^2+a^2)(pa-2c)^2+(pc^2+a^2+b^2)(pa-2b)^2 = (a^2+pb^2+pc^2)((pa-2b)^2+(pa-2c)^2)+(1-p)((pab-2b^2)^2+(pac-2c$$

$$\geq (a^2+pb^2+pc^2)\frac{1}{2}(2b-2c)^2+(1-p)\frac{1}{2}(2b^2-2c^2+pac-pbc)^2 \geq (a^2+\frac{1}{2}(b+c)^2).2.(b-c)^2+\frac{1}{2}(1-p)(b-c)^2(2b+2c-pa)^2.$$

We need prove

$$2a^2 + p(b+c)^2 + \frac{1}{2}(1-p)(2b+2c-pa)^2 \geq (2+p)(1-p)(b+c)^2$$

or

$$(4+p^2-p^3)a^2 + 2p^2(b+c)^2 \geq 4p(1-p)a(b+c)$$

, which is true because $(4+p^2-p^3)a^2+2p^2(b+c)^2 \geq 4a^2+2p^2(b+c)^2 \geq 4p(1-p)a(b+c)$. The inequality was proved for $0 < p \leq 1$.

Second case: $1 < p \leq \frac{3}{2}$. From the expression of $S_b + S_c$ in the first case, We see that $S_a + S_b \geq 0, S_b + S_c \geq 0, S_c + S_a \geq 0$ for $1 < p$. it suffits to prove $S_b \geq 0$, which is true because $b \geq c$ and

$$pb^2+c^2+a^2+(1-p)(pa^2+b^2+c^2) = b^2+(2-p)c^2+(1+p-p^2)a^2 \geq b^2+(1+p-p^2)b^2 \geq 0.$$

The inequality was proved in this case.

Third case: $\frac{3}{2} \leq p$. For this case, We rewrite again as

$$S_c = (pc^2+a^2+b^2)(pb-2a)^2+(2+p)(c^2-a^2)[c^2-b^2+(2-p)(a^2+b^2+pb^2)] = K_c+(2+p)(c^2-a^2)(c^2-b^2)$$

with

$$K_c = (pc^2+a^2+b^2)(pb-2a)^2 + (4-p^2)(c^2-a^2)(a^2+b^2+pb^2).$$

We also have another form of K_c as

$$K_c = (pc^2+a^2+b^2)(pa-2b)^2 + (4-p^2)(c^2-b^2)(b^2+a^2+pa^2).$$

(it is similar for K_a and K_b) Then

$$\sum S_c(a-b)^2 = \sum K_c(a-b)^2 + \sum (2+p)(c^2-a^2)(c^2-b^2)(a-b)^2.$$

Because
$$\sum(c^2-a^2)(c^2-b^2)(a-b)^2=(a-b)^2(b-c)^2(c-a)^2\ge 0$$
, We need prove
$$K_c(a-b)^2+K_b(a-c)^2+K_a(b-c)^2\ge 0.$$
Clearly that K_b is always nonnegative (from $a\ge b\ge c$) and $K_a\ge 0$ for $p\le 2$, $K_c\ge 0$ for $p\ge 2$ (where We used both two forms of them). We have
$$K_b+K_c=(pb^2+c^2+a^2)(pa-2c)^2+(pc^2+a^2+b^2)(pa-2b)^2-(4-p^2)(b^2-c^2)^2K_b+K_a$$
$$=(pb^2+c^2+a^2)(pc-2a)^2+(pa^2+c^2+b^2)(pc-2b)^2-(4-p^2)(b^2-a^2)^2$$
From this, if $p\ge 2$ then the inequality is clearly true. if $2\ge p\ge\frac{3}{2}$, then $K_a\ge 0$ and We must prove $K_b+K_c\ge 0$, which is true because
$$(pb^2+c^2+a^2)(pa-2c)^2+(pc^2+a^2+b^2)(pa-2b)^2=\frac{1}{2}(p+2)(b^2+c^2)(pa-2c)^2+\frac{1}{2}(p+2)(b^2+c^2)(pa-2b)^2$$
$$=\frac{1}{2}(p+2)(b^2+c^2)((pa-2c)^2+(pa-2b)^2)\ge\frac{1}{2}(p+2)(b^2+c^2).\frac{1}{2}(2b-2c)^2=(p+2)(b^2+c^2)(b-c)^2\ge(4-p^2)(b^2-c^2)^2.$$
The inequality was proved in the last case.
342.
Let x,y,z,t be positive real number such that $max(x,y,z,t)\le\sqrt{5}min(x,y,z,t)$.
Prove that:
$$\frac{xy}{5x^2-y^2}+\frac{yz}{5y^2-z^2}+\frac{zt}{5z^2-t^2}+\frac{tx}{5t^2-x^2}\ge 1$$
Solution: From
$$max(x,y,z,t)\le min\sqrt{5}min(x,y,z,t)$$
We have
$$5x^2-y^2,5y^2-z^2,5z^2-t^2,5t^2-x^2\ge 0.$$
Setting
$$a=\frac{x}{y},b=\frac{y}{z},c=\frac{z}{t},d=\frac{t}{x}$$
We have $abcd=1$
The inequality can rewrite:
$$\frac{a}{5a^2-1}+\frac{b}{5b^2-1}+\frac{c}{5c^2-1}+\frac{d}{5d^2-1}\ge 1.$$
We have:
$$\frac{a}{5a^2-1}\ge\frac{1}{a^3+a^2+a+1}<=>\frac{(a-1)^2(a^2+3a+1)}{(5a^2-1)(a^3+a^2+a+1)}\ge 0$$
(true)
We will prove that:
$$\frac{1}{(1+a)(1+a^2)}+\frac{1}{(1+b)(1+b^2)}+\frac{1}{(1+c)(1+c^2)}+\frac{1}{(1+d)(1+d^2)}\ge 1.$$
Without loss of generality assume that $a\ge b\ge c\ge d$
. Then from Chebyshev's inequality We have that
$$\frac{1}{(1+a)(1+a^2)}+\frac{1}{(1+b)(1+b^2)}+\frac{1}{(1+c)(1+c^2)}\ge$$

$$\geq \frac{1}{3}\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right)\left(\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}\right).$$

Lemma (Vasile Cirtoaje): if$a \geq b \geq c \geq d$ and $abcd = 1$
then it holds that $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \geq \frac{3}{1+\sqrt[3]{abc}}$.
Solution of the Lemma: We know that
$\frac{1}{1+a}+\frac{1}{1+b} \geq \frac{2}{1+\sqrt{ab}}$.
So, it suffices to prove that $\frac{1}{1+c}+\frac{2}{1+\sqrt{ab}} \geq \frac{3}{1+\sqrt[3]{abc}}$.
Let us denote by

$$x = \sqrt{ab}, y = \sqrt[3]{abc} \Longrightarrow c = \frac{y^3}{x^2}$$

Substituting them to the above inequality We get that

$$\frac{1}{1+c}+\frac{2}{1+\sqrt{ab}}-\frac{3}{1+\sqrt[3]{abc}} = \frac{x^2}{x^2+y^3}+\frac{2}{1+x}-\frac{3}{1+y},$$

which reduces to the inequality

$$\frac{(x-y)^2\left[2y^2-y+x(y-2)\right]}{(1+x)(1+y)(x^2+y^3)}$$

, which is obvious since

$$2y^2-y+(y-2)x \geq 2y^2-y+(y-2)y^3 = y(y-1)(y^2-y+1) \geq 0.$$

Back to our inequality now, from the above lemma We deduce that:

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c} \geq \frac{3}{1+\sqrt[3]{abc}}$$

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2} \geq \frac{3}{1+\sqrt[3]{a^2b^2c^2}}.$$

For convenience denote by k the $\sqrt[3]{abc}$.
Therefore We have that

$$\frac{1}{(1+a)(1+a^2)}+\frac{1}{(1+b)(1+b^2)}+\frac{1}{(1+c)(1+c^2)} \geq \frac{3}{(1+k)(1+k^2)}.$$

Thus it remains to prove that

$$\frac{3}{(1+k)(1+k^2)}+\frac{1}{(1+d)(1+d^2)} \geq 1$$

But $abcd = 1$.
So, the last fraction is of the form

$$\frac{1}{\left(1+\frac{1}{k^3}\right)\left(1+\frac{1}{k^6}\right)}.$$

After that We get

$$\frac{1}{\left(1+\frac{1}{k^3}\right)\left(1+\frac{1}{k^6}\right)}+\frac{3}{(1+k)(1+k^2)} \geq 1$$

Conclusion follows from the obvious inequality

$$\frac{(k-1)^2(2k^4+k^3+k+2)}{(k^3+1)(k^6+1)} \geq 0$$

Q.E.D
The Enquality holds when $x = y = z = t = 1$.

Remark: Let $x_1, x_2, ..., x_n$ be positive real number such that $max(x_1, x_2, .., x_n) \leq \sqrt{5}min(x_1, x_2, ..., x_n)$..Prove that:

$$\frac{x_1x_2}{5x_1^2 - x_2^2} + \frac{x_2x_3}{5x_2^2 - x_3^2} + ... + \frac{x_nx_1}{5x_n^2 - x_1^2} \geq 1$$

243., Given $a, b, c \geq 0$. Prove that:

$$\frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{a^2}{a^2+bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab}$$

Solution:
1.

$$\sum \frac{2a^2}{(a+b)(a+c)} - \sum \frac{a^2}{a^2+bc} = \sum \frac{a^2(a-b)(a-c)}{(a+b)(a+c)(a^2+bc)} \geq 0$$

$$\frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \sum \frac{2a^2}{(a+b)(a+c)} = \frac{2\sum ab(a+b)}{(a+b)(b+c)(c+a)}$$

Assume that $a + b + c = 1$ and put $q = ab + bc + ca, r = abc$, then the inequality becomes

$$\frac{1}{4q} \geq \frac{q-3r}{q-r}$$

$$\Leftrightarrow \frac{q-r}{q-3r} \geq 4q$$

$$\Leftrightarrow \frac{2r}{q-3r} \geq 4q - 1$$

By Schur's inequality for third degree, We have $r \geq \frac{4q-1}{9}$, then

$$\frac{2r}{q-3r} \geq \frac{2r}{q - \frac{4q-1}{3}} = \frac{6r}{1-q}$$

it suffices to show that

$$6r \geq (4q-1)(1-q)$$

But this is just Schur's inequality for fourth degree

$$\sum a^4 + abc\sum a \geq \sum ab(a^2+b^2)$$

We have done.
2.
Suppose $a + b + c = 3$. We need to prove:

$$f(r) = 4q^4 - 9q^3 + 24qr^2 - 54q^2r - 72r^2 - 243r + 216qr \leq 0$$

$$f'(r) = 48qr - 54q^2 - 144r - 243 + 216q$$

$$f''(r) = 48(q-3) \leq 0, so f'(r) \leq f'(0) = -54q^2 - 144 + 216q \leq 0$$

So, with $q \leq \frac{9}{4}, f(r) \leq f(0) = q^3(4q-9) \leq 0$ With $q \geq \frac{9}{4}$, We have:

$$f(r) \leq f(\frac{4q-9}{3}) \leq 0 (\text{trues with } q \geq \frac{9}{4})$$

4, Let $a+b+c=1$; $a,b,c\geq 0$. Prove that:

$$\frac{1}{2a-5b^2}+\frac{1}{2b-5c^2}+\frac{1}{2c-5a^2}\geq\frac{3}{(a^2+b^2+c^2)^2}$$

Solution:
By Cauchy-Schwarz 's inequality, We have:

$$LHS\geq\frac{9}{2(a+b+c)^2-5\sum a^2}\geq\frac{3}{(a^2+b^2+c^2)(6\sum a^2-5\sum a^2)}=\frac{3}{(\sum a^2)^2}$$

Q.E.D

244., Let $x,y,z\geq 0$ and $x+y+z=1$. Prove that:

$$27(x^3+yz)(y^3+xz)(z^3+xy)\geq 64x^2y^2z^2$$

Solution:
it's the following ineq of a ill-know ineq :

$$(a^2+3)(b^2+3)(c^2+3)\geq 64\ \ \forall a+b+c=3$$

$$\prod(\frac{3x^4}{xyz}+3)\geq 64$$

Setting :$3x=a,3y=b,3z=c$ By am-gm , We have :

$$LHS(1)\geq(a^4+3)(b^4+3)(c^4+3)\geq\frac{1}{64}((a^2+3)(b^2+3)(c^2+3))^2\geq 64$$

Note : it's better if you think more about classical ineq before use modern tech 245., Find the best value of k to this ineq is truefor all $a,b,c\geq 0, abc=1$

$$k(\sum\frac{a^2+b^2}{c^2})+\sum\frac{ab}{c^2}-\frac{3}{4}\geq\sum\frac{c^2}{ab}+\sum\frac{a+b}{2c}$$

Solution.
1.
with $a=b=c$ We find $k=\frac{5}{8}$
Let $a+b+c=p, ab+bc+ca=q, abc=r$. We have:

$$ineq\leftrightarrow\frac{5}{8}\sum a^2b^2(a^2+b^2)+\sum a^3b^3-\frac{3a^2b^2c^2}{4}\geq\ abc(a^3+b^3+c^3)+\frac{1}{2}\sum ab(a+b)$$

$$\leftrightarrow\frac{5}{8}(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)+\sum a^3b^3-\frac{3a^2b^2c^2}{4}-abc(a^3+b^3+c^3)-\frac{1}{2}\sum ab(a+b)-\frac{15}{8}\geq 0$$

$$\leftrightarrow\frac{5}{8}(p^2-2q)(q^2-2p)+(q^3-3pq+3)-(p^3-3pq+3)-\frac{1}{2}(pq-3)-\frac{21}{8}\geq 0$$

$$\leftrightarrow\frac{5p^2q^2}{8}-\frac{q^3}{4}-\frac{9p^3}{4}+2pq-\frac{9}{8}\geq 0$$

Follow Schur:

$$r\geq\ \frac{(4q-p^2)(p^2-q)}{6p}\rightarrow 1\geq\ \frac{(4q-p^2)(p^2-q)}{6p}$$

2.

$$\frac{5}{8}(\sum\frac{a^2+b^2}{c^2})+\sum\frac{ab}{c^2}-\frac{3}{4}\geq\sum\frac{c^2}{ab}+\sum\frac{a+b}{2c}$$

$$\Leftrightarrow 5p^2q^2-10p^3-10q^3+16pq-9\geq 0$$

setting

$$f(p) = 5p^2q^2 - 10p^3 - 10q^3 + 16pq - 9$$

$$f'(p) = 10pq^2 - 30p^2 + 16q \geq 0$$

$$\Rightarrow f(p) \geq f(\sqrt{3q}) = (\sqrt{q} - \sqrt{3})(5q^2\sqrt{q} + 5\sqrt{3}q^2 + 15q\sqrt{q} + \sqrt{3}q + 3q + 3\sqrt{3})$$

by Am-Gm

$$\Rightarrow q \geq 3 \Rightarrow f(p) \geq f(\sqrt{3q}) \geq 0$$

"="

$$\Leftrightarrow (a, b, c) = (1, 1, 1)$$

Perhaps, it is the **Solution** which is used "pqr tech" that We said thank your **Solution** our ineq

$$\Leftrightarrow \sum c^2(a^2 - b^2 - 2ac - 2bc)^2 \geq 0$$

it is not natural, We know that.
246.
, Let $x, y, z \geq 0$ and $x + y + z = 1$. Prove that

$$27(x^3 + yz)(y^3 + xz)(z^3 + xy) \geq 64x^2y^2z^2$$

Solution:
it's the following ineq of a ill-know ineq :

$$(a^2 + 3)(b^2 + 3)(c^2 + 3) \geq 64 \ \ \forall a + b + c = 3$$

Tranvanluan's ineq is equivalent to :

$$\prod(\frac{3x^4}{xyz} + 3) \geq 64(1)$$

Setting : $3x = a, 3y = b, 3z = c$ By am-gm, We have :

$$LHS(1) \geq (a^4 + 3)(b^4 + 3)(c^4 + 3) \geq \frac{1}{64}((a^2 + 3)(b^2 + 3)(c^2 + 3))^2 \geq 64$$

$=>$ Q.E.D

2.
Lemma:

$$r \leq \frac{q^2(1 - q)}{2(2 - 3q)}$$

and

$$r \geq \frac{(4q - 1)(1 - q)}{6}$$

$$27(x^3 + yz)(y^3 + xz)(z^3 + xy) \geq 64x^2y^2z^2$$

$$\Leftrightarrow 27r^3 + 27q^4 - 54q^2r + 125r^2 + 108qr^2 + 27r - 108rq \geq 0$$

setting

$$f(r) = 27r^3 + 27q^4 - 54q^2r + 125r^2 + 108qr^2 + 27r - 108rq$$

the first case:

$$81r^2 - 54q^2 + 250r + 216rq + 27 - 108q \geq 0$$

$$f'(r) = 81r^2 - 54q^2 + 250r + 216rq + 27 - 108q \geq 0$$

$$\Rightarrow f(r) \geq f(\frac{(4q-1)(1-q)}{6}) = \frac{-1}{72}(3q-1)(192q^5 - 1808q^4 + 476q^3 - 267q^2 + 518q - 83) \geq 0$$

the second case:

$$81r^2 - 54q^2 + 250r + 216rq + 27 - 108q \leq 0$$

$$f'(r) = 81r^2 - 54q^2 + 250r + 216rq + 27 - 108q \leq 0$$

$$\Rightarrow f(r) \geq f(\frac{q^2(1-q)}{2(2-3q)}) = \frac{q^2(3q-1)(9q^6 + 192q^5 + 1061q^4 - 3490q^3 + 4064q^2 - 2160q + 432)}{8(-2+3q)^3} \geq 0$$

We have done Wink

247.,

Let $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 3$. Prove that

$$ab + bc + ca \leq abc + 2$$

Solution:

Put

$$f(a; b; c) = ab + bc + ca - abc.$$

To suppose

$$c = mina; b; c$$

We have

$$f(a; b; c) - f(\sqrt{\frac{a^2+b^2}{2}}; \sqrt{\frac{a^2+b^2}{2}}; c)$$

$$= (ab - \frac{a^2+b^2}{2}) + [c(a+b) - c\sqrt{2(a^2+b^2)}] - [abc - \frac{c(a^2+b^2)}{2}]$$

$$= \frac{-(a-b)^2}{2} - \frac{c(a-b)^2}{a+b+\sqrt{2(a^2+b^2)}} + \frac{c(a-b)^2}{2}$$

$$= (a-b)^2(\frac{c}{2} - \frac{1}{2} - \frac{c}{a+b+\sqrt{2(a^2+b^2)}}) \leq 0$$

$$\rightarrow f(a; b; c) \leq f(\sqrt{\frac{a^2+b^2}{2}}; \sqrt{\frac{a^2+b^2}{2}}; c)$$

$$\rightarrow f(a; b; c) \leq f(t; t; t) = 2$$

Q.E.D

. 248.Let $a_1, a_2, a_3, ..., a_n$ be n non-negative real numbers, such that $a_1 + a_2 + ...a_n = 1$. Prove that

$$a_1a_2 + a_2a_3 + a_3a_4 + ...a_{n-1}a_n \leq \frac{1}{4}$$

Solution:

it does not work when $n = 1$. You need $n \geq 2$. When $n = 2$, and $a_1 + a_2 = 1$, We do have $a_1a_2 \leq \frac{1}{4}$. The **Solution** is easy and i'll omit it here.

Assume that for some $n \geq 2, a_1a_2 + \cdots + a_{n-1}a_n \leq \frac{1}{4}$ whenever $a_1 + \cdots + a_n = 1$.

Let $a_1 + a_2 + \cdots + a_n + a_{n+1} = 1$. WLOG, assume that $a_n \leq a_{n-1}$.

Then if $A = a_n + a_{n+1}$ We have $a_1 + \cdots + a_{n-1} + A = 1; hencea_1a_2 + \cdots + a_{n-1}A \leq \frac{1}{4}$.

$$\therefore\ a_1a_2 + \cdots + a_{n-1}a_n + a_na_{n+1} \leq a_1a_2 + \cdots + a_{n-1}\left(a_n + a_{n+1}\right) \leq \frac{1}{4}.$$

Hence the result is true by induction.

249.
, Let $x, y, z > 0$ and $xyz = 1$. Prove that

$$\frac{x^3}{x^4+1} + \frac{y^3}{y^4+1} + \frac{z^3}{z^4+1} \le \frac{3}{2}$$

Solution:
Let 1.

$$f(x,y,z) = \frac{x^3}{x^4+1} + \frac{y^3}{y^4+1} + \frac{z^3}{z^4+1} - \frac{3}{2}$$

Then

$$f(x,y,z) - f(x,\sqrt{yz},\sqrt{yz}) \le 0$$

But

$$f(x,\sqrt{yz},\sqrt{yz}) = f(\frac{1}{t^2},t,t) = -\frac{3t^{12}-4t^9+3t^8-2t^6+3t^4-4t^3-2t^3+3}{2(t^8+1)(t^4+1)} \le 0$$

and inequality is prove.
2.
setting : $x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$ the inequality becomes:

$$\sum_{cyc} \frac{a^3b}{a^4+b^4} \le \frac{3}{2}$$

We have:

$$2LHS \le \sum_{cyc} \frac{2a^3b}{a^2b^2+\sqrt{\frac{a^8+b^8}{2}}} \le \sum_{cyc} \frac{a^3b}{ab\sqrt{\sqrt{\frac{a^8+b^8}{2}}}} = \sum_{cyc} (\frac{2a^8}{a^8+b^8})^{1/4} \le \sum_{cyc} \sqrt{\frac{2a^4}{a^4+b^4}} \le 3 = 2RHS$$

the last inequality is true since it's Vasc's
. 250., Let x, y, z are non-negative numbers which not two of them equal to 0. Prove that:

$$\sqrt{\frac{x}{y+z}} + \sqrt{\frac{y}{x+z}} + \sqrt{\frac{z}{y+x}} \ge 2\sqrt{1+\frac{xyz}{(x+y)(x+z)(y+z)}}$$

$$<=> \sum x(x+y)(x+z) + 2\sum(x+y)\sqrt{xy(x+z)(y+z)} \ge 4(\sum x)(\sum xy)$$

By am-gm +schur, We hvae : +

$$LHS \ge \sum x(x+y)(x+z) + 12xyz + 2\sum xy(x+y)$$

$$+x(x+y)(x+z)+12xyz+2\sum xy(x+y)-4(\sum x)(\sum xy) = \sum x^3+3xyz-\sum xy(x+y) \ge 0$$

With this problem We have 2 way to solved it
The way 1: it is similar to mitdac123 s**Solution**
The way2 (me)

$$(\sum a^2(b+c)).LHS^2 \ge (a+b+c)^3*$$

let

$$a+b+c = 1 = p, \sum ab = \frac{1-q^2}{3}, abc = r$$

We will prove

$$<=> \frac{4(1-q)^2-3(1-q^2)}{36(1-q^2)-9} \leq r$$

Use

$$r \leq \frac{1+q)^2(1-2q)}{27}$$

We will prove this ineq

$$12(1-q)^2(1+q)-9(1-q) \leq (1+q)(1-2q)(4(1-q^2)-1)$$

$$<=> q^2(4q-1)^2 \geq \ 0$$

251., Prove if $a, b, c > 0$ then

$$(ab+bc+ca)\sum a^3b^3 \geq (-a+b+c)(a-b+c)(a+b-c)(a^3+b^3+c^3)(a^2+b^2+c^2)$$

Assume that: $a+b+c=1$

$$<=> f(r) = -45qr^2+24r^2+69q^2r+11r-58qr+q^4-9q+26q^2+1-24q^3 \geq 0$$

$$f'(r) = -90qr+48r+69q^2+11-58q \geq 0$$

$$=> f(r) \geq f(\frac{4q-1}{9}) = \frac{1}{27}(q-1)(3q-1)(9q^2-8q+2) \geq 0$$

Q.E.D

252. Let $a, b, c > 0$. Prove that

$$(ab+bc+ca)^3 \geq (-a+b+c)(a-b+c)(a+b-c)(a+b+c)^3$$

Assume $p=1$

$$r \geq \frac{4q-1}{9}$$

$$<=> (3q-1)(3q^2+q-1) \geq 0$$

Q.E.D

253. Let x, y, z are positive numbers. Prove that:

$$\frac{x^5}{y^2+z^2}+\frac{y^5}{x^2+z^2}+\frac{z^5}{y^2+x^2} \geq \frac{3(x^6+y^6+z^6)}{2(x^3+y^3+z^3)}$$

Solution:
By AM-GM ,We have

$$\frac{x^5}{y^2+z^2} \geq \frac{x^6}{2\sqrt{(\frac{x^2+y^2+z^2}{3})^3}}$$

And

$$\sum_{cyc}\frac{x^5}{y^2+z^2} \geq \frac{(x^6+y^6+z^6)^2}{\sum(x^7y^2+x^7z^2)} \geq \frac{3(x^6+y^6+z^6)}{2(x^3+y^3+z^3)}$$

where the last inequality is true by AM-GM too.
4.25, Let $x, y, z, t \in R$ and $x+y+z+t = x^7+y^7+z^7+t^7 = 0$. Find

$$S = t(t+x)(t+y)(t+z)$$

Setting : $x+y=-z-t=k$

$$=> x^7+(k-x)^7=(k+t)^7-t^7$$

$$<=> k(x+t)(-t+x-k)=0$$

$$<=> (t+z)(t+y)(t+x)=0$$

$$=> S=0$$

Q.E.D

255, Let $a,b,c>0$. Prove that:

$$\frac{abc}{a^3+b^3+c^3}+\frac{a^2+b^2+c^2}{ac+ab+bc}\geq\frac{4}{3}$$

Solution:
Let

$$p=a+b+c, q=ab+bc+ca, r=abc.$$

Assume $p=1$

$$<=> (3q-1)(10q-9r-3)\geq 0$$

By schur We have:

$$9r\geq 4q-1$$

$$(3q-1)(10q-9r-3)\geq 2(3q-1)^2\geq 0$$

Done Smile

$$\sum(a-b)^2\left(\frac{1}{ab+bc+ca}-\frac{a+b+c}{3(a^3+b^3+c^3)}\right)\geq 0$$

wich is true because

$$3(a^3+b^3+c^3)\geq(a+b+c)(ab+bc+ca)$$

Q.E.D

256., Prove that if $a,b,c\geq 0$ then

$$\sum a\sqrt{a^2-ab+b^2}\sqrt{a^2-ac+c^2}\leq a^3+b^3+c^3$$

Solution:

$$\left(\sum_{\text{cyclic}} a\sqrt{a^2-ab+b^2}\sqrt{a^2-ac+c^2}\right)^2=\left(\sum_{\text{cyclic}}\sqrt{a^3-a^2b+b^2a}\sqrt{a^3-a^2c+c^2a}\right)^2\leq(a^3+b^3+c^3)^2$$

if are you mean that

$$\sum_{cyc}\sqrt{(a^3-a^2b+b^2a)(a^3-a^2c+c^2a)}\leq\frac{1}{2}\sum_{cyc}(a^3-a^2b+b^2a+a^3-a^2c+c^2a)==a^3+b^3+c^3$$

=> Q.E.D

257., Let $a,b,c>0$. Prove that

$$\sum\frac{(a+b)^2}{c(2c+a+b)}\geq 3$$

Solution:
Let $a+b+c=3$. Then

$$\sum_{cyc}\frac{(a+b)^2}{c(2c+a+b)} \ge 3 \Leftrightarrow \sum_{cyc}\left(\frac{(3-c)^2}{c(3+c)}-1\right) \ge 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\frac{1-c}{c(3+c)} \ge 0 \Leftrightarrow \sum_{cyc}\left(\frac{1-c}{c(3+c)}+\frac{c-1}{4}\right) \ge 0 \Leftrightarrow \sum_{cyc}\frac{(c-1)^2(4+c)}{c(3+c)} \ge 0$$

Q.E.D

258, Let $a,b,c>0$. Prove that

$$\sum \frac{(a+b)^2}{c(2c+a+b)} \ge 2\sum \frac{a}{b+c}$$

Solution:
Setting :

$$x=\frac{a+b}{c};y=\frac{c+a}{b};z=\frac{b+c}{a}$$

By cauchy-schwarz, We can prove :

$$\sum \frac{x^2}{2+x} \ge \frac{(x+y+z)^2}{6+\sum x} \ge \frac{\sum x}{2} \ge 2(\frac{1}{x}) \text{ (The last ineq is easy}$$

$$<=> \sum \frac{a+b}{c} \ge 4\sum \frac{a+b}{a+b+2c}$$

Setting :

$$x=\frac{b+c}{2a};y=\frac{c+a}{2b};z=\frac{a+b}{2c}$$

$$<=> \sum x \ge 2\sum \frac{x}{x+1}$$

We have $xyz \ge 1$

$$=> LHS \ge \sum \sqrt{x} \ge \sum \frac{2x}{x+1}$$

Q.E.D

259. Let $a,b,c \in \mathbb{R}^+$. Prove that

$$\sum_{cyclic} a\sqrt{7a^2+9b^2} \ge \frac{4}{3}(a+b+c)^2$$

Solution:
By the Cauchy Schwarz inequality, We have

$$\sum a\sqrt{7a^2+9b^2} \ge \frac{1}{4}\sum a(7a+9b) = \frac{1}{4}(7\sum a^2+9\sum ab)$$

it suffices to prove that

$$\frac{1}{4}(7\sum a^2+9\sum ab) \ge \frac{4}{3}(\sum a)^2$$

$$\Leftrightarrow 21\sum a^2+27\sum ab \ge 16\sum a^2+32\sum ab$$

$$\Leftrightarrow 5\sum a^2 \ge 5\sum ab$$

$$<=> (a-b)^2+(b-c)^2+(c-a)^2 \geq 0$$

which is true

260., Let $a,b,c \in \mathbb{R}^+$ and $ab+bc+ca=abc$. Prove that

$$\sum_{cyclic} \frac{1}{\sqrt{ab}-1} \leq \frac{3}{2}$$

Solution:

Setting $a:=\sqrt{ab}, b:=\sqrt{bc}, c=\sqrt{ca}$ then $a^2+b^2+c^2=abc$

And the inequality becomes

$$\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1} \leq \frac{3}{2}$$

$$\sum \frac{1}{a-1} = \frac{\sum (b-1)(c-1)}{(a-1)(b-1)(c-1)} = \frac{(ab+bc+ca)-2(a+b+c)+3}{abc-(ab+bc+ca)+a+b+c-1}$$

Setting $p=a+b+c, q=ab+bc+ca, r=abc$ then We need to show that

$$3p^2+7p-11q-9 \geq 0$$

By Schur ineq We have

$$q \leq \frac{p^3+9p^2}{18+4p}$$

it suffices us to show that

$$3p^2+7p-11\left(\frac{p^3+9p^2}{18+4p}\right)-9 \geq 0$$

$$\Leftrightarrow p^3-17p^2+90p-162 \geq 0$$

$$\Leftrightarrow (p-9)\left(p^2-8p+18\right) \geq 0$$

Which is true by

$$p^2-8p+18 \geq 0 and p-9 \geq 0$$

We have done

261. Let $a,b,c>0$ such at $abc=1$. Prove that:

$$\frac{a\sqrt{b+c}}{b+c+1}+\frac{b\sqrt{c+a}}{c+a+1}+\frac{c\sqrt{a+b}}{a+b+1} \geq \sqrt{2}$$

Solution:

Another **Solution**. Applying AM-GM inequality, We have:

$$1 \leq \frac{\sqrt{a}(b+c)}{2}.$$

Thus, We only need to prove that:

$$\sum_{sym} \frac{a}{\sqrt{b+c}(\sqrt{a}+2)} \geq \frac{\sqrt{2}}{2}$$

using CS inequality, We have:

$$\sum_{sym} \frac{a}{\sqrt{b+c}(\sqrt{a}+2)} \geq \frac{(a+b+c)^2}{2\sum a\sqrt{b+c}+\sum a\sqrt{a(b+c)}} \geq \frac{(a+b+c)^2}{2\sqrt{2(\sum a)(\sum ab)}+\sqrt{(\sum a)(\sum ab(a+b))}}$$

Setting $t^2 = a+b+c; u^2 = ab+bc+ca$. Rewrite inequality, We need to prove:

$$\sqrt{2}t^3 - 2\sqrt{2}u \geq \sqrt{t^2u^2 - 3}$$

By AM-GM ineq, $u^2 <= t^2/3$. Thus,

$$(\sqrt{2}t^3 - 2\sqrt{2}u)^2 - t^2u^2 + 3 \geq \frac{5}{3}t^6 - \frac{8\sqrt{3}}{3}t^5 + \frac{8}{3}t^4 + 3 \geq 0$$

Using AM-GM:

$$\frac{5}{3}t^6 + \frac{8}{3}t^4 + 3 \geq 24\sqrt[24]{\frac{t^{122}}{3^{37}}} \geq \frac{8\sqrt{3}}{3}t^5 (t \geq \sqrt{3})$$

261.
Let $a,b,c > 0$ and $a^3+b^3+c^3 = 3$. Prove that

$$\frac{a^2}{3-a^2} + \frac{b^2}{3-b^2} + \frac{c^2}{3-c^2} \geq \frac{3}{2}$$

Solution:
Using AM-GM inequality We have

$$\frac{a^2}{3-a^2} = \frac{a^3}{a(3-a^2)} = \frac{a^3}{\sqrt{a^2(3-a^2)^2}} = \frac{a^3}{2.\sqrt{a^2.\frac{3-a^2}{2}.\frac{3-a^2}{2}}} \geq \frac{a^3}{2}$$

Hence We have :

$$\frac{a^2}{3-a^2} + \frac{b^2}{3b^2} + \frac{c^2}{3-c^2} \geq \frac{a^2+b^2+c^2}{2} = \frac{3}{2}$$

Q.E.D

262, Let $a,b,c > 0$ such that $a+b+c+1 = 4abc$. Prove that

$$\frac{1}{a^4+b+c} + \frac{1}{b^4+c+a} + \frac{1}{c^4+a+b} \leq \frac{3}{a+b+c}$$

Solution:
The inequality comes from: if $a,b,c > 0$ such that $a+b+c \geq 3$ then

$$\frac{1}{a^4+b+c} + \frac{1}{b^4+c+a} + \frac{1}{c^4+a+b} \leq \frac{3}{a+b+c}$$

This is a very known result.
We have :

$$a^4+b+c = \frac{(a^2)^2}{1} + \frac{(b^2)^2}{b^3} + \frac{(c^2)^2}{c^3} \geq \frac{(a^2+b^2+c^2)^2}{1+b^3+c^3}$$

$$\Leftrightarrow \sum \frac{1}{a^4+b+c} \leq \frac{3+2(a^3+b^3+c^3)}{(a^2+b^2+c^2)^2}$$

We have to prove that :

$$\frac{3+2(a^3+b^3+c^3)}{(a^2+b^2+c^2)^2} \leq \frac{3}{a+b+c}(*)$$

with : $a,b,c > 0$ and $a+b+c+1 = 4abc$ $a+b+c = p; ab+bc+ca = q; abc = r$ We have :

$$4abc = a+b+c+1 \geq 4\sqrt[4]{abc} \Leftrightarrow abc = r \geq 1$$

$$(*) \Leftrightarrow (p^2-3q)^2 + (q^2-3p) + 2(q^2-3pr) \geq 0$$

We have:

$$(p^2-3q)^2+(q^2-3p)+2(q^2-3pr)\geq (p^2-3q)^2+3(q^2-3pr)\geq 0$$

Q.E.D

263. Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that

$$\sqrt{\frac{8a(b+c)+9bc}{(2b+c)(b+2c)}}+\sqrt{\frac{8b(c+a)+9ca}{(2c+a)(c+2a)}}+\sqrt{\frac{8c(a+b)+9ab}{(2a+b)(a+2b)}}\geq 5$$

264. Let $a,b>0$. Prove that

$$\frac{a}{\sqrt{a^2+3b^2}}+\frac{b}{\sqrt{b^2+3a^2}}\geq 1$$

Solution:
Setting:

$$x=\frac{a}{\sqrt{a^2+3b^2}} \quad \text{and} \quad y=\frac{b}{\sqrt{b^2+3a^2}}$$

that lead to the condition:

$$8(xy)^2+x^2+y^2=1$$

Assume that

$$4xy\leq (x+y)^2\leq 1$$

then We have

$$4(xy)^2\geq 1$$

(from the above condition, you can check it properly). As a result, We have the contradiction.

By Holder

$$\left(\frac{a}{\sqrt{a^2+3b^2}}+\frac{b}{\sqrt{b^2+3a^2}}\right)^2(a(a^2+3b^2)+b(b^2+3a^2))\geq (a+b)^3 == a(a^2+3b^2)+b(b^2+3a^2).$$

Q.E.D

265., Let a,b,c be positive real numbers. Prove that

$$\frac{\sqrt{bc+4ab+4ac}}{b+c}+\frac{\sqrt{ca+4bc+4ba}}{c+a}+\frac{\sqrt{ab+4ca+4cb}}{a+b}\leq \frac{3}{2\sqrt{2}}\left(\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}}+\sqrt{\frac{a+b}{c}}\right)$$

Solution:
Since

$$\frac{\sqrt{4(bc+4ab+4ac)}}{b+c}=\sqrt{\frac{16a}{b+c}+\frac{4bc}{(b+c)^2}}\leq\sqrt{\frac{16a+b+c}{b+c}}$$

it suffices to prove that

$$\frac{3}{\sqrt{2}}\sum\sqrt{\frac{b+c}{a}}\geq\sum\sqrt{\frac{16a+b+c}{b+c}}$$

Squaring both sides, We have

$$\left(\sum\sqrt{\frac{b+c}{a}}\right)^2=\sum\frac{b+c}{a}+2\sum\sqrt{\frac{(a+b)(a+c)}{bc}}\geq\sum\frac{b+c}{a}+2\sum\frac{a+\sqrt{bc}}{\sqrt{bc}}$$

$$= \sum \frac{b+c}{a} + 2\sum \frac{a}{\sqrt{bc}} + 6 \geq \sum \frac{b+c}{a} + 4\sum \frac{a}{b+c} + 6$$

$$= \sum a\left(\frac{1}{b} + \frac{1}{c}\right) + 4\sum \frac{a}{b+c} + 6 \geq 8\sum \frac{a}{b+c} + 6$$

and

$$\left(\sum \sqrt{\frac{16a+b+c}{b+c}}\right)^2 = \sum \frac{16a+b+c}{b+c} + 2\sum \sqrt{\frac{(16a+b+c)(16b+c+a)}{(a+c)(b+c)}} \leq$$

$$\sum \frac{16a+b+c}{b+c} + \sum \left(\frac{16a+b+c}{a+c} + \frac{16b+c+a}{b+c}\right)$$

$$= 18\sum \frac{a}{b+c} + 54$$

Hence, it suffices to prove that

$$\frac{9}{2}\left(8\sum \frac{a}{b+c} + 6\right) \geq 18\sum \frac{a}{b+c} + 54$$

$$\Leftrightarrow \sum \frac{a}{b+c} \geq \frac{3}{2}$$

This is Nesbitt's inequality. Equality holds if and only if a = b = c. We have done.

266. Let $a, b, c \geq 0; ab + bc + ca = 1$. Prove that

$$\sum \frac{1}{a+b} \geq \frac{5}{2}$$

Solution:
WLOG

$$c = min(a, b, c)$$

Case 1. if

$$c = 0 \rightarrow ab = 1$$

$$LHS = \frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} \geq \frac{5}{2}$$

$$<=> (\sqrt{a} - \sqrt{b})^2(2a + 2b - ab) \geq 0$$

it is true because $ab = 1$
. Case 2..if $a, b, c \neq 0$.
Setting

$$x = \frac{ab}{c}; y = \frac{ca}{b}; z = \frac{bc}{a}$$

$$\rightarrow LHS \geq \frac{(ab + bc + ca)^2}{2abc} \geq \frac{5}{2}$$

it is true.
267. Let a, b, c be nonegative real numbers such that $a^2 + b^2 + c^2 = 1$. Prove that

$$\frac{a}{\sqrt{1+bc}} + \frac{b}{\sqrt{1+ca}} + \frac{c}{\sqrt{1+ab}} \leq \frac{3}{2}$$

Solution:
Applying Cauchy inequality, We have

$$\left(\sum \frac{a}{\sqrt{1+bc}}\right)^2 \leq \left(\sum a\right)\left(\sum \frac{a}{1+bc}\right)$$

We have the following result:

$$\sum \frac{a}{1+bc} \le \sum \frac{a}{(a+b)(a+c)}$$

Actually, We have

$$LHS - RHS = \sum \frac{a((a+b)(a+c) - a^2 - b^2 - c^2 - bc)}{(1+bc)(a+b)(a+c)} = \sum \frac{ab(a-b) - ca(c-a)}{(1+bc)(a+b)(a+c)} =$$

$$\sum \frac{ab(a-b)}{(1+bc)(a+b)(a+c)} - \sum \frac{ab(a-b)}{(1+ac)(b+a)(b+c)} = -\sum \frac{ab(a-b)^2(1-c^2)}{(a+b)(b+c)(c+a)(1+ac)(1+bc)} \le 0$$

Hence, it is sufficient to prove that

$$\left(\sum a\right)\left(\sum \frac{a}{(a+b)(a+c)}\right) \le \frac{9}{4}$$

$$\Leftrightarrow (a+b)(b+c)(c+a) \ge \frac{8}{9}(ab+bc+ca)(a+b+c)$$

By AM - GM inequality, it is true.

268. Let $a, b, c > 0$. Prove that

$$(a+b)\left(\frac{a}{b} + \frac{b}{c}\right) + \frac{1}{b} \ge 2\left(\frac{a}{b} + \sqrt{\frac{b}{c}}\right)$$

Solution:
Divide by b to get

$$(\frac{a}{b} + 1)(\frac{a}{b} + \frac{b}{c}) + \frac{1}{b^2} \ge \frac{2}{b}(\frac{a}{b} + \sqrt{\frac{b}{c}})$$

Let

$$\frac{a}{b} = x, \frac{b}{c} = y, and \frac{1}{b} = z$$

Then We have to show

$$(x+1)(x+y) + z^2 \ge 2z(x + \sqrt{y})$$

$$x^2 + x + y + xy + (z - x - \sqrt{y})^2 \ge (x + \sqrt{y})^2$$

$$x + xy + (z - x - \sqrt{y})^2 \ge 2x\sqrt{y}$$

$$x(1 - \sqrt{y})^2 + (z - x - \sqrt{y})^2 \ge 0$$

Which is obviously true. Equality holds when $b = c$ and $a + b = 1$.

269. Let $x, y, z, k > 0$. Prove that

$$\frac{x^2+y^2}{z+k} + \frac{y^2+z^2}{x+k} + \frac{z^2+x^2}{y+k} \ge \frac{3}{2}(x+y+z-k)$$

Solution:
Using Cauchy-Schwarz's inequality We have,

$$2(x+y+z+3k)(\frac{x^2+y^2}{z+k} + \frac{y^2+z^2}{x+k} + \frac{z^2+x^2}{y+k})$$

$$= [2(z+k) + 2(x+k) + 2(y+k)](\frac{x^2}{z+k} + \frac{y^2}{z+k} + \frac{y^2}{x+k} + \frac{z^2}{x+k} + \frac{z^2}{y+k} + \frac{x^2}{y+k})$$

$$\geq 4(x+y+z)^2$$

Now let $x+y+z=a$ then We only need to prove

$$4a^2 \geq 3(a+3k)(a-k)$$

$$\Leftrightarrow (a-3k)^2 \geq 0, \quad \text{which is obviously true.}$$

using AM-GM :

$$\frac{x^2+y^2}{z+k}+\frac{k+z}{2} \geq x+y$$

Q.E.D

270, Let $a,b,c>0$. Prove that

$$ab+bc+ca \leq a^4+b^4+c^4+\frac{3}{4}.$$

Solution:
Bu Am-GM's inequality, We have:

$$a^4+\frac{1}{4} \geq a^2$$

and cyclic. So

$$a^4+b^4+c^4+\frac{3}{4} \geq a^2+b^2+c^2 \geq ab+bc+ca.$$

$$(a^4+b^4+1/8+1/8)/4 \geq \sqrt[4]{a^4b^4/64}$$

Q.E.D
271, Let $a,b,c>0$ and $a+b+c=3$. Prove that

$$\frac{a^2+1+2b^2}{a+1}+\frac{b^2+1+2c^2}{b+1}+\frac{c^2+1+2a^2}{c+1} \geq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})$$

Solution:

$$\frac{a^2+1+2b^2}{a+1}+\frac{b^2+1+2c^2}{b+1}+\frac{c^2+1+2a^2}{c+1} \geq \sqrt{2}(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2})$$

$$a^2+b^2 \geq 2ab$$

and

$$b^2+1 \geq 2b$$

so

$$a^2+1+2b^2 \geq 2ab+2b=2b(a+1)$$

So then

$$LHS \geq 2(a+b+c)$$

So We have to prove that:

$$a+b+c \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca}, \quad \text{Which is trivial}$$

272, Let $a,b,c>0$. Prove that

$$\frac{1}{(a^3-1)^2}+\frac{1}{(b^3-1)^2}+\frac{1}{(c^3-1)^2}<\frac{1}{9}\left[\frac{1}{a^2(a-1)^2}+\frac{1}{b^2(b-1)^2}+\frac{1}{c^2(c-1)^2}\right]$$

Solution:
All We need to do is to prove

$$9a^2(a-1)^2 \le (a^3-1)^2$$

it deduces to prove that

$$9a^2 \le (a^2+a+1)^2(*)$$

Moreover, We use the identity

$$A^2-B^2=(A+B)(A-B)$$

$(*)$ becomes $(a-1)^2(a^2+4a+1) \ge 0$ which is trivial. Finally, the equality cannot hold obviously so We have the strict inequality. Q.E.D

273. Given $a,b,c \ge 0$. Prove that:

$$\sqrt{\frac{b^2-bc+c^2}{a^2+bc}}+\sqrt{\frac{a^2-ab+b^2}{c^2+ab}}+\sqrt{\frac{c^2-ca+a^2}{b^2+ac}}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2} \ge 4$$

Solution:
This inequality is valid, nguoivn and it follows from applying AM-GM as follow:

$$\sqrt{\frac{b^2-bc+c^2}{a^2+bc}} \ge \frac{2(b^2-bc+c^2)}{(a^2+bc)+(b^2-bc+c^2)}=\frac{2(b^2-bc+c^2)}{a^2+b^2+c^2}$$

274. Given that $a,b,c \ge 0$ and $a+b+c=3$. Prove that

$$a+ab+2abc \le \frac{9}{2}$$

You're right, shaam. We can show that

$$a+ab+2abc \le \frac{9}{2}$$

as follow: Replacing $b=3-a-c$, then We have to prove

$$a+a(3-a-c)+2ac(3-a-c) \le \frac{9}{2},$$

or equivalently,

$$f(a)=(2c+1)a^2+(2c^2-5c-4)a+\frac{9}{2} \ge 0.$$

We see that f(a) is a quadratic polynomial of a with the highest coefficient is positive. Moreover, its disciminant is

$$\Delta=(2c^2-5c-4)^2-18(2c+1)=(2c-1)^2(c^2-4c-2) \le 0, as 0 \le c \le 3.$$

Therefore, $f(a) \ge 0$ and our **Solution** is completed. Equality holds if and only if $a=\frac{3}{2}, b=1, c=\frac{1}{2}$.

$$2a+ab+abc=a(2+b(c+1)) \le a(2+(\frac{b+c+1}{2})^2) \le 9$$
$$a(8+(b+c+1)^2)=a(8+(5-a)^2)=33a-10a^2+a^3 \le 36 \iff$$
$$-a^3+10a^2-33a+36=(4-a)(3-a)^2 \ge 0$$

which is obviously true. Equality if $a=3, b=1, c=0$.

275. Given $a,b,c>0$ and $abc=1$. Prove that

$$\frac{(a^2+1)(b^2+1)(c^2+1)}{(a+1)(b+1)(c+1)} \geq \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}$$

Solution:
1.

$$\frac{x^2+1}{x+1} \geq \sqrt[3]{\frac{x^3+1}{2}}$$
$$\Leftrightarrow (x-1)^4(x^2+x+1) \geq 0$$

and

$$(a^2+bc)(1+\frac{bc}{c}) \geq (a+b)^2$$

so

$$(a^2+bc)(b^2+ca)(c^2+ab) \geq abc(a+b)(b+c)(c+a)$$
$$(a^3+1)(b^3+1)(c^3+1) \geq (a+b)(b+c)(c+a)$$

The stronger is trues and very easy:

$$(1+a)^3(1+b)^3(1+c)^3 \geq \ 64(a+b)(b+c)(c+a)$$

$$\prod_{cyc}(1+a)^{16} = \left(\prod_{cyc}(1+a)(1+b)\right)^8 = \left(\prod_{cyc}(1+ab+a+b)\right)^8 \geq$$

$$\geq 8^8\left(\prod_{cyc}(1+ab)(a+b)\right)^4 = 8^8\prod_{cyc}(a+b)^4\left(\prod_{cyc}(1+ab)(1+ac)\right)^2 =$$

$$= 8^8\prod_{cyc}(a+b)^4\left(\prod_{cyc}(1+a+a(b+c))\right)^2 \geq 8^{10}\prod_{cyc}(a+b)^5\prod_{cyc}(1+a),$$

which is true. 2.

$$(a+b)(1+ab) \leq \ \frac{(1+a)^2(1+b)^2}{4}$$

Because of:

$$(1+ab)(1+bc)(1+ca) = (1+a)(1+b)(1+c)$$

Because

$$abc=1$$

Q.E.D

276. Given $a,b,c \geq 0$. Prove that:

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2} \geq \frac{5}{2(ab+bc+ca)}$$

Solution:
We have :

$$\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2} \geq \frac{4}{2c^2+(a+b)^2-2ab}$$

$a \geq b \geq c$; assume $a+b=1$. We have : $c(1-c) \leq x = ab \leq \frac{1}{4}$
We have to prove that:

$$(x+c)(\frac{1}{1-2x}+\frac{4}{2c^2-2x+1}) \geq \frac{5}{2}$$

$$\Leftrightarrow f(x) = -40x^2 + x(24c^2-20c+30) + 4c^3 - 10c^2 + 10c - 5 \geq 0$$

We have :

$$f^{'}(x) = -80x + 24c^2 - 20c + 30$$

$$f^{''}(x) = -80 < 0$$

$$f^{'}(x) \geq f^{'}(\frac{1}{4}) = 24c^2 - 20c + 10 > 0$$

We have :

$$f(x) \geq f(\frac{1}{4}) = c((2c-1)^2+4) \geq 0$$

We have done
Assume : $c = min\{a;b;c\}$ We have :

$$a^2+b^2 \leq x^2+y^2, b^2+c^2 \leq y^2, c^2+a^2 \leq x^2$$

$$ab+bc+ca \geq xy$$

With :

$$x = a+\frac{c}{2}; y = b+\frac{c}{2}$$

We have to prove that :

$$\frac{xy}{x^2+y^2}+\frac{x^2+y^2}{xy} \geq \frac{5}{2}$$

By AM-GM We have :

$$\frac{xy}{x^2+y^2}+\frac{x^2+y^2}{xy} = (\frac{xy}{x^2+y^2}+\frac{x^2+y^2}{4xy}) + \frac{3(x^2+y^2)}{4xy} \geq \frac{5}{2}$$

We have done.

277., Given $a,b,c \geq 0$. Prove that

$$(a^3b^3+b^3c^3+c^3a^3)[(a+b)(b+c)(c+a)-8abc] \geq abc(a-b)^2(b-c)^2(c-a)^2$$

Solution:

$$[c(a-b)^2+b(a-c)^2+a(b-c)^2][\frac{1}{c^3}+\frac{1}{b^3}+\frac{1}{a^3}] \geq (\frac{a-b}{c}+\frac{b-c}{a}+\frac{c-a}{b})^2 = \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}$$

and We get

$$(a^3b^3+b^3c^3+c^3a^3)[(a+b)(b+c)(c+a)-8abc] \geq abc(a-b)^2(b-c)^2(c-a)^2$$

278.Let a,b,c be positive real number .Prove that:

$$(a^3b^3+b^3c^3+c^3a^3)[(a+b)(b+c)(c+a)-8abc] \geq 9abc(a-b)^2(b-c)^2(c-a)^2$$

Solution:
Replacing a,b,c by $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ respectively, then our inequality becomes

$$(a^3+b^3+c^3)[(a+b)(b+c)(c+a)-8abc] \geq 9(a-b)^2(b-c)^2(c-a)^2$$

Now, assume that $c=\min\{a,b,c\}$ then We have

$$a^3+b^3+c^3 \geq a^3+b^3$$

$$(a+b)(b+c)(c+a)-8abc=2c(a-b)^2+(a+b)(a-c)(b-c) \geq (a+b)(a-c)(b-c),$$

and

$$(a-b)^2(a-c)^2(b-c)^2 \leq ab(a-b)^2(a-c)(b-c)$$

Therefore, We can reduce our inequality to

$$(a^3+b^3)(a+b)(a-c)(b-c) \geq 9ab(a-b)^2(a-c)(b-c),$$

or

$$(a^3+b^3)(a+b) \geq 9ab(a-b)^2$$

which is equivalent to

$$(a^2-4ab+b^2)^2 \geq 0$$

which is trivial

279. Given $a,b,c>0$ and $abc=1$. Prove that:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq a+b+c+\frac{(a^2c+b^2a+c^2b-3)^2}{ab+bc+ca}$$

Solution:
it is equivalent to

$$\sum_{cyc}(a^3-3a^2b+6a^2c-4abc) \geq 0, \quad \text{which is obvious.}$$

$$\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right)\left(\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}\right)$$
$$\geq \left(\frac{a-b}{b}+\frac{b-c}{c}+\frac{c-a}{a}\right)^2$$

Q.E.D

280, Given $a,b,c \geq 0$. Prove that:

$$(a+2b)^2(b+2c)^2(c+2a)^2 \geq 27abc(a+2c)(b+2a)(c+2b)$$

Solution:

$$(a+2b)(b+2c)=ab+bc+bc+b^2+b^2+bc+bc+ca+ca=b(a+2c)+b(c+2b)+c(b+2a) \geq 3\sqrt[3]{b^2c(a+2c)(c+2b)(b+2a)}$$

and similar,We have

$$(a+2b)^2(b+2c)^2(c+2a)^2 \geq 27abc(a+2c)(b+2a)(c+2b)$$

Q.E.D

281.
Let a,b,c are three positive real numbers. Prove that

$$\frac{(a+b-c)^2}{c^2+(b+a)^2}+\frac{(b+c-a)^2}{a^2+(b+c)^2}+\frac{(c+a-b)^2}{b+(c+a)^2} \geq \frac{3}{5}$$

Solution:
because the ineq is homogenuous, We can assume that $a+b+c=1$
ineq

$$\Leftrightarrow \sum(\frac{(1-2a)^2}{a^2+(1-a)^2}-\frac{1}{5}\geq 0 \Leftrightarrow \sum\frac{(3a-1)(3a-2)}{a^2+(1-a)^2}\geq 0$$

We can apply Chebyshev inequality ,We have

$$LHS\geq\frac{1}{3}(3a+3b+3c-3)(\sum\frac{3a-2}{a^2+(1-a)^2})\geq 0$$

Be cau se $(a+b+c=1)$.

282, Let a,b,c are real numbers. Prove that

$$(a+b)^4+(b+c)^4+(c+a)^4\geq\frac{4}{7}(a+b+c)^4$$

Solution:
by holder inequality, We get:

$$\left(\sum(a+b)^4\right)(1+1+1)(1+1+1)(1+1+1)\geq(2(a+b+c))^4$$

So

$$\left(\sum(a+b)^4\right)\geq\frac{16}{27}(a+b+c)^4\geq\frac{4}{7}(a+b+c)^4$$

Q.E.D

283.
Given $a,b,c>0$ and $a+b+c=3$. Prove that:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a^2}+3\sqrt{abc}\geq 6$$

Solution:
Using a ill-known result:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}$$

(We can prove it easily by Am-Gm)
So,

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{3(a^2+b^2+c^2)}{ab+bc+ca}$$

Besides, We have:

$$\sqrt{abc}\geq\frac{8abc}{(a+b)(b+c)(c+a)}$$

$$<=>(a+b)^2(b+c)^2(c+a)^2\geq\frac{64}{27}abc(a+b+c)^3$$

(trues again by Am-Gm)
So, We obtain:

$$\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{8abc}{(a+b)(b+c)(c+a)}\geq 2$$

284.Given $a,b,c>0$. Prove that:

$$\frac{(a-b)^2}{c^2}+\frac{(b-c)^2}{a^2}+\frac{(c-a)^2}{b^2}\geq\frac{(a-b)^2(b-c)^2(c-a)^2}{3a^2b^2c^2}$$

Solution:
it's equivalent to:

$$abc(\sum a^3 + 3abc - \sum ab(a+b)) \geq \ 0 \quad \text{(obviously trues)}$$

if $abc(a-b)(b-c)(c-a) \geq 0$. Prove that

$$\frac{(b-c)^3}{a^3} + \frac{(c-a)^3}{b^3} + \frac{(a-b)^3}{c^3} \leq \ \frac{3abc}{(a-b)(b-c)(c-a)}$$

if $abc(a-b)(b-c)(c-a) \leq 0$. Prove that:

$$\frac{(b-c)^3}{a^3} + \frac{(c-a)^3}{b^3} + \frac{(a-b)^3}{c^3} \geq \ \frac{3abc}{(a-b)(b-c)(c-a)}$$

And in fact, We proved the stronger:

$$\frac{(a-b)^2}{c^2} + \frac{(b-c)^2}{a^2} + \frac{(c-a)^2}{b^2} \geq \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}$$

Setting

$$x = \frac{b-c}{a}, y = \frac{c-a}{b}, z = \frac{c-a}{b}$$

We get $x + y + z + xyz = 0$
We have

$$x^2 + y^2 + z^2 = (x+y+z)^2 - 2(xy+yz+zx) = x^2y^2z^2 - 2(xy+yz+zx)$$

We have:

$$a, b, c \geq 0,$$

We have

$$xy + yz + zx \geq 0$$

and $x^2 + y^2 + z^2 \geq x^2y^2z^2$ or

$$\frac{(b-c)^2}{a^2} + \frac{(c-a)^2}{b^2} + \frac{(a-b)^2}{c^2} \geq \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}$$

and the inequalities is true if and only $xy + yz + zx \leq 0$, and then

$$\frac{(b-c)^2}{a^2} + \frac{(c-a)^2}{b^2} + \frac{(a-b)^2}{c^2} \leq \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2b^2c^2}$$

We have $xy + yz + zx \leq k(x+y+z)^2, k = \frac{1}{3}$ is the best. And

$$x^2 + y^2 + z^2 - 2(xy+yz+zx) \geq x^2y^2z^2 - \frac{2(x+y+z)^2}{3} = \frac{r^2}{3}$$

So in inequalities

$$\frac{(b-c)^2}{a^2} + \frac{(c-a)^2}{b^2} + \frac{(a-b)^2}{c^2} \geq \frac{(a-b)^2(b-c)^2(c-a)^2}{ka^2b^2c^2}$$

and $k = \frac{1}{3}$ is best.
285. Prove that:

$$\frac{2(a^2+b^2)}{(a+b)^2} + \frac{2(b^2+c^2)}{(b+c)^2} + \frac{2(c^2+a^2)}{(c+a)^2} \geq 3 + \frac{(a-c)^2}{a^2+b^2+c^2}$$

We can write it into:

$$\sum \frac{(a-b)^2}{(a+b)^2} \geq \frac{(a-c)^2}{a^2+b^2+c^2}$$

Using CS, We have:

$$LHS \geq \frac{(a-c)^2}{(a+b)^2+(b+c)^2} + \frac{(a-c)^2}{(a+c)^2} \geq \frac{4(a-c)^2}{\sum (a+b)^2} \geq RHS$$

(By AM-GM).

286. Give $a, b, c > 0$ and $ab + bc + ca = 3$. Prove that:

$$\frac{1}{\sqrt{1+(a+b)^3}-1} + \frac{1}{\sqrt{1+(b+c)^3}-1} + \frac{1}{\sqrt{1+(a+c)^3}-1} \geq \frac{3}{2}$$

Solution:

$$1+x^3 = (1+x)(1-x+x^2) \leq (1+\frac{x^2}{2})^2$$

let $x = a+b, x = b+c, x = c+a$,We get:

$$\sqrt{1+(a+b)^3} - 1 \leq 1 + \frac{(a+b)^2}{2} - 1 = \frac{(a+b)^2}{2}$$

And finally,We need to prove :

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \geq \frac{3}{4}$$

or

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \geq \frac{9}{4(ab+bc+ca)}$$

This is iran TST96.

287. Let a, b, c be positive numbers such that: $ab + bc + ca + abc = 4$. Prove that:

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \frac{a+b+c}{\sqrt{2}}$$

Solution:
By Horder We have :

$$(\sum \frac{a}{\sqrt{b+c}})^2.(\sum a(b+c)) \geq (a+b+c)^3$$

We have to prove that :

$$a+b+c \geq ab+bc+ca$$

With a, b, c be positive numbers such that: $ab + bc + ca + abc = 4$ it is VMO 1996. We have done.
288.
Given $a, b, c \geq 0$. Prove that:

$$\frac{a^2}{a^2+ab+b^2} + \frac{b^2}{b^2+bc+c^2} + \frac{c^2}{c^2+ca+a^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \leq 2$$

Solution:

$$\frac{a^2}{a^2+ab+b^2} + \frac{b^2}{b^2+bc+c^2} + \frac{c^2}{c^2+ca+a^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \leq 2$$

$$<=> 2-\frac{ab+ac+bc}{a^2+b^2+c^2}-\sum_{cyc}\frac{a^2}{a^2+ab+c^2}=\frac{(ab+ac+bc)\sum(a^5c-a^3c^2b)}{(a^2+b^2+c^2)\prod(a^2+ab+b^2)}.$$

it is easy by AM-GM 's inequality, Prove that work for reader.
288.
Let $a,b,c>0$ such that $ab+bc+ca=3$. Prove that

$$3\sqrt{(a+b)(b+c)(c+a)} >= 2(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}) \ge 6\sqrt{2}$$

Solution:
★ First , We prove the left ineq . Setting :$x=ab; y=bc; z=ca => x+y+z=3$ By cauchy-schwarz , We need to prove :

$$9(x+y)(y+z)(z+x) \ge 8(x+y+z)(xy+yz+zx)(\text{Right})$$

★, Next, We prove the right ineq : By Am-Gm , We need to prove :

$$(a+b)(b+c)(c+a) \ge 8$$

$$<=> 3(a+b+c)-abc \ge 8(\text{Right because } ab+bc+ca=3)$$

289.
Let $x,y,z>0$. Prove that :

$$3(\sqrt{x(x+y)(x+z)}+\sqrt{y(y+z)(y+x)}+\sqrt{z(z+x)(z+y)})^2 \le 4(x+y+z)^3$$

Solution:
By Cauchy-Schwarz ineq , We have :

$$LHS \le 3(x+y+z)(\sum x^2+yz+zx+xy)$$

Then We prove that :

$$4(x+y+z)^2 \ge 3[(x+y+z)^2+xy+yz+zx]$$

$$\leftrightarrow \sum(x-y)^2 \ge 0$$

289.
Let $a,b,c \ge 0$ such that $a+b+c=3$. Prove that:

$$(a^2b+b^2c+c^2a)+2(ab^2+bc^2+ca^2)+3abc \le 12$$

Solution:
use this ineq :

$$a,b,c \ge 0; a+b+c=3$$

$$a^2b+b^2c+c^2a+abc \le 4$$

it is inequality beautifl famous and beautiful.
291.
, Let a,b,c be positive real numbers such that $a+b+c=abc$. Prove that:

$$\sqrt{(1+a^2)(1+b^2)}+\sqrt{(1+b^2)(1+c^2)}+\sqrt{(1+c^2)(1+a^2)} \ge 4+\sqrt{(1+a^2)(1+b^2)(1+c^2)}$$

Solution:
Setting :$a=\frac{1}{x^2}; b=\frac{1}{y^2}; c=\frac{1}{z^2}$ By Am-Gm We have :

$$(x^2+z^2)(y^2+z^2) \ge (xy+z^2)^2$$

$$<=> \sum a^2b+b^2a \geq 6abc$$

it is true by AM-GM . Q.E.D

292.

Let $a, b, c \geq 0$. Prove that

$$\frac{(a+b+c)^3}{abc}+\sqrt[3]{\frac{ab+bc+ca}{a^2+b^2+c^2}} \geq 28$$

Solution:

By AM-GM inequality, We get

$$3abc \leq \frac{(ab+bc+ca)^2}{a+b+c}$$

it suffices to prove that

$$\frac{3(a+b+c)^4}{(ab+bc+ca)^2}+\sqrt[3]{\frac{ab+bc+ca}{a^2+b^2+c^2}} \geq 28$$

Notice that (by AM-GM inequality for two numbers)

$$\frac{(a+b+c)^2}{3(ab+bc+ca)}+\sqrt[3]{\frac{ab+bc+ca}{a^2+b^2+c^2}} \geq \frac{2(a+b+c)}{\sqrt{3}\sqrt[6]{(ab+bc+ca)^2(a^2+b^2+c^2)}}$$

and (AM-GM inequality for three numbers)

$$(ab+bc+ca)\cdot(ab+bc+ca)\cdot(a^2+b^2+c^2) \leq \frac{1}{27}(a+b+c)^6$$

Thus

$$\frac{(a+b+c)^2}{3(ab+bc+ca)}+\sqrt[3]{\frac{ab+bc+ca}{a^2+b^2+c^2}} \geq \frac{2(a+b+c)}{\sqrt{3}\sqrt[6]{\frac{1}{27}(a+b+c)^6}}=2$$

it suffices to prove that

$$\frac{3(a+b+c)^4}{(ab+bc+ca)^2} \geq \frac{(a+b+c)^2}{3(ab+bc+ca)}+26$$

it is true because

$$\frac{3(a+b+c)^4}{(ab+bc+ca)^2} \geq \frac{9(a+b+c)^2}{ab+bc+ca}=\frac{(a+b+c)^2}{3(ab+bc+ca)}+\frac{26(a+b+c)^2}{3(ab+bc+ca)} \geq \frac{(a+b+c)^2}{3(ab+bc+ca)}+26$$

We have done.

293.

Let $a, b, c \geq 0$. Prove that

$$\frac{a}{b(a^2+2b^2)}+\frac{b}{c(b^2+2c^2)}+\frac{c}{a(c^2+2a^2)} \geq \frac{3}{ab+bc+ca}$$

Solution:

$$a=\frac{1}{a}, b=\frac{1}{b}, c=\frac{1}{c}, \quad \text{the inequality becomes}$$

$$\sum \frac{b^2}{c(2a^2+b^2)} \geq \frac{3}{a+b+c}$$

By the Cauchy Schwarz inequality, We get

$$LHS \geq \frac{(\sum a^2)^2}{\sum b^2c(2a^2+b^2)}$$

it suffices to prove that

$$(\sum a^2)^2(\sum a) \geq 3\sum b^2c(2a^2+b^2)$$

$$\Leftrightarrow \sum a^5 + \sum ab^4 + 2\sum a^3b^2 + 2\sum a^2b^3 \geq 2\sum a^4b + 4\sum a^2b^2c$$

By the AM-GM inequality, We get

$$\sum a^5 + \sum a^3b^2 = \sum a^3(a^2+b^2) \geq 2\sum a^4b$$

$$\sum ab^4 + \sum a^2b^3 = \sum (ab^4 + c^2a^3) \geq 2\sum a^2b^2c$$

$$\sum a^3b^2 + \sum a^2b^3 = \sum a^3(b^2+c^2) \geq 2\sum a^3bc \geq 2\sum a^2b^2c$$

Adding up these inequalities, We get the result.

294.

Let $a,b,c>0$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{3a}{a^2+2bc}+\frac{3b}{b^2+2ca}+\frac{3c}{c^2+2ab}$$

Solution:

the inequality is equivalent to :

$$\sum (a-b)^2 \left(\frac{c(a-b)^2+ab(a+b+c)}{(a^3+2abc)(b^3+2abc)}\right) \geq 0$$

The inequality is equalivent with

$$(ab+bc+ca)\left(a^2+2bc\right)\left(b^2+2ca\right)\left(c^2+2ab\right)$$

$$\geq abc\left(\sum 3a\left(b^2+2ca\right)\left(c^2+2ab\right)\right)$$

By setting $q=ab+bc+ca, r=abc$ and assume $a+b+c=1$ We have

$$\sum 3a\left(b^2+2ca\right)\left(c^2+2ab\right) = 6q^2-27rq$$

$$\left(a^2+2bc\right)\left(b^2+2ca\right)\left(c^2+2ab\right) = 2q^3+27r^2-18rq+4r$$

So the inequality becomes

$$q\left(2q^3+27r^2-18rq+4r\right) \geq r\left(6q^2-27rq\right)$$

$$\left(27a^2b^2c^2+2abc+(ab+bc+ca)^3-12abc\,(ab+bc+ca)\right) \geq 0$$

So We need to prove that

$$27a^2b^2c^2+2abc+(ab+bc+ca)^3 \geq 12abc\,(ab+bc+ca)$$

But it's easy, by AM-GM We have

$$abc = abc\,(a+b+c)^2 \geq 3abc\,(ab+bc+ca)$$

$$27a^2b^2c^2 + abc + (ab + bc + ca)^3 \geq 9abc\,(ab + bc + ca)$$

Setting

$$a = \frac{1}{a}, b = \frac{1}{b}, c = \frac{1}{c}, \quad \text{the inequality is equivalent to}$$

$$\sum a \geq 3abc \sum \frac{1}{2a^2 + bc}$$

$$\Leftrightarrow \sum \frac{a(a^2 - bc)}{2a^2 + bc} \geq 0$$

$$\Leftrightarrow 3 \sum \frac{a^3}{2a^2 + bc} \geq \sum a$$

By the Cauchy Schwarz inequality, We get

$$\sum \frac{a^3}{2a^2 + bc} \geq \frac{(\sum a^2)^2}{2 \sum a^3 + 3abc}$$

it suffices to prove that

$$3(\sum a^2)^2 \geq (\sum a)(2 \sum a^3 + 3abc)$$

Assume $a + b + c = 1$, setting $q = ab + bc + ca, r = abc$, the inequality becomes

$$3(1 - 2q)^2 \geq 2 - 6q + 9r$$

Since $q^2 \geq 3r$, it suffices to show

$$3(1 - 2q)^2 \geq 2 - 6q + 3q^2$$

$$\Leftrightarrow 3 - 12q + 12q^2 \geq 2 - 6q + 3q^2$$

$$\Leftrightarrow (1 - 3q)^2 \geq 0$$

295:

Let $a, b, c \geq 0$ such that $a + b + c = 3$. Prove that

$$\sqrt{\frac{a^3}{a^2 + 8b^2}} + \sqrt{\frac{b^3}{b^2 + 8c^2}} + \sqrt{\frac{c^3}{c^2 + 8a^2}} \geq 1$$

Solution:

By Am-Gm ,We have :

$$LHS \geq \sum \frac{6a^2}{9a + a^2 + 8b^2} \geq \frac{6(\sum a^2)^2}{\sum 3(b + c)a^3 + 4a^4 + 8b^2a^2}$$

We need to prove :

$$6(\sum a^2)^2 \geq 3 \sum (b + c)a^3 + 4(\sum a^2)^2$$

$$<=> 2(\sum a^2)^2 \geq 3 \sum (b + c)a^3$$

$$<=> \sum (a - b)^2(a^2 - ab + b^2) \geq 0$$

Q.E.D

296.

Let $a, b, c \geq 0$. Prove that

$$\frac{(a + b + c)^3}{abc} + \sqrt[3]{\frac{ab + bc + ca}{a^2 + b^2 + c^2}} \geq 28$$

by AM-GM We have :

$$\sqrt[3]{\frac{a^2+b^2+c^2}{ab+ac+bc}} \leq \frac{a^2+b^2+c^2}{3(ab+ac+bc)} + \frac{2}{3}$$
$$= \frac{3(a+b+c)^2}{ab+ac+bc}$$

so :

$$\sqrt[3]{\frac{ab+ac+bc}{a^2+b^2+c^2}} \geq \frac{ab+ac+bc}{3(a+b+c)^2}$$

now We have to prove :

$$\frac{(a+b+c)^3}{abc} + \frac{ab+ac+bc}{3(a+b+c)^2} \geq 28$$

since the inequality is homogenous assume that : $a+b+c=1$
so We have to prove :

$$\frac{1}{abc} + \frac{1}{3}(ab+ac+bc) \geq 28$$

by schur We have :

$$\frac{1}{abc} \geq \frac{9}{ab+ac+bc}$$

so We have to prove :

$$(ab+ac+bc)^2 + 27 \geq 84(ab+ac+bc)$$

wich is true because $ab+ac+bc \leq \frac{1}{3}$.

WLOG assume $a+b+c=3$, put $t=ab+bc+ca => a^2+b^2+c^2=9-2t$

$$LSH = \frac{27}{abc} + \frac{t}{\sqrt[3]{t.t.(9-2t)}} \geq \frac{27}{abc} + \frac{t}{3} \geq \frac{27}{abc} + 3\sqrt[3]{a^2b^2c^2} \text{ (by AM-GM)}$$
$$= \frac{1}{3}(3\sqrt[3]{a^2b^2c^2} + 2\frac{1}{abc}) + (27-\frac{2}{3})\frac{1}{abc} \geq \frac{5}{3} + 27 - \frac{2}{3} = 28$$

(by AM-GM) $=>$ Q.E.D
!
*A nother result,same **Solution**:

$$\frac{(a+b+c)^3}{abc} + \sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}} \geq 28 \quad \text{while } a,b,c>0$$

*Some general problems: Let a,b,c>0. Find the best constant of k for inequalitis:

$$1/\frac{(a+b+c)^3}{abc} + (\frac{ab+bc+ca}{a^2+b^2+c^2})^k \geq 28$$
$$2/\frac{(a+b+c)^3}{abc} + k\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}} \geq 27+k$$

297.
Let $a,b,c \geq 0$. Prove that

$$\frac{a^3}{a^3+(a+b)^3} + \frac{b^3}{b^3+(b+c)^3} + \frac{c^3}{c^3+(c+a)^3} \geq \frac{1}{3}$$

Solution:
Let $a+b+c=3$, We get $a<3$, note that

$$\frac{a^3}{a^3+(3-a)^3} - \frac{1}{9} - \frac{4}{9}(a-1) = \frac{-1}{3}\frac{(a-3)(a-1)^2}{a^2-3a+3} \geq 0$$

Another **Solution** :

$$\frac{a^3}{a^3+(b+c)^3} \geq (\frac{a^2}{a^2+b^2+c^2})^2$$

298.

Let $a, b, c > 0$ such that $abc = 1$. Prove that:

$$\sum \frac{a}{a^3+2} \leq 1$$

Solution:

By AM-GM We have:

$$a^3+1+1 \geq 3a => \sum \frac{a}{a^3+2} \leq 3.\frac{1}{3} = 1$$

And here is our **Solution**:

by AM-GM We have:

$$\sqrt[3]{\frac{ab+bc+ca}{a^2+b^2+c^2}} \geq \frac{ab+bc+ca}{a^2+b^2+c^2},$$

and:

$$abc \leq \frac{(ab+bc+ca)^2}{3(a+b+c)},$$

so We have to prove that:

$$\frac{3(a+b+c)^4}{(ab+bc+ca)^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \geq 28,$$

We put $a+b+c = x$ and $ab+ac+bc = y$ with $x \geq y$, so We find that We have to prove that:

$$3x^3+12x^2y-16xy^2+y^3 \geq 0,$$

which is only AM-GM.

299.

A nother result,same **Solution**:

$$\frac{(a+b+c)^3}{abc} + \sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}} \geq 28 \quad \text{while } a, b, c > 0$$

Solution:

1. We have

$$\frac{ab+bc+ca}{a^2+b^2+c^2} \leq 1$$

then

$$\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}} \geq \frac{ab+bc+ca}{a^2+b^2+c^2}$$

So, We only need to prove that

$$\frac{(a+b+c)^3}{abc} + \frac{ab+bc+ca}{a^2+b^2+c^2} \geq 28$$

To prove this inequality, We can use the known $(ab+bc+ca)^2 \geq 3abc(a+b+c)$. and so We only need to prove

$$\frac{3(a+b+c)^4}{(ab+bc+ca)^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \geq 28$$

To prove this inequality, We just setting $x = \frac{ab+bc+ca}{a^2+b^2+c^2} \le 1$, then it becomes

$$3\left(\frac{1}{x}+2\right)^2 + x \ge 28$$

$$\Leftrightarrow 3(1+2x)^2 + x^3 \ge 28x^2$$

$$\Leftrightarrow x^3 - 16x^2 + 12x + 3 \ge 0$$

$$\Leftrightarrow (x-1)(x^2 - 15x - 3) \ge 0$$

which is true since $x \le 1$.

2.
WLOG assume $a+b+c=3$, put

$$t = ab+bc+ca \ge 3\sqrt[3]{a^2b^2c^2} => a^2+b^2+c^2 = 9-2t$$

$$LSH = \frac{27}{abc} + \sqrt{\frac{t^3}{t.t.(9-2t)}} \ge \frac{27}{abc} + \sqrt{\frac{t^3}{27}} \ge \frac{27}{abc} + abc = \frac{26}{abc} + \frac{1}{abc} + abc \ge 28$$

300.
Let $a,b,c>0$. Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge \sqrt{\frac{a^3+b^3}{a+b}} + \sqrt{\frac{b^3+c^3}{b+c}} + \sqrt{\frac{c^3+a^3}{c+a}}$$

Solution:
Using the lemma

$$\sum \frac{a^2}{b} \ge \frac{15(a^2+b^2+c^2)}{2(a+b+c)} - \frac{3}{2}(a+b+c)$$

and the Cauchy Schwarz inequality, We get

$$\sum \sqrt{\frac{a^3+b^3}{a+b}} = \sum \sqrt{a^2-ab+b^2} \le \sqrt{3(2\sum a^2 - \sum a)}$$

it suffices to prove that

$$\left(\frac{15(a^2+b^2+c^2)}{2(a+b+c)} - \frac{3}{2}(a+b+c)\right)^2 \ge 3(2\sum a^2 - \sum ab)$$

Assume $a+b+c=1$ and setting $x = ab+bc+ca \le \frac{1}{3}$, We have to prove

$$(\frac{15}{2}(1-2x) - \frac{3}{2})^2 \ge 3(2(1-2x)-x)$$

$$\Leftrightarrow (6-15x)^2 \ge 3(2-5x)$$

$$\Leftrightarrow 36 - 180x + 225x^2 \ge 6 - 15x$$

$$\Leftrightarrow 30 - 165x + 225x^2 \ge 0$$

$$\Leftrightarrow 10 - 55x + 75x^2 \ge 0$$

$$\Leftrightarrow 2 - 11x + 15x^2 \ge 0$$

$$\Leftrightarrow (1-3x)(2-5x) \ge 0$$

301.
, Let $a, b, c \geq 0$ such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{a}{4-a} + \frac{b}{4-b} + \frac{c}{4-c} \leq 1$$

Solution:

$$\sum_{cyc} \frac{a}{4-a} \leq 1 \Leftrightarrow \sum_{cyc} \left(\frac{1}{3} - \frac{a}{4-a} + \frac{2(a^2-1)}{9} \right) \geq 0 \Leftrightarrow \sum_{cyc} \frac{(a-1)^2(2-a)}{4-a} \geq 0.$$

Q.E.D

302. Let $a, b, c \geq 0$. Prove that

$$\frac{a^3}{a^3 + (a+b)^3} + \frac{b^3}{b^3 + (b+c)^3} + \frac{c^3}{c^3 + (c+a)^3} \geq \frac{1}{3}$$

Solution:
it is equivalent to

$$\sum \frac{1}{1 + (1+k)^3} \geq \frac{1}{3} \text{ with } klm = 1$$

Now make the subtitution $k = \frac{yz}{x^2}$ and now We have to prove that

$$\sum \frac{x^6}{x^6 + (x^2 + yz)^3} \geq \frac{1}{3}$$

By Cauchy Swartz We have

$$\sum \frac{x^6}{x^6 + (x^2 + yz)^3} \geq \frac{(x^3 + y^3 + z^3)^2}{\sum x^6 + (x^2 + yz)^3}$$

So We have to prove that

$$\frac{(x^3 + y^3 + z^3)^2}{\sum x^6 + (x^2 + yz)^3} \geq \frac{1}{3}$$

which is equivalent to

$$x^6 + y^6 + z^6 + 5(x^3y^3 + y^3z^3 + z^3x^3) \geq 3xyz(x^3 + y^3 + z^3) + 9x^2y^2z^2$$

By AM-GM We have

$$x^6 + x^3y^3 + x^3z^3 \geq 3x^4yz$$

and similar for the others and also by AM-GM

$$3(x^3y^3 + y^3z^3 + z^3x^3) \geq 9x^2y^2z^2$$

303.
Let $a, b, c \geq 0$ such that $a + b + c = 3$. Prove that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{9}{2\sqrt{3(ab+bc+ca)}}$$

Solution:

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{9}{2\sqrt{3(ab+bc+ca)}} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{a^2}{b+c}-\frac{b+c}{4}\right) \geq \frac{(a+b+c)^2}{2\sqrt{3(ab+bc+ca)}}-\frac{a+b+c}{2} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\frac{(2a-b-c)(2a+b+c)}{b+c} \geq \frac{2(a+b+c)\left(a+b+c-\sqrt{3(ab+ac+bc)}\right)}{\sqrt{3(ab+ac+bc)}} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{(a-b)(2a+b+c)}{b+c}-\frac{(c-a)(2a+b+c)}{b+c}\right) \geq$$

$$\geq \frac{2(a+b+c)\sum(a-b)^2}{3(ab+ac+bc)+(a+b+c)\sqrt{3(ab+ac+bc)}} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\left(\frac{(a-b)(2a+b+c)}{b+c}-\frac{(a-b)(2b+a+c)}{a+c}\right) \geq$$

$$\geq \frac{2(a+b+c)\sum(a-b)^2}{3(ab+ac+bc)+(a+b+c)\sqrt{3(ab+ac+bc)}} \Leftrightarrow \sum_{cyc}(a-b)^2S_c \geq 0,$$

$$S_c=\frac{1}{(a+c)(b+c)}-\frac{1}{3(ab+ac+bc)+(a+b+c)\sqrt{3(ab+ac+bc)}}.$$

Let

$$a \geq b \geq c. Then S_b \geq 0, S_c \geq 0 and (a-c)^2 \geq (b-c)^2.$$

Thus,

$$\sum_{cyc}(a-b)^2S_c \geq (a-c)^2S_b+(b-c)^2S_a \geq (b-c)^2(S_b+S_a) \geq 0,$$

which is true because $S_b+S_a \geq 0 \Leftrightarrow$

$$\Leftrightarrow \frac{a+b+2c}{(a+b)(a+c)(b+c)} \geq \frac{2}{3(ab+ac+bc)+(a+b+c)\sqrt{3(ab+ac+bc)}},$$ which obviously.

304.
Let $a,b,c>0$. Prove that

$$\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2} \geq \frac{1}{\sqrt{a^2-ab+b^2}}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{\sqrt{c^2-ca+a^2}}$$

Solution:
1.)Notice that by the AM-GM, We have

$$\sum\frac{a}{b^2} \geq \sum\frac{1}{a}$$

Hence, it suffices to prove that

$$\sum\frac{1}{a} \geq \sum\frac{1}{\sqrt{a^2-ab+b^2}}$$

By the Cauchy Schwarz inequality, We have

$$\left(\sum\frac{1}{\sqrt{a^2-ab+b^2}}\right)^2 \leq \left(\sum\frac{1}{ab}\right)\left(\sum\frac{ab}{a^2-ab+b^2}\right)$$

By AM-GM, We have

$$\sum\frac{ab}{a^2-ab+b^2} \leq 3$$

$$\sum\frac{1}{ab} \leq \frac{1}{3}\left(\sum\frac{1}{a}\right)^2$$

Multiplying these inequalities, We can get the result.

2)
First by Holder We have

$$\left(\frac{a}{b^2}+\frac{b}{c^2}+\frac{c}{a^2}\right)(ab+bc+ca)^2 \geq (a+b+c)^3$$

And by Cauchy Schwarz We have

$$\sum \frac{1}{\sqrt{a^2-ab+b^2}} \leq \sqrt{3\left(\sum \frac{1}{a^2-ab+b^2}\right)}$$

it suffices us to show that

$$\frac{(a+b+c)^3}{(ab+bc+ca)^2} \geq \sqrt{3\left(\sum \frac{1}{a^2-ab+b^2}\right)}$$

Assume $a+b+c=1$ and $q=ab+bc+ca, r=abc$ then We have

$$\prod \left(a^2-ab+b^2\right) = -3q^3+q^2+10rq-8r^2-3r$$

$$\sum \left(a^2-ab+b^2\right)\left(b^2-bc+c^2\right) = 7q^2-5q-2r+1$$

the inequality equalivents to

$$f(r) = -8r^2+\left(6q^4+10q-3\right)r-21q^6+15q^5-3q^4-3q^3+q^2 \geq 0$$

We see that if

$$q \leq 0,2954 then 6q^4+10q-3 \leq 0$$

with the lemma

$$r \leq \frac{q^2(1-q)}{2(2-3q)}$$

We have

$$f(r) \geq f\left(\frac{q^2(1-q)}{2(2-3q)}\right) = \frac{-360q^8+744q^7-574q^6+176q^5+3q^4-13q^3+2q^2}{2(2-3q)^2} \geq 0$$

Note that $q \leq \frac{1}{3}$ so the inequality has proved.
if $q \geq 0,294$ then We see that

$$f(r) \geq \min\left\{f(0), f\left(\frac{1}{27}\right)\right\} \geq 0$$

Note that when $r=\frac{1}{27}$ We have $q^2 \geq 3pr \Leftrightarrow q \geq \frac{1}{3}$. So q must be $\frac{1}{3}$.

92, Let $a,b,c \geq 0$. Prove that

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{9}{2\sqrt{3(ab+bc+ca)}}$$

By Cauchy Schwarz We have

$$\left(\sum \frac{a^2}{b+c}\right)\left(\sum a^2\right) \geq \left(\sum \frac{a^2}{\sqrt{b+c}}\right)^2$$

And by Holder We have

$$\left(\sum \frac{a^2}{b+c}\right)\left(\sum a^2\right) \ge \left(\sum \frac{a^2}{\sqrt{b+c}}\right)^2$$

So it suffices us to show that

$$\frac{\left(\sum a^2\right)^3}{\left(\sum a^2\left(b+c\right)\right)\left(\sum a^2\right)} \ge \frac{9}{2\sqrt{3\left(ab+bc+ca\right)}}$$

Setting $q = ab+bc+ca, r = abc$ We rewrite the inequality

$$2\sqrt{3q}\left(9-2q\right)^2 \ge 9\left(3q-3r\right)$$

By Schur inequality We have $r \ge \max\left\{0, \frac{4q-9}{3}\right\}$
if $q \le \frac{9}{4} \Rightarrow r \ge 0$ then We need to show

$$2\sqrt{3q}\left(9-2q\right)^2 \ge 27q$$

$$\Leftrightarrow \left(q-6,75\right)\left(q-2,701\right) \ge 0$$

The inequality has proved. in the other way if $q \ge \frac{9}{4} \Rightarrow r \ge \frac{4q-9}{3}$ then We need to show that

$$2\sqrt{3q}\left(9-2q\right)^2 \ge 9\left(9-q\right)$$

$$\Leftrightarrow \left(q-3\right)\left(q-0,0885\right)\left(q-5,487\right) \ge 0$$

which is true so We have done. The equality holds if $a=b=c$.

305.
Let $a,b,c \ge 0$ and $ab+bc+ca=3$. Prove that

$$a+b+c \ge abc + 2\sqrt{\frac{a^2b^2+b^2c^2+c^2a^2}{3}}$$

Solution:
Setting $p = a+b+c$ and $r = abc$ then the inequality becomes

$$p - r \ge \sqrt{\frac{9-2pr}{3}}$$

$$\Leftrightarrow 3p^2 - 4pr + 3r^2 - 9 \ge 0$$

$$\Delta' = 4p^2 - 3\left(3p^2-9\right) = 27 - 5p^2 \le 0$$

And the equality does not hold.

306.
Let a,b,c be positive real numbers such that $abc=1$. Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{2}{(1+a)(1+b)(1+c)} \ge 1$$

Solution:
We use the fact that if $a,b \ge 1$ or $a,b \le 1$ hence We have :$(a-1)(b-1) \ge 0$ this mean that : $ab+1 \ge a+b$ hence:

$$\frac{2}{(1+a)(1+b)(1+c)} = \frac{2}{(1+c)(1+ab+a+b)} \ge \frac{c}{(1+c)^2}$$

but We also have :

$$\frac{1}{(a+1)^2}+\frac{1}{(1+b)^2}=\frac{ab(a-b)^2+(1+ab)^2}{(ab+1)(a+1)^2(b+1)^2}+\frac{1}{ab+1}\geq\frac{c}{c+1}$$

thus We have :

$$LHS\geq\frac{c}{c+1}+\frac{1}{(c+1)^2}+\frac{c}{(c+1)^2}=1=RHS$$

Setting :

$$xy=\frac{1}{(1+a)^2};yz=\frac{1}{(1+b)^2};zx=\frac{1}{(1+c)^2}$$

Next, use contractdition. We need to prove :

$$\prod(4\sqrt{\frac{mn}{(n+p)(m+p)}}-1)\leq 1$$

Which is true by Am-Gm.

307.

Let $a,b,c\geq 0$; $ab+bc+ca=3$. Prove that:

$$\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\geq\frac{3}{2}$$

We will prove it br pqr

$$<=>p^2\geq 12+3r^2-2pr$$

if $p^2\geq 12(1)$, easy to prove that :

$$RHS\leq 12\leq LHS$$

if $p^2\leq 12$,We have :

$$r\geq\frac{p(12-p^2)}{9}$$

$$<=>(12-p^2)(p^4+24p^2-27)\geq 0$$

Let $p=a+b+c,q=ab+bc+c=3,r=abc$

$$LHS=\frac{q^2-2pr+2p^2-4q+3}{(q-1)^2+(p-r)^2}$$

$$\frac{q^2-2pr+2p^2-4q+3}{(q-1)^2+(p-r)^2}\geq\frac{3}{2}\iff\frac{2p^2-2pr}{4+(p-r)^2}\geq\frac{3}{2}$$

$$\iff 4p^2-4pr\geq 12+3(p-r)^2=12+3p^2-6pr+3r^2$$

$$\iff p^2+2pr\geq 12+3r^2$$

from Am-GM ;

$$(a+b+c)(ab+bc+ca)\geq 9abc\therefore p\geq 3r$$

$$\therefore pr\geq 3r^2$$

from Am-Gm ;

$$p^2q+3pr\geq 4q^2\rightarrow p^2+pr\geq 12$$

$$\therefore$$

$$p^2+2pr\geq 12+3r^2$$

$$\therefore LHS \geq \frac{3}{2}$$

308.
Let $a,b,c>0$. Prove that:

$$(a^2+b^2+c^2)(a^4+b^4+c^4) \geq ((a-b)(b-c)(c-a))^2$$

Solution:

$$LHS=(\sum_{cyc}c^2)(\sum_{cyc}(a^2)^2) \geq (a^2c+ab^2+bc^2)^2 \geq (a^2c+ab^2+bc^2-(b^2c+a^2b+ac^2))^2 = RHS$$

Because:

$$a^2c+b^2a+c^2b-a^2b-b^2c-c^2a=(b-a)(c-a)(c-b)$$

Q.E.D

309
, For a,b,c real numbers such that $a+b+c=1$. Prove that

$$\sum \frac{1}{1+a^2} \leq \frac{27}{10}$$

Solution:

$$<=> \sum \frac{a^2}{a^2+(a+b+c)^2} \geq \frac{3}{10}$$

We have :$(a+b+c)^2 \leq (|a|+|b|+|c|)^2$ Setting :$x=|a|;y=|b|;z=|c|$ Assume that :$x+y+z=1$ By cauchy-schwarz ,We need to prove :

$$10(x^2+y^2+z^2)^2 \geq 3(x^2+y^2+z^2)+3\sum x^4$$

$$<=> 17q^2-11q-6r+2 \geq 0$$

By Am-Gm We have :$r \leq \frac{q}{9}$

$$LHS \geq 17q^2-11q-\frac{2q}{3}+2=\frac{2}{3}(3q-1)(17q-6) \geq 0$$

Q.E.D

310
, Find the maximum $k=const$ the inequality is right

$$\Big(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{b+a}+1\Big)^2 \geq k\frac{a^2+b^2+c^2}{ab+bc+ac}+\frac{25}{4}-k$$

Solution:
Try $a=b;c=0$, We have :$\frac{11}{4} \geq k$
To prove : if $a,b,c \geq 0$ then :

$$(\sum \frac{a}{b+c}+1)^2 \geq \frac{11}{4}.\frac{a^2+b^2+c^2}{ab+bc+ca}-\frac{11}{4}+\frac{25}{4}$$

$a,b,c \geq 0$ and no two of which are zero . Prove that :

$$(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+1)^2 \geq \frac{11(a^2+b^2+c^2)}{4(ab+bc+ca)}+\frac{7}{2}$$

Assume that :$a+b+c=1$

$$<=> (\frac{1-q+2r}{q-r})^2 \geq \frac{11}{4q}-2$$

$$\bigstar.4q \leq 1$$

We prove :

$$\frac{(1-q)^2}{q^2} \geq \frac{11}{4q}-2$$

$$\leftrightarrow (4q-1)(3q-4) \geq 0$$

(Right because

$$q \leq \frac{1}{4})$$

$$\bigstar.4q \geq 1$$

Use schur, We have :

$$r \geq \frac{4q-1}{9} \longrightarrow LHS \geq (\frac{q-7}{5q+1})^2$$

We prove that :

$$(\frac{q-7}{5q+1})^2 \geq \frac{11}{4q}-2$$

$$\leftrightarrow (3q-1)(17q-11)(4q-1) \geq 0$$

Which is obvious true because

$$\frac{1}{3} \geq q \geq \frac{1}{4}$$

311.
Let a,b,c are positive numbers such that $a+b+c=3$. Prove that

$$\sum_{cyc} \frac{a}{b^2+2c} \geq 1$$

Solution:

$$\sum_{cyc} \frac{a}{b^2+2c} \geq (a+b+c)^2 \frac{1}{ab^2+bc^2+ca^2+2(ab+bc+ca)} \geq 1$$

$$<=> a^2+b^2+c^2 \geq ab^2+bc^2+ca^2$$

The last is equivalent to:

$$(a+b+c)(a^2+b^2+c^2)-3(ab^2+bc^2+ca^2)$$

$$\iff \sum a^3+\sum a^2b \geq 2\sum ab^2$$

which is always true

$$\bigstar.Lemma : a+b+c=3; a,b,c \geq 0$$

$$=> ab^2+bc^2+ca^2+abc \leq 4$$

$$LHS \geq \frac{9}{\sum ab^2+2\sum ab}$$

We need to prove that :

$$9 \geq (\sum ab^2 + abc) + 2\sum ab - abc$$

Use this lemma and schur

$$(abc \geq \frac{4(ab+bc+ca)-9}{3})$$

Setting :$x = ab + bc + ca$

$$5 + abc \geq 5 + \frac{4(ab+bc+ca)-9}{3}$$

We have to prove: $3 \geq (\sum ab)$. Which is obvious true.

312.
Let $a, b, c > 0$ and $a^2 + b^2 + c^2 = 3$. Prove that

$$\sum \frac{1}{a^3+2} \geq 1$$

Solution:

$$\sum \frac{1}{a^3+2} \geq 1 \Leftrightarrow \sum_{cyc} \left(\frac{1}{a^3+2} - \frac{1}{3} + \frac{a^2-1}{6} \right) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc} \frac{a^2(a+2)(a-1)^2}{a^3+2} \geq 0.$$

We prove it by contractdition The ineq equivalent to :

$$4 \geq a^3b^3c^3 + \sum a^3b^3$$

By contractdition, We must prove this ineq: if $a, b, c \geq 0$ safity that:

$$4 = abc + ab + bc + ca.then : \sum (ab)^{\frac{2}{3}} \geq 3$$

Exist $m, p, n \geq 0$ safity that :

$$a = \frac{2m}{n+p}; b = \frac{2n}{m+p}; c = \frac{2p}{m+n}.$$

So We must prove :

$$\sum (\sqrt[3]{\frac{2a}{b+c}})^2 \geq 3$$

Q.E.D

313.
, Problem if a, b, c and d are positive real numbers such that $a + b + c + d = 4$. Prove that

$$\frac{a}{1+b^2c} + \frac{b}{1+c^2d} + \frac{c}{1+d^2a} + \frac{d}{1+a^2b} \geq 2$$

Solution:

$$\sum \frac{a}{1+b^2c} \geq \frac{(a+b+c+d)^2}{a+b+c+d+\sum ab^2c}$$

Hence it remains to show that

$$ab^2c + bc^2d + cd^2a + da^2b \leq 4$$

We have

$$ab^2c + bc^2d + cd^2a + da^2b = (ab + cd)(ad + bc) \leq \left(\frac{ab + bc + cd + da}{2}\right)^2$$

$$= \left(\frac{(a + c)(b + d)}{2}\right)^2 \leq \frac{1}{4}\left(\frac{a + b + c + d}{2}\right)^4 = 4$$

A similar problem posted in the same topic, proven in a similar way as ill; but the **Solution** isn't quite obvious at first glance: 314.

, Problem Let a, b, c and d be non-negative numbers such that $a + b + c + d = 4$. Prove that

$$a^2bc + b^2cd + c^2da + d^2ab \leq 4$$

Solution The left side of this inequality cannot be factorized as We did in the previous one. But We do see that it can be written as

$$ac(ab + cd) + bd(ad + bc) \leq 4$$

i'd be done if We could make $ab + cd$ appear on the left instead of $ad + bc$. So let's assume that $ad + bc \leq ab + cd$. Then We have

$$ac(ab + cd) + bd(ad + bc) \leq (ac + bd)(ab + cd) \leq \left(\frac{ac + bd + ab + cd}{2}\right)^2 =$$

$$\left(\frac{(a + d)(b + c)}{2}\right)^2 \leq \frac{1}{4}\left(\frac{a + b + c + d}{2}\right)^4 = 4$$

and i're done! Now it remains to deal with the case $ab + cd \leq ad + bc$. But due to the symmetry in the expression this case is easily dealt with in exactly the same way:

$$ac(ab + cd) + bd(ad + bc) \leq (ac + bd)(ad + bc) \leq \left(\frac{ac + bd + ad + bc}{2}\right)^2 =$$

$$\left(\frac{(a + b)(c + d)}{2}\right)^2 \leq \frac{1}{4}\left(\frac{a + b + c + d}{2}\right)^4 = 4$$

Thus We are done! Some harder problems: 315.

1) if a, b, c are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}$$

Solution

Note that

$$\frac{1}{a} + 6b = \frac{7ab + bc + ca}{a}$$

Hence our inequality becomes

$$\sum \sqrt[3]{bc(7ab + bc + ca)} \leq \frac{1}{(abc)^{\frac{2}{3}}}$$

From Holder's inequality We have

$$\sum \sqrt[3]{bc(7ab + bc + ca)} \leq \sqrt[3]{\left(\sum a\right)^2 \left(9 \sum bc\right)}$$

Hence it remains to show that

$$9(a + b + c)^2(ab + bc + ca) \leq \frac{1}{(abc)^2} \iff [3abc(a + b + c)]^2 \leq (ab + bc + ca)^4$$

Which is obviously true since

$$(ab+bc+ca)^2 \ge 3abc(a+b+c) \iff \sum a^2(b-c)^2 \ge 0$$

2)Show that for all positive real numbers a,b and c the following inequality holds:

$$\frac{(b+c)(a^4-b^2c^2)}{ab+ac+2bc}+\frac{(c+a)(b^4-c^2a^2)}{bc+ba+2ca}+\frac{(a+b)(c^4-a^2b^2)}{ca+cb+2ab} \ge 0.$$

Solution:
Without loss of generality, assume that $a \ge b \ge c$. Since $a^4 - b^2c^2 \ge 0$ and

$$ab+ac+2bc \le bc+ba+2ca,$$

we have

$$\frac{(b+c)(a^4-b^2c^2)}{ab+ac+2bc} \ge \frac{(b+c)(a^4-b^2c^2)}{bc+ba+2ca}.$$

Similarly, since $c^4 - a^2b^2 \le 0$ and $ca+cb+2ab \ge bc+ba+2ca$, we have

$$\frac{(a+b)(c^4-a^2b^2)}{ca+cb+2ab} \ge \frac{(a+b)(c^4-a^2b^2)}{bc+ba+2ca}.$$

Therefore, it suffices to prove that

$$\frac{(b+c)(a^4-b^2c^2)}{bc+ba+2ca}+\frac{(c+a)(b^4-c^2a^2)}{bc+ba+2ca}+\frac{(a+b)(c^4-a^2b^2)}{bc+ba+2ca} \ge 0,$$

which reduces to the obvious inequality

$$bc(b+c)(b-c)^2+ca(c+a)(c-a)^2+ab(a+b)(a-b)^2 \ge 0.$$

The proof is completed. Equality occurs if and only if $a=b=c$.
3.

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^{2/3}$$

Proof:
Let $a+b+c=3u$, $ab+ac+bc=3v^2$ (where $v>0$), $abc=w^3$ and

$$\sqrt[3]{\frac{a^2+b^2+c^2}{ab+ac+bc}}=p.$$

Then $p \ge 1$, $u^2=\frac{p^3+2}{3}v^2$ and

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^{2/3} \Leftrightarrow \sum_{cyc} a^2c \ge 3p^2w^3 \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}(a^2b+a^2c) \ge 6p^2w^3+\sum_{cyc}(a^2b-a^2c) \Leftrightarrow$$

$$\Leftrightarrow 9uv^2-3w^3 \ge 6p^2w^3+(a-b)(a-c)(b-c).$$

But $9uv^2-3w^3 \ge 6p^2w^3$ is true because

$$(a-b)^2(a-c)^2(b-c)^2 \ge 0 \Leftrightarrow w^6-2(3uv^2-2u^3)w^3+4v^6-3u^2v^4 \le 0,$$

which gives

$$w^3 \le 3uv^2 - 2u^3 + 2\sqrt{(u^2-v^2)^3}.$$

Thus, we need to prove here that

$$3uv^2 \ge (1+2p^2)(3uv^2 - 2u^3 + 2\sqrt{(u^2-v^2)^3}).$$

But

$$3uv^2 \ge (1+2p^2)(3uv^2 - 2u^3 + 2\sqrt{(u^2-v^2)^3}) \Leftrightarrow$$
$$\Leftrightarrow 3 \ge (1+2p^2)\left(3 - \frac{2(p^3+2)}{3} + 2\sqrt{\frac{(p^3-1)^3}{9(p^3+2)}}\right) \Leftrightarrow$$
$$\Leftrightarrow (p-1)^2(2p^6+p^5+p^4-3p^3-p^2+2p+1) \ge 0,$$

which is obvious. Hence, enough to prove that

$$9uv^2 - 3w^3 \ge 6p^2w^3 + \sqrt{(a-b)^2(a-c)^2(b-c)^2}.$$

But

$$9uv^2 - 3w^3 \ge 6p^2w^3 + \sqrt{(a-b)^2(a-c)^2(b-c)^2} \Leftrightarrow$$
$$\Leftrightarrow (3uv^2 - (1+2p^2)w^3)^2 \ge 3(-w^6 + 2(3uv^2-2u^3)w^3 - 4v^6 + 3u^2v^4) \Leftrightarrow$$
$$\Leftrightarrow (1+p^2+p^4)w^6 - 3(2uv^2 - u^3 + uv^2p^2)w^3 + 3v^6 \ge 0 \Leftrightarrow$$
$$\Leftrightarrow (1+p^2+p^4)w^6 - 3(2 - \frac{p^3+2}{3} + p^2)\sqrt{\frac{p^3+2}{3}}v^3w^3 + 3v^6 \ge 0 \Leftrightarrow$$
$$\Leftrightarrow (1+p^2+p^4)w^6 - (4+3p^2-p^3)\sqrt{\frac{p^3+2}{3}}v^3w^3 + 3v^6 \ge 0.$$

For $t \ge 1$ we obtain:

$$4+3p^2-p^3 \le 0 \Leftrightarrow p \ge 3.356...$$

Id est, for $p \ge 3.357$ the original inequality is proved.
Let $1 \le p < 3.357$. Thus, it remains to prove that

$$\left((4+3p^2-p^3)\sqrt{\frac{p^3+2}{3}}\right)^2 - 12(1+p^2+p^4) \le 0,$$

which is equivalent to

$$(p-1)^2(p^7-4p^6-2p^4+8p^3-8p-4) \le 0,$$

which is true for all

$$1 \le p < 3.357.$$

4.
Let a,b,c $>$0,. Setting

$$M = max\left\{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)^2, 8\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right)\right\}$$

Prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \ge \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{M}{3}$$

Solution:

First Ineq :

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\ge\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{1}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)^2$$

Proof: Let

$$X=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

Hence,$X\ge 3$,$LHS\ge\frac{1}{3}X^2$

$$\frac{1}{3}X^2\ge X+\frac{1}{3}(X-3)^2$$

$$\iff X^2\ge 3X+X^2-6X+9$$

$$\iff 3X\ge 9$$

That is true.

Second Ineq :

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\ge\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{4\sum_{cyc}(a-b)^2}{3(a^2+b^2+c^2)}$$

Proof : Notice that

$$\frac{4\sum_{cyc}(a-b)^2}{3(a^2+b^2+c^2)}\le\sum_{cyc}\frac{4(a-b)^2}{3(a^2+b^2)}=4-\sum_{cyc}\frac{8ab}{3(a^2+b^2)}$$

Now it suffices to prove that

$$\sum_{cyc}\frac{a^2}{b^2}+\sum_{cyc}\frac{8ab}{3(a^2+b^2)}\ge\sum_{cyc}\frac{a}{b}+4$$

Using AM-GM

$$(1)\frac{2}{3}\left(\frac{a^2+b^2}{b^2}+\frac{4ab}{a^2+b^2}\right)\ge\frac{2}{3}\left(4\sqrt{\frac{a}{b}}\right)$$

$$(2)\frac{1}{3}\left(\frac{a^2}{b^2}+2\sqrt{\frac{a}{b}}\right)\ge\frac{a}{b}$$

5.

If a, b, c are positive real numbers, then

$$\frac{a}{a+\sqrt{(a+2b)(a+2c)}}+\frac{b}{b+\sqrt{(b+2c)(b+2a)}}+\frac{c}{c+\sqrt{(c+2a)(c+2b)}}\le\frac{3}{4}.$$

Solution:

Now we prove this ineq:

$$\frac{a}{a+\sqrt{(a+2b)(a+2c)}}\le\frac{a}{4(a+b)}+\frac{a}{4(a+c)}$$

$$\Leftrightarrow(2a+b+c)(a+\sqrt{(a+2b)(a+2c)})\ge 4(a+b)(a+c)$$

Denote

$$\frac{a+b}{a}=x,\frac{a+c}{a}=y$$

Notice that $x+y\ge 2$

Rewrite:

$$(x+y)(1+\sqrt{(2x-1)(2y-1)}\ge 4xy$$

$$\Leftrightarrow x+y-\frac{4xy}{x+y}\ge x+y-1-\sqrt{(2x-1)(2y-1)}$$

$$\Leftrightarrow (x-y)^2(\frac{1}{x+y}-\frac{1}{x+y-1+\sqrt{(2x-1)(2y-1)}})\geq 0$$

Notice :

$$x+y-1+\sqrt{(2x-1)(2y-1)}\geq x+y$$

$$\Leftrightarrow \sqrt{(2x-1)(2y-1)}\geq 1 \Leftrightarrow 2xy\geq x+y$$

$$\Leftrightarrow 2(a+b)(a+c)\geq a(2a+b+c)\Leftrightarrow ab+ac+2bc\geq 0$$

which is true .
So this ineq is true. Similarly, add these ineqs , we have Q.E.D
316.
Problem Let $a,b,c>0$ such that $a+b+c=1$. Prove that

$$\frac{\sqrt{a^2+abc}}{c+ab}+\frac{\sqrt{b^2+abc}}{a+bc}+\frac{\sqrt{c^2+abc}}{b+ca}\leq \frac{1}{2\sqrt{abc}}$$

Solution
Note that

$$\sum \frac{\sqrt{a^2+abc}}{c+ab}=\sum \frac{\sqrt{a(c+a)(a+b)}}{(b+c)(c+a)}$$

Therefore our inequality is equivalent to

$$\sum \frac{\sqrt{a^2+abc}}{c+ab}=\sum \frac{\sqrt{a(c+a)(a+b)}}{(b+c)(c+a)}$$

By AM-GM,

$$\sum \frac{\sqrt{a(c+a)(a+b)}}{(b+c)(c+a)}\leq \frac{a+b+c}{2\sqrt{abc}}$$

$$\sum a(a+b)\sqrt{bc(c+a)(a+b)}\leq \frac{1}{2}(a+b+c)(a+b)(b+c)(c+a)$$

Now

$$\sum \frac{\sqrt{a(c+a)(a+b)}}{(b+c)(c+a)}\leq \frac{a+b+c}{2\sqrt{abc}}$$

$$\sum a(a+b)\sqrt{bc(c+a)(a+b)}\leq \frac{1}{2}(a+b+c)(a+b)(b+c)(c+a)$$

which was what We wanted. ■ Another one with square-roots and fractions.
317.
Let $a,b,c>0$. Prove that

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}\leq \sqrt{3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$$

Solution
From Cauchy-Schwarz inequality We have

$$\sum \sqrt{\frac{a}{b+c}}=\frac{\sum \sqrt{a(a+b)(c+a)}}{\sqrt{(a+b)(b+c)(c+a)}}\leq \sqrt{\frac{(\sum a)(\sum a^2+3\sum bc)}{(a+b)(b+c)(c+a)}}$$

Therefore it remains to show that

$$2abc\left(\sum a\right)\left(\sum a^2+3\sum bc\right)\leq 3\left(\sum ab^2\right)\left(\sum a^2b+\sum ab^2+2abc\right)$$

Let $p = \sum a^2b, q = \sum ab^2$. Then this is equivalent to

$$2abc(a+b+c)^3 + 2abc(p+q+3abc) \le 3q(p+q+2abc)$$
$$\Leftrightarrow\ 2abc\sum a^3 + 6abc(p+q+2abc) + 2abc(p+q+3abc) \le 3q(p+q+2abc)$$
$$\Leftrightarrow\ 2abc\sum a^3 + 8abcp + 8abcq + 18(abc)^2 \le 3pq + 3q^2 + 6abcq$$
$$\Leftrightarrow\ 2abc\sum a^3 + 8abcp + 2abcq + 18(abc)^2 \le 3\sum a^3b^3 + 3abc\sum a^3 + 9(abc)^2 + 3q^2$$
$$\Leftrightarrow\ 8abcp + 2abcq + 9(abc)^2 \le 3\sum a^3b^3 + abc\sum a^3 + 3q^2$$

Now verify that

$$q^2 \ge 3abcp\ \Leftrightarrow\ \sum a^2b^4 \ge abc\sum a^2b\ \Leftrightarrow\ \sum b^2(ab-c^2)^2 \ge 0$$

which is obviously true. Thus $q^2 \ge 3abcp$ and $q^2 \ge 3abcq$ (the latter follows directly from AM-GM), which imply $3q^2 \ge 8abcp + abcq$. Therefore it remains to show that

$$abcq + 9(abc)^2 \le 3\sum a^3b^3 + abc\sum a^3$$

which follows from adding the following inequalities, of which the former follows from AM-GM and the latter from Rearrangement:

$$3\sum a^3b^3 \ge 9(abc)^2 abc\sum a^3 \ge abcq$$

Hence We are done. Q.E.D ■ in the **Solution**s to the last few problems, one may rise the question: why do We break up the square-roots in that specific way? For example in the fourth problem one could apply AM-GM for bc and $(c+a)(a+b)$ instead of $b(c+a) and c(a+b)$. Here are our thoughts on this: while trying to get a stronger bound, it's always worth it to end up with a form which is much less, as less as possible, than the upper bound of the problem (especially in these sort of cases while using AM-GM or Cauchy-Schwarz). Hence in accordance with the majorization inequality, We try to derive an expression where the degrees of the terms minorize as much as possible. For example, if We used AM-GM for $4bc$ and $(c+a)(a+b)$ i'd get $[2,0,0]$ and $[1,1,0]$ terms. But if We use it on $b(c+a)$ and $c(a+b)$ We get all $[1,1,0]$ terms, which in the long run could possibly be useful. The same idea goes for the other problems as ill..

318.
1) if $a, b, c > 0$. show that :

$$\sum \frac{a}{b^2+bc+c^2} \ge \frac{3}{a+b+c}$$

good prob: now by cs :

$$\sum a(b^2+bc+c^2).\sum \frac{a}{b^2+bc+c^2} \geqslant \left(\sum a\right)^2$$

it's enough to show that:

$$\frac{\left(\sum a\right)^2}{\sum a(b^2+bc+c^2)} \geqslant \frac{3}{\sum a} \Leftrightarrow \sum a^3 \geqslant 3abc$$

Q.E.D
2)
Given a,b,c>0. Prove that:

$$3(a^3+b^3+c^3)+2abc \geq 11\sqrt{(\frac{a^2+b^2+c^2}{3})^3}$$

Solution:
$a+b+c=x$, $ab+bc+ca=y$, $abc=z$
The inequality is equivalent to

$$3x^3-9xy+11z \geq 11\sqrt{\left(\frac{x^2-2y}{3}\right)^3}$$

$$f(z)=3x^3-9xy+11z-11\sqrt{\left(\frac{x^2-2y}{3}\right)^3}$$

$$f''(z)=0$$

which means $f(z)$ gets its maximum and minimum values when two of $\{a,b,c\}$ are equal or one of them is zero. By homogeneity there are two cases:

$$1)\ c=0$$

$$3(a^3+b^3) \geq \frac{3}{2\sqrt{2}} \cdot (a^2+b^2)^{\frac{3}{2}} > \frac{11}{3\sqrt{3}} \cdot (a^2+b^2)^{\frac{3}{2}}$$

$$2)\ b=c=1$$

$$3(a^3+2)+2a \geq 11\sqrt{\left(\frac{a^2+2}{3}\right)^3} \iff$$

$$(a-1)^2(122a^4+244a^3-36a^2+656a+4) \geq 0$$

it is easy to show that $122a^4+244a^3-36a^2+656a+4>0$ for all $a>0$ and the **Solution** is done.
319.
Let a,b and c ate non-negative numbers such that $ab+ac+bc \neq 0$. Prove that:

$$\frac{a}{4b^2+bc+4c^2}+\frac{b}{4a^2+ac+4c^2}+\frac{c}{4a^2+ab+4b^2} \geq \frac{1}{a+b+c}$$

Solution:
1)
Using Cauchy Schwarts, We need prove:

$$(a^2+b^2+c^2)^2(a+b+c) \geq\ 4\sum\ a^2b^2(a+b)+abc\sum\ a^2$$

$$<=>\sum\ a^5+\sum\ ab(a^3+b^3)+2abc\sum\ ab \geq\ 2\sum\ a^2b^2(a+b)+abc\sum a^2$$

Using Schur which degree 5, We have:

$$\sum a^5 + abc\sum a^2 \geq \sum ab(a^3+b^3)$$

So, We only need to prove:

$$\sum ab(a^3+b^3) + abc\sum ab \geq \sum a^2b^2(a+b) + abc\sum a^2$$

$$<=> [ab(a+b) - \frac{abc}{2}](a-b)^2 \geq 0$$

Easily to see that $Sb, Sc, Sa+Sb \geq 0$. We have done

2)

$$LHS \geq \frac{\left(a^3+b^3+c^3\right)^2}{4\sum a^3b^3(a+b) + abc\sum a^4}$$

We need to prove that:

$$\left(a^3+b^3+c^3\right)^2(a+b+c) \geq 4\sum a^3b^3(a+b) + abc\sum a^4$$

it equivalent to:

$$\sum a^7 + \sum ab(a^5+b^5) + 2abc\sum a^2b^2 \geq 2\sum a^3b^3(a+b) + abc\sum a^4$$

$$<=> \sum (a-b)^2\left(a^5+b^5-c^5-2(a+b)^2abc+4ab(a+b)(a^2+ab+b^2)\right)$$

it's very easily to prove that

$$S_c; S_b; S_b+S_a > 0 if a \geq b \geq c$$

We have done.

320:

, if a,b,c be nonnegative real numbers such that $a+b+c=3$, then

$$\frac{1}{(a+b)^2+6} + \frac{1}{(b+c)^2+6} + \frac{1}{(c+a)^2+6} \geq \frac{3}{10}$$

Solution:

$$\sum_{cyc} \frac{1}{(a+b)^2+6} \geq \frac{3}{10} \Leftrightarrow \sum_{cyc}\left(\frac{1}{(3-a)^2+6} - \frac{1}{10} - \frac{1}{25}(a-1)\right) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc} \frac{(a-1)^2(5-2a)}{a^2-6a+15} \geq 0$$

Thus, our inequality is proven for $\max\{a,b,c\} \leq 2.5$.

Let $2.5 < a \leq 3$. Hence, $0 \leq b+c < 0.5$. But

$$\left(\frac{1}{x^2-6x+15}\right)'' = \frac{6(x^2-6x+7)}{(x^2-6x+15)^3} > 0 for 0 \leq x < 0.5.$$

Hence,

$$\sum_{cyc} \frac{1}{a^2-6a+15} \geq \frac{2}{\left(\frac{b+c}{2}\right)^2 - 6\cdot\frac{b+c}{2} + 15} + \frac{1}{a^2-6a+15} =$$

$$= \frac{2}{\left(\frac{3-a}{2}\right)^2 - 6 \cdot \frac{3-a}{2} + 15} + \frac{1}{a^2-6a+15} = \frac{8}{a^2+6a+33} + \frac{1}{a^2-6a+15}$$

id est, it remains to prove that

$$\frac{8}{a^2+6a+33} + \frac{1}{a^2-6a+15} \geq \frac{3}{10}$$

which is equivalent to $(a-1)^2(3-a)(a+5) \geq 0$, which is true for $2.5 < a \leq 3$. Done!

321.
This is the strongest of this form

$$\sum \frac{a}{4b^3+abc+4c^3} \geq \frac{1}{a^2+b^2+c^2}$$

Solution:
By the way, the following reasoning

$$\sum_{cyc} \frac{a}{4b^3+abc+4c^3} = \sum_{cyc} \frac{a^2(ka+b+c)^2}{(4b^3a+a^2bc+4c^3a)(ka+b+c)^2} \geq$$

$$\geq \frac{\left(\sum(ka^2+2ab)\right)^2}{\sum(4b^3a+a^2bc+4c^3a)(ka+b+c)^2}$$

gives a wrong inequality

$$\frac{\left(\sum(ka^2+2ab)\right)^2}{\sum(4b^3a+a^2bc+4c^3a)(ka+b+c)^2} \geq \frac{1}{a^2+b^2+c^2}$$

for all real k.

You can try to use Cauchy Schwarz like this, arqady.

$$\left(\sum \frac{a}{4b^3+abc+4c^3}\right)\left(\sum \frac{a(4b^3+abc+4c^3)}{(b+c)^2}\right) \geq \left(\sum \frac{a}{b+c}\right)^2.$$

322.
, The following inequality is true too. Let a, b and c are non-negative numbers such that $ab+ac+bc \neq 0$. Prove that

$$\frac{a^3}{2b^3-abc+2c^3} + \frac{b^3}{2a^3-abc+2c^3} + \frac{c^3}{2a^3-abc+2b^3} \geq 1$$

where all denominators are positive.

Solution:

$$LHS \geq \frac{(a^3+b^3+c^3)^2}{4\sum a^3b^3 - abc(a^3+b^3+c^3)}$$

We need to prove that:

$$\sum a^6 + abc(a^3+b^3+c^3) \geq 2\sum a^3b^3$$

by Schur and AM-GM ineq,We get:

$$LHS \geq \sum ab(a^4+b^4) \geq 2\sum a^3b^3$$

We have done!

Also by

$$\sum a^6 + abc(a^3 + b^3 + c^3) \geq 2\sum a^3b^3$$

We have :

$$a^3 + b^3 + c^3 \geq 3abc$$

So We have to prove that:

$$\sum a^6 + 3a^2b^2c^2 \geq 2\sum a^3b^3$$

Puting $a^2 = x$ and cyclic We have Schur for $n = 3$ So

$$\sum a^6 + 3a^2b^2c^2 \geq \sum_{sym} a^4b^2$$

So We now have to prove that:

$$\sum_{sym} a^4b^2 \geq 2\sum a^3b^3$$

which is Muirhead for the triples $(4, 2, 0) \succ (3, 3, 0)$......

323.

For all a, b, c be nonnegative real numbers, We have

$$\frac{a}{a^2 + 2bc} + \frac{b}{b^2 + 2ca} + \frac{c}{c^2 + 2ab} \leq \frac{2(a^2 + b^2 + c^2)}{(a + b + c)^2}\left(\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a}\right)$$

Solution:

We have

$$\sum \frac{a}{a^2 + 2bc} \leq \frac{a + b + c}{ab + bc + ca} \quad \text{(ill-known result)}$$

and

$$\sum \frac{1}{b + c} \geq \frac{(\sqrt{a} + \sqrt{b} + \sqrt{c})^2}{\sum a(b + c)} = \frac{(\sqrt{a} + \sqrt{b} + \sqrt{c})^2}{2(ab + bc + ca)}$$

(Cauchy Schwarz) Hence, it suffices to prove that

$$\frac{a + b + c}{ab + bc + ca} \leq \frac{(a^2 + b^2 + c^2)(\sqrt{a} + \sqrt{b} + \sqrt{c})^2}{(a + b + c)^2(ab + bc + ca)},$$

or

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})^2(a^2 + b^2 + c^2) \geq (a + b + c)^3.$$

Luckily, this is Holder inequality, and so it is valid

We need prove that:

$$\frac{2(a^2 + b^2 + c^2)}{(a + b + c)^2}\left(\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a}\right) \geq \frac{a + b + c}{ab + bc + ac}$$

Using Cauchy Schwarts, We only need to prove:

$$2(a^2 + b^2 + c^2)(ab + bc + ca) \geq \ (a + b + c).\sum \ ab(a + b)$$

$$<=> \sum \ ab(a - b)^2 \geq \ 0$$

(obviously trues)

We have:
$$LHS \leq \frac{a+b+c}{ab+bc+ac}$$
We need prove that:
$$\frac{2(a^2+b^2+c^2)}{(a+b+c)^2}\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) \geq \frac{a+b+c}{ab+bc+ac}(*)$$
Let: $p=a+b+c=1; q=ab+bc+ac \leq \frac{1}{3}; r=abc$ So
$$(*) <=> \frac{2(1-2q)(1+q)}{q-r} \geq \frac{1}{q}$$
$$<=> q-2q^2-4q^3+r \geq 0$$
Case 1:
$$q \leq \frac{1}{4}$$
So:
$$q-2q^2-4q^3+r \geq 2q(\frac{1}{4}-q)+4q(\frac{1}{16}-q^2)+\frac{q}{4} \geq 0$$
We have done! Case 2:
$$\frac{1}{3} \geq q \geq \frac{1}{4}$$
We have:
$$r \geq \frac{4q-1}{9} - bychur$$
So:
$$9(q-2q^2-4q^3+r) \geq 9q+4q-1-18q^2-4q^3 = (1-12q^2-10q)(3q-1) \geq 0$$
it's true because
$$\frac{1}{3} \geq q \geq \frac{1}{4}$$
324,
if a, b, c be sidelengths of a triangle, then
$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}} \leq \sqrt{\frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}+\frac{a+b+c}{2}}$$
Solution:
By cauchy-swarchz.We have:
$$LHS^2 \leq (a+b+c)(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b})$$
We need prove that:
$$\frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}+\frac{a+b+c}{2} \geq (a+b+c)(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b})$$
We have:
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \leq \frac{1}{2}+\frac{3(a^3+b^3+c^3)}{(a+b+c)(a^2+b^2+c^2)}$$
Remark:
1) if a, b, c be sidelengths of a triangle, then
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \leq \frac{1}{2}+\frac{3(a^3+b^3+c^3)}{(a+b+c)(a^2+b^2+c^2)}$$

2), if a, b, c be sidelengths of a triangle, then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\leq\frac{7}{6}+\frac{a^3+b^3+c^3}{(a+b+c)(ab+bc+ca)}$$

3), if a, b, c be nonnegative real numbers, then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{5}{4}+\frac{3(a^3+b^3+c^3)}{4(a+b+c)(ab+bc+ca)}$$

325.
Let $a, b, c \geq 0; a+b+c=1$. Prove that:

$$\sum\frac{a+bc}{b+c}+\frac{9abc}{4(ab+bc+ac)}\geq\frac{9}{4}$$

Solution:
1)
We have:
$\frac{a+bc}{b+c}=\sum\frac{(a+b)(a+c)}{b+c}=(a+b)(b+c)(c+a)(\sum\frac{1}{(b+c)^2})\geq\frac{9}{4}.\frac{(a+b)(b+c)(c+a)}{ab+bc+ca}=$

$$\frac{9}{4}.\frac{(a+b)(b+c)(c+a)}{(ab+bc+ca)(a+b+c)}$$

and:
$$\frac{9}{4}.\frac{abc}{ab+bc+ca}=\frac{9}{4}.\frac{abc}{(ab+bc+ca)(a+b+c)}$$

Q.E.D
2)
Setting:

$$ab+bc+ac=x;\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}=y$$

So:
$$xy\geq\frac{9}{4}$$

Because:
$$a+b+c=1=>abc\leq\frac{x}{9}$$

inequality
$$<=>xy+abc(\frac{9}{4x}-y)\geq\frac{9}{4}$$

$$LHS\geq xy+\frac{x}{9}(\frac{9}{4x}-y)\geq\frac{9}{4}$$

$$<=>\frac{8}{9}xy+\frac{1}{4}\geq\frac{9}{4}$$

Q.E.D
326.
Let a, b, c be nonnegative real numbers such that $max(a, b, c)\leq 4min(a, b, c)$. Prove that

$$2(a+b+c)(ab+bc+ca)^2\geq 9abc(a^2+b^2+c^2+ab+bc+ca)$$

Solution:

$$<=>S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\geq 0$$

With:

$$S_a = \frac{4ab+4ac-5bc}{36bc(ab+bc+ac)}; S_b = \frac{4ab+4bc-5ac}{36ac(ab+bc+ac)}; S_c = \frac{4bc+4ac-5ab}{36ab(ab+bc+ac)}$$

* $a \geq b \geq c => a \leq 4c$

$$=> S_a \geq 0; S_b \geq 0; S_a + S_c = \frac{4b(c-a)^2 + 2ca(2c+2a-b)}{36abc(ab+bc+ac)} \geq 0$$

Your **Solution** is corect. in our **Solution**, We have ($assuma \geq b \geq c$):

$$S_a \geq S_b \geq S_c and S_b + S_c = 4a(b-c)^2 + 2bc(2b+2c-a) \geq 0$$

Equality holds for $a = b = c$ or $a = 4b = 4c$
Q.E.D
327.
Given that $a, b, c \geq 0$. Prove that;

$$\left(\frac{a}{b+c}\right)^2 + \left(\frac{b}{c+a}\right)^2 + \left(\frac{c}{a+b}\right)^2 \geq \frac{3(a^2+b^2+c^2)}{4(ab+bc+ca)}$$

Solution:
suppose that $a \geq b \geq c$, by arrangement inequality We have

$$(\frac{a}{b+c})^2 + (\frac{b}{c+a})^2 + (\frac{c}{a+b})^2$$

$$\geq \frac{1}{3}(a^2+b^2+c^2)[(\frac{1}{b+c})^2 + (\frac{1}{c+a})^2 + (\frac{1}{a+b})^2]$$

then it's only to prove that

$$(ab+bc+ca)[(\frac{1}{b+c})^2 + (\frac{1}{c+a})^2 + (\frac{1}{a+b})^2] \geq \frac{9}{4}$$

which is obvious now

it equivalent to

$$4abc\sum_{cyc}\frac{a}{(b+c)^2} + 4\sum\frac{a^3}{b+c} \geq 3(a^2+b^2+c^2)$$

By AM-GM

$$\sum\frac{a}{(b+c)^2} \geq \frac{9}{4(a+b+c)}$$

We only need to prove

$$\frac{9abc}{a+b+c} + 4\sum\frac{a^3}{b+c} \geq 3(a^2+b^2+c^2)$$

$$\Leftrightarrow 9abc + 4\sum\frac{a^4}{b+c} + a^3+b^3+c^3 \geq 3\sum bc(b+c)$$

By AM-GM

$$\sum\frac{a^4}{b+c} \geq \frac{a^3+b^3+c^3}{2}$$

it suffices to show that

$$a^3+b^3+c^3+3abc \geq ab(a+b)+bc(b+c)+ca(c+a)$$

it's Schur!

328.
Let a, b, c be nonnegative real number, no two of which are zero. Prove that

$$(a)\quad \frac{a(2a-b-c)}{(b+c)^2}+\frac{b(2b-c-a)}{(c+a)^2}+\frac{c(2c-a-b)}{(a+b)^2}+2\geq\frac{2(a^2+b^2+c^2)}{ab+bc+ca}$$

$$(b)\quad \frac{a(a-b-c)}{(b+c)^2}+\frac{b(b-c-a)}{(c+a)^2}+\frac{c(c-a-b)}{(a+b)^2}+\frac{3}{2}\geq\frac{3(a^2+b^2+c^2)}{4(ab+bc+ca)}$$

329.
Let be $a, b, c > 0$ such that $ab+bc+ca=1$. Show that :

$$\frac{a}{a+\sqrt{b^2+1}}+\frac{b}{b+\sqrt{c^2+1}}+\frac{c}{c+\sqrt{a^2+1}}\leq 1$$

Solution:
Because

$$a^2+1=a^2+bc+ab+ca\geq 2a\sqrt{bc}+ab+ca=a(\sqrt{b}+\sqrt{c})^2$$

hence

$$\sum\frac{a}{a+\sqrt{b^2+1}}\leq\sum\frac{a}{a+\sqrt{a}(\sqrt{b}+\sqrt{c})}=1$$

Q.E.D
330.
1) Prove that for all a, b, c be nonnegative real numbers, We have

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{2}{3}\frac{(a+b+c)^2}{ab+bc+ca}\left(\frac{a^2}{2a+b+c}+\frac{b^2}{2b+c+a}+\frac{c^2}{2c+a+b}\right)$$

2) For all a, b, c be nonnegative real numbers. Prove that

$$\frac{a^4}{(b+c)^2}+\frac{b^4}{(c+a)^2}+\frac{c^4}{(a+b)^2}+\frac{ab+bc+ca}{4}\geq\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}$$

3) Let a, b, c be sidelengths of a triangle. Prove that

$$\sum_{cyc}\frac{1}{a^2+bc}\geq\frac{3}{4}(a^2+b^2+c^2)\left(\frac{1}{a^3b+b^3c+c^3a}+\frac{1}{ab^3+bc^3+ca^3}\right)$$

4) if a, b, c be sidelengths of a triangle. Prove that

$$\frac{1}{a^2+bc}+\frac{1}{b^2+ca}+\frac{1}{c^2+ab}\geq\frac{2(a+b+c)^2}{3(a^2+b^2+c^2)}\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)$$

5)Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a(a+b)(a+c)}{(b+c)^3}+\frac{b(b+c)(b+a)}{(c+a)^3}+\frac{c(c+a)(c+b)}{(a+b)^3}\geq\frac{(a+b+c)^4}{6(ab+bc+ca)^2}$$

6), Let a, b, c be nonnegative real numbers. Prove that

$$\sum_{cyc}\frac{1}{(b+c)(b^2+bc+c^2)}\geq\frac{4}{(a+b)(b+c)(c+a)}$$

331.
Let a, b, c be non-negative real numbers such that $a+b+c=3$. Prove that

$$\frac{a}{a+2bc}+\frac{b}{b+2ca}+\frac{c}{c+2ab}\geq 1$$

Solution:

$$\frac{a}{a+2bc}+\frac{b}{b+2ca}+\frac{c}{c+2ab}=\sum_{cyclic}\frac{a^2}{a^2+2abc}\geq\frac{(a+b+c)^2}{a^2+b^2+c^2+6abc}$$

from am-gm

$$(a+b+c)(ab+bc+ca)\geq 9abc, so ab+bc+ca\geq 3abc$$

$$\therefore\frac{(a+b+c)^2}{a^2+b^2+c^2+6abc}\geq\frac{a^2+b^2+c^2+6abc}{a^2+b^2+c^2+6abc}=1$$

Q.E.D

332.

Let a,b,c be non-negative real numbers such that $a+b+c=3$. Prove that

$$\frac{a}{2a+bc}+\frac{b}{2b+ca}+\frac{c}{2c+ab}\leq 1$$

Solution:

$$\sum_{cyclic}\frac{a}{2a+bc}\leq 1\longleftrightarrow\sum_{cyclic}\frac{2a}{2a+bc}\leq 2$$

$$\longleftrightarrow\sum_{cyclic}(1-\frac{2a}{2a+bc})\geq 1$$

by Cauchy-Schwarz ;

$$\sum_{cyclic}\frac{bc}{2a+bc}=\sum_{cyclic}\frac{(bc)^2}{2abc+b^2c^2}\geq\frac{(ab+bc+ca)^2}{6abc+a^2b^2+b^2c^2+c^2a^2}$$

$$=\frac{a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)}{6abc+a^2b^2+b^2c^2+c^2a^2}=1$$

Q.E.D

. 333.

Let $a,b,c>0$, $\sum\frac{1}{a^2+1}=2$. Prove:

$$ab+bc+ac\leq\frac{3}{2}$$

Solution:

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}=2\Leftrightarrow\frac{a^2}{1+a^2}+\frac{b^2}{1+b^2}+\frac{c^2}{1+c^2}=1$$

by cauchy:

$$(a^2+1+b^2+1+c^2+1)(\frac{a^2}{1+a^2}+\frac{b^2}{1+b^2}+\frac{c^2}{1+c^2})\geq(a+b+c)^2\Rightarrow\frac{3}{2}\geq(ab+bc+ca)$$

334.

Let be $a,b,c\in(0,\infty)$ such that $ab+bc+ca=1$. Show that :

$$9a^2b^2c^2+abc(\sqrt{1+a^2}+\sqrt{1+b^2}+\sqrt{1+c^2})\leq 1$$

Solution:

We have: $1+a^2=(a+b)(a+c)$, by AM-GM We have

$$LSH\leq 9ab.bc.ca+abc(a+b+b+c+c+a)\leq\frac{1}{3}(ab+bc+ca)^3+\frac{2}{3}(ab+bc+ca)^2=1$$

Q.E.D

335.

Let a, b, c be the side lengths of a triangle. Prove that

$$\sqrt[3]{\frac{a^2+b^2}{2c^2}}+\sqrt[3]{\frac{b^2+c^2}{2a^2}}+\sqrt[3]{\frac{c^2+a^2}{2b^2}} \geq \frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}$$

336.

for all $a, b, c \geq 0$ We have following inequality

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \leq \frac{2}{3}(a^2+b^2+c^2)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)$$

337.

Let $a, b, c \geq 0$. Prove the following inequality:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{2}{3}(ab+bc+ca)\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)$$

Solution:

$$\sum \frac{3a}{b+c} \geq \sum\left[\frac{2ab}{(a+b)^2}+\frac{2c}{a+b}\right]$$

$$\leftrightarrow \sum \frac{a}{b+c} \geq \sum \frac{2ab}{(a+b)^2}$$

$$\leftrightarrow \sum \frac{ab+ac-2bc}{(b+c)^2} \geq 0$$

$a \geq b \geq c$; $ab+ac-2bc \geq bc+ba-2ca \geq ca+cb-2ab$

$$\frac{1}{(b+c)^2} \geq \frac{1}{(c+a)^2} \geq \frac{1}{(a+b)^2}$$

Applying Chebyshev inequality.

338.

Let $a, b, c \geq 0$. Prove the following inequality:

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2 \geq (ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right]$$

339.

, Prove that for all a, b, c be nonnegative real numbers, We have

$$a^3+b^3+c^3+6abc \geq a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

Solution:

it's equivalent to

$$(a-b)^2(a-c)^2(b-c)^2+\sum_{cyc} ab(a^2+ab+b^2-c^2)(a-b)^2 \geq 0$$

which is obviously true.

340.

, 1) Prove that for all a, b, c be nonnegative real numbers, We have

$$\frac{b+c-a}{5a^2+4bc}+\frac{c+a-b}{5b^2+4ca}+\frac{a+b-c}{5c^2+4ab} \geq \frac{1}{a+b+c}$$

2), For a, b, c be nonnegative real numbers, We have

$$\frac{1}{\sqrt{a^2+bc}}+\frac{1}{\sqrt{b^2+ca}}+\frac{1}{\sqrt{c^2+ab}} \geq \frac{\sqrt{2}}{\sqrt{a^2+b^2+c^2}}+\frac{2}{\sqrt{ab+bc+ca}}$$

3)

$$\sum_{cyc}\frac{1}{\sqrt{a^2+bc}} \leq \sum_{cyc}\frac{\sqrt{2}}{a+b}$$

340.
Let a,b,c be nonnegative real numbers. Prove that

$$\sum_{cyc}\frac{1}{\sqrt{a^2+bc}} \leq \sum_{cyc}\frac{\sqrt{2}}{a+b}.$$

Solution:
By the Cauchy schwarz inequality, We have

$$\left(\sum\frac{1}{\sqrt{a^2+bc}}\right)^2 \leq \left(\sum\frac{1}{(a+b)(a+c)}\right)\left(\sum\frac{(a+b)(a+c)}{a^2+bc}\right)$$

$$=\frac{2\sum a}{(a+b)(b+c)(c+a)}\left(\sum\frac{a(b+c)}{a^2+bc}+3\right)$$

it suffices to show that

$$\frac{2\sum a}{(a+b)(b+c)(c+a)}\left(\sum\frac{a(b+c)}{a^2+bc}+3\right) \leq 2\left(\sum\frac{1}{a+b}\right)^2$$

$$\Leftrightarrow \sum\frac{a(b+c)}{a^2+bc}+3 \leq \frac{(\sum a^2+3\sum ab)^2}{(a+b)(b+c)(c+a)\sum a}$$

$$\Leftrightarrow \sum\frac{a(b+c)}{a^2+bc}-3 \leq \frac{\sum a^4-\sum a^2b^2}{(a+b)(b+c)(c+a)\sum a} \Leftrightarrow \sum(a-b)(a-c)\left(\frac{1}{a^2+bc}+\frac{1}{(b+c)(a+b+c)}\right) \geq 0$$

Due to symmetry, We may assume $a \geq b \geq c$, since $a-c \geq \frac{a}{b}(b-c)$. it suffices to show that

$$a\left(\frac{1}{a^2+bc}+\frac{1}{(b+c)(a+b+c)}\right) \geq b\left(\frac{1}{b^2+ca}+\frac{1}{(a+c)(a+b+c)}\right)$$

$$\Leftrightarrow c(a^2-b^2)[(a-b)^2+ab+bc+ca] \geq 0$$

which is trivial. Equality holds if and only if $a=b=c$.

341. Let a, b, c be non-negative real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{a}{\sqrt{a^2+3bc}}+\frac{b}{\sqrt{b^2+3ca}}+\frac{c}{\sqrt{c^2+3ab}}$$

Solution:
After using Cauchy Schwarz, We can see that the inequality follows from

$$\sum\frac{a}{b+c} \geq \sum\frac{a(b+c)}{a^2+3bc},$$

that is

$$\sum\frac{a(a^2+bc-b^2-c^2)}{(b+c)(a^2+3bc)} \geq 0,$$

or

$$\sum\frac{a^3(b+c)-a(b^3+c^3)}{(b+c)^2(a^2+3bc)} \geq 0.$$

Without loss of generality, We can assume that $a \geq b \geq c$, then

$$a^3(b+c) - a(b^3+c^3) \geq 0 \geq c^3(a+b) - c(a^3+b^3),$$

and

$$\frac{1}{(b+c)^2(a^2+3bc)} \geq \frac{1}{(c+a)^2(b^2+3ca)} \geq \frac{1}{(a+b)^2(c^2+3ab)}.$$

it follows that

$$\frac{a^3(b+c) - a(b^3+c^3)}{(b+c)^2(a^2+3bc)} \geq \frac{a^3(b+c) - a(b^3+c^3)}{(c+a)^2(b^2+3ca)},$$

and

$$\frac{c^3(a+b) - c(a^3+b^3)}{(a+b)^2(c^2+3ab)} \geq \frac{c^3(a+b) - c(a^3+b^3)}{(c+a)^2(b^2+3ca)}.$$

Therefore

$$\sum \frac{a^3(b+c) - a(b^3+c^3)}{(b+c)^2(a^2+3bc)} \geq \frac{\sum(a^3(b+c) - a(b^3+c^3))}{(c+a)^2(b^2+3ca)} = 0.$$

Our **Solution** is completed.

infact, We have the following inequality for all $a, b, c \geq 0$

$$\frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{a}{\sqrt{a^2+3bc}} + \frac{b}{\sqrt{b^2+3ca}} + \frac{c}{\sqrt{c^2+3ab}}$$

342.
Let equation:

$$(x+1).lnx - x.ln(x+1) = 0.$$

Prove that this equation have only one root

Solution:

f(x) = (x + 1)lnx - xln(x + 1)

$$f'(x) = ln(\frac{x}{x+1}) + \frac{x+1}{x} - \frac{x}{x+1}$$

Setting :

$$t = \frac{x}{x+1} < 1$$

$$g(x) = lnt + \frac{1}{t} - t$$

$$g'(x) = \frac{1}{t} - \frac{1}{t^2} - 1 \leq 0 \forall t \in R$$

$$=> g(x) \geq g(1) = 0$$

$$=> f'(x) \geq 0$$

$$=> ...$$

But We have :

$$f(...) = ... > 0$$

343.
, For positive a,b and c such that $a+b+c=3$. Prove that:

$$\frac{a^2}{2a+b^2}+\frac{b^2}{2b+c^2}+\frac{c^2}{2c+a^2}\geq 1$$

Solution:

$$\sum\frac{a^2}{2a+b^2}=\sum\frac{a^4}{2a^3+a^2b^2}\geq\frac{(\sum a^2)^2}{2\sum a^3+\sum a^2b^2}$$

So We must prove that

$$\sum a^4+\sum a^2b^2\geq 2\sum a^3 or \sum a^4+3\sum a^2b^2\geq 2\sum a^3b+a^3c$$

and We think the last inequality is true
Of course the last is was true Wink .Because :

$$<=>\sum(a-b)^4\geq 0$$

$$\sum\frac{a^2}{a+2b^2}=\sum\frac{a^4}{a^3+2a^2b^2}\geq\frac{(a^2+b^2+c^2)^2}{(a^3+b^3+c^3)+2(a^2b^2+b^2c^2+c^2a^2)}\geq 1$$

So it's enough to prove that :

$$a^4+b^4+c^4\geq a^3+b^3+c^3\Leftrightarrow 3(a^4+b^4+c^4)\geq(a^3+b^3+c^3)(a+b+c)$$

which reduces to :

$$2\sum a^4\geq\sum(a^3b+ab^3)$$

which is just Muirhead.Or you can also prove this last inequality by AM-GM.

344.
Let $a,b,c>0$ and $abc=1$. Prove that

$$\frac{a^2}{1+2ab}+\frac{b^2}{1+2bc}+\frac{c^2}{1+2ca}\geq\frac{a}{ab+b+1}+\frac{b}{bc+c+1}+\frac{c}{ca+a+1}$$

Solution:
Because

$$abc=1$$

We Setting:

$$a=\frac{x}{y}b=\frac{y}{z}c=\frac{z}{x}$$

We have to prove that:

$$\sum_{cyc}\frac{x^2z}{y^2(2x+z)}\geq\sum_{cyc}\frac{(\frac{xz}{y})}{x+y+z}$$

$$(x+y+z)\sum_{cyc}\frac{x^2z}{y^2(2x+z)}\geq\sum_{cyc}\frac{xz}{y}$$

$$\sum_{cyc}(\frac{x^2z}{y^2}+\frac{2x^2z}{y(z+2x)}+\frac{x^2z^2}{y^2(z+2x)})\geq 2\sum_{cyc}\frac{xz}{y}$$

By Cauchy

$$\frac{x^2}{y} * \frac{z}{y} \geq \frac{(2x-y)z}{y} = \frac{2xz}{y} - z$$

$$\frac{9x^2}{(\frac{y(2x+z)}{z})} \geq 6x - \frac{y(2x+z)}{z} = 6x - y - \frac{2xy}{z}$$

$$\frac{z}{y}\frac{9x^2}{(\frac{y(2x+z)}{z})} \geq \frac{z}{y}(6x - \frac{y(2x+z)}{z}) = \frac{6xz}{y} - 2x - z$$

So

$$\sum_{cyc}(\frac{x^2z}{y^2} + \frac{2x^2z}{y(z+2x)} + \frac{x^2z^2}{y^2(z+2x)}) \geq \sum_{cyc}(\frac{20xz}{9y} - \frac{2x}{9}) \geq 2\sum_{cyc}\frac{xz}{y}$$

Let me try another **Solution** MSetting :

$$a = \frac{x}{y}; b = \frac{y}{z}; c = \frac{z}{x}$$

$$LHS = \sum \frac{\frac{x^2y^2}{z^2}}{2xz + z^2}$$

$$RHS = \frac{\sum \frac{xy}{z}}{x+y+z}$$

By cauchy-schwarz, We can prove

$$(x+y+z)^2 LHS = (\sum 2xz + z^2)(\sum \frac{\frac{x^2y^2}{z^2}}{2xz+z^2}) \geq (\sum \frac{xy}{z})^2$$

Now, We need to prove :

$$(\sum \frac{xy}{z})^2 \geq (x+y+z)(\sum \frac{xy}{z})$$

$$<=> \sum \frac{xy}{z} \geq x+y+z$$

Which is obvious true by Am-Gm

345.

Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a}{2b+1} + \frac{b}{2c+1} + \frac{c}{2a+1} \leq \frac{1}{abc}$$

Solution:

1)

★*Lemma* :

Let $a, b, c \geq 0$ such that $a+b+c=3$ We have:

$$a^2c + c^2b + b^2c \leq 4 - abc$$

it is easy to prove if We assume that

$$a(a-b)(c-b) \leq 0.$$

Now We are coming back above inequality:

★ Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a}{2b+1} + \frac{b}{2c+1} + \frac{c}{2a+1} \leq \frac{1}{abc}$$

Expanding and numeratoring We have the inequality is equivalent to:

$$7 + 4abc\sum ab + 4\sum ab \geq 2\sum a^2b^2 + 13abc + 4abc(a^2c + c^2b + b^2c)$$

Applying above lemma, We only need prove that:

$$7 + 4abc\sum ab + 4\sum ab \geq 2\sum a^2b^2 + 13abc + 4abc(4 - abc)$$

Now,We will solve it easily by p, q, r technique. in fact, We put $q = ab + bc + ca, r = abc$ and note that $q \leq 3$. The inequality is equivalent to:

$$f(r) = 4r^2 + (4q - 17)r - 2q^2 + 4q + 7$$

Using Schur's inequality We have:

$$\frac{12q - 27}{9} \leq r \leq \frac{q}{3}$$

Now, We will use derivative method to prove that $f(r) \geq 0$.

$$\bullet The first case : 4q + 8r - 17 \geq 0 then$$

$$f'(r) = 4q + 8r - 17 \geq 0 \to f(r) \geq f\left(\frac{12q - 27}{9}\right) = \frac{94(3 - q)^2}{9} \geq 0$$

• The second case: $4q + 8r - 17 \leq 0$ then

$$f'(r) = 4q + 8r - 17 \leq 0 \to f(r) \geq f\left(\frac{q}{3}\right) = \frac{(3 - q)(2q + 21)}{9} \geq 0, \forall 0 \leq q \leq 3$$

2)

★ Lemma :

$$a + b + c = 3, a, b, c \geq 0$$

then :

$$a^2b + b^2c + c^2a + abc \leq 4 \text{ (it can prove by Am-Gm)}$$

Expanding the ineq :

$$4(\sum a^2c)abc + 4abc + (\sum a^2)abc \leq 4\sum ab + 7$$

$$LHS \leq 20abc + (\sum a^2)abc - 4a^2b^2c^2$$

We need to prove:

$$20abc + (\sum a^2)abc \leq 4(a^2b^2c^2 + 1) + 4\sum ab + 3$$

Easy to prove :

$$a^2b^2c^2 + 1 \geq 2abcand : \sum ab \geq 3abc$$

The ineq become:

$$3 \geq (\sum a^2)abc$$

But :

$$9LHS = (\sum a^2)[3(a + b + c)abc] \leq (\sum a^2)(ab + bc + ca)^2 \leq 27$$

Another **Solution**(but similar) Wink

$$<=> \sum \frac{2a^2bc}{2b + 1} \leq 2$$

$$<=> \sum \frac{a^2c}{2b+1} \geq \sum a^2c - 2$$

By cauchy-schwarz , We can prove:

$$LHS \geq \frac{(\sum a^2c)^2}{2(\sum a)abc + \sum a^2c}$$

So ,We need to prove :

$$(\sum a^2c)^2 \geq (\sum a^2c - 2)(6abc + \sum a^2c)$$

$$<=> \sum a^2c + 6abc \geq 3abc(\sum a^2c)$$

$$<=> 6abc \geq (3abc-1)(\sum a^2c)$$

$$<=> 5abc + 3a^2b^2c^2 \geq (3abc-1)(\sum a^2c + abc)$$

Why the above lemma , We can prove :

$$RHS \leq 12abc - 4$$

$$LHS = 5abc + 3(a^2b^2c^2+1) + 1 - 4 \geq 12abc - 4$$

Q.E.D

346.

, Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that ★

$$\frac{a}{2b+1} + \frac{b}{2c+1} + \frac{c}{2a+1} \leq \frac{1}{abc}$$

★

$$\frac{1}{a(2ab+1)} + \frac{1}{b(2bc+1)} + \frac{1}{c(2ca+1)} \leq \frac{1}{a^2b^2c^2}$$

★

$$\frac{1}{3} \leq \frac{a}{(b+2)^2} + \frac{b}{(c+2)^2} + \frac{c}{(a+2)^2} \leq \frac{1}{3abc}$$

★

$$\frac{1}{3} \leq \frac{1}{a(ab+2)^2)} + \frac{1}{b(bc+2)^2} + \frac{1}{c(ca+2)^2} \leq \frac{1}{3a^2b^2c^2}$$

347.

Let $a, b, c \geq 0$ such that $a+b+c=1$. Prove that

$$\sqrt{\frac{a+2b}{a+2c}} + \sqrt{\frac{b+2c}{b+2a}} + \sqrt{\frac{c+2a}{c+2b}} \geq 3$$

Solution:

Setting :$x = a+2b; y = b+2c; z = c+2a; x, y, z \in [0; \frac{3}{2}]$

Assume that :$a+b+c=1 => x+y+z=3$

$$<=> \sum \sqrt{\frac{x}{2-x}} \geq 3$$

$$[\sum x^2(2-x)]LHS^2 \geq (x+y+z)^3$$

We need to prove:

$$(x+y+z)^3 \geq 3[2(x+y+z)(x^2+y^2+z^2) - 3x^3 - 3y^3 - 3z^3]$$

$$<=> 4\sum x^3 - 3\sum xy(x+y) + 6xyz \geq 0$$

it's easier than schur.

$$x^3 + y^3 + z^3 + 3 \geq 2(x^2 + y^2 + z^2)$$

$$\iff (1 + x^3 - 2x^2 + x - 1) + (1 + y^3 - 2y^2 + y - 1) + (1 + z^3 - 2z^2 + z - 1) \geq 0$$

$$\iff x(x-1)^2 + y(y-1)^2 + z(z-1)^2 \geq 0$$

348.

1)

$$\sqrt{\frac{a+2b}{a+2c}} + \sqrt{\frac{b+2c}{b+2a}} + \sqrt{\frac{c+2a}{c+2b}} \geq 3; \forall a, b, c \geq 0$$

2)

$$\sqrt{\frac{a+2b}{c+2b}} + \sqrt{\frac{b+2c}{a+2c}} + \sqrt{\frac{c+2a}{b+2a}} \geq 3; \forall a, b, c \geq 0$$

348.

Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a+2b+3}{c+2b+3} + \frac{b+2c+3}{a+2c+3} + \frac{c+2a+3}{b+2a+3} \geq 3$$

Solution:

assume $c = min(a, b, c)$ and set $c + 1 = z, a + 1 = z + m, b + 1 = z + n$ with $m, n \geq 0$

Setting $x = a + 1, y = b + 1, z = c + 1$, then the inequality becomes

$$\sum \frac{x+2y}{z+2y} \geq 3$$

$$\Leftrightarrow \sum (x+2y)(2z+x)(2x+y) \geq 3(2x+y)(2y+z)(2z+x)$$

$$\Leftrightarrow \sum (x+2z)(2x^2+5xy+2y^2) \geq 3(9xyz + 4\sum x^2y + 2\sum xy^2)$$

$$\Leftrightarrow 2\sum x^3 + 9\sum x^2y + 6\sum xy^2 + 30xyz \geq 3(9xyz + 4\sum x^2y + 2\sum xy^2)$$

$$\Leftrightarrow 2\sum x^3 + 3xyz \geq 3\sum x^2y$$

By Schur's inequality, We get

$$\sum x^3 + 3xyz \geq \sum x^2y + \sum xy^2$$

it suffices to prove that

$$\sum x^3 + \sum xy^2 \geq 2\sum x^2y$$

$$\Leftrightarrow \sum x(x-y)^2 \geq 0$$

which is true.

349.

And general problems:

$$1,\ \frac{a+2b+k}{c+2b+k} + \frac{b+2c+k}{a+2c+k} + \frac{c+2a+k}{b+2a+k} \geq 3$$

for

$$k \geq 0$$

$$2,\ \frac{a+mb+n}{c+mb+n}+\frac{b+mc+n}{a+mc+n}+\frac{c+ma+n}{b+ma+n} \geq 3$$

for

$$m, n \geq 0$$

Solution:
1/ Setting $x = a + \frac{k}{3}, y = b + \frac{k}{3}, z = c + \frac{k}{3}$, the inequality becomes

$$\sum \frac{x+2y}{z+2y} \geq 3$$

it is the previous.
2/ Setting $x = a + \frac{n}{m+1}, y = b + \frac{n}{m+1}, z = c + \frac{n}{m+1}$, the inequality beomces

$$\sum \frac{x+our}{z+our} \geq 3$$

This inequality is not always true.

350.
Prove if $a, b, c > 0$; $abc = 1$ then

$$\frac{a^3+b^3+c^3}{\sqrt{3abc}} >= \sum \frac{a^2}{\sqrt{b+c}} >= \frac{\sqrt[6]{abc}(a+b+c)}{\sqrt{2}}$$

Solution:
Assume that :$abc = 1$
★ :

$$\sum a^3+b^3+c^3 \geq \sum a^2\sqrt[4]{a} = \sum \frac{a^2}{\sqrt[4]{bc}} \geq \sum \frac{\sqrt{2}a^2}{\sqrt{b+c}}$$

Done
★

$$3\sum \frac{a^2}{\sqrt{b+c}} \geq (\sum a^2)(\sum \frac{1}{\sqrt{b+c}}) \geq (a+b+c)^2 \frac{3\sqrt{3}}{\sqrt{2(a+b+c)}} \geq ..$$

351.
Let $a, b, c > 0$ and $abc = 1$. Prove that

$$a^3+b^3+c^3-3 \geq \sum \frac{a}{b+c} - \frac{3}{2}$$

Solution:
The inequality is equivalent to

$$\sum S_c(a-b)^2 \geq 0$$

with

$$S_a = a+b+c - \frac{1}{(a+b)(a+c)} > 0$$

similar with S_b, S_c We get the same result.so the inequality has been proved.

$$a^3+b^3+c^3 \geq a\sqrt{a}+b\sqrt{b}+c\sqrt{c} = \sum \frac{a}{\sqrt{bc}} \geq \sum \frac{2a}{b+c} \geq \sum \frac{a}{b+c} + \frac{3}{2}$$

By the hypothesis, We have

$$a^3+b^3+c^3=\sum\frac{a^2}{bc}\geq 4.\sum(\frac{a}{b+c})^2\geq\frac{4}{3}.(\sum\frac{a}{b+c})^2\geq\sum\frac{a}{b+c}+\frac{3}{2}$$

352.
This inequality is true ,but very easy:

$$a^3+b^3+c^3+\frac{3}{2}\geq 3\sum\frac{a}{b+c}$$

$$2LHS\geq 3\sum a^2\geq 3\sum a\sqrt{a}=3\sum\frac{a}{\sqrt{bc}}\geq 2RHS$$

=>done By schur

$$a^3+b^3+c^3+3abc\geq\sum ab(a+b)$$

$$abc=1-->a^3+b^3+c^3+3\geq\sum\frac{a+b}{c}\geq 4\sum\frac{a}{b+c}$$

353.
Let $a,b,c>0$ such that $abc=1$. Prove that

$$\sum\frac{1}{(1+a)^2}+\frac{1}{1+b+c+a}\geq 1$$

Solution:
Easy expand

$$<=>q^2-2qp+p^3-5p-3\geq 0$$

$$\Delta'=p^2+3+5p-p^3=-(p-3)(1+p)^2\leq 0$$

354.
Let $a,b,c\geq 0$. Prove that:

$$\frac{1}{5(a^2+b^2)-ab}+\frac{1}{5(b^2+c^2)-bc}+\frac{1}{5(c^2+a^2)-ca}\geq\frac{1}{a^2+b^2+c^2}$$

Solution:
. We need prove

$$\sum\frac{5c^2+ab}{5(a^2+b^2)-ab}\geq 2$$

Using Cauchy Schwarz,We have:

$$\sum\frac{5c^2+ab}{5(a^2+b^2)-ab}\geq\frac{(5\sum a^2+\sum ab)^2}{\sum(5(a^2+b^2)-ab)(5c^2+ab)}(1)$$

Setting $q=ab+bc+ca, r=abc, p=a+b+c=1$. We have

$$(1)\Leftrightarrow(5\sum a^2+\sum ab)^2\geq\sum(5(a^2+b^2)-ab)(5c^2+ab)$$

$$(5-9q)^2\geq 2(-108r+39q^2+5q)$$

$$\Leftrightarrow(q-1)(3q-1)\geq 0$$

355.
, Let $a,b,c>0$ such that $ab+bc+ca=3$. Prove that

$$3\sqrt{(a+b)(b+c)(c+a)}\geq 2(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a})\geq 6\sqrt{2}$$

Solution:
★ First, We prove the left ineq. Setting :$x = ab; y = bc; z = ca => x + y + z = 3$
By cauchy-schwarz , We need to prove :

$$9(x + y)(y + z)(z + x) \geq 8(x + y + z)(xy + yz + zx)(Right)$$

★, Next, We prove the right ineq :
By Am-Gm, We need to prove

$$(a + b)(b + c)(c + a) \geq 8$$

$$<=> 3(a + b + c) - abc \geq 8(Right because ab + bc + ca = 3)$$

356.
, Let $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 3$. Prove that:

$$(2 - a)(2 - b)(2 - c) \geq \frac{25}{27}$$

Let be $c = Min(a, b, c)$ then $c \in [0, 1]$. We have:

$$2(2-a)(2-b) = 8-4(a+b)+2ab = 8-4(a+b)+(a+b)^2-a^2-b^2 = (a+b-2)^2+4-a^2-b^2 \geq c^2+1$$

Thus

$$f(c) = (2 - c)\left(\frac{c^2 + 1}{2}\right)$$

$$f'(c) = 0 \to c = \frac{1}{3}, c = 1 Hence, Min f(c) = Min\left(f(0), f(1), f\left(\frac{1}{3}\right)\right) = \frac{25}{27}$$

The equality holds if and only if $a = \frac{5}{3}, b = \frac{1}{3}, c = \frac{1}{3}$ or any cyclic permutations.

357.
Let $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{9}{2(a + b + c)} \leq \sum \frac{a}{b + c}$$

Solution:
We have

$$\sum \frac{a}{b + c} = \sum \frac{a^2}{ab + ac} \geq \frac{(a + b + c)^2}{2(ab + bc + ac)}$$

then in view the problem :

$$a^2 + b^2 + c^2 = 3 -> s^2 = 2p + 3$$

We obsever

$$s^3 + s^3 + 27 \geq 9s^2$$

$$<=> (3 - s)(9 + 3s - 2s^2) \geq 0$$

We have

$$\sum \frac{a + b}{c} \geq \frac{18}{a + b + c}$$

358.
Let $a, b, c > 0$ such that $a^2 + b^2 + c^2 + d^2 = 1$. Prove that

$$(1 - a)(1 - b)(1 - c)(1 - d) \geq \frac{(ab + cd)(ac + bd)}{4}$$

By CBS
$$\sqrt{(a^2+b^2)(c^2+d^2)} \geq ac+bd$$
We have
$$2(1-a)(1-d) = (a+d-1)^2 + b^2 + c^2 \geq b^2 + c^2$$
$$2(1-a)(1-d) \geq b^2 + c^2$$
$$2(1-b)(1-c) \geq a^2 + d^2$$
$$\sqrt{(a^2+d^2)(b^2+c^2)} \geq (ab+cd)$$
$$\sqrt{(a^2+d^2)(b^2+c^2)} \geq (ac+bd)$$
359., Let AB is a triangle have $A; B; C < 90$ Prove that
$$tan^2A + 3tan^2B + 15tan^2C \geq 36$$
Let $tanA = 3x; tanB = 2y; tanC = z(x; y; z > 0)$ **Solution**:
by AM - GM:
$$3x^2 + 4y^2 + 5c^2 \geq 12\sqrt[12]{x^6y^8z^{10}}$$
$$3x + 2y + z \geq 6\sqrt[6]{x^3y^2z}$$
$$---> (3x^2 + 4y^2 + 5z^2)(3x + 2y + z) \geq 72xyz$$
$$---> 3(3x^2 + 4y^2 + 5z^2) \geq 36$$
Q.E.D
360.
if x, y, z are three nonnegative reals, then prove that
$$\sum_{\text{cyc}} \sqrt{(z+x)(x+y)} \geq x+y+z+\sqrt{3} \cdot \sqrt{yz+zx+xy},$$
where the $\sum_{\text{cyc}}$ sign means cyclic summation.
Applying the Conway substitution theorem (http://www.mathlinks.ro/Forum/viewtopic.php?t=2958 post 3) to the reals x, y, z (in the role of u, v, w), We see that, since the numbers $y+z, z+x, x+y$ and $yz+zx+xy$ are all nonnegative, We can conclude that there exists a triangle ABC with sidelengths $a = BC = \sqrt{y+z}, b = CA = \sqrt{z+x}, c = AB = \sqrt{x+y}$ and area $S = \frac{1}{2}\sqrt{yz+zx+xy}$.
Now,
$$\sum \sqrt{(z+x)(x+y)} = \sum \sqrt{z+x} \cdot \sqrt{x+y} = \sum b \cdot c = bc + ca + ab;$$
$$x+y+z = \frac{1}{2}\left((y+z)+(z+x)+(x+y)\right)$$
$$= \frac{1}{2}\left(\left(\sqrt{y+z}\right)^2 + \left(\sqrt{z+x}\right)^2 + \left(\sqrt{x+y}\right)^2\right)$$
$$= \frac{1}{2}\left(a^2+b^2+c^2\right);$$
$$\sqrt{3} \cdot \sqrt{yz+zx+xy} = 2\sqrt{3} \cdot \frac{1}{2}\sqrt{yz+zx+xy} = 2\sqrt{3} \cdot S.$$
Hence, the inequality in question,
$$\sum \sqrt{(z+x)(x+y)} \geq x+y+z+\sqrt{3} \cdot \sqrt{yz+zx+xy},$$

becomes

$$bc + ca + ab \geq \frac{1}{2}\left(a^2 + b^2 + c^2\right) + 2\sqrt{3} \cdot S.$$

Multiplication by 2 transforms this into

$$2\left(bc + ca + ab\right) \geq \left(a^2 + b^2 + c^2\right) + 4\sqrt{3} \cdot S,$$

or, equivalently,

$$2\left(bc + ca + ab\right) - \left(a^2 + b^2 + c^2\right) \geq 4\sqrt{3} \cdot S.$$

Using the notation

$$Q = (b-c)^2 + (c-a)^2 + (a-b)^2$$

this rewrites as

$$\left(a^2 + b^2 + c^2\right) - Q \geq 4\sqrt{3} \cdot S,$$

what is equivalent to

$$a^2 + b^2 + c^2 \geq 4\sqrt{3} \cdot S + Q.$$

But this is the ill-known Hadwiger-Finsler inequality

$$\sum_{cyc} \sqrt{(z+x)(z+y)} \geq x + y + z + \sqrt{3(xy + xz + yz)} \Leftrightarrow$$

$$\Leftrightarrow 2\left(x + y + z - \sqrt{3(xy + xz + yz)}\right) - \sum_{cyc}\left(\sqrt{z+x} - \sqrt{z+y}\right)^2 \geq 0.$$

But

$$2\left(x + y + z - \sqrt{3(xy + xz + yz)}\right) - \sum_{cyc}\left(\sqrt{z+x} - \sqrt{z+y}\right)^2 =$$

$$= \sum_{cyc}(x-y)^2\left(\frac{1}{x + y + z + \sqrt{3(xy + xz + yz)}} - \frac{1}{\left(\sqrt{z+x} + \sqrt{z+y}\right)^2}\right) =$$

$$= \sum_{cyc}\frac{(x-y)^2\left(z + 2\sqrt{(z+x)(z+y)} - \sqrt{3(xy + xz + yz)}\right)}{\left(x + y + z + \sqrt{3(xy + xz + yz)}\right)\left(\sqrt{z+x} + \sqrt{z+y}\right)^2} =$$

$$= \sum_{cyc}\frac{(x-y)^2\left(z + \frac{4z^2 + xy + xz + yz}{2\sqrt{(z+x)(z+y)} + \sqrt{3(xy+xz+yz)}}\right)}{\left(x + y + z + \sqrt{3(xy + xz + yz)}\right)\left(\sqrt{z+x} + \sqrt{z+y}\right)^2} \geq 0.$$

361.

Prove if $a, b, c > \frac{1}{4}$ such that $\sum\sqrt{a} \leq 3$ then

$$\sum\sqrt{a^2 + 3} \geq a + b + c + 3$$

Solution:

$$\sqrt{a^2 + 3} - (a+1) \geq f_{(\sqrt{x})} denote a > \frac{1}{4}$$

$$\sqrt{a^2 + 3} \geq a - \sqrt{a} + 2 with a. \frac{1}{4}$$

$$\sqrt{a^2 + 3} \geq a - \sqrt{a} + 2$$

$$< - > (4a-1)(a-1)^2 \geq 0$$

362.
Let $a, b, c > 0$ such that a+b+c=1 Prove that

$$1 \leq \sum \frac{(a+b)b}{c+a}$$

Solution:

$$1 \leq \sum_{cyc} \frac{(a+b)b}{c+a} \Leftrightarrow \sum_{cyc} (a^4 - a^2b^2 + a^3c - a^2bc) \geq 0, \text{ which obviously true.}$$

$$\sum \frac{b(a+b)}{c+a} \geq \sum a$$

$$<-> \sum \frac{a(a+b+c) + b(a+b+c-a-c)}{c+a} \geq 2 \sum a$$

$$<-> \sum \frac{a+b}{a+c} \geq 3$$

Q.E.D
363.
Let ABC is a triangle. Prove that

$$\frac{1}{sin\frac{A}{2} sin\frac{B}{2} sin\frac{C}{2}} \geq \sum \sqrt{\frac{cos\frac{B-C}{2}}{sin\frac{B}{2} sin\frac{C}{2}}} + 2$$

Solution:

$$(*) <-> \sum \frac{a}{p-a} \geq \sum \sqrt{\frac{b+c}{p-a}}$$

because

$$\sum \frac{a}{p-a} \geq \sum \frac{c+b}{a}$$

BY CBS
Let $x = p - a > 0 : y = p - b > 0; z = p - c > 0$ We have

$$\sum \frac{x+y}{z} \geq 4 \sum \frac{x}{y+z} \geq \frac{2x+y+z}{y+z}$$

$$--> \sum \frac{a}{p-a} \geq \sum \frac{b+c}{a}$$

$$--> (\sum \frac{a}{p-a})^2 \geq (\sum \frac{a}{o-a})(\sum \frac{b+c}{a}) \geq (\sum \sqrt{\frac{b+c}{p-a}})^2$$

Q.E.D
364.
Prove that the sides a, b, c of any triangle suck that $a^2 + b^2 + c^2 = 3$ satisfy the inequality

$$\sum \frac{a}{a^2+b+c} \geq 1$$

Solution:

$$LHS \leq \frac{\sum_{cyc} a(1+b+c)}{(a+b+c)^2} = \frac{a+b+c+2ab+2bc+2ca}{3+2ab+2bc+2ca} \leq 1$$

365.
Prove if $a, b, c > 0$ then

$$\sum (a+b)(b+c)\sqrt{a-b+c} \geq 4(a+b+c)\sqrt{(-a+b+c)(a-b+c)(a+b-c)}$$

Solution:

$$b+c-a = x^2, a+c-b = y^2, a+b-c = z^2$$

We want to prove

$$\sum x(2y^2+z^2+x^2)(2z^2+y^2+x^2) \geq 16xyz(x^2+y^2+z^2)$$

$$\sum x(2y^2+z^2+x^2)(2z^2+y^2+x^2) = \sum x((x^4+y^2z^2)+(2y^4+2y^2z^2)+(2z^4+2y^2z^2)+(3x^2y^2+3x^2z^2)) \geq$$

$$\sum (8x^3yz + 4y^3zx + 4z^3xy) = 16xyz(x^2+y^2+z^2)$$

The inequality is equivalent to

$$\sum \frac{(a+b)(b+c)}{\sqrt{(b+c-a)(c+a-b)}} \geq 4(a+b+c)$$

From AM-GM We get

$$\frac{(a+b)(b+c)}{\sqrt{(b+c-a)(c+a-b)}} \geq \frac{2(a+b)(b+c)}{b+c-a+c+a-b} = \frac{(a+b)(b+c)}{c}$$

Therefore it remains to show that

$$\sum \frac{(a+b)(b+c)}{c} \geq 4(a+b+c) \qquad (1)$$

Since the sequences $\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\}$ and$\{(c+a)(a+b), (a+b)(b+c), (b+c)(c+a)\}$ are oppositely sorted, from Rearrangement We get

$$\sum \frac{(a+b)(b+c)}{c} \geq \sum \frac{(a+b)(b+c)}{b} = a+b+c+\frac{ca}{b}$$

Therefore it remains to show that

$$\sum \frac{ca}{b} \geq a+b+c$$

which follows from Rearrangement

$$\sum \frac{ca}{b} \geq \sum \frac{ca}{c} = a+b+c$$

366.
Let $a, b, c > 0$ such that $a^2+b^2+c^2 = 3$. Prove that

$$\sum \frac{a+b}{\sqrt{a+b-c}} \geq 6$$

P2: if $a, b, c > 0$ then

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3\sqrt[4]{\frac{a^4+b^4+c^4}{3}}$$

By Holder ,We have

$$(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a})^2(a^2b^2+b^2c^2+c^2a^2) \geq (a^2+b^2+c^2)^3$$

and

$$\frac{(a^2+b^2+c^2)^6}{(a^2b^2+b^2c^2+c^2a^2)^2} \ge 27(a^4+b^4+c^4)$$

Let $x^2=-a+b+c;.......$ then $a=\frac{y^2+z^2}{2};....$ By P2 We have done

367.

Prove if $a,b,c>0$ then

$$\frac{a^2}{b^3+c^3}+\frac{b^2}{c^3+a^3}+\frac{c^2}{a^3+b^3} \ge \frac{3\sqrt[2]{3}}{2\sqrt[2]{(a^2+b^2+c^2}}$$

Solution:

We assume $a^2+b^2+c^2=3$ then the inequality becomes

$$\frac{a^2}{b^3+c^3}+\frac{b^2}{c^3+a^3}+\frac{c^2}{a^3+b^3} \ge \frac{3}{2}$$

Note that for $a,b,c \ge 0$ and $a+b+c=3$ then

$$a^{\frac{3}{2}}b+b^{\frac{3}{2}}c+c^{\frac{3}{2}}a \le 3$$

By the Cauchy Schwarz We get

$$\frac{a^2}{b^3+c^3}+\frac{b^2}{c^3+a^3}+\frac{c^2}{a^3+b^3} \ge \frac{\left(a^2+b^2+c^2\right)^2}{\sum a^2b^3+\sum a^3b^2} \ge \frac{9}{6}=\frac{3}{2}$$

Let $a^2+b^2+c^2=3$. Then We need to prove that

$$\sum_{cyc}\frac{a^2}{b^3+c^3} \ge \frac{3}{2}.$$

But

$$\sum_{cyc}\frac{a^2}{b^3+c^3}=\sum_{cyc}\frac{a^4}{b^3a^2+c^3a^2} \ge \frac{9}{\sum(a^3b^2+a^3c^2)}.$$

id est, it remains to prove that

$$\sum_{cyc}(a^3b^2+a^3c^2) \le 6.$$

But

$$\sum_{cyc}(a^3b^2+a^3c^2) \le 6 \Leftrightarrow \sum_{cyc}a^3(3-a^2) \le 6 \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}(a^5-3a^3+2) \ge 0 \Leftrightarrow \sum_{cyc}(a^5-3a^3+2+2(a^2-1)) \ge 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}a^2(a+2)(a-1)^2 \ge 0.$$

368.

, Let $a,b,c>0$ such that $ab+bc+ca=1$. Find min:

$$M=\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-2(a^2+b^2+c^2)$$

Solution:

We can solve it by the lenma:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge \frac{(a^2+b^2+c^2)(a+b+c)}{ab+bc+ca}$$

369.
Let $a, b, c > 0$ $abc = 1$. Prove that

$$(a+b)(b+c)(c+a) + 7 \geq 5(a+b+c)$$

Solution:

$$\Leftrightarrow (a+b+c)(ab+bc+ca) + 6 \geq 5(a+b+c)$$

$$\Leftrightarrow ab+bc+ca+\frac{6}{a+b+c} \geq 5$$

setting

$$F(a,b,c) = ab+bc+ca+\frac{6}{a+b+c}$$

$$F(a,b,c) - F(a,\sqrt{bc},\sqrt{bc}) = (\sqrt{b}-\sqrt{c})^2(a - frac6(a+b+c)(a+2\sqrt{bc}))$$

assume

$$a = max(a,b,c) \Rightarrow F(a,b,c) \geq F(a,\sqrt{bc},\sqrt{bc})$$

thus, We need prove

$$F(1/t^2, t, t) \geq 5$$

$$(a+b)(b+c)(c+a) + 7 \geq 5(a+b+c)$$

$$\Longleftrightarrow (a+b+c)(5-ab-bc-ca) \leq 6.$$

oh, after an hour for it, We have an interesting **Solution** Very Happy with:

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$$

and continue with AM-GM
Similar to it, We have:

$$(a+b)(b+c)(c+a) + 3n - 8 \geq n(a+b+c)(n \geq 3)$$

370.
Let $a, b > 0$. Prove that

$$\frac{a}{\sqrt{a^2+3b^2}} + \frac{b}{\sqrt{b^2+3a^2}} \geq 1$$

Solution:
By Holder

$$\left(\frac{a}{\sqrt{a^2+3b^2}} + \frac{b}{\sqrt{b^2+3a^2}}\right)^2 (a(a^2+3b^2)+b(b^2+3a^2)) \geq (a+b)^3 =$$

$$= a(a^2+3b^2)+b(b^2+3a^2).$$

Q.E.D
371.
Let $a, b, c > 0$ such that $a^2+b^2+c^2 = 3$. Prove that

$$a^3(a+b) + b^3(b+c) + c^3(c+a) \geq 6$$

Solution:
We have

$$(a+b+c)(a^3+b^3+c^3) \geq (a^2+b^2+c^2)^2$$

and

$$3(a^3c+b^3a+c^3c)\le(a^2+b^2+c^2)^2$$

Q.E.D.
372.
Let $a,b,c>0$. Prove that

$$\sum\sqrt{\frac{2a}{b+c}}\le\sqrt{3(\sum\frac{a}{b})}$$

ill, you can easily prove by AM-GM that:

$$\sqrt{3(\sum\frac{a}{b})}\ge 3.$$

So it suffices to show that:

$$\sum\sqrt{\frac{2a}{b+c}}\le 3,$$

which is a known Vasc inequality.

373.
1)Find the best positive constant k such that the following inequality's right

$$\frac{ab}{ab+k(k-a)}+\frac{bc}{bc+k(k-b)}+\frac{ca}{ca+k(k-c)}\le 1$$

for all positive numbers a,b,c such that $a^2+b^2+c^2=1$. is $k=\frac{2\sqrt{3}}{3}$

2) Let a,b,c be positive number such that $a+b+c=1$. Prove that
3)

$$\frac{b(b+c-2a)}{3ab+2b+c}+\frac{c(c+a-2b)}{3bc+2c+a}+\frac{a(a+b-2c)}{3ca+2a+b}\ge 0$$

4)

$$\frac{ab}{3ab+2b+c}+\frac{bc}{3bc+2c+a}+\frac{ca}{3ca+2a+b}\le\frac{1}{4}$$

5) Let a,b,c be positive numbers such that $a+b+c=1$. Prove that

$$\frac{ab}{3ab+2b+c}+\frac{bc}{3bc+2c+a}+\frac{ca}{3ca+2a+b}\le\frac{1}{4}$$

6) Let x,y,z be positive numbers such that $x^2+y^2+z^2=6$ and A,B,C are three angles of an acute triangle. Prove that

$$\sum_{cyc}\frac{1}{1+yzcosA+xyz^2cosAcosB}\ge 1$$

7) Let a,b,c be positive number such that $\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1$. Prove that

$$\frac{2a-1}{1+2a+4ab}+\frac{2b-1}{1+2b+4bc}+\frac{2c-1}{1+2c+4ca}\le 0$$

8) Let a,b,c be positive numbers. Prove that

$$(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})\ge\frac{(a+2b)^2}{a^2+b(c+a)}+\frac{(b+2c)^2}{b^2+c(a+b)}+\frac{(c+2a)^2}{c^2+a(b+c)}$$

Solution:

$$LHS = \sum_{cyc}(1 + \frac{b}{c} + \frac{b}{a}) = \sum_{cyc} \frac{a^2}{a^2} + \frac{4b^2}{bc + ba} \geq \sum_{cyc} \frac{a + 2b^2}{a^2 + bc + ba} = RHS$$

374.
Let a, b, c be positive numbers. Prove that

$$\frac{a^3}{a^2 + b^2} + \frac{b^3}{b^2 + c^2} + \frac{c^3}{c^2 + a^2} \geq \frac{b + c}{2} + \frac{ca^2}{c^2 + a^2}$$

Solution:

$$\frac{a^3}{a^2 + b^2} + \frac{b^3}{b^2 + c^2} + \frac{c^3}{c^2 + a^2} \geq \frac{a + b + c}{2}$$

and an inequality very strong

$$\frac{a^3}{a^2 + b^2} + \frac{b^3}{b^2 + c^2} + \frac{c^3}{c^2 + a^2} \geq \frac{\sqrt{3(a^2 + b^2 + c^2)}}{2}$$

$$\sum_{cyc} \frac{a^3}{a^2 + b^2} = \sum_{cyc} a - \frac{ab^2}{a^2 + b^2} \geqslant \sum_{cyc} a - \frac{ab^2}{2ab} = \frac{1}{2} \sum_{cyc} a$$

it remains to prove

$$\frac{a + b + c}{2} \geqslant \frac{b + c}{2} + \frac{ca^2}{c^2 + a^2}$$

which is equivalent to $ac^2 + a^3 \geqslant 2ca^2$ which is true by AM-GM

375.
Let $x, y, z > -1$. Prove that

$$\frac{1 + x^2}{1 + y + z^2} + \frac{1 + y^2}{1 + z + x^2} + \frac{1 + z^2}{1 + x + y^2} \geq 2.$$

Solution:
As $x \leq \frac{1+x^2}{2}$ We have

$$\sum \frac{1 + x^2}{1 + y + z^2} \geq \sum \frac{2(1 + x^2)}{(1 + y^2) + 2(1 + z^2)}$$

Denoting $1 + x^2 = a$ and so on We have to prove that

$$\sum \frac{a}{b + 2c} \geq 1$$

but Cauchy tells us

$$\sum \frac{a}{b + 2c} \sum a(2b + c) \geq \left(\sum a\right)^2$$

and as

$$\left(\sum a\right)^2 \geq 3(ab + bc + ca) = \sum a(2b + c)$$

We have the result.

376.
Solve the equation:

$$\sqrt{13x^2 + 8x + 5} + \sqrt{29x^2 - 24x + 5} = 2\sqrt{12x^2 + 4x - 1}$$

squaring both sides of your equation and simplifying We get

$$2\sqrt{13x^2+8x+5}\sqrt{29x^2-24x+5} = 6x^2+32x-14$$

squaring again and factoring We have

$$16(23x^2+12x-6)(2x-1)^2 = 0$$

The only **Solution** which fulfills our equation is $x = \frac{1}{2}$

$$\sqrt{13x^2+8x+5}+\sqrt{29x^2-24x+5} = 2\sqrt{12x^2+4x-1}$$
$$\sqrt{(3x+2)^2+(2x-1)^2}+\sqrt{(5x-2)^2+(2x-1)^2} = \sqrt{(8x)^2-(4x-2)^2}$$

Note that:

$$\sqrt{(3x+2)^2+(2x-1)^2} \geq |3x+2|$$
$$\sqrt{(5x-2)^2+(2x-1)^2} \geq |5x-2|$$
$$\sqrt{(8x)^2-(4x-2)^2} \leq |8x|$$

Therefore:

$$\sqrt{(3x+2)^2+(2x-1)^2}+\sqrt{(5x-2)^2+(2x-1)^2} \geq |3x+2|+|5x-2| \geq |(3x+2)+(5x-2)| = |8x| \geq \sqrt{(8x)^2-(4x-2)^2}$$

with equality occuring when

$$(2x-1)^2 = (4x-2)^2 = 0 \Rightarrow \boxed{x = \frac{1}{2}}.$$

377,

Let ABC be an acute triangle. Prove that

$$\frac{cos^4\frac{A}{2}}{sin^2\frac{A}{2}}+\frac{cos^4\frac{B}{2}}{sin^2\frac{B}{2}}+\frac{cos^4\frac{C}{2}}{sin^2\frac{C}{2}} \geq 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2})$$

Solution:

Let $b+c-a = x, a+c-b = yanda+b-c = z$. Hence,

$$\sum_{cyc}\frac{\cos^4\frac{A}{2}}{\sin^2\frac{A}{2}} \geq 4\sum_{cyc}\cos^2\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\frac{\left(1+\frac{b^2+c^2-a^2}{2bc}\right)^2}{4\cdot\frac{1-\frac{b^2+c^2-a^2}{2bc}}{2}} \geq \sum_{cyc}\left(1+\frac{b^2+c^2-a^2}{2bc}\right)\sqrt{\left(1+\frac{a^2+c^2-b^2}{2ac}\right)\left(1+\frac{a^2+b^2-c^2}{2ab}\right)} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\frac{(b+c-a)^2}{bc(a+b-c)(a+c-b)} \geq \sum_{cyc}\frac{(b+c-a)\sqrt{(a+b-c)(a+c-b)}}{abc\sqrt{bc}} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\frac{(b+c-a)^2}{bc(a+b-c)(a+c-b)} \geq \sum_{cyc}\frac{(b+c-a)\sqrt{(a+b-c)(a+c-b)}}{abc\sqrt{bc}} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}\frac{(y+z)x^2}{2yz} \geq \sum_{cyc}\sqrt{\frac{4x^2yz}{(x+y)(x+z)}} \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}(x^3y+x^3z) \geq 2xyz\sum_{cyc}\sqrt{\frac{4x^2yz}{(x+y)(x+z)}}$$

which is true because

$$\sum_{cyc}(x^3y+x^3z)\ge 2xyz(x+y+z)$$

and

$$\sum_{cyc}\frac{1}{x+y+z}\sqrt{\frac{4x^2yz}{(x+y)(x+z)}}=\sum_{cyc}\frac{y+z}{2(x+y+z)}\sqrt{\frac{16x^2yz}{(x+y)(x+z)(y+z)^2}}\le$$

$$\le\sqrt{\sum_{cyc}\frac{8x^2yz}{(x+y+z)(x+y)(x+z)(y+z)}}=\sqrt{\frac{8xyz}{(x+y)(x+z)(y+z)}}\le 1$$

id est, your inequality is proven.

378.

Let a,b,c be positive number such that $a+b+c=abc$. Prove that

$$\frac{3\sqrt{3}}{4}\le\frac{bc}{a(1+bc)}+\frac{ca}{b(1+ca)}+\frac{ab}{c(1+ab)}\le\frac{a+b+c}{4}$$

Solution:

use the inequality:

$$\frac{4}{x+y}\le\frac{1}{x}+\frac{1}{y}$$

We obtain:

$$\frac{4bc}{2a+b+c}\le\frac{bc}{a+b}+\frac{bc}{a+c}$$
$$\frac{4ac}{2b+a+c}\le\frac{ca}{a+b}+\frac{ac}{b+c}$$
$$\frac{4ab}{2c+a+b}\le\frac{ab}{b+c}+\frac{ab}{a+c}$$

From Titu's lemma, We have:

$$\sum\frac{ab}{c+abc}\ge\frac{(\sum\sqrt{ab})^2}{a+b+c+3abc}=\frac{(\sum\sqrt{ab})^2}{4abc}\ge\frac{3\sqrt{3}abc}{4abc}=\frac{3\sqrt{3}}{4}$$

379.

, Let a,b,c be positive number. Prove that

$$\sum_{cyc}\frac{a^2}{\sqrt{a^4+24b^3c^3}}\ge\frac{3}{5}$$

Solution:

or equivalently:

$$P=\sum_{cyc}\frac{a}{\sqrt{a^2+24bc}}\ge\frac{3}{5}$$

Then Hölder gives us:

$$\left(\sum_{cyc}a^3+24abc\right)P^2\ge\left(\sum_{cyc}a\right)^3$$

so it suffices to prove

$$25\left(\sum_{cyc}a\right)^3\ge 9\left(\sum_{cyc}a^3+24abc\right)$$

which is obvious upon expanding

Another **Solution**:

$$\sum \frac{a^2}{\sqrt{a^4+24b^2c^2}} = \sum \sqrt{\frac{a^4}{a^4+24b^2c^2}} = \sum \frac{1}{\sqrt{1+\frac{24a^2b^2c^2}{a^6}}}$$

We consider the function

$$f(x) = \frac{1}{\sqrt{1+\frac{24a^2b^2c^2}{x^6}}}$$

$$f''(x) \geq 0$$

So

$$f(a)+f(b)+f(c) \geq 3f(\frac{a+b+c}{3})$$

So the

$$LHS \geq 3\sum \frac{1}{\sqrt{1+\frac{3^6*24a^2b^2c^2}{(a+b+c)^6}}}$$

So We have to prove that:

$$\sum \frac{1}{\sqrt{1+\frac{3^6*24a^2b^2c^2}{(a+b+c)^6}}} \geq \frac{1}{5} \Leftrightarrow (a+b+c)^6 \geq 3^6a^2b^2c^2$$

which is true .

380.

, Let a,b,c be positive number such that $a+b+c=1$. Find the minimum of

$$P = \frac{a}{1+b^2+c^2} + \frac{b}{1+c^2+a^2} + \frac{c}{1+a^2+b^2}$$

Solution:

bu Cauchy-Schwarz,

$$S = P\left(a+b+c+\sum ab(a+b)\right) \geq (a+b+c)^2 = 1$$

by Schur We can prove that,

$$\sum ab(a+b) \leq a^3+b^3+c^3+3abc \leq a^3+b^3+c^3+6abc$$

$$\Leftrightarrow 4\sum ab(a+b) \leq a^3+b^3+c^3+3\sum ab(a+b)+6abc = (a+b+c)^3 = 1$$

$$\Leftrightarrow \sum ab(a+b) \leq \frac{1}{4}$$

then

$$P\left(1+\frac{1}{4}\right) \geq S \geq 1$$

and finally

$$P_{min} = \frac{4}{5}$$

381.

, Let a,b,c be satisfying $\frac{1}{\sqrt{2}} < a,b,c < 1$ and

$$a^4+b^4+c^4+4a^2b^2c^2 = 2(a^2b^2+b^2c^2+c^2a^2)$$

Prove that
$$\sum_{cyc}\frac{a^4}{1-a^2}\geq 4abc(a+b+c)$$
Solution:
As $a^2b^2+b^2c^2+c^2a^2\geq abc(a+b+c)$ We can even prove the more stronger inequality still holds:
if $\frac{1}{2}<a,b,c<1$ such that
$$4abc=2(ab+bc+ca)-a^2-b^2-c^2.$$
Then
$$\frac{a^2}{1-a}+\frac{b^2}{1-b}+\frac{c^2}{1-c}\geq 4(ab+bc+ca).$$
This inequality follows from the following inequality
$$8(a^2+b^2+c^2)-(ab+bc+ca)\geq\frac{21abc(a+b+c)}{2(ab+bc+ca)-a^2-b^2-c^2}$$
if a,b,c are the sidelengths of a triangle.

382.
,Let $a,b,c\in\mathbb{R}_+$ and
$$\sum_{cyc}\frac{a^3+2a}{a^2+1}=\frac{9}{2}$$
Prove that
$$\sum_{cyc}\frac{1}{a}\geq 3$$
Solution:
We have
$$\frac{9}{2}=\sum a+\sum\frac{a}{a^2+1}soWehave\sum a\geq 3$$
but
$$\frac{9}{2}=\sum\frac{1}{a}+\sum\frac{a^4+a^2-1}{a^3+a}=\sum a+\sum f(a)$$
where
$$f(a)=\sum\frac{a^4+a^2-1}{a^3+a}$$
but
$$f''(a)=\frac{-2(6a^4+3a^2+1)}{a^3(a^2+1)^3}<0foreverya>0$$
so We have
$$\sum f(a)\leq 3f(\frac{a+b+c}{3})\leq 3f(1)=\frac{3}{2}$$
so We have
$$\sum\frac{1}{a}\geq 3$$
383.
Let a,b and c are non-negative numbers such that $ab+ac+bc=3$. Prove that:
$$1)\ a^2+b^2+c^2+3abc\geq 6$$

Solution:
Using Schur's inequality

$$4(ab+bc+ca)(a+b+c)-9abc \leq (a+b+c)^3$$

Then

$$(a+b+c)^2 \geq 12-\frac{9abc}{a+b+c} and$$

$$LHS=(a+b+c)^2-6+3abc \geq 6-\frac{9abc}{a+b+c}+3abc \geq RHS$$

because

$$(a+b+c)^2 \geq 3(ab+bc+ca)$$

$$2)\ a^4+b^4+c^4+15abc \geq 18$$

assume that: $a+b+c=p, ab+bc+ca=q, abc=r$ so $q=3$ and We have to prove that:

$$r \geq \frac{p^2(12-p^2)}{15+4p}$$

case1:$if p^2>12$ the ineq is true
case2: $if p^2 \leq 12$
remember this schur ineq

$$r \geq \frac{(p^2-3)(12-p^2)}{6p}$$

We will prove that

$$\frac{(p^2-3)(12-p^2)}{6p} \geq \frac{p^2(12-p^2)}{15+4p}$$

which is equivalent to $(p-3)(2p^2-9p-15) \leq 0$ (which is obvious true for all $\leq p \leq \sqrt{12}$).

384.
Let ABC be an acute triangle. Prove that:

$$\cos A+\cos B+\cos C \leq \sqrt{\sin^2 A+\sin^2 B+\sin^2 C}$$

Solution:
Let

$$\cos A=\sqrt{\frac{bc}{(b+a)(c+a)}}$$

So We have to prove that:

$$\sqrt{2\sum ab(a+b)+6abc} \geq \sum \sqrt{ab(a+b)}$$

which is equivalent to:

$$\sum \sqrt{\frac{a+b}{c}} \leq \sqrt{2\sum \frac{a+b}{c}+6}$$

$$cosA+cosB+cosC \leq \sqrt{3-(cos^2A+cos^2B+cos^2C)}$$

$$(cosA+cosB+cosC)^2+(cos^2A+cos^2B+cos^2C) \leq 3$$

By Cauchy,We have

$$(cosA+cosB+cosC)^2+(cos^2A+cos^2B+cos^2C) \leq \frac{4}{3}(cos^2A+cos^2B+cos^2C)$$

it remains to prove
$$cos^2A + cos^2B + cos^2C \le \frac{9}{4}$$
it's obvious.....

385.
Let $a, b, c \ge 0$ satisfy $a + b + c = 1$. Prove that
$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) \le \frac{1}{32}$$
Solution:
let
$$f(a, b, c) = (a^2 + b^2)(b^2 + c^2)(c^2 + a^2)$$
let$c = max(a, b, c)$; We have
$$f(a, b, c) \le f(a + b, 0, c)$$
(which is equivalent)
$$ab(-4abc^2 + a^3b + ab^3 - 4a^2c^2 - 4b^2c^2 - 2c^4) \le 0$$
We will prove that
$$f(a + b, 0, c) = f(1 - c, 0, c) \le \frac{1}{2}$$
which is equivalent to
$$\frac{1}{32} * (16c^4 - 32c^3 + 20c^2 - 4c - 1))(-1 + 2c)^2 \le 0$$
remember that
$$16c^4 - 32c^3 + 20c^2 - 4c - 1 = 4(2c^2 - 2c + \frac{1 - \sqrt{5}}{4})(2c^2 - 2c + \frac{1 + \sqrt{5}}{4}) \ge 0$$
for every $c \in [0, 1]$.

386.
Let $a, b, c > 0$ such that $4abc = a + b + c + 1$. Prove that
$$\frac{b^2 + c^2}{a} + \frac{c^2 + a^2}{b} + \frac{a^2 + b^2}{c} \ge 2(ab + bc + ca)$$
Solution:
By AM-GM's inequality,We have:
$$LHS \ge \frac{2bc}{a} + \frac{2ca}{b} + \frac{2ab}{c} = \frac{2}{abc}((ab)^2 + (bc)^2 + (ca)^2)$$
But
$$(ab)^2 + (bc)^2 + (ca)^2 \ge \frac{1}{3}(ab + bc + ca)^2$$
$$\Longrightarrow LHS \ge \frac{2}{3abc}(ab + bc + ca)^2$$
Thus, it is enough to prove
$$\frac{2}{3abc}(ab + bc + ca)^2 \ge 2(ab + bc + ca)$$
$$\Leftrightarrow ab + bc + ca \ge 3abc.$$

indeed, from the condition, by AM - GM, We obtain:

$$4abc = a + b + c + 1 \geq 4\sqrt[4]{abc} \Longrightarrow abc \geq 1$$

$$\Longrightarrow a + b + c \geq 4abc - 1 \geq 3abc$$

But

$$(ab + bc + ca)^2 \geq 3abc(a + b + c) \geq 9a^2b^2c^2 \Longrightarrow ab + bc + ca \geq 3abc.$$

The result is lead as follow.

387.
Let a, b, c be positive real numbers,prove that

$$\sum \left(\frac{a+b}{c}\right) \geq 4\sum \left(\frac{a}{b+c}\right) + 4\sum \left(\frac{a(b-c)^2}{(b+c)^3}\right)$$

Solution:

$$\sum \left(\frac{a+b}{c}\right) \geq 4\sum \left(\frac{a}{b+c}\right) + \frac{(x+y)(y+z)(z+x) - 8xyz}{(x+y)(y+z)(z+x)}$$

$$<=> \sum a(b-c)^2 (\frac{1}{bc(b+c} - \frac{4}{(b+c)}) \geq 0$$

Q.E.D
388.
Let be $x, y, z \in \mathbb{R}_+$. Show that :

$$\left(x^2 + \frac{3}{4}\right)\left(y^2 + \frac{3}{4}\right)\left(z^2 + \frac{3}{4}\right) \geq \sqrt{(x+y)(y+z)(z+x)}$$

Solution:
Because

$$\left(x^2 + \frac{3}{4}\right)\left(y^2 + \frac{3}{4}\right) \geq x + y$$

We have

$$x^2 + \frac{3}{4} = \left(x^2 + \frac{1}{4}\right) + \frac{1}{2} \geq 2\sqrt{\frac{1}{2}\left(x^2 + \frac{1}{4}\right)}$$

Similarly We obtains

$$\left(x^2 + \frac{3}{4}\right)\left(y^2 + \frac{3}{4}\right)\left(z^2 + \frac{3}{4}\right) \geq 8\sqrt{\frac{1}{8}\left(x^2 + \frac{1}{4}\right)\left(y^2 + \frac{1}{4}\right)\left(z^2 + \frac{1}{4}\right)}$$

So ineq at first equivalent to:

$$\prod \left(4x^2 + 1\right) \geq 8\prod (x + y)$$

Then apply Cauchy-Shwar We have

$$\left(4x^2 + 1\right)\left(1 + 4y^2\right) \geq 4\left(x + y\right)^2$$

similarly and multiply We have finished.
389.

Let x, y, z be the non-negative real number satisfying $(x+y+z)^2+xy+yz+zx=2$. Prove that

$$\frac{x+y}{\sqrt{z^2+xy+1}}+\frac{y+z}{\sqrt{x^2+yz+1}}+\frac{z+x}{\sqrt{y^2+zx+1}} \geq \frac{3\sqrt{2}}{2}$$

Solution:
We can let $a=x+y, b=y+z, c=z+x$ then We have $ab+bc+ca=2$ and We have to prove that:

$$\sum_{cyc} \frac{a}{\sqrt{a^2+3bc}} \geq \frac{3}{2}(1)$$

Just use Holder, let the LHS be S, then by Holder

$$S^2(\sum a(a^2+3bc)) \geq (a+b+c)^3$$

So We have to show

$$4(a+b+c)^3 \geq 9(a^3+b^3+c^3+9abc)$$

which is obvious by Muirhead. We will must prove that:

$$2(ab(a+b)+bc(b+c)+ca(c+a)) \geq a^3+b^3+c^3+9abc$$

$$<=> (3c-a-b)(a-b)^2+(3a-b-c)(b-c)^2+(3b-a-c)(c-a)^2 \geq 0(*)$$

Suppose $a \geq b \geq \geq c$ We have:

$$3b-a-c \geq 3b-c-(b+c)=2(b-c) \geq 0,,(c-a)^2 \geq (a-b)^2+(b-c)^2$$

then

$$LHS \geq (3c-a-b+3b-a-c)(a-b)^2+(3b-a-c+3a-b-c)(b-c)^2 \geq 0$$

$$\frac{x+y+z}{xyz}(\sum \frac{x^2y^2}{(x+y)^2}) \geq \frac{9}{4}$$

but:

$$LHS \geq \frac{x+y+z}{xyz} * \frac{(\sum xy)^2}{\sum(x+y)^2}$$

remember that:

$$\sum(x+y)^2 \leq \frac{4}{3}(x+y+z)^2$$

$$LHS \geq \frac{x+y+z}{xyz} * \frac{3}{4} * \frac{(\sum xy)^2}{(\sum x)^2} = \frac{3}{4} * \frac{(\sum xy)^2}{xyz(x+y+z)} \geq \frac{9}{4}$$

390.
1) For any triangle with sides a, b, c. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a) \geq 0$$

Solution:
Let $a=x+y, b=y+z, c=z+x$; after expanding,We need to prove that:

$$\sum_{cyc} x^3y \geq (x+y+z)xyz$$

2) For all positive real numbers a, b and c

$$\sqrt{\frac{2a(b+c)}{(a+b)(a+c)}}+\sqrt{\frac{2b(c+a)}{(b+c)(b+a)}}+\sqrt{\frac{2c(a+b)}{(c+a)(c+b)}} \leq \frac{a+b+c}{\sqrt[3]{abc}}$$

$$LHS \leq 3 \leq RHS$$

391.
Given $a, b, c \geq\ 0$ satisfy $a+b+c=6$. Prove that:

$$(11+a^2)(11+b^2)(11+c^2)+120abc \geq\ 4320$$

Equality occurs when $(a,b,c)=(1,2,3)$.
Solution:

$$LHS-RHS=11(ab+bc+ca-11)^2+(abc-6)^2+121(a+b+c)^2-36 \geq 4356-36=4320$$

Equality occurs $iffa+b+c=6, ab+bc+ca=11$ and $abc=6ora=1, b=2, c=3$ or any symmetric permutation.

$$\frac{a^3+b^3+c^3}{abc}+9\frac{ab+bc+ac}{a^2+b^2+c^2} \geq 12$$

$$LHS-RHS=(\sum a^2-\sum ab)(\sum \frac{1}{ab}-\frac{9}{\sum ab}) \geq 0$$

it is instantly solved by SOS

$$S_a=S_b=S_c=\frac{(a+b+c)(a^2+b^2+c^2)-9abc}{2abc(a^2+b^2+c^2)} \geq 0$$

392., let a, b, c be positive numbers such that: $abc=a+b+c+2$. Prove that

$$abc-2(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq 5$$

Solution:
We need to prove that:

$$abc-\frac{2\sum ab}{abc} \geq 5$$

$$(\sum a+2)^2-2(\sum ab) \geq 5(\sum a+2)$$

$$\sum a^2 \geq \sum a+6$$

remember that:

$$abc=a+b+c+2then \sum a \geq 6$$

$$\sum (a^2+4) \geq \sum 4a$$

$$3\sum a \geq 18$$

Q.E.D
393. Let $x, y, z>0$ and $\sum x^2+2xyz=1$. Prove that:

$$3xyz \leq 2\sum x^2y^2$$

Solution:
Let $x=\sqrt{\frac{ab}{(a+c)(b+c)}}, y=\sqrt{\frac{ca}{(c+b)(a+b)}}, z=\sqrt{\frac{ab}{(a+c)(b+c)}}$ This ineq becomes:

$$\sum\ \frac{a}{b+c} \geq\ \frac{3}{2}$$

394.
, 1)Prove that in any triangle ABC exists the relation

$$\sqrt{\frac{b+c-a}{a}}+2\cdot\sum\frac{a}{b+c}\geq 6$$

Solution:
the inequality

$$<=>\sum\sqrt{\frac{b+c-a}{a}}\geq 2\sum\frac{b+c-a}{b+c}$$

Remember that:

$$(b+c)\geq 2\sqrt{b+c-a}*\sqrt{a}$$

so We get:

$$\sqrt{\frac{b+c-a}{a}}\geq 2\frac{b+c-a}{b+c}$$

2) Let a,b,c be the length of the sides of triangle ABC. S is the area of ABC and $0<a<b<c$. Prove that:

$$\sum(\frac{ab(a-b)}{(b-c)(c-a)})^2\geq 4S$$

Solution:
use Cauchy, and this inequality:

$$abc\geq\frac{4}{3}*\sqrt{a^2+b^2+c^2}*S$$

And We will prove this inequality:

$$\sqrt[3]{\frac{27abc}{(a-b)^2(b-c)^2(c-a)^2}}*\frac{4}{3}*\sqrt{a^2+b^2+c^2}\geq 1$$

395.
Let a,b,c be the lengths of the sides of triangle ABC and R is the circumradius ang r is the inradius of triangle ABC. Prove this inq

$$\frac{(a+b+c)^2}{2(ab+bc+ca)-(a^2+b^2+c^2)}+\frac{2r}{R}\geq 4$$

t is really a complicated **Solution** and of course it is not necessary for this one Razz

$$\Leftrightarrow\frac{(a+b+c)^2}{2(ab+bc+ca)-(a^2+b^2+c^2)}+\frac{(a+b-c)(a+c-b)(b+c-a)}{abc}\geq 4$$

Let $a+b-c=x, b+c-a=y, c+a-b=z$ the inequality is equivalent to:

$$\frac{(x+y+z)^2}{xy+yz+zx}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 4$$

$$\Longleftrightarrow\frac{x^2+y^2+z^2}{xy+yz+zx}+\frac{8xyz}{(x+y)(y+z)(z+x)}\geq 2$$

396.
, Let be $a,b,c>0$. Prove that :

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\frac{2ab+2bc+2ca}{a^2+b^2+c^2}\geq 11$$

Solution:

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq \frac{(a+b+c)^2}{ab+bc+ca} = 2+\frac{a^2+b^2+c^2}{ab+bc+ca}$$

$$LHS \geq 6+\frac{a^2+b^2+c^2}{ab+bc+ca}+2(\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{ab+bc+ca}{a^2+b^2+c^2}) \geq 11$$

From Cauchy

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{(a+b+c)^2}{ab+bc+ca}.$$

Let $x = a^2+b^2+c^2, y = ab+bc+ca$. Then it remains to show that

$$\frac{3(x+2y)}{y}+\frac{2y}{x} \geq 11$$

which in turn is equivalent to

$$(x-y)(3x-2y) \geq 0$$

which is obviously true since $x \geq y$.

For your ineq,We can prove easily. For example:

$$2(\frac{a}{b}+\frac{b}{c}+\frac{c}{a})+\frac{2(ab+bc+ca)}{a^2+b^2+c^2} \geq \frac{2(a^2+b^2+c^2)}{ab+bc+ca}+\frac{2(ab+bc+ca)}{a^2+b^2+c^2}+4 \geq 6$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{8(ab+bc+ca)}{3(a^2+b^2+c^2)} \geq \frac{17}{3}$$

397.

Let $a, b, c > 0$. Prove that

$$\sum \frac{\sqrt{a^2+bc}}{b+c} \geq \frac{3\sqrt{2}}{2}$$

Solution:

With this one, use Holder ineq, We need to prove:

$$(\sum a^2+\sum ab)^3 \geq \frac{9}{2}\sum (a^2+bc)^2(b+c)^2(1)$$

And pqr works here, of course, not so nice (notice that the equality of (1) occurs when $a=b=c$ or $a=b; c=0$.

398.

, Let $a, b, c \in [0,1]$. Prove that

$$C = \sum \frac{a}{1+bc}+abc \leq \frac{5}{2}$$

Solution:

Assume $a \geq b \geq c$

$$=> C \leq \frac{a}{1+bc}+\frac{b+c}{1+bc}+abc \leq \frac{a}{1+bc}+abc+1 = V$$

$$V = \frac{a(bc^2+bc+1)}{1+bc}+1 \leq 5/2$$

(Because $a, b, c \in [0,1]$.

399.
, Let $a, b, c \in R$ such that $abc = 1$. Find max :

$$P = \sum \frac{1}{a+b+4}$$

Solution:
We have

$$\sum \frac{1}{a+b+1} \leq 1$$

with $a, b, c > 0$ such that $abc = 1$ and

$$\frac{4}{a+b+4} \leq \frac{1}{a+b+1} + \frac{1}{3}$$

400.
Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 \geq 12$. Find the minimum of:

$$S = \frac{x^6}{xy + 2\sqrt{1+z^3}} + \frac{y^6}{yz + 2\sqrt{1+x^3}} + \frac{z^6}{zx + 2\sqrt{1+y^3}}$$

Solution:
We have

$$2\sqrt{1+x^3} \leq x^2 + 2$$

then, use CS

$$=> min = 96/5$$

401.
Let $a, b, c \geq 0$. Prove that

$$\prod (a^4 + 7a^2 + 10) \geq 216(a+b+c)^3$$

Solution:
We have

$$(a^2+2)(b^2+2)(c^2+2) \geq 3(a+b+c)^2$$

and our ineq:

$$(a^2+5)(b^2+5)(c^2+5) \geq 72(a+b+c)$$

C-S lemma

$$a^4 + 7a^2 + 10 \geq 6(a^3+2)$$

$$(a^3+2)(b^3+2)(c^3+2) \geq (a+b+c)^3$$

it's very easy C-S.

342.
Let a, b, c be positive numbers such that $a + b + c = 3$. Prove that

$$abc + \frac{12}{ab+bc+ac} \geq 5$$

$$abc \geq (b+c-a)(c+a-b)(a+b-c)$$

and then We have

$$abc \geq \frac{4}{3}\sum bc - 3$$

$$abc+\frac{12}{ab+bc+ca}\geq 4(\frac{ab+bc+ca}{3}+\frac{3}{ab+bc+ca})-3\geq 4.2-3=5$$

402.

if a,b,c are non-negative numbers, prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\geq (ab+bc+ca)^3.$$

Because We have

$$a^2+ab+b^2\geq \frac{3\left(a+b\right)^2}{4}$$

so

$$\left(a^2+ab+b^2\right)\left(b^2+bc+c^2\right)\left(c^2+ca+a^2\right)\geq \frac{27\left(a+b\right)^2\left(b+c\right)^2\left(c+a\right)^2}{64}$$

$$\geq \frac{1}{3}\left(a+b+c\right)^2\left(ab+bc+ca\right)^2\geq (ab+bc+ca)$$

403.

Let $a,b,c\geq 0$ and $a+b+c=3$. Prove that

$$4\geq a^2b+b^2c+c^2a+abc$$

Solution:

WLOG $a+b+c=3$

$$if\{p,q,r\}=\{a,b,c\},p\geq q\geq r$$

then as

$$pq\geq pr\geq qr,$$

$$a^2b+b^2c+c^2a+abc=a(ab)+b(bc)+c(ca)+b(ac)\leq p(pq)+2q(pr)+r(qr)=\frac{1}{2}(2q)(p+r)(p+r)$$

$$=\frac{1}{2}(\frac{(2q)+(p+r)+(p+r)}{3})^3=\frac{2^3}{2}=4.$$

404.

Prove that for all positive real numbres a,b,c

$$(a^2+2)(b^2+2)(c^2+2)\geq 9(ab+ac+bc)$$

Solution:

WLOG $(a-1)(b-1)\geq 0$ We have

$$(a^2+1+1)(1+b^2+1)\geq (1+1+1)(a^2+b^2+1)(byTchebychef'sinequality.)$$

and

$$(a^2+b^2+1)(1+1+c^2)\geq (a+b+c)^2\geq 3(ab+bc+ca)$$

405.

, Let $a,b,c>0$ and $abc\leq 1$. Find minimum of

$$A=\frac{bc}{a^2b+a^2c}+\frac{ca}{b^2a+b^2c}+\frac{ab}{c^2a+c^2b}$$

$$[\frac{bc}{a^2(b+c)}+\frac{ac}{b^2(a+c)}+\frac{ab}{c^2(a+b)}]\cdot[\frac{b+c}{bc}+\frac{a+b}{ab}+\frac{a+c}{ac}]\geq (\frac{1}{a}+\frac{1}{b}+\frac{1}{c})^2$$

406.

Let $x,y,z>0$ and $xyz=1$. Prove that

$$\frac{x^2}{1+y}+\frac{y^2}{1+z}+\frac{z^2}{1+x}\geq \frac{3}{2}$$

Solution:
by CS We have
$$\frac{x^2}{1+y}+\frac{y^2}{1+z}+\frac{z^2}{1+x}\geq\frac{(x+y+z)^2}{3+x+y+z}\geq\frac{3}{2}$$
since for $X=x+y+z$ We have $2X^2-3X-9\geq 0 for X\geq 3$.

We can use Cauchy- Schwartz to solve this problem: We have
$$\frac{x^2}{1+y}+\frac{1+y}{4}\geq x$$

407.
$\forall a,b,c,d>0$ and $a,b,c,d\in\mathbb{R}$ and $a^2+b^2+c^2+d^2=1$. Prove that
$$(1-a)(1-b)(1-c)(1-d)\geq abcd$$

Note that
$$(1-a)(1-b)\geq cd$$
Since $\frac{c^2+d^2}{2}\geq cd$, it suffices to prove that
$$(1-a)(1-b)\geq\frac{c^2+d^2}{2}=\frac{1-(a^2+b^2)}{2}$$
$$\iff 1-(a+b)+ab\geq\frac{1}{2}-\frac{a^2+b^2}{2}$$
$$\iff 2-2(a+b)+2ab\geq 1-(a^2+b^2)$$
$$\iff 1-2(a+b)+2ab+(a^2+b^2)\geq 0$$
$$\iff 1-2(a+b)+(a+b)^2\geq 0$$
$$\iff [(a+b)-1]^2\geq 0$$
Similarly, $(1-c)(1-d)\geq ab$.

408.
Let a,b,c be positive real number. Prove that
$$\sqrt[3]{\frac{a^3}{a^3+(b+c)^3}}++\sqrt[3]{\frac{b^3}{b^3+(b+c)^3}}+\sqrt[3]{\frac{c^3}{c^3+(b+c)^3}}\geq 1$$

Solution:
$$\sqrt[3]{\frac{a^3}{a^3+(b+c)^3}}\geq\frac{a^2}{a^2+b^2+c^2}$$
We could use the same tehnic as in here. So , from Holder We have :
$$\left(\sum\frac{a}{\sqrt[3]{a^3+(b+c)^3}}\right)^3\left(\sum a(a^3+(b+c)^3)\right)\geq(a+b+c)^4$$
it is enough to prove that
$$(a+b+c)^4\geq\sum a(a^3+(b+c)^3)=(a+b+c)(\sum a^3+6abc)$$

409.
Let a, b, c and x, y, z be non-negative numbers such that $a+b+c=x+y+z$. Prove that

$$ax(a+x)+by(b+y)+cz(c+z) \geq 3(abc+xyz).$$

Solution:
Apply CBS

$$(a^2x+b^2y+c^2z)(yz+zx+xy) \geq xyz(a+b+c)^2$$

But

$$(a+b+c)^2=(x+y+z)^2 \geq 3(xy+yz+zx)$$

therefore

$$a^2x+b^2y+c^2z \geq 3xyz$$

Similarly

$$ax^2+by^2+cz^2 \geq 3abc$$

Adding up these inequalities yields the desired rezult.

450.
Let a, b, c be positive real number. Prove that:

$$\frac{a}{\sqrt[3]{a^3+26abc}}+\frac{b}{\sqrt[3]{b^3+26abc}}+\frac{c}{\sqrt[3]{c^3+26abc}} \geq 1$$

Solution:
By Holder We have

$$\left(\sum \frac{a}{\sqrt[3]{a^3+26abc}}\right)^3\left(\sum a\left(a^3+26abc\right)\right) \geq \left(\sum a\right)^4$$

So it is enough to show that

$$\left(\sum a\right)^4 \geq \sum a\left(a^3+26abc\right)$$

$$\Leftrightarrow 4\left(\sum ab\left(a^2+b^2\right)\right)+6\left(\sum a^2b^2\right) \geq 14abc\left(a+b+c\right)$$

which is true.

451
, Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{\sqrt{a(a+bc)}}{b+ca}+\frac{\sqrt{b(b+ca)}}{c+ab}+\frac{\sqrt{c(c+ab)}}{a+bc} \leq \frac{1}{2\sqrt{abc}}$$

Solution:

$$LHS \leq \sum \frac{\sqrt{a(a+bc)}}{2\sqrt{abc}}$$

but

$$\sum \sqrt{a(a+bc)} \leq \sqrt{(a+b+c)(a+b+c+ab+bc+ca)} \leq 2(B-C-s)$$

so We get

$$\frac{\sqrt{a(a+bc)}}{2\sqrt{abc}} \leq \frac{1}{\sqrt{abc}}$$

by AM-GM We have :

$$\sum \frac{\sqrt{a(a+bc)}}{b+ca} \leq \frac{1}{2} \sum \frac{a+a+bc}{b+ca}$$

and We have :

$$\sum \frac{a+a+bc}{b+ca} = \sum \frac{2a+bc}{b(a+b+c)+ca} = \sum \frac{2a+bc}{(a+b)(b+c)}$$

now the inequality become :

$$\sum (2a+bc)(a+c) \leq \frac{(a+b)(a+c)(b+c)}{2\sqrt{abc}}$$

452.

Prove that for all non-negative real numbers a, b, c We have

$$\frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2} + \frac{2(ab+bc+ca)}{a^2+b^2+c^2} \geq 3.$$

Solution:

Standard Vornicu-Schur will do: using

$$x^2+y^2+z^2-xy-yz-zx = \sum_{cycl} (x-y)(x-z)$$

the inequality is equivalent to

$$x(a-b)(a-c)+y(b-c)(b-a)+z(c-a)(c-b) \geq 0$$

where

$$x = \frac{(a+b)(a+c)}{a^2b^2+b^2c^2+c^2a^2} - \frac{2}{a^2+b^2+c^2}$$

(and similarly for y and z). Note that

$$(a^2+ab+bc+ca)(a^2+b^2+c^2) \geq (ab+bc+ca)(a^2+b^2+c^2)$$

$$\geq a^3(b+c)+a(b^3+c^3)+bc(b^2+c^2) \geq 2(a^2b^2+b^2c^2+c^2a^2)$$

by the AM-GM inequality so that $x, y, z \geq 0$. And clearly, $x \geq y$ if We assume $a \geq b \geq c$.

Suppose that $c = min\{a; b; c\}$ We can rewrite this inequality :

$$(a-b)^2[\frac{(a+b)^2}{a^2b^2+b^2c^2+c^2a^2} - \frac{2}{a^2+b^2+c^2}] + (a-c)(b-c)[\frac{(a+c)(b+c)}{a^2b^2+b^2c^2+c^2a^2} - \frac{2}{a^2+b^2+c^2}] \geq 0$$

From $c = min\{a; b; c\}$,We have

$$\frac{(a+b)^2}{a^2b^2+b^2c^2+c^2a^2} - \frac{2}{a^2+b^2+c^2} \geq 0$$

and

$$\frac{(a+c)(b+c)}{a^2b^2+b^2c^2+c^2a^2} - \frac{2}{a^2+b^2+c^2} \geq 0$$

Here is another **Solution** with AM-GM

We have that

$$\frac{a^4+b^4+c^4}{a^2b^2+b^2c^2+c^2a^2} + 2\frac{ab+bc+ca}{a^2+b^2+c^2} \geq 3\sqrt[3]{\frac{(a^4+b^4+c^4)(ab+bc+ca)^2}{(a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)^2}}$$

So We have to show that

$$(a^4+b^4+c^4)(ab+bc+ca)^2 \geq (a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)^2$$

which is equivalent to

$$\sum(abc^2(a-b)(a^3-b^3))+abc(a^5+b^5+c^5-abc(a^2+b^2+c^2)) \geq 0$$

453,
Let a,b,c be real numbers. Prove that:

$$a^2+b^2+c^2-ab-bc-ca \geq 3(a-b)(a-c)$$

the inequality becomes

$$\sum(a^2-ab-ac+bc) \geq 3(a-b)(b-c)$$

$$\sum(a-b)(a-c) \geq 3(a-b)(b-c)$$

$$(a-b)(a-c)+(c-a)(c-b) \geq 4(a-b)(b-c)$$

$$(a-c)(a-b-c+b) \geq 4(a-b)(b-c)$$

$$(a-b+b-c)^2 \geq 4(a-b)(b-c)$$

it's quivalent to $(x+y)^2 \geq 4xy$ which is true for all reals number x,y.

454.
★ Let a,b,c,d are real number such that $ad-bc=\sqrt{3}$. Prove that:

$$a^2+b^2+c^2+d^2+ac+bd \geq 3$$

Proff:
We will prove that:

$$a^2+b^2+c^2+d^2+ac+bd \geq (ad-bc)\sqrt{3}$$

$$<=> (a+\frac{c}{2}-\frac{d\sqrt{3}}{2})^2+(b+\frac{d}{2}+\frac{c\sqrt{3}}{2})^2 \geq 0$$

455, For any three positive reals a,b,c. Prove the inequality

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b} \geq a+b+c$$

Solution:
it's equivalent to;

$$\frac{a^2+ab+ac+bc}{b+c}+\frac{b^2+ab+ac+bc}{a+c}+\frac{c^2+ab+ac+bc}{a+b} \geq 2(a+b+c)$$

$$\Leftrightarrow \frac{(a+b)(a+c)}{(b+c)}+\frac{(a+b)(b+c)}{(a+c)}+\frac{(a+c)(b+c)}{(a+b)} \geq (a+b)+(a+c)+(b+c)$$

Let $a+b=x, a+c=y, b+c=z$ We have to show that

$$\frac{xy}{z}+\frac{yz}{x}+\frac{xz}{y} \geq x+y+z$$

equivalent to;

$$(xy)^2+(yz)^2+(xz)^2 \geq xyz(x+y+z)$$

With true by Am-Gm

$$\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}-(a+b+c)=\frac{(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2}{2(b+c)(c+a)(a+b)}.$$

456.
Let $a+b+c=1, a,b,c \geq 0$. Prove that

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{a+c}}+\frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3}{2}}$$

Solution:
Another way:

$$\sum \frac{a}{\sqrt{b+c}}=\sum \frac{a^2}{(a\sqrt{b+c}} \geq \frac{(a+b+c)^2}{(a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{b+a}}=\frac{1}{(a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{b+a}}$$

So now We try to solve that:

$$\frac{1}{a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{b+a}} \geq \sqrt{\frac{3}{2}}$$

Equivalent to

$$(a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{b+a} \leq \sqrt{\frac{2}{3}}$$

By Cauchy Schwarz We have:

$$(a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{b+a}=\sqrt{a}\sqrt{ab+ac}+\sqrt{b}\sqrt{bc+ab}+\sqrt{c}\sqrt{ac+bc} \leq$$

$$\sqrt{(}2((a+b+c)((ab+bc+ca)=\sqrt{2(ab+bc+ca)}=\sqrt{2(ab+bc+ca)}$$

Since

$$(ab+bc+ca \leq \frac{(a+b+c)^2}{3}=\frac{1}{3}$$

So We have:

$$(a\sqrt{b+c}+b\sqrt{a+c}+c\sqrt{b+a} \leq \sqrt{\frac{2}{3}}$$

$$\sum \frac{a}{\sqrt{b+c}} \geq \frac{1}{3}\sum a \sum \frac{1}{\sqrt{b+c}} \Rightarrow \frac{1}{3}\frac{3\sqrt{3}}{\sqrt{2(a+b+c)}}=\sqrt{\frac{3}{2}}$$

457.
Assume that a,b,c are positive reals satisfying $a+b+c \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$. Prove that

$$\frac{a^3c}{b(a+c)}+\frac{b^3a}{c(a+b)}+\frac{c^3b}{a(b+c)} \geq \frac{3}{2}$$

Solution:
first use holder or the general of cauchy We have:

$$\left(\frac{a^3c}{b(a+c)}+\frac{b^3a}{c(a+b)}+\frac{c^3b}{a(b+c)}\right)(2a+2b+2c)(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}) \geq (a+b+c)^3$$

so:

$$\frac{a^3c}{b(a+c)}+\frac{b^3a}{c(a+b)}+\frac{c^3b}{a(b+c)} \geq \frac{a+b+c}{2}$$

but We also have:

$$a+b+c \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$$

458.

Let a, b, c be positive real numbers. Prove the inequality

$$\frac{1}{a\left(b+1\right)}+\frac{1}{b\left(c+1\right)}+\frac{1}{c\left(a+1\right)} \geq \frac{3}{1+abc}$$

Solution:

$$\begin{aligned}
&(1+abc)\left(\frac{1}{a\left(1+b\right)}+\frac{1}{b\left(1+c\right)}+\frac{1}{c\left(1+a\right)}\right)+3 \\
&=\frac{1+abc+a+ab}{a+ab}+\frac{1+abc+b+bc}{b+bc}+\frac{1+abc+c+ca}{c+ca} \\
&=\frac{1+a}{ab+a}+\frac{b+1}{bc+b}+\frac{c+1}{ca+c}+\frac{b(c+1)}{b+1}+\frac{c(a+1)}{c+1}+\frac{a(b+1)}{a+1} \\
&\geq \frac{3}{\sqrt[3]{abc}}+3\sqrt[3]{abc} \geq 6.
\end{aligned}$$

The inequality is equivalent to

$$\sum_{cyc}\frac{abc+1}{a(1+b)} \geq 3$$

$$\sum_{cyc}\left(\frac{1+a}{a(1+b)}+\frac{abc+ab}{a(1+b)}\right) \geq 6$$

This ineq is right .($usingAM_GM3times$).

We have

$$\frac{1+abc}{a+ab}=\frac{1+a+ab+abc}{a+ab}-1=\frac{1+a}{a(1+b)}+\frac{b(1+c)}{1+b}-1.$$

Hence, rewrite the inequality in the form:

$$\frac{1+a}{a(1+b)}+\frac{b(1+c)}{1+b}+\frac{1+b}{b(1+c)}+\frac{c(1+a)}{1+c}+\frac{1+c}{c(1+a)}+\frac{a(1+b)}{1+a} \geq 6$$

$$ab(b+1)(ca-1)^2+bc(c+1)(ab-1)^2+ca(a+1)(bc-1)^2 \geq 0 which is true$$

$$\frac{1}{a(b+1)}+\frac{1}{b(c+1)}+\frac{1}{c(a+1)} \geq \frac{3}{1+abc} \Leftrightarrow \sum_{cyc}(bc^2+bc)(ab-1)^2 \geq 0.$$

$$\frac{1+abc}{a(b+1)}+1=\frac{1+a+ab+abc}{a(b+1)}=\frac{1+a}{a(1+b)}+\frac{b(1+c)}{1+b}$$

So

$$\sum \frac{1+a}{a(1+b)}+\frac{b(1+c)}{1+b} \geq 6$$

true by Am - Gm

459

$$\frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy}+\frac{x+y}{2}$$

Solution:

.

$$\left(\sqrt{\frac{x^2+y^2}{2}}-\sqrt{xy}\right)^2 \geq 0 \Rightarrow \frac{x^2+y^2}{2}+xy \geq 2\sqrt{xy\frac{x^2+y^2}{2}} \Rightarrow$$

$$(x+y)^2 \geq \frac{x^2+y^2}{2}+xy+2\sqrt{xy\frac{x^2+y^2}{2}} = \left(\sqrt{\frac{x^2+y^2}{2}}+\sqrt{xy}\right)^2 \Rightarrow$$

$$x+y \geq \sqrt{\frac{x^2+y^2}{2}}+\sqrt{xy} \Rightarrow \frac{(x-y)^2}{x+y} \leq \frac{(x-y)^2}{\sqrt{\frac{x^2+y^2}{2}}+\sqrt{xy}} \Rightarrow$$

$$\frac{x+y}{2}+\sqrt{xy} \leq \frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}}$$

$$\frac{2xy}{x+y}+\sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy}+\frac{x+y}{2} \Leftrightarrow \frac{(x-y)^2}{2(\sqrt{\frac{x^2+y^2}{2}}+\sqrt{xy})}-\frac{(x-y)^2}{2(x+y)} \geq 0 \Leftrightarrow$$

$$\Leftrightarrow x+y-\sqrt{\frac{x^2+y^2}{2}}-\sqrt{xy} \geq 0 \Leftrightarrow (\sqrt{x}-\sqrt{y})^2-(\sqrt{\frac{x^2+y^2}{2}}-\sqrt{xy}) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (\sqrt{x}-\sqrt{y})^2-\frac{(\sqrt{x}-\sqrt{y})^2(\sqrt{x}+\sqrt{y})^2}{2(\sqrt{\frac{x^2+y^2}{2}}+\sqrt{xy})} \geq 0 \Leftrightarrow \sqrt{2(x^2+y^2)}-x-y \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (x-y)^2 \geq 0$$

460, find the minimal of expression P

$$P(a,b) = \frac{(a+\sqrt{a^2+b^2})^3}{ab^2}$$

$$a^2+b^2 = a^2+8.(b^2/8) \geq 9(a^2.b^{16}/8^8)^{1/9}$$

and

$$a+3(a.b^8/8^4)^{1/9} \geq 4(a.b^2/8)^{1/3}$$

461, For all nonnegative real numbers a, b and c, no two of which are zero,

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2} \geq \frac{(a+b+c)^4}{4(a^2+b^2+c^2)(ab+bc+ca)^2}$$

This inequality follows from

$$(a^2+b^2+c^2)\left(\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\right) \geq \left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2$$

and

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

462.

, , For all nonnegative real numbers a,b and c. prove:

$$\sum_{cyc} a^2 \sum_{cyc} a(b+c) \sum_{cyc} a(b+c) \sum_{cyc} \frac{1}{(b+c)^2} \geq (a+b+c)^4$$

Solution:

But what about the following:

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2} \geq \frac{3(a+b+c)^2}{8(ab+bc+ca)}\left(\frac{1}{ab+bc+ca}+\frac{1}{a^2+b^2+c^2}\right)$$

463.

, For all nonnegative real numbers a,b and c, no two of which are zero,

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\geq\frac{3\sqrt{3abc(a+b+c)}(a+b+c)^2}{4(ab+bc+ca)^3}$$

it's obviously trues because of Am-Gm, We have:

$$(\sum a^2+\sum ab)^2.\sum ab\leq\ 108 with a+b+c=3$$

Replacing $a,b,c by \frac{1}{a},\frac{1}{b},\frac{1}{c}$ respectively, We have to prove that

$$\sum\frac{a^2b^2}{(a+b)^2}\geq\frac{3\sqrt{3(ab+bc+ca)}(ab+bc+ca)^2}{4(a+b+c)^3}.$$

Now, using Cauchy Schwarz inequality, We hav

$$\sum\frac{a^2b^2}{(a+b)^2}\geq\frac{(ab+bc+ca)^2}{(a+b)^2+(b+c)^2+(c+a)^2}=\frac{(ab+bc+ca)^2}{2(a^2+b^2+c^2+ab+bc+ca)}$$

it suffices to prove that

$$\frac{(ab+bc+ca)^2}{2(a^2+b^2+c^2+ab+bc+ca)}\geq\frac{3\sqrt{3(ab+bc+ca)}(ab+bc+ca)^2}{4(a+b+c)^3}$$

or equivalently,

$$2(a+b+c)^3\geq 3\sqrt{3(ab+bc+ca)}(a^2+b^2+c^2+ab+bc+ca)$$

that is

$$4(a+b+c)^6\geq 27(ab+bc+ca)(a^2+b^2+c^2+ab+bc+ca)^2$$

By AM-GM, We see that

$$27(ab+bc+ca)(a^2+b^2+c^2+ab+bc+ca)^2\leq$$

$$\frac{1}{2}\left(2(ab+bc+ca)+(a^2+b^2+c^2+ab+bc+ca)+(a^2+b^2+c^2+ab+bc+ca)\right)^3=$$

$$4(a+b+c)^6$$

464.

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\geq\frac{1}{ab+bc+ca}+\frac{2}{a^2+b^2+c^2}$$

1st **Solution**. (also in pvthuan's book, page 62)

By Cauchy inequality,

$$\sum_{cyc}(b+c)^2(2a^2+bc)\sum_{cyc}\frac{1}{2a^2+bc}\geq 4(a+b+c)^2$$

it remains to show that

$$\sum_{cyc}(b+c)^2(2a^2+bc)\leq 4(a^2+b^2+c^2)(ab+bc+ca)$$

which is easy.

465.

Let $a,b,c>0$ such that $a+b+c=3abc$. Prove that

$$\sum\frac{1}{a+b}\leq\frac{3}{2}$$

Solution: Set:

$$a = \frac{1}{x}; b = \frac{1}{y}; c = \frac{1}{z}; xy + yz + zx = 3$$

$$LHS = \sum \frac{xy}{x+y} \le \frac{\sqrt{xy}}{2} \le \frac{3}{2}$$

275, Prove if $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 3$ then

$$\sum \frac{a^3}{\sqrt{a^4 - b^4 + b^2}} \ge 3$$

The condition of this ineq didn't show $a^4 - b^4 + b^2 \ge 0$. But if $a^4 - b^4 + b^2 \ge 0$(and others expression) , We can prove M

$$(a^2 + b^2 + c^2)LHS^2 \ge (a^2 + b^2 + c^2)^3 = 27$$

$$=> LHS \ge 3$$

466.

Prove that if a, b, c are nonnegative real numbers, We have

$$\sqrt[3]{\frac{a(a+b)}{a^2+2b^2}} + \sqrt[3]{\frac{b(b+c)}{b^2+2c^2}} + \sqrt[3]{\frac{c(c+a)}{c^2+2a^2}} \ge 3\sqrt[3]{\frac{2abc}{(a^3+b^3+c^3)}}$$

Solution:

$$\sqrt[3]{\frac{a(a+b)}{a^2+2b^2}} + \sqrt[3]{\frac{b(b+c)}{b^2+2c^2}} + \sqrt[3]{\frac{c(c+a)}{c^2+2a^2}} \ge 3\sqrt[3]{\frac{2abc}{a^3+b^3+c^3}}$$

By Am-Gm ,We need to prove :

$$(a^3 + b^3 + c^3)^3 \ge abc(a^2 + b^2 + c^2)^3$$

it can prove by Am-Gm

$$LHS \ge 3\sqrt[9]{\frac{abc(a+b)(b+c)(c+a)}{(a^2+2b^2)(b^2+2c^2)(c^2+a^2)}} \ge 3\sqrt[9]{\frac{8a^2b^2c^2}{(a^2+2b^2)(b^2+2c^2)(c^2+a^2)}}$$

So We need to prove:

$$(a^3 + b^3 + c^3)^3 \ge abc(a^2 + b^2 + c^2)^3$$

But

$$a^3 + b^3 + c^3 \ge 3abc$$

and

$$3(a^3 + b^3 + c^3) \ge (a^2 + b^2 + c^2)^3$$

467.

, Let a, b, c be positive reall number satisfyin $abc = a + b + c$. Prove the following inequality :

$$\frac{a}{1+a^2} + \frac{b}{1+b^2} + \frac{c}{1+c^2} \le \frac{3\sqrt{3}}{4}$$

Solution:

$$\sum \frac{a}{1+a^2} \le \frac{3\sqrt{3}}{4}$$

Setting :

$$a = \frac{1}{x}; b = \frac{1}{y}; c = \frac{1}{z} => xy + yz + zx = 1$$

By Am-Gm ,We can prove :

$$\frac{1}{x} + x \geq \frac{2}{3x} + \frac{2}{\sqrt{3}}; \forall x > 0$$

So We need to prove:

$$\frac{x}{\sqrt{3} + x} \geq \frac{3}{4}$$

$$<=> \frac{1}{\sqrt{3} + x} \geq \frac{3\sqrt{3}}{4}$$

But it's true by Am-Gm.

468.

Let $a, b, c > 0$, $ab + bc + ca = 3$. Prove that :

$$\frac{a}{\sqrt{5b + 4c}} + \frac{b}{\sqrt{5c + 4a}} + \frac{c}{\sqrt{5a + 4b}} \geq 1$$

Solution:

$$27LHS^2 = [\sum a(5b + 4c)]LHS^2 \geq (a + b + c)^3$$

We need to prove:

$$(a + b + c)^3 \geq 9(ab + bc + ca)$$

But it's true by Am-Gm Smile.

469.

Given $a, b, c > 0$. Prove that:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \ 3.\sqrt[4]{\frac{a^3 + b^3 + c^3}{3abc}} (*)$$

Solution:

.

$$Let \frac{b}{c} = x, \frac{c}{a} = y, \frac{a}{b} = z.$$

Then We have : $xyz = 1$ and :

$$(*) \Leftrightarrow x + y + z \geq 3\sqrt[4]{\frac{1}{3}(\frac{y}{z} + \frac{z}{x} + \frac{x}{y})}$$

$$\Leftrightarrow (x + y + z)^4 \geq 27(xy^2 + yz^2 + zx^2)$$

But We have a inequality :

$$27(xy^2 + yz^2 + zx^2) \leq 4(x + y + z)^3 - 27$$

Therefore We have only prove that :

$$(x + y + z)^4 \geq 4(x + y + z)^3 - 27$$

$$\Leftrightarrow (x + y + z - 3)^2[(x + y + z)^2 + 2(x + y + z) + 3] \geq 0$$

it's right Wink .

Suppose

$$abc = 1; Puta = \frac{y}{x}; b = \frac{z}{y}; c = \frac{x}{z}$$

This inequality become:

$$(x^3 + y^3 + z^3)^4 \geq \ 27xyz(x^6y^3 + y^6z^3 + z^6x^3)(*)$$

Put $m = x^3, n = y^3, p = z^3, (*)$ become:

$$(m + n + p)^12 \geq \ 3^12mnp(m^2n + n^2p + p^2m)^3(**)$$

Using yhe ill-known result:

$$27(m^2n + n^2p + p^2m + mnp) \geq \ 4(m + n + p)^3$$

and AM-GM inequality, We have $(**)$.

470, For positive number x, y, z such that $x + y + z = 1$. Prove that

$$\frac{xy}{\sqrt{xy + yz}} + \frac{yz}{\sqrt{yz + zx}} + \frac{zx}{\sqrt{zx + xy}} \leq \frac{\sqrt{2}}{2}$$

$$\sum_{cyc} \frac{xy}{xy + xz + yz} \sqrt{\frac{2(xy + xz + yz)^2}{(xy + yz)(x + y + z)^2}} \leq 1$$

But

$$\sum_{cyc} \frac{xy}{xy + xz + yz} * \sqrt{\frac{2(xy + xz + yz)^2}{(xy + yz)(x + y + z)^2}} \leq \sqrt{\sum_{cyc} \frac{2x(xy + xz + yz)}{(x + z)(x + y + z)^2}}$$

We have to prove that

$$\sum_{cyc} \frac{2x}{x + z} \leq \frac{(x + y + z)^2}{xy + xz + yz}$$

$$<=> \sum_{cyc} \frac{2x^2z}{x + z} \leq x^2 + y^2 + z^2$$

$$<=> \frac{x^2 + y^2 + z^2}{2} \leq \frac{x^4}{x^2 + xz} + \frac{y^4}{y^2 + xy} + \frac{z^4}{z^2 + yz}$$

but We have that

$$\frac{x^4}{x^2 + xz} + \frac{y^4}{y^2 + xy} + \frac{z^4}{z^2 + yz} \geq \frac{(x^2 + y^2 + z^2)^2}{x^2 + y^2 + z^2 + xy + xz + yz} \geq \frac{x^2 + y^2 + z^2}{2}$$

$$\sum_{cyc} \frac{xy}{\sqrt{xy + yz}} = \sum_{cyc} \sqrt{\frac{x^2y}{x + z}} = \sum_{cyc} \left(\frac{x + y}{2} \cdot \sqrt{\frac{4x^2y}{(x + y)^2(x + z)}} \right) \leq \sqrt{\sum_{cyc} \frac{2x^2y}{(x + y)(x + z)}}$$

Remain to prove that

$$\sum_{cyc} \frac{2x^2y}{(x + y)(x + z)} \leq \frac{1}{2}$$

But

$$\sum_{cyc} \frac{2x^2y}{(x + y)(x + z)} \leq \frac{1}{2} \Leftrightarrow \sum_{cyc} 4x^2y(y + z) \leq (x + y)(x + z)(y + z)(x + y + z) \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}(x^3y + x^3z - 2x^2y^2) \geq 0.$$

471.
Let $a, b, c > 0$ such that $a + b + c = 3$. Prove that:

$$abc + a^2b + b^2c + c^2a \leq 4$$

Solution:
Because

$$a^2b + b^2c + c^2a \leq \frac{4}{27}(a + b + c)^3 - abc$$

472, Let $a + b + c + d = 4$ and $a, b, c, d \geq 0$. Prove that

$$a^2bc + b^2cd + c^2da + d^2ab \leq 4$$

it is necessary to prove, that

$$(a + b + c + d)^4 \geq 64(a^2bc + b^2cd + c^2da + d^2ab)$$

if $a = \min\{a, b, c, d\}$ and $b = a + x, c = a + y, d = a + z$ then it is killing. But it is very ugly.

Let $p, q, r, s = a, b, c, d$ and $p \geq q \geq r \geq s$. Then by rearrangement inequality,

$$a^2bc + b^2cd + c^2da + d^2ab = a(abc) + b(bcd) + c(cda) + d(dab)$$

$$\leq p(pqr) + q(pqs) + r(prs) + s(qrs) = (pq + rs)(pr + qs)$$

$$\leq (\frac{pq + rs + pr + qs}{2})^2 = \frac{1}{4}((p + s)(q + r))^2$$

$$\leq \frac{1}{4}((\frac{p + q + r + s}{2})^2)^2$$

$$= 4.$$

Equality holds $\iff$ $q = r = 1$ and $p + s = 2$. So equality holds if two of them are equal to 1. Applying this, We can get the equality conditions $(a, b, c, d) = (1, 1, 1, 1), (2, 1, 1, 0)$ or any cyclic forms.

And by this idea, We can solve that

$$a^2b + b^2c + c^2a \leq \frac{4}{27} if a + b + c = 1$$

which was from Canada. if $\{p, q, r\} = \{a, b, c\}, p \geq q \geq r$, then as $pq \geq pr \geq qr$,

$$a^2b + b^2c + c^2a = a(ab) + b(bc) + c(ca) \leq p(pq) + q(pr) + r(qr)$$

$$= q(p^2 + pr + r^2) \leq q(p + r)^2 = \frac{1}{2}(2q)(p + r)(p + r)$$

$$\leq \frac{1}{2}(\frac{(2q) + (p + r) + (p + r)}{3})^3$$

$$= \frac{1}{2}(\frac{2}{3})^3 = \frac{4}{27}.$$

473.
Let $a, b, c > -1$. Prove that:

$$\sum \frac{1 + a^2}{1 + b + c^2} \geq 2$$

Solution:
We have : $a, b, c > -1$ therefore

$$1+b+c^2, 1+c+a^2, 1+a+b^2 \geq 0$$

So

$$b^2+1 \leq 2b$$

$$\Rightarrow 2(1+b+c^2) \leq (1+b^2)+2(1+c^2)$$

$$2(1+c+a^2) \leq (1+c^2)+2(1+a^2)$$

$$2(1+a+b^2) \leq (1+a^2)+2(1+b^2)$$

So,

$$\sum \frac{1+a^2}{1+b+c^2} \geq \sum \frac{2x}{y+2z} = \sum \frac{2x^2}{xy+2xz}$$

where $x = 1+a^2, y = 1+b^2, z = 1+c^2$. Clearly $x, y, z > 0$. Other, by Cauchy-Schwarzt We have :

$$\sum \frac{2x^2}{xy+2xz} \geq \frac{2(x+y+z)^2}{3(xy+yz+zx)} \geq 2$$

474, Let $x, y, z \geq 0$ be such that $x^2+y^2+z^2 = 1$. Prove that

$$1 \leq \frac{z}{1+xy} + \frac{y}{1+xz} + \frac{z}{1+xy} \leq \sqrt{2}$$

Firstly, We prove that:

$$(a+b+c)^2 \leq 2(1+bc)^2(1)$$

indeed, We have :

$$(1) \Leftrightarrow 2(ab+bc+ca) \leq 1+4bc+2b^2c^2 \Leftrightarrow 2a(b+c) \leq a^2+(b+c)^2+2b^2c^2$$

$$\Leftrightarrow (b+c-a)^2+2b^2c^2 \geq 0(true)$$

Therefore,

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \leq \frac{\sqrt{2}a}{a+b+c} + \frac{\sqrt{2}b}{a+b+c} + \frac{\sqrt{2}c}{a+b+c} = \sqrt{2}$$

Other, We have :

$$a+abc \leq a+\frac{a(b^2+c^2)}{2} = 1-\frac{(a-1)^2(a+2)}{2} \leq 1$$

therefore,

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \geq a^2+b^2+c^2 = 1$$

We are done

$$1 \leq \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab}$$

$$(a+b+c)^2 = \left(\sum_{cyc} \sqrt{a}\sqrt{1+bc} \cdot \frac{\sqrt{a}}{\sqrt{1+bc}}\right)^2 \leq \left(\sum_{cyc} \frac{a}{1+bc}\right)\left(\sum_{cyc} a(1+bc)\right)$$

$$\sum_{cyc} \frac{a}{1+bc} \geq \frac{(a+b+c)^2}{a+b+c+3abc} \geq 1$$

$$p^2-p-3r\geq 0$$

475, Let $a,b,c\geq 0$ satisfying $\sum a^2=1$. Prove that:

$$\sum\frac{a}{1+bc}\geq 1$$

$$\sum_{cyc}\frac{a}{1+bc}\geq\sum_{cyc}\frac{a}{1+\frac{b^2+c^2}{2}}=\sum_{cyc}\frac{a}{1+\frac{1-a^2}{2}}=$$

$$=1+\sum_{cyc}\left(\frac{2a}{3-a^2}-\frac{1}{3}-\left(a^2-\frac{1}{3}\right)\right)=1+\sum_{cyc}\frac{a(a+2)(a-1)^2}{3(3-a^2)}\geq 1.$$

$$\sum\frac{a}{1+bc}\geq\frac{1}{3}\sum a\sum\frac{1}{1+bc}\geq\frac{3\sum a}{3+\sum ab}\geq\frac{6\sum a}{5+(\sum a)^2}$$

And $1\leq\sum a\leq\sqrt{3}$

$$\Rightarrow\frac{6\sum a}{5+(\sum a)^2}\geq 1\Leftrightarrow(\sum a)^2-6\sum a+5\leq 0\Leftrightarrow(\sum a-1)(\sum a-5)\leq 0, Right$$

by cauchy

$$\sum\frac{a}{1+bc}\geq\frac{(a^2+b^2+c^2)^2}{a^3+b^3+c^3+\sum a^3bc}=\frac{1}{a^3+b^3+c^3+abc}$$

it remains to show that

$$\frac{1}{a^3+b^3+c^3+abc}\geq 1$$

which is obviously true from

$$(a^2+b^2+c^2)^3\geq(a^3+b^3+c^3+abc)^2$$

(by muirhead , AM-GM)

$$(a^2+2)^3\geq(a^3+a+2)^2\geq a(a^2+1)^2$$

But after cancelling the degree 6 term the right side has larger degree so it is incorrect. Try again you should look yourself After expanding

$$L.H.S=a^6+6a^4+12a^2+8\geq 2a^5+4a^4+2a^3+10a^2+2a+7 by AM-GM$$

and

$$2a^5+4a^4+2a^3+10a^2+2a+7>a^5+2a^3+a=a(a^2+1)^2$$

476, Let $a,b,c\geq 0$. Prove that:

$$\sum\frac{1}{a^2+ab+b^2}\geq\frac{9}{(\sum a)^2}$$

Solution : We have :

$$a^2+ab+b^2=(a+b+c)^2-(ab+bc+ca)-(a+b+c)c$$

$$b^2+bc+c^2=(a+b+c)^2-(ab+bc+ca)-(a+b+c)b$$

$$c^2+ca+a^2=(a+b+c)^2-(ab+bc+ca)-(a+b+c)a$$

Suppose $a+b+c=1$. We have :

$$(*)\leftrightarrow\sum\frac{1}{1-(ab+bc+ca)-a}\geq 9$$

$$\leftrightarrow 1-4(ab+bc+ca)+9abc \geq 0$$

$$\leftrightarrow (a+b+c)^2-4(ab+bc+ca)(a+b+c)+9abc \geq 0$$

$$\leftrightarrow a^3+b^3+c^3+3abc \geq a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$$

it is true because it is inequality 's Schur. We are done.

Try

$$\sum_{cyc} \frac{1}{a^2+ab+b^2} = \sum_{cyc} \frac{(3c+a+b)^2}{(3c+a+b)^2(a^2+ab+b^2)} \geq$$

$$\geq \frac{25(a+b+c)^2}{\sum(3c+a+b)^2(a^2+ab+b^2)}.$$

We show

$$\frac{25(a+b+c)^2}{\sum(3c+a+b)^2(a^2+ab+b^2)} \geq 9$$

pqr technique works here. We put $a+b+c=1, \sum ab = q, abc = r$. The ineq becomes

$$q(3q-1)^2+9r+1-4q$$

after expanding Very Happy And it is obivious since $9r+1-4q \geq 0$(schur ineq) The equality holds when $a=0, b=c=\frac{1}{2}, a=b=c$.

477, Solve the equation

$$\sqrt{x+\sqrt{x^2-1}} = \frac{9\sqrt{2}}{4}(x-1)\sqrt{x-1}$$

Solution:
Let $a=\sqrt{x-1}, b=\sqrt{x+1}$. Then $b^2-a^2=2$. There fore :

$$\begin{cases} b^2-a^2=2 \\ 2(a+b)=9a^3 \end{cases}$$

$$\leftrightarrow \begin{cases} b^2=a^2+2 \\ 2b=9a^3-2a \end{cases}$$

$$\leftrightarrow \begin{cases} b^2=a^2+2 \\ 81a^6-36a^4-8=0 \end{cases}$$

$$\leftrightarrow \begin{cases} b^2=a^2+2 \\ a^2=\frac{2}{3} \end{cases}$$

$$\leftrightarrow \begin{cases} b^2=\frac{8}{3} \\ 2b=9a^3-2a \end{cases}$$

$$\leftrightarrow x=\frac{5}{3}$$

478.
, Let a,b,c be nonnegative real numbers such that $a+b+c=1$. Prove that

$$\sum a\sqrt{8b^2+c^2} \leq 1$$

$$\sum a\sqrt{4b^2+c^2} \leq \frac{3}{4}$$

Note that

$$3b + c - \frac{3bc}{2b+c} \geq \sqrt{8b^2 + c^2}$$

Hence, it suffices to show that

$$(\sum a)^2 \geq \sum a(3b + c - \frac{3bc}{2b+c})$$

$$\Leftrightarrow 3abc \sum \frac{1}{2b+c} + \sum a^2 - 2\sum ab \geq 0$$

The Cauchy Schwarz inequality gives us

$$\sum \frac{1}{2b+c} \geq \frac{3}{a+b+c}$$

it suffices to show that

$$\frac{9abc}{a+b+c} + \sum a^2 - 2\sum ab \geq 0$$

$$\Leftrightarrow \sum a^3 + 3abc \geq \sum bc(b+c)$$

which is true by Schur.

479.
Let $a, b, c > 0$.Prove that:

$$\sum a\sqrt{4b^2 + c^2} \leq \frac{3}{4}$$

Solution
Using two lemma.

$$1)\ 2b + c - \frac{2bc(2b+c)}{4b^2 + 3bc + c^2} \geq \sqrt{4b^2 + c^2}$$

$$2)\ 8\sum \frac{2b+c}{4b^2 + 3bc + c^2} \geq \frac{27}{a+b+c}$$

480,
Let $a, b, c, d \geq 0$ $a^2 + b^2 + c^2 + d^2 = 4$. Prove that:

$$a^3 + b^3 + c^3 + d^3 \leq 8$$

Solution:
Squaring the both sides We need to prove that:

$$(a^3 + b^3 + c^3 + d^3)^2 \leq (a^2 + b^2 + c^2 + d^2)^3.$$

Using CBS We infer that:

$$(a^3 + b^3 + c^3 + d^3)^2 \leq (a^2 + b^2 + c^2 + d^2)(a^4 + b^4 + b^4 + d^4)$$

So We need only to prove that:

$$a^4 + b^4 + c^4 + d^4 \leq (a^2 + b^2 + c^2 + d^2)^2$$

We have :

$$a^2 + b^2 + c^2 + d^2 = 4$$

$$0 \leq a, b, c, d \leq 2$$

$$0 \leq a^3 \leq 2a^2, 0 \leq b^3 \leq 2b^2, 0 \leq c^3 \leq 2c^2, 0 \leq d^3 \leq 2d^2$$

$$0 \leq a^3 + b^3 + c^3 + d^3 \leq 2(a^2 + b^2 + c^2 + d^2) = 8$$

481,

Let $x, y, z > 0$. Prove that :

$$3(\sqrt{x(x+y)(x+z)} + \sqrt{y(y+z)(y+x)} + \sqrt{z(z+x)(z+y)})^2 \leq 4(x+y+z)^3$$

Solution:

By Cauchy-Schwarz ineq , We have :

$$LHS \leq 3(x+y+z)(\sum x^2 + yz + zx + xy)$$

Then We prove that :

$$4(x+y+z)^2 \geq 3[(x+y+z)^2 + xy + yz + zx]$$

$$\leftrightarrow \sum (x-y)^2 \geq 0$$

482

, Prove that for any reals x, y, z which satisfy condition $x^2 + y^2 + z^2 = 2$ We have

$$x + y + z) \leq xyz + 2$$

Solution:

We have :

$$2 = x^2 + y^2 + z^2 \geq 2yz or yz \leq 1$$

By Bunhiacopsky We have :

$$[x(1-yz) + y + z]^2 \leq [x^2 + (y+z)^2][(1-yz)^2 + 1] \leq (2+2yz)(y^2z^2 - 2yz + 1) \leq 4$$

$(because yz \leq 1)$ There fore:

$$|x + y + z - xyz| \leq 2$$

or

$$x + y + z \leq 2 + xyz$$

483.

, For all nonnegative real numbers a, b and c, no two of which are zero,

$$\frac{a(b+c)(a^2-bc)}{a^2+bc} + \frac{b(c+a)(b^2-ca)}{b^2+ca} + \frac{c(a+b)(c^2-ab)}{c^2+ab} \geq 0$$

Solution:

$$\frac{1}{2}\left(\sum_{cyc} \frac{1}{ab}\right) \geq \sum_{cyc} \frac{1}{c^2+ab}$$

$$\frac{c^4(a^2-b^2)^2 + b^4(c^2-a^2)^2 + c^4(a^2-b^2)^2}{(a^2+bc)(b^2+ca)(c^2+ab)} \geq 0$$

Setting $A = LHS$, then We see that

$$A = \sum \frac{a^3(b+c)}{a^2+bc} - abc \sum \frac{b+c}{a^2+bc} \geq \sum \frac{a^3(b+c)}{a^2+bc} - abc \sum \frac{1}{a} =$$

$$\sum \frac{a^3(b+c)}{a^2+bc} - \sum \frac{a(b+c)}{2} = \frac{1}{2}\sum \frac{a(b+c)(a^2-bc)}{a^2+bc} = \frac{1}{2}A$$

which shows that $A \geq 0$.

484.

, For any positive real numbers a, b and c

$$\sqrt{\frac{a(b+c)}{a^2+bc}} + \sqrt{\frac{b(c+a)}{b^2+ca}} + \sqrt{\frac{c(a+b)}{c^2+ab}} \leq \sqrt{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)}$$

Solution:

From Cauchy We have

$$\left(\sum \sqrt{\frac{a(b+c)}{a^2+bc}}\right)^2 \leq (\sum \sqrt{a}) \left(\sum \frac{\sqrt{a}(b+c)}{a^2+bc}\right)$$

Now all We have to prove is

$$\sum \frac{\sqrt{a}(b+c)}{a^2+bc} \leq \sum \frac{1}{\sqrt{a}}$$

which is equivalent

$$\sum \frac{(a-b)(a-c)}{\sqrt{a}(a^2+bc)} \geq 0$$

which is Vornicu Schur.

From this idea We should square the inequality and then use that

$$\frac{ab(c+a)(c+b)}{(a^2+bc)(b^2+ca)} \leq 1$$

for example .Then you will have to prove that

$$\frac{(a+b)(a+c)}{a^2+bc} + \frac{(b+a)(b+c)}{b^2+ca} + \frac{(c+a)(c+b)}{c^2+ab} \leq \sum (\frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}})$$

Solution:2

We have the inequality is equivalent to

$$\left(\sum \sqrt{\frac{a(b+c)}{a^2+bc}}\right)^2 \leq \left(\sum \sqrt{a}\right)\left(\sum \frac{1}{\sqrt{a}}\right)$$

$$\Leftrightarrow \sum \frac{a(b+c)}{a^2+bc} + 2\sum \sqrt{\frac{ab(a+c)(b+c)}{(a^2+bc)(b^2+ca)}} \leq \left(\sum \sqrt{a}\right)\left(\sum \frac{1}{\sqrt{a}}\right)$$

We can easily prove that

$$\sum \sqrt{\frac{ab(a+c)(b+c)}{(a^2+bc)(b^2+ca)}} \leq 3$$

So, it suffices to prove that

$$\Leftrightarrow \sum \frac{a(b+c)}{a^2+bc} + 6 \leq \left(\sum \sqrt{a}\right)\left(\sum \frac{1}{\sqrt{a}}\right)$$

To prove this ineq, We only need to prove that

$$\frac{a+b}{\sqrt{ab}} - \frac{c(a+b)}{c^2+ab} - 1 \geq 0$$

But this is trivial, because

$$\frac{a+b}{\sqrt{ab}}-\frac{c(a+b)}{c^2+ab}-1=(a+b)\left(\frac{1}{\sqrt{ab}}-\frac{c}{c^2+ab}\right)-1\geq 2\sqrt{ab}\left(\frac{1}{\sqrt{ab}}-\frac{c}{c^2+ab}\right)-1=\frac{\left(c-\sqrt{ab}\right)^2}{c^2+ab}\geq 0$$

We are done.

485.

, For any positive real numbers a, b and c,

$$\frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab}\leq \frac{1}{2}\sqrt{27+(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

Solution:

The inequality is equivalent to

$$\sum \frac{a^2(b+c)^2}{(a^2+bc)^2}+2\sum \frac{ab(b+c)(c+a)}{(a^2+bc)(b^2+ca)}\leq \frac{15}{2}+\frac{1}{4}\left(\sum \frac{b+c}{a}\right)$$

Notice that

$$(a^2+bc)(b^2+ca)-ab(b+c)(c+a)=c(a+b)(a-b)^2$$

then

$$2\sum \frac{ab(b+c)(c+a)}{(a^2+bc)(b^2+ca)}\leq 6(1)$$

Other hand,

$$\sum \frac{a^2(b+c)^2}{(a^2+bc)^2}\leq \sum \frac{a^2(b+c)^2}{4a^2bc}=\frac{1}{4}\sum\left(\frac{b}{c}+\frac{c}{b}+2\right)$$

From (1) and (2) We have done! Besides, by the sam ways, We have a nice **Solution** for an old problem:

$$\sum \sqrt{\frac{a(b+c)}{a^2+bc}}\leq \sqrt{\left(\sum \sqrt{a}\right)\left(\sum \frac{1}{\sqrt{a}}\right)}$$

486,

Let $a, b, c > 0$ and $abc = 1$. Prove that

$$\sum \frac{a-1}{b+c}\geq 0$$

Solution:

$$\leftrightarrow (a+b+c-1)(\sum \frac{1}{b+c})\geq 3$$

By Am-Gm , We can prove :

$$LHS\geq (a+b+c-1)\frac{9}{2(a+b+c)}$$

So We need to prove this ineq :

$$3(a+b+c-1)\geq 2(a+b+c)$$

$$<=> a+b+c\geq 3$$

301, Let a, b, c be nonnegative real numbers. Prove that

$$\frac{3}{\sqrt{2}}\cdot\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}\leq \sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\leq \frac{3}{\sqrt{2}}$$

$$(\sum\sqrt{\frac{a}{a+b}})^2\le 2(a+b+c)(\sum\frac{a}{(a+b)(a+c)})=4\frac{(a+b+c)(ab+bc+ca)}{(a+b+c)(ab+bc+ca)-abc}\le\frac{9}{2}$$

487,

Given $a,b,c\ge 0$ and $a+b+c=8$. Prove that:

$$268+12a^2b^2c^2\ge\ ab(a-b)^2+bc(b-c)^2+ca(c-a)^2$$

Solution:

By "pqr"

$$\Leftrightarrow 12(r-1)^2+\frac{4}{9}(q^2-40)^2\ge 0$$

488.

Given $a,b,c>0$ satisfy $a+b+c=3$. Prove that

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}+\frac{9abc}{4}\ge\frac{21}{4}$$

Below is our first attempt, which is indirect but fairly short: Rewrite the inequality as

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{9}{4}\ge\frac{21}{4abc}$$

Put $x=a^{-1},y=b^{-1}andz=c^{-1}$. Then $xy+yz+zx=3xyz$ and the inequality becomes

$$4(x^2+y^2+z^2)+9\ge 7(xy+yz+zx)$$

or after homogenizing

$$4(x+y+z)^2+\frac{81(xyz)^2}{(xy+yz+zx)^2}\ge 15(xy+yz+zx)$$

Without loss of generality, assume $x+y+z=1$. Put $xy+yz+zx=(1-q^2)/3$. Then as in

$$xyz\ge\frac{(1+q)^2(1-2q)}{27}$$

it remains to show that

$$4\left(\frac{(1-q^2)}{3}\right)^2+81\left(\frac{(1+q)^2(1-2q)}{27}\right)^2\ge 15\left(\frac{1-q^2}{3}\right)^3$$

or

$$4(1-q^2)^2+(1+q)^4(1-2q)^2-5(1-q^2)^3\ge 0$$

But this is reduced to

$$q^2(1+q)^2(1-3q)^2\ge 0$$

489.

, Using Cauchy Schwarts and Am-Gm, We will need to prove:

$$\sum\frac{a^4+b^4}{a^2+b^2}+\frac{2(ab+bc+ca)^2}{a^2+b^2+c^2}\ge\ (a+b+c)^2$$

First squaring

$$\Leftrightarrow\sum\frac{a^4+b^4}{a^2+b^2}+4\sum\frac{a^2b^2}{\sqrt{(a^2+b^2)(b^2+c^2)}}\ge(\sum a)^2$$

Then use

$$(a^2+b^2)(b^2+c^2)\le(a^2+2b^2+c^2)^2$$

and Cauchy Schwartz. The ineq turns into the form nguoivn gave.

$$\sum \frac{(a^2+b^2-ca-cb)^2(a-b)^2}{a^2+b^2} \geq 0$$

450.

, Let a,b,c be positive numbers such that: $a+b+c=1$. Prove that:

$$\frac{a}{b^2+b}+\frac{b}{c^2+c}+\frac{c}{a^2+a} \geq \frac{36(a^2+b^2+c^2)}{ab+bc+ca+5}$$

We can prove your problem by one result of hungkhtn and vacs is if $a+b+c=1, a,b,c$ be positive numbers then

$$a^2b+b^2c+c^2a+abc \leq \frac{4}{27}$$

uses cauchuy-schawrs We have:

$$\sum \frac{a}{b^2+b} \geq \frac{(a+b+c)^2}{\sum ab+\sum b^2a} \geq \frac{36(a^2+b^2+c^2)}{ab+bc+ca+5}$$

let $q=\sum ab, r=abc$ We have

$$36\sum a^2(\sum ab^2+\sum ab) \geq 36\sum a^2(\frac{4}{27}-abc+\sum ab) \geq \sum ab+5$$

$$<=> q+5 \geq 36(1-2q)(\frac{4}{27}-r+q)(1)$$

becase

$$1-2q \geq 1-\frac{2}{3} \geq 0$$

so uses schur third degree We have

$$r \geq \frac{1-4q}{9}$$

supposing

$$(1) <=> q+5 \geq 36(1-2q)(\frac{4}{27}-\frac{4q-1}{9}+q) <=> (1-3q)(40q+13) \geq 0$$

equality when $a=b=c=\frac{1}{3}$.

2)

$$\frac{a}{b^2+b}+\frac{b}{c^2+c}+\frac{c}{a^2+a} \geq \frac{3}{4}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq \frac{36(a^2+b^2+c^2)}{ab+bc+ca+5}$$

How about the stronger, Toan Smile

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{30(a^2+b^2+c^2)}{3(a+b+c)^2+ab+bc+ca}$$

We use the lemma:

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+1 \geq \frac{21(a^2+b^2+c^2)}{(a+b+c)^2}$$

451.

, Let be $a,b,c>0$. Show that :

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9+\frac{2((a-b)^2+(b-c)^2+(c-a)^2)}{ab+bc+ca}$$

Solution:

$$<=> S_c(a-b)^2+S_a(b-c)^2+S_b(a-c)^2 \geq 0$$

With:

$$S_c = \frac{1}{ab} - \frac{2}{ab+bc+ca}; ...$$

Assume:

$$a \geq b \geq c.$$

$$So: S_a; S_b \geq 0 - easyRazz$$

We have:

$$S_a + S_c = \frac{a^2(b+c)+c^2(a+b)-2abc}{abc(ab+bc+ca)} \geq 0$$

306, if x, y, z are reals and $x^2+y^2+z^2 = 2$. Prove that

$$x+y+z \leq xyz+2$$

WLOG $x \leq y \leq z \Rightarrow xy \leq 1$. By Cauchy-Schwartz

$$(xyz-(x+y+z))^2 = (z(xy-1)-x-y)^2 \leq (z^2+(x+y)^2)((xy-1)^2+1) = (2+2xy)(2-2xy+(xy)^2) = 4-2(xy)^2(1-xy)$$

452

.For any positive reals a, b, c such that $\sum a = 1$

$$(ab+bc+ca)(\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a}) \geq \frac{3}{4} holds$$

Solution:

by AM-GM;Schwarz;Holder ineqlities,We obtain:

$$\sum \frac{a}{b^2+b} = \sum \left(\frac{a}{b} - \frac{a}{b+1} \right) \geq$$

$$\sum \left(\frac{a}{b} - \frac{\sqrt[4]{3^3}a}{4\sqrt[4]{b}} \right) = \sum \frac{a}{b} - \frac{\sqrt[4]{3^3}}{4} \sum \sqrt[4]{\frac{a}{b} * a * a * a}$$

$$\geq \sum \frac{a}{b} - \frac{\sqrt[4]{3^3}}{4} \sqrt[4]{\sum \frac{a}{b}} \geq \frac{3}{4} \sum \frac{a}{b} \geq \frac{3}{4(ab+bc+ca)}$$

453.

, Let $a, b, c \geq 0$. Prove that

$$\sum (\frac{a}{b+c})^3 + \frac{9abc}{(a+b)(b+c)(c+a)} \geq \sum \frac{a}{b+c}$$

$$q = xy+yz+zx, p = x+y+z, r = xyz$$

Solution:

$$\sum (\frac{a}{b+c})^3 + \frac{9abc}{(a+b)(b+c)(c+a)} \geq \sum \frac{a}{b+c}$$

$$<=> p^3 + 12r + 3pr \geq 16p$$

$$<=> p^3 + (3r-16)p + 12r \geq 0$$

$$f(p) = p^3 + (3r-16)p + 12r$$

$$f'(p) = 3p^2 + 3r - 16$$

$$f'(p) = 0 <=> p = \sqrt{(16-3r)/3} \geq \sqrt{13/3}$$

$$=> f'(p) > 0 with p \in [3; 4]$$

$$=> Minf(p) = minf(3)$$

$$if p \geq 4$$

We have

$$r \geq \frac{16p - p^3}{4p + 9}$$

454.

For $a, b, c > 0$ and $ab + bc + ca = 3$. Prove that :

$$3 + \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} \geq \frac{a + b^2c^2}{b + c} + \frac{b + c^2a^2}{c + a} + \frac{c + a^2b^2}{a + b} \geq 3$$

By cauchy-swarchz:

$$\frac{a + b^2c^2}{b + c} + \frac{b + c^2a^2}{c + a} + \frac{c + a^2b^2}{a + b} \geq \frac{(a + b + c)^2}{2(ab + bc + ca)} + \sum \frac{b^2c^2}{b + c} \geq \frac{a + b + c}{2} + \sum \frac{b^2c^2}{b + c}$$

By Am-Gm We have:

$$\sum (\frac{a^2b^2}{a + b} + \frac{a + b}{4}) \geq \sum ab = 3$$

By Am-Gm and Cauchy Schwarts, We can prove easily the stronger:

$$\frac{5(a^2 + b^2 + c^2)}{6} + \frac{1}{2} \geq \frac{a + b^2c^2}{b + c} + \frac{b + c^2a^2}{c + a} + \frac{c + a^2b^2}{a + b}$$

The first, using our old result:

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} + \frac{1}{2} \geq \sum \frac{a}{b + c}$$

Besides, by CS and Am-Gm:

$$\sum \frac{a^2b^2}{a + b} \leq \sqrt{\sum a^2b^2 . \sum \frac{a^2b^2}{(a + b)^2}} \leq \sqrt{\frac{1}{3}.(a^2 + b^2 + c^2)^2 . \sum \frac{ab}{4}} = \sum \frac{a^2 + b^2 + c^2}{2}$$

Add 2 inequalities, We have our stronger

455.

, Let x, y, z be postive real numbers such that $xyz = x + y + z + 2$. Prove that:

$$2(\sqrt{xy} + \sqrt{yz} + \sqrt{zx}) \leq x + y + z + 6$$

Solution:

The inequality is enquivalent to:

$$(\sum \sqrt{x})^2 - (x + y + z) \leq x + y + z + 6 or \sum \sqrt{x} \leq \sqrt{2(x + y + z + 3)}$$

Denote

$$x = \frac{b + c}{a}, y = \frac{c + a}{b}, z = \frac{a + b}{c}$$

Therefore, We just need to prove:

$$\sum \sqrt{\frac{b + c}{a}} \leq \sqrt{2(\sum \frac{b + c}{a} + 3)} = \sqrt{2(a + b + c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})} = \sqrt{[\sum (b + c)](\frac{1}{a} + \frac{1}{b} + \frac{1}{c})}$$

But this is obviously true due to Cauchy-Schwartz, which ends our **Solution**.

457.

, Given a, b, c are prositive real numbers $a^2+b^2+c^2=1$. Find max of P:

$$P=\frac{ab}{1+c^2}+\frac{bc}{1+a^2}+\frac{ca}{1+b^2}$$

Solution:

We think it trues by AM-GM:

$$\sum\frac{ab}{a^2+c^2+b^2+c^2}\leq\sum\frac{ab}{2\sqrt{(a^2+c^2)(b^2+c^2)}}\leq\frac{1}{4}\sum(\frac{a^2}{a^2+c^2}+\frac{b^2}{c^2+b^2})$$

anh also true by cauchy Schwarz:

$$\sum\frac{ab}{a^2+c^2+b^2+c^2}\leq\sum\frac{(a+b)^2}{4(a^2+c^2+b^2+c^2)}\leq\frac{1}{4}\sum(\frac{a^2}{a^2+c^2}+\frac{b^2}{c^2+b^2})$$

Q.E.D

458.

, Let $a, b, c>0$ and

$$\frac{1}{a}+\frac{1}{2b}+\frac{1}{3c}\geq 3;\frac{1}{2b}+\frac{1}{3c}\geq 2;\frac{1}{3c}\geq 1.$$

Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq 14$$

Solution:

Setting :

$$\frac{1}{3c}=x+1;\frac{1}{2b}+\frac{1}{3c}=2+y;\frac{1}{a}+\frac{1}{2b}+\frac{1}{3c}=z+3(x,y,z\geq 0)$$

$$=>\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-14=2(z+3y+5x)+(x-y)^2+(y-z)^2+x^2\geq 0$$

Then We have done Mr. it holds when $x=y=z=0$

$$=>a=1;b=\frac{1}{2};c=\frac{1}{3}$$

We have

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=(\frac{1}{a^2}+\frac{1}{4b^2}+\frac{1}{9c^2})+3(\frac{1}{4b^2}+\frac{1}{9c^2})+5(\frac{1}{9c^2})\geq 3(\frac{\frac{1}{a}+\frac{1}{2b}+\frac{1}{3c}}{3})^2+6(\frac{\frac{1}{2b}+\frac{1}{3c}}{3})^2+5(\frac{1}{3c})^2\geq 3+6+5=14.$$

459.

Let $x, y, z>0$. Prove that :

$$3(\sqrt{x(x+y)(x+z)}+\sqrt{y(y+z)(y+x)}+\sqrt{z(z+x)(z+y)})^2\leq 4(x+y+z)^3$$

By Cauchy-Schwarz ineq , We have :

$$LHS\leq 3(x+y+z)(\sum x^2+yz+zx+xy)$$

Then We prove that :

$$4(x+y+z)^2\geq 3[(x+y+z)^2+xy+yz+zx]$$

$$\leftrightarrow\sum(x-y)^2\geq 0$$

460,

Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$8\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+9\geq 10(a^2+b^2+c^2).$$

Equality holds for $a=2, b=c=\frac{1}{2}$.

Solution:
Setting:

$$a+b+c=p=3; ab+bc+ca=\frac{p^2-t^2}{3}; abc=r$$

$$\leftrightarrow 8.\frac{9-t^2}{r}+27-10(2t^2+9)\geq 0(3\geq t\geq 0)$$

We have this ineq :

$$r\leq\frac{(3-t)^2(3+2t)}{27}.$$

Then the ineq becomes one varible .

461.
Given that $a,b,c\geq 0$. Prove that,

$$(a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 4\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}+5$$

Find the maximal k, roof:
so that the following inequality holds:

$$(a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq k\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}+9-k$$

We can find, $k=4$ is best constant. .

Simple calcultation, We will set

$$a=\frac{1}{a}, b=\frac{1}{b}, c=\frac{1}{c}$$

then our inequality becomes

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq\frac{4(a^2b^2+b^2c^2+c^2a^2)}{abc(a+b+c)}+5$$

$$(ab+bc+ca)(a+b+c)^2\geq 4(a^2b^2+b^2c^2+c^2a^2)+5abc(a+b+c)$$

$$ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\geq 0$$

This **Solution** also shows that the best constant is k = 4. So, your statement is valid, shaam.

We can also prove it by Muirhead inequality: Our inequality is equivolent to

$$(a+b+c)(ab+ac+bc)^2\geq 4abc(a^2+b^2+c^2)+5abc(ab+ac+bc)$$

$$\Leftrightarrow\sum_{sym}a^3b^2+5\sum_{cyclic}a^2b^2c+\sum_{sym}a^3bc\geq 2\sum_{sym}a^3bc+5\sum_{cyclic}a^2b^2c$$

$$\Leftrightarrow\sum_{sym}a^3b^2\geq\sum_{sym}a^3bc$$

which is right by Muirhead inequality.

it is only the following indentity Mr. Green

$$(a-b)^2(b-c)^2(c-a)^2\geq 0$$

which is the strongest 3-variables inequatily

We can also solve it easily by SOS with

$$Sa = \frac{1}{bc} - \frac{2}{ab+bc+ca}...$$

$$4\frac{a^2+b^2+c^2}{ab+bc+ac} + 5 = 4\frac{(a+b+c)^2}{ab+bc+ac} - 3$$

$$(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) + 3 = \sum_{cyc}(\frac{a}{b}+\frac{b}{a}+2) = \sum_{cyc}\frac{(a+b)^2}{ab} \geq 4\frac{(a+b+c)^2}{ab+bc+ac}$$

How about

$$(a+b-2c)^2(b+c-2a)^2(c+a-2b)^2 \geq 0$$

Because:

$$(a-b)^2(b-c)^2(c-a)^2 \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}(a^4b^2 + a^4c^2 + 2a^3b^2c + 2a^3bc^2) \geq \sum_{cyc}(2a^3b^3 + 2a^4bc + 2a^2b^2c^2)$$

462.,
Given $a, b, c \geq 0$. Prove that:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)}$$

Solution:
it follows that

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \leq 2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - 1\right)$$

And We can deduce our inequality to

$$\frac{a^3}{(b+c)^3} + \frac{b^3}{(c+a)^3} + \frac{c^3}{(a+b)^3} + \frac{abc}{(a+b)(b+c)(c+a)} \geq 2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - 1\right)^2$$

Setting $x = \frac{2a}{b+c}, y = \frac{2b}{c+a}, z = \frac{2c}{a+b}$, then $xy + yz + zx + xyz = 4$ and our inequality becomes

$$x^3 + y^3 + z^3 + xyz \geq 4(x+y+z-2)^2$$

Now, We denote $p = x+y+z, q = xy+yz+Zx, r = xyz$, then $q + r = 4$ and our inequality is equivalent to

$$p^3 - 3pq + 4r \geq 4(p-2)^2$$

$$p^3 - 3p(4-r) + 4r \geq 4(p-2)^2$$

$$(p-4)(p^2+4) + (3p+4)r \geq 0$$

if $p \geq 4$, it is trivial. if $4 \geq p \geq 3$, applying Schur's inequality, We obtain $r \geq \frac{p(4q-p^2)}{9}$, hence

$$4 = q + r \geq q + \frac{p(4q-p^2)}{9}$$

it follows that

$$q \leq \frac{p^3+36}{4p+9}$$

and We obtain

$$r = 4 - q \geq 4 - \frac{p^3+36}{4p+9} = \frac{p(16-p^2)}{4p+9}$$

We have to prove

$$(p-4)(p^2+4)+(3p+4)\cdot\frac{p(16-p^2)}{4p+9}\ge 0$$

$$p(p+4)(3p+4)-(4p+9)(p^2+4)\ge 0$$

$$7p^2-p^3-36\ge 0$$

$$(p-3)(12+4p-p^2)\ge 0$$

which is obviously true because $4\ge p\ge 3$.
This completes our **Solution**. Equality holds if and only if $a=b=c$ or $a=b, c=0$ and its cyclic permutations. 463.
Given $a,b,c\ge 0$. Prove that:

$$(\frac{a}{b+c})^3+(\frac{b}{c+a})^3+(\frac{c}{a+b})^3+\frac{5abc}{(a+b)(b+c)(c+a)}\ge\frac{a^2+b^2+c^2}{ab+bc+ca}$$

Solution:
The following inequality is stronger than it

$$\left(\frac{a}{b+c}\right)^3+\left(\frac{b}{c+a}\right)^3+\left(\frac{c}{a+b}\right)^3+\frac{9abc}{(a+b)(b+c)(c+a)}\ge\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

used a nice lenma:

$$\sum\frac{a}{b+c}\ge\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{4abc}{(a+b)(b+c)(c+a)}$$

After expand, it's become:

$$abc(a^2+b^2+c^2-ab-bc-ca)\ge 0$$

And We think it's an useful lenma because notice that it's stronger than:

$$\sum\frac{a}{b+c}\ge\frac{(a+b+c)^2}{2(ab+bc+ca)}$$

We dont need to expand here, nguoWe vn. We rewrite it as follow:

$$\sum\frac{a[a(b+c)+bc]}{b+c}\ge a^2+b^2+c^2+\frac{4abc(ab+bc+ca)}{(a+b)(b+c)(c+a)}$$

$$a^2+b^2+c^2+abc\sum\frac{1}{b+c}\ge a^2+b^2+c^2+\frac{4abc(ab+bc+ca)}{(a+b)(b+c)(c+a)}$$

$$\sum\frac{1}{b+c}\ge\frac{4(ab+bc+ca)}{(a+b)(b+c)(c+a)}$$

which is obviously true by Cauchy Schwarz because

$$\sum\frac{1}{b+c}\ge\frac{9}{2(a+b+c)}$$

and

$$\frac{4(ab+bc+ca)}{(a+b)(b+c)(c+a)}\le\frac{9}{2(a+b+c)}$$

We meant:

$$\sum\frac{a}{b+c}\ge\frac{a^2+b^2+c^2}{ab+bc+ca}+\frac{4abc}{(a+b)(b+c)(c+a)}$$

$$<=> \frac{(a+b+c)(a^2+b^2+c^2)-abc}{(a+b)(b+c)(c+a)} \geq \frac{a^2+b^2+c^2}{ab+bc+ca}$$

$$<=> (a+b+c)(a^2+b^2+c^2)(ab+bc+ca)-abc\sum ab \geq [(a+b+c)(ab+bc+ca)-abc](a^2+b^2+c^2)$$

$$<=> abc(a^2+b^2+c^2-ab-bc-ca) \geq 0$$

Q.E.D

464.

Let a, b, c be nonnegative real numbers, not all are zero. Prove that

$$\frac{a}{a+b+7c} + \frac{b}{b+c+7a} + \frac{c}{c+a+7b} + \frac{2}{3} \cdot \frac{ab+bc+ca}{a^2+b^2+c^2} \leq 1$$

Equality holds if and only if $(a, b, c) \sim (1, 1, 1), (2, 1, 0), (1, 0, 0)$ **Solution**:

Because

$$\frac{a}{a+b+c} - \frac{a}{a+b+7c} = \frac{6ca}{(a+b+c)(a+b+7c)},$$

it suffices to prove that

$$\sum \frac{ca}{a+b+7c} \geq \frac{(a+b+c)(ab+bc+ca)}{9(a^2+b^2+c^2)}$$

if $a = b = 0$ or $b = c = 0$ or $c = a = 0$, the inequality becomes equality. For $a+b > 0, b+c > 0, c+a > 0$, applying the Cauchy Schwarz inequality, We get

$$\sum \frac{ca}{a+b+7c} \geq \frac{(ab+bc+ca)^2}{\sum ca(a+b+7c)}.$$

The inequality is reduced to

$$9(ab+bc+ca)(a^2+b^2+c^2) \geq (a+b+c)(7\sum a^2b + \sum ab^2 + 3abc),$$

or

$$\sum a^3b + 4\sum ab^3 - 4\sum a^2b^2 \geq abc\sum a,$$

that is

$$\sum ab(a-2b)^2 \geq abc\sum a.$$

Applying the Cauchy Schwarz inequality again, We get

$$\left[\sum ab(a-2b)^2\right]\left(\sum c\right) \geq \left[\sum \sqrt{abc}(a-2b)\right]^2 = abc(a+b+c)^2,$$

hence

$$\sum ab(a-2b)^2 \geq abc\sum a$$

and our **Solution** is completed.

465.

, Let $a, b, c > 0$. Prove that

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \geq \frac{4}{3}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

its easy

$$(a+\frac{1}{a})\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \geq 4(c+\frac{1}{c})$$

$$(a+\frac{1}{a})\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\geq 4(b+\frac{1}{b})$$

$$(a+\frac{1}{a})\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\geq 4(a+\frac{1}{a})$$

After summing We will get:

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\geq \frac{4}{3}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

466.
, Let $a, b, c > 0$. Prove that

$$\frac{1}{(1+\sqrt{ab})^2}+\frac{1}{(1+\sqrt{bc})^2}+\frac{1}{(1+\sqrt{ca})^2}\geq \frac{3+a+b+c}{(1+a)(1+b)(1+c)}$$

$$(1+\sqrt{ab})^2\leq (1+a)(1+b)$$

$$\frac{1}{(1+\sqrt{ab})^2}+\frac{1}{(1+\sqrt{bc})^2}+\frac{1}{(1+\sqrt{ca})^2}\geq \sum\frac{1}{(1+a)(1+b)}=\frac{3+a+b+c}{(1+a)(1+b)(1+c)}$$

467.
, Let p, q, r be positive numbers. Prove or disprove

$$9+p+q+r\leq \sqrt{6(p^2+q^2+r^2)+15(p+q+r)+18}+\frac{18}{p+q+r+3}$$

Solution:
Let

$$p+q+r=aandp^2+q^2+r^2=bthenb\geq \frac{a^2}{3},$$

The inequality is equivalent after squaring to:

$$6b(a^2+6a+9)+27a+81\geq a^4+9a^3+54a^2$$

replacing b by $\frac{a^2}{3}$ in the LHS it suffices to prove that:

$$a^4+3a^3+27a+81\geq 36a^2$$

which is clearly true by AM-GM.
468.
, if a, b, c are three positive numbers such that $abc=1$. Let $S=a^2+b^2+c^2, S'=ab+bc+ca$. Prove or disprove

$$(S'+1)(S'+12)\leq 2(S+3)(S+2)$$

We know that $S\geq S'$, then replacing S by S' in the RHS it suffices to prove that $S'\geq 3$ which is clearly true by AM-GM whith $abc=1$.

327, if $a, b, c\geq 1$. Prove or disprove

$$3(\sqrt{a-1}+\sqrt{b-1}+\sqrt{b-1})\leq a+b+c+\sqrt{3(a+b+c-3)}\leq \frac{3\sqrt{3}}{2}\sqrt{a^2+b^2+c^2}$$

Solution:
(*) We will prove first that

$$3(\sqrt{a-1}+\sqrt{b-1}+\sqrt{b-1})\leq a+b+c+\sqrt{3(a+b+c-3)}$$

By AM-GM We have:

$$(\sqrt{a-1}+\sqrt{b-1}+\sqrt{b-1}) \leq \sqrt{3((a-1)+(b-1)+(c-1))} = \sqrt{3(a+b+c-3)}(1)$$

also We have that:

$$a = (a-1)+1 \geq 2\sqrt{a-1}$$

similarly for b and c,then adding cyclically the three inequalities We have:

$$a+b+c \geq 2(\sqrt{a-1}+\sqrt{b-1}+\sqrt{b-1})(2)$$

adding the inequalities (1) and (2) We get the desired result.
(*) Let us prove now that

$$a+b+c+\sqrt{3(a+b+c-3)} \leq \frac{3\sqrt{3}}{2}\sqrt{a^2+b^2+c^2}$$

it's easy to see from AM-GM that:

$$\frac{3\sqrt{3}}{2}\sqrt{a^2+b^2+c^2} \geq \frac{3}{2}.(a+b+c)$$

it suffices to prove that:

$$\frac{3}{2}.(a+b+c) \geq a+b+c+\sqrt{3(a+b+c-3)}$$

which is equivalent to:

$$(a+b+c-6)^2 \geq 0$$

469.
, Let $a,b,c \in R$.Prove

$$3(a+b+c) \leq \sqrt{25(a^2+b^2+c^2)+2(ab+bc+ca)} \leq 3\sqrt{3(a^2+b^2+c^2)}$$

Prove of LHS
Solution: The inequality is equivalent to

$$9(a+b+c)^2 \leq 24(a^2+b^2+c^2)+(a+b+c)^2.$$

$$\Leftrightarrow (a+b+c)^2 \leq 3(a^2+b^2+c^2)$$

Prove of RHS
The inequality is equivalent to

$$24(a^2+b^2+c^2)+(a+b+c)^2 \leq 27(a^2+b^2+c^2).$$

$$\Leftrightarrow (a+b+c)^2 \leq 3(a^2+b^2+c^2)$$

470,
Let x,y be two positive numbers such that $xy=1$. Prove or disprove

$$\frac{(\sqrt{x}+\sqrt{y})(x+y)(x^2+y^2)}{\sqrt[4]{x}+\sqrt[4]{y}} \geq 4.$$

We have:

$$\sqrt{x}+\sqrt{y} \geq \frac{(\sqrt[4]{x}+\sqrt[4]{y})^2}{2} \geq \sqrt[4]{x}+\sqrt[4]{y}$$

But $(x+y)(x^2+y^2)\geq 4$, then We have Q.E.D

your inequality is true because it is equivalent to

$$\frac{(x+1)(x^2+1)(x^4+1)}{x^{\frac{13}{4}}(\sqrt{x}+1)}\geq 4$$

and this is equivalent to

$$x^7+x^6+x^5+x^4+x^3+x^2+x+1-4x^{\frac{15}{4}}-4x^{\frac{13}{4}}\geq 0$$

This true by AM-GM, since

$$\frac{x^7+x^3+x^2+x}{4}\geq \sqrt[4]{x^{13}}$$

and

$$\frac{x^6+x^5+x^4+1}{4}\geq \sqrt[4]{x^{15}}$$

We have:

$$x+y\geq \frac{(\sqrt{x}+\sqrt{y})^2}{2}\geq \frac{(\sqrt[4]{x}+\sqrt[4]{y})^4}{8}$$

And

$$(\sqrt{x}+\sqrt{y})(x^2+y^2)\geq 4$$

Then,We have. Equality holds when $x=y=1$.

471.
Let $a,b,c>0$ .Prove that:

$$\frac{(\sqrt{x}+\sqrt{y})(x+y)(x^2+y^2)}{(\sqrt[4]{x}+\sqrt[4]{y})^4}\geq \frac{\sqrt{x^3y^3}}{\sqrt{xy}+1}$$

Solution:

$$LHS\geq \frac{a^5}{2}withxy=a^4$$

So,We only prove that:

$$a^5(a^2+1)\geq 2a^6 <=> a^5(a-1)^2\geq 0$$

Which is obvious true. Equality holds when $x=y=1$ 472.
, Let $a,b,c>0$ such that $a^4+b^4+c^4=3$. Prove that

$$\sum_{cyc}\frac{a^3-b}{b+c}\geq 0$$

Solution:
ineq

$$<=> \sum(a^3-b)(a+b)(a+c)\geq 0$$

$$<=> \sum a^5+\sum a^4(b+c)+abc(a^2+b^2+c^2)-(a+b+c)(ab+bc+ca)-(a^2b+b^2c+c^2a)\geq 0$$

$$<=> (a^4+b^4+c^4)(a+b+c)+abc(a^2+b^2+c^2)-(a+b+c)(ab+bc+ca)-(a^2b+b^2c+c^2a)\geq 0)1)$$

We have:

$$a^3+b^3+c^3\geq a^2b+b^2c+c^2a$$

We need prove that:

$$LHS(1) \geq (a^4+b^4+c^4)(a+b+c)+abc(a^2+b^2+c^2)-[a^3+b^3+c^3+(a+b+c)(ab+bc+ca)] \geq 0$$

$$<=> (a+b+c-abc)(a^4+b^4+c^4-a^2-b^2-c^2) \geq 0$$

it's always true because $a^4+b^4+c^4=3$
473.
, Prove that for all positive numbers p, q

$$\sqrt{pq} \leq \left(\frac{p+q}{\sqrt{p}+\sqrt{q}}\right)^2.$$

Solution:
Since

$$p^2+q^2 \geq \sqrt{pq}(p+q),$$

adding $2pq$ both sides, We have

$$(p+q)^2 \geq \sqrt{pq}(p+2\sqrt{pq}+q)(p+q)^2 \geq \sqrt{pq}(\sqrt{p}+\sqrt{q})^2$$

The given inequality is equivalent to

$$p^2+q^2 \geq \sqrt{pq}(p+q)$$

squaring this We get

$$p^4+q^4-p^3q-pq^3 \geq 0$$

this is true because it is equivalent with

$$(p-q)^2(p^2+pq+q^2) \geq 0.$$

very easy, this is our **Solution**,

$$\left(\frac{p+q}{\sqrt{p}+\sqrt{q}}\right)^2 \geq \left(\frac{\sqrt{p}+\sqrt{q}}{2}\right)^2 \geq \sqrt{pq}$$

474,
Let $u, v \in R^+$. Prove that

$$1+2\frac{\sqrt{u}+\sqrt{v}}{u+v} \leq \frac{1+u+v}{\sqrt{uv}}$$

Solution:
We have to prove that:

$$\sqrt{uv}+\frac{2\sqrt{u}v+2\sqrt{v}u}{u+v} \leq 1+u+v$$

but

$$\sqrt{uv} \leq \frac{u+v}{2}$$

so We will prove the stronger:

$$u^2+v^2+2uv+2(u+v) \geq 4u\sqrt{v}+4v\sqrt{u}$$

but:

$$u^2+v+uv+u \geq 4u\sqrt{v}; v^2+u+uv+v \geq 4v\sqrt{u}$$

adding these ineqs

2) First, We need to prove:

$$u^2+v^2 \geq \sqrt{uv}(u+v).$$

Proving above inequality:
By c-s inequality,

$$2(u^2+v^2) \geq (u+v)^2 \Rightarrow u^2+v^2 \geq \frac{u+v}{2}(u+v) \Rightarrow u^2+v^2 \geq \sqrt{uv}(u+v)$$

Hence, adding up u + v, We have

$$u^2+v^2+u+v \geq (\sqrt{uv}+1)(u+v) \Leftrightarrow \frac{u}{u+v}(1+u)+\frac{v}{u+v}(1+v) \geq \sqrt{uv}+1 \Leftrightarrow 2+u+v \geq$$

$$\sqrt{uv}+1+\frac{v}{u+v}(1+u)+\frac{u}{u+v}(1+v)$$

$$\left[\text{Add both sides } \frac{v}{u+v}(1+u)+\frac{u}{u+v}(1+v)\right] \Leftrightarrow 1+u+v \geq \sqrt{uv}+\frac{v}{u+v}2\sqrt{u}+\frac{u}{u+v}2\sqrt{v}[\text{By am-gm}]$$

$$newline \Leftrightarrow \frac{1+u+v}{\sqrt{uv}} \geq 1+2\frac{\sqrt{u}+\sqrt{v}}{u+v}$$

which is equivalent to

$$1+2\frac{\sqrt{u}+\sqrt{v}}{u+v} \leq \frac{1+u+v}{\sqrt{uv}}.$$

The equality is $u=v=1$

475.
Let $a,b \in (0,1]$. Prove that

$$\frac{(a+b)^{2a}+(a+b)^{2b}}{2^{a+b+1}} \geq (ab)^{\frac{b}{a}+\frac{a}{b}}$$

Solution:

$$\frac{(a+b)^{2a}+(a+b)^{2b}}{2^{a+b+1}} \geq (ab)^{\frac{b}{a}+\frac{a}{b}}$$

$$\Leftrightarrow \frac{(a+b)^{2a}+(a+b)^{2b}}{2^{a+b}} \geq 2(ab)^{\frac{b}{a}+\frac{a}{b}}$$

Applying AM-GM:

$$LHS=\frac{(a+b)^{2a}+(a+b)^{2b}}{2^{a+b}} \geq \frac{4^a(ab)^a+4^b(a+b)^b}{2^{a+b}}$$

$$=\frac{2^a}{2^b}.(ab)^a+\frac{2^b}{2^a}.(ab)^b \geq 2.\sqrt{\frac{2^a}{2^b}.(ab)^a.\frac{2^b}{2^a}.(ab)^b}=2(ab)^{\frac{a+b}{2}}$$

Since $a,b \in (0;1] \Rightarrow (ab) \in (0;1](1)$ And We have:

$$\frac{a+b}{2} \leq 1 < 2 \leq \frac{a}{b}+\frac{b}{a}(2)$$

From (1) and (2)

$$\Rightarrow (ab)^{\frac{a+b}{2}} \geq (ab)^{\frac{a}{b}+\frac{b}{a}} \Rightarrow 2(ab)^{\frac{a+b}{2}} \geq 2(ab)^{\frac{a}{b}+\frac{b}{a}} \Rightarrow LHS \geq RHS$$

We think this inequality is very iak, equality holds when $a=b=1$.

476.
For any positive real numbers a, b and c, prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c \geq \frac{6(a^2+b^2+c^2)}{a+b+c}$$

Solution:

$$\sum_{cyc}(\frac{a^2}{b}+b-2a) \geq 2(\frac{3(a^2+b^2+c^2)}{a+b+c}-a-b-c)$$

$$\Leftrightarrow \sum_{cyc}\frac{(a-b)^2}{b} \geq 2\sum_{sym}\frac{(a-b)^2}{a+b+c}$$

$$\Leftrightarrow \sum_{cyc}(a-b)^2(\frac{1}{b}-\frac{2}{a+b+c}) \geq 0$$

Easy to see that We only need to check the case $a \geq b \geq c$. So $S_c \geq 0, S_a \geq 0$. Easy to prove

$$S_a+2S_b, S_c+2S_b \geq 0$$

Notive that if $S_a+S_b+S_c \geq 0$ then We can assume that $S_a+S_b \geq 0$. Let $x=a-b, y=b-c$ then

$$S_c x^2+S_a y^2+S_b(x+y)^2=(S_a+S_b)y^2+2S_b xy+(S_b+S_c)x^2$$

Because

$$\Delta'=S_b^2-(S_b+S_a)(S_b+S_c)=-(S_aS_b+S_bS_c+S_cS_a) \leq 0$$

So We are done. But this case is little using

We enough to prove $S_a+S_b+S_c \geq 0$ and $S_aS_b+S_aS_c+S_bS_c \geq 0$. But

$$S_a+S_b+S_c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{6}{a+b+c} \geq 0 and S_aS_b+S_aS_c+S_bS_c \geq 0 \Leftrightarrow$$

$$\Leftrightarrow a^3+b^3+c^3-a^2b-a^2c-b^2a-b^2c-c^2a-c^2b+6abc \geq 0,$$

which obvuously true.

477,
if $a_1, a_2, ..., a_n$ are nonnegative numbers such that $a_1^2+a_2^2+...+a_n^2=n$, then

$$\sum\frac{1}{a_1^2+1} \leq \frac{n^3}{2(\sum a_1)^2}.$$

Solution:

$$\frac{a_1^2}{2a_1^2}+\frac{a_2^2}{a_1^2+a_2^2}+\cdots+\frac{a_n^2}{a_1^2+a_n^2} \geq \frac{(a_1+\cdots+a_n)^2}{na_1^2+(a_1^2+\cdots+a_n^2)}$$

Adding this and similar inequalities, the result follows. Variation on the same theme: if $a_1, a_2, ..., a_n$ are nonnegative numbers such that $a_1^2+a_2^2+...+a_n^2=n$, then

$$\sum_{k=1}^{n}\frac{1}{a_k^2+n-1} \leq \frac{n^2}{(a_1+\cdots+a_n)^2}$$

478,
Prove that for all positive real numbers a, b and c,

$$\frac{bc}{a^2+bc}+\frac{ca}{b^2+ca}+\frac{ab}{c^2+ab}\le\frac{1}{2k+1}\left(3k+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

where k=1/2.

Solution:
rewrite the inequality as

$$\sum_{cycl}\left(\frac{2bc}{a^2+bc}-1\right)\le\frac{1}{2k+1}\sum_{cycl}\left(\frac{2a}{b+c}-1\right)$$

$$\Leftrightarrow\sum_{cycl}\frac{bc-a^2}{a^2+bc}\le\frac{1}{2k+1}\sum_{cycl}\frac{(a-b)^2}{(a+c)(b+c)}$$

We have

$$\sum_{cycl}\frac{bc-a^2}{a^2+bc}=\frac{1}{2}\sum_{cycl}\frac{(a+c)(b-a)+(a+b)(c-a)}{a^2+bc}=\frac{1}{2}\sum_{cycl}\frac{(a-b)^2(ab-c^2)}{(a^2+bc)(b^2+ca)}$$

Thus the original inequality is equivalent to

$$\sum_{cycl}(a-b)^2\left(\frac{2}{(a+c)(b+c)}+\frac{(2k+1)(c^2-ab)}{(a^2+bc)(b^2+ca)}\right)\ge 0$$

For $0\le k\le\frac{1}{2}$, it suffices to show that

$$(a^2+bc)(b^2+ca)\ge ab(a+c)(b+c)$$

345, Let a, b, c be positive real number. Prove that

$$(a+b+c)[\frac{a}{(2a+b+c)(b+c)}+\frac{b}{(2b+c+a)(c+a)}+\frac{c}{(2c+a+b)(a+b)}]\ge\frac{9}{8}.$$

Let $a+b=z, a+c=y$ and $b+c=x$. Hence,

$$(a+b+c)[\frac{a}{(2a+b+c)(b+c)}+\frac{b}{(2b+c+a)(c+a)}+\frac{c}{(2c+a+b)(a+b)}]\ge\frac{9}{8}\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\frac{(x+y+z)(y+z-x)}{x(y+z)}\ge\frac{9}{2}\Leftrightarrow\sum_{cyc}(x+y+z)\left(\frac{1}{x}-\frac{1}{y+z}\right)\ge\frac{9}{2}\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}(x-y)^2\left(\frac{1}{xy}-\frac{1}{2(z+x)(z+y)}\right)\ge 0$$

which obviously true.

Since the inequality is homogenuous We suppose a+b+c=1 and the inequality is equivalent to:

$$\sum\frac{a}{1-a^2}\ge\frac{9}{8}$$

Now considering the function $f:(0,1)->\mathbb{R}, f(x)=\frac{x}{1-x^2}$ (f is convex) from Jensen's inequality We have :

$$\sum\frac{a}{1-a^2}=\sum f(a)\ge 3\cdot f\left(\frac{a+b+c}{3}\right)=3\cdot f\left(\frac{1}{3}\right)=\frac{9}{8}$$

Let $a+b+c=3$
$$ineq <=> \frac{a}{9-a^2}+\frac{b}{9-b^2}+\frac{c}{9-c^2}\geq\frac{3}{8}$$
We have
$$\frac{a}{9-a^2}\geq\frac{5a}{32}-\frac{1}{32}$$
this ineq
$$<=> \frac{(a-1)^2(5a+9)}{32(9-a^2)}\geq 0 it is true$$
So
$$\frac{a}{9-a^2}+\frac{b}{9-b^2}+\frac{c}{9-c^2}\geq\frac{5(a+b+c)}{32}-\frac{3}{32}=\frac{3}{8}$$
346, if a,b,c are positive prove that
$$\frac{a}{(2a+b+c)(b+c)}+\frac{b}{(2b+a+c)(a+c)}+\frac{c}{(2c+a+b)(a+b)}\geq\frac{9}{8(a+b+c)}$$
Let $a+b+c=1$
$$=> (1) <=> \sum a/(1-a^2)\geq 9/8 <=> \sum a^2/(a-a^3)\geq 9/8$$
We know that
$$a_1^2/b_1+a_2^2/b_2+a_3^2/b_3\geq(a_1+a_2+a_3)^2/(b_1+b_2+b_3) =>$$
$$\sum a^2/(a-a^3)\geq 1/(1-a^3-b^3-c^3)\geq 9/8$$
Becous $(a^3+b^3+c^3)/3\geq((a+b+c)/3)^3$

Here is another **Solution**
$$\sum\frac{a}{(2a+b+c)(b+c)}=\frac{1}{2}\sum\left[\frac{1}{b+c}-\frac{1}{2a+b+c}\right]\geq$$
$$\frac{1}{2}\left[\sum\frac{1}{b+c}-\frac{1}{2}\sum\frac{1}{b+c}\right]=\frac{1}{4}\sum\frac{1}{b+c}\geq\frac{9}{8(a+b+c)}$$
where the first inequality follows from the ill-known fact
$$\sum\frac{1}{x+y}\leq\frac{1}{2}\sum\frac{1}{x}$$
(put x=b+c, y=c+a and z=a+b) and the second from CBS inequality.

479.
, For any positive real numbers a,b and c,
$$\frac{ab^2}{c(c+a)}+\frac{bc^2}{a(a+b)}+\frac{ca^2}{b(b+c)}\geq\frac{3}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c}$$
Solution:
This inequality, We can prove just use the followings
$$\sum a^2+6\geq\frac{3}{2}(a+b+c+ab+bc+ca)\qquad\forall a,b,c>0, abc=1$$
$$(a+b+c)^2\geq 3(ab+bc+ca)\qquad\forall a,b,c>0$$

Now, Let We post our **Solution** for it

Lemma. if $x, y, z \geq 0$ such that $xyz = 1$ then

$$x^2 + y^2 + z^2 + 6 \geq \frac{3}{2}(x + y + z + xy + yz + zx)$$

We can prove it by mixing variable method.

Back to the original problem

Setting $a = \frac{1}{x}, b = \frac{1}{y}, z = \frac{1}{z}$ then the inequality becomes

$$\sum \frac{x^2}{y^2(z+x)} \geq \frac{3\sum x^2y^2}{2xyz\sum xy}$$

$$\Leftrightarrow \sum \frac{x^2}{y^2} + \sum \frac{x^2}{y(z+x)} \geq \frac{3(\sum x^2y^2)(\sum x)}{2xyz\sum xy}$$

Using the above lemma, We get

$$\sum \frac{x^2}{y^2} \geq \frac{3}{2}\sum \frac{x^2+y^2}{xy} - 6$$

and the Cauchy Schwarz inequality gives us

$$\sum \frac{x^2}{y(z+x)} \geq \frac{(\sum x)^2}{2\sum xy}$$

it suffices to show that

$$\frac{3}{2}\sum \frac{x^2+y^2}{xy} - 6 + \frac{(\sum x)^2}{2\sum xy} \geq \frac{3(\sum x^2y^2)(\sum x)}{2xyz\sum xy}$$

$$\Leftrightarrow \frac{7(\sum x^2 - \sum xy)}{2\sum xy} \geq 0$$

which is true.

480,

Prove that, for any positive real numbers a, b and c,

$$\frac{b^2+c^2}{a^2+bc} + \frac{c^2+a^2}{b^2+ca} + \frac{a^2+b^2}{c^2+ab} \geq 2 + \frac{a^2+b^2+c^2}{ab+bc+ca}$$

Solution:

$$\sum_{cyc} \frac{b^2+c^2}{a^2+bc} = \sum_{cyc} \frac{a^2+b^2+c^2}{a^2+bc} - \sum_{cyc} \frac{a^2}{a^2+bc}$$

$$\frac{a^2}{a^2+bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab} \leq \frac{(a+b+c)^2}{2(ab+bc+ca)}$$

Note that

$$\sum_{cycl} \frac{(b+c)^2}{a^2+bc} = \sum_{cycl} \frac{b^2+c^2}{a^2+bc} + \sum_{cycl} \frac{2bc}{a^2+bc}$$

$$= \sum_{cycl} \frac{b^2+c^2}{a^2+bc} + 2\sum_{cycl}\left(1 - \frac{a^2}{a^2+bc}\right)$$

$$= 6 + \sum_{cycl} \frac{b^2+c^2}{a^2+bc} - 2\sum_{cycl} \frac{a^2}{a^2+bc}$$

481.
Let $a, b, c >$ .Prove that:

$$\sum_{cycl} \sqrt{\frac{b+c}{a}} \geq \sqrt{6} \frac{a+b+c}{\sqrt{ab+bc+ca}}$$

Solution:

$$\sum \frac{b+c}{a} + \sum \sqrt{\frac{b+c}{a}} \sqrt{\frac{c+a}{b}} \geq 6 \frac{(\sum a)^2}{\sum ab}$$

then We used Cauchy

$$(\frac{b+c}{a})(\frac{c+a}{b}) = (1+\frac{c}{a})(1+\frac{c}{b}) \geq (1+\frac{c}{\sqrt{ab}})^2$$

1st **Solution**: Chebyshev and Cauchy
2nd **Solution**: Holder and AM-GM
Very nice inequality We will follow the second hint Razz (We like Holder) By holder We have

$$(\sum_{cycl} \sqrt{\frac{b+c}{a}})^2 (\sum (a(b+c)^2)) \geq 8(a+b+c)^3$$

After some trivial munipulation We have to prove that

$$4(a+b+c)(ab+bc+ca) \geq 3 \sum (a(b+c)^2)$$

which is equivalent to

$$\sum (a^2b + b^2a) \geq 6abc$$

which is a plain AM-GM QED

WLOG $a \geq b \geq c$ then $a+b \geq a+c \geq b+c$ and

$$\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a}$$

so by Chebyshev

$$\sum \sqrt{\frac{b+c}{a}} \geq \frac{1}{3} \sum (\frac{1}{\sqrt{a}}) \sum (\sqrt{b+c})$$

This way do you use Chebychev?

$$\sum_{cycl} \sqrt{\frac{b+c}{a}} = \sum_{cycl} \frac{b+c}{\sqrt{a(b+c)}} \cdots$$

482,
For positive reals a, b, c. Prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \leq (((b+c)(c+a)(a+b))^{\frac{2}{3}} - (abc)^{\frac{2}{3}})^3.$$

Solution:
Rewrite the inequality as

$$\left(1+\sqrt[3]{\frac{(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)}{a^2b^2c^2}}\right)^3 \leq \frac{(a+b)^2(b+c)^2(c+a)^2}{a^2b^2c^2}$$

By Holder inequality,

$$\left(1+\sqrt[3]{\prod_{cycl}\frac{a^2+ab+b^2}{ab}}\right)^3 \le \prod_{cycl}\left(1+\frac{a^2+ab+b^2}{ab}\right)=\prod_{cycl}\frac{(a+b)^2}{ab}$$

Nice. in fact, our **Solution** uses the similar method.

$$\prod(a^2+ab+b^2)=\prod((a+b)^2-ab)\le (RHS), since \prod(p_i-q_i)\le ((\prod p_i)^{\frac{1}{3}}-(\prod q_i)^{\frac{1}{3}})^3$$

holds by Holder.

483,
For positive reals a, b, c prove that

$$\frac{4(a+b+c)(bc+ca+ab)}{(b+c)^2+(c+a)^2+(a+b)^2}\le ((2a+b)(2b+c)(2c+a))^{\frac{1}{3}}.$$

Solution:

$$((a+b+c)(bc+ca+ab))^3=((a+b)(b+c)(c+a)+abc)^3\le((a+b)^3+a^3)((b+c)^3+b^3)((c+a)^3+c^3)$$
$$=(2a+b)(2b+c)(2c+a)(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)$$

so We need to show that

$$8(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\le(a^2+b^2+c^2+bc+ca+ab)^3$$

484.
Let x, y, z be positive real numbers. Prove that

$$\sqrt{\frac{x}{x+y}}+\sqrt{\frac{y}{y+z}}+\sqrt{\frac{z}{z+x}}\le\frac{3\sqrt{2}}{2}$$

Solution:

$$LHS=\frac{\sqrt{x(y+z)(z+x)}+\sqrt{y(z+x)(x+y)}+\sqrt{z(x+y)(y+z)}}{\sqrt{(x+y)(y+z)(z+x)}}$$
$$\le\sqrt{\frac{(x(y+z)+y(z+x)+z(x+y))(x+y+y+z+z+x))}{(x+y)(y+z)(z+x)}}$$
$$=\sqrt{4\cdot\frac{(xy+yz+zx)(x+y+z)}{(x+y)(y+z)(z+x)}}$$
$$=2\cdot\sqrt{\frac{(x+y)(y+z)(z+x)+xyz}{(x+y)(y+z)(z+x)}}$$
$$=2\cdot\sqrt{1+\frac{xyz}{(x+y)(y+z)(z+x)}}$$
$$\le 2\cdot\sqrt{1+\frac{1}{8}}$$
$$=\frac{3\sqrt{2}}{2}.$$

485.
, Let a, b, c be positive real numbers. Prove that:

$$\sqrt[3]{\frac{a^2+bc}{b^2+c^2}}+\sqrt[3]{\frac{b^2+ca}{c^2+a^2}}+\sqrt[3]{\frac{c^2+ab}{a^2+b^2}} \geq \frac{9\sqrt[3]{abc}}{a+b+c}$$

Solution:
By the AM-GM inequality,

$$\frac{a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)}{a^2+bc} = \frac{a(b^2+c^2)}{a^2+bc}+b+c \geq 3\sqrt[3]{\frac{abc(b^2+c^2)}{a^2+bc}}$$

Hence

$$\sum_{cycl}\sqrt[3]{\frac{a^2+bc}{abc(b^2+c^2)}} \geq \frac{3(a^2+b^2+c^2+ab+bc+ca)}{a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)} \geq \frac{9}{a+b+c}$$

where We use the fact

$$(a+b+c)(a^2+b^2+c^2+ab+bc+ca)-3\sum_{cycl}a(b^2+c^2) = \sum_{cycl}a(a-b)(a-c) \geq 0$$

486.
, Let a, b, c be positive real numbers. Prove that:

$$a^3+b^3+c^3+ab^2+bc^2+ca^2 \geq 2(a^2b+b^2c+c^2a)$$

Solution:

$$a^3+b^3+c^3+ab^2+bc^2+ca^2 = a(a^2+b^2)+b(b^2+c^2)+c(c^2+a^2) \geq \geq 2(a^2b+b^2c+c^2a)$$

$$a^3+ab^2 \geq 2a^2b$$

by AM-GM, and i're done.

$$a^3+b^3+c^3+ab^2+bc^2+ca^2 \geq 2\sqrt{(a+b+c)(a^3b^2+b^3c^2+c^3a^2)}$$

487.
Let a, b, c be positive. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq (a+b+c)\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}.$$

Solution:
By Cauchy inequality,

$$\sum_{cycl}b(c+a)^2\sum_{cycl}\frac{a^2}{b} \geq (a^2+b^2+c^2+ab+bc+ca)^2$$

it suffices to show that

$$(a^2+b^2+c^2+ab+bc+ca)^2 \geq (a+b+c)\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}\sum_{cycl}b(c+a)^2$$

Since

$$a^2+b^2+c^2+ab+bc+ca \geq \frac{2}{3}(a+b+c)^2 \geq \frac{a+b+c}{2(ab+bc+ca)}\sum cb(c+a)^2$$

$$a^2+b^2+c^2+ab+bc+ca \geq \frac{2}{3}(a+b+c)^2 \geq \frac{a+b+c}{2(ab+bc+ca)}b(c+a)^2$$

$$a^2+b^2+c^2+ab+bc+ca \geq \frac{2}{3}(a+b+c)^2 \geq \frac{a+b+c}{2(ab+bc+ca)}\sum b(c+a)^2$$

it remains to prove

$$a^2+b^2+c^2+ab+bc+ca \geq 2(ab+bc+ca)\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}} = 2\sqrt{(a^2+b^2+c^2)(ab+bc+ca)}$$

which is true by the AM-GM inequality.

Here is another **Solution** We found .Hope to like it
By Cauchy Swartz

$$\sum \frac{a^2}{b} \geq \frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a}$$

So it suffices to prove that

$$(a^2+b^2+c^2)^3(ab+bc+ca) \geq (a^2b+b^2c+c^2a)^2(a+b+c)^2$$

By Vasc's inequality

$$(a^2+b^2+c^2)^2 \geq 3(a^3b+b^c+c^3a)$$

and

$$3(a^2+b^2+c^2) \geq (a+b+c)^2$$

So it remains to show that

$$(a^3b+b^3c+c^3a)(ab+bc+ca) \geq (a^2b+b^2c+c^2a)^2$$

which is true by Cauchy -Swartz inequality .

488.
, Prove that for all positive real numbers a, b and c. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq \frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}$$

Solution:
by cross-multiplication We get

$$\sum a^3+\sum a^2c+\sum \frac{a^3c}{b} \geq \sum a^3+\sum a^2b+\sum a^2c$$

or equivalently

$$\sum a^4c^2 \geq \sum a^3b^2c$$

which is true by Muirhead (or alternatively notice that

$$a^4c^2+b^4a^2 \geq 2a^3b^2c$$

from AM-GM and We are done).

489,
Let x, y, z positive real numbers such that $x^2+y^2+z^2=1$. Prove that

$$xyz+\sqrt{\sum x^2y^2} \geq \frac{4}{3}\sqrt{xyz(x+y+z)}$$

Solution:
if a,b and c are positive reals such that $a^{-2}+b^{-2}+c^{-2}=1$, then

$$1+\sqrt{a^2+b^2+c^2}\geq\frac{4}{3}\sqrt{ab+bc+ca}$$

Homogenizing:

$$1+\sqrt{(a^2+b^2+c^2)(a^{-2}+b^{-2}+c^{-2})}\geq\frac{4}{3}\sqrt{(ab+bc+ca)(a^{-2}+b^{-2}+c^{-2})}$$

Without loss of generality, assume a+b+c=1. Note that

$$\sum_{cycl}\left(\frac{2}{3}-a\right)^2=a^2+b^2+c^2$$

By Cauchy-Schwarz inequality,

$$\sqrt{(a^2+b^2+c^2)(a^{-2}+b^{-2}+c^{-2})}=\sqrt{\sum_{cycl}\left(\frac{2}{3}-a\right)^2\sum_{cycl}a^{-2}}=\sum_{cycl}\frac{1}{a}\left(\frac{2}{3}-a\right)$$

$$\frac{2}{3}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-3=\frac{2}{3}\sum_{cycl}\frac{b+c}{a}-1$$

it remains to show that

$$\sum_{cycl}\frac{b+c}{a}\geq 2\sqrt{(ab+bc+ca)(a^{-2}+b^{-2}+c^{-2})}$$

which is not hard !!

490.
Let a,b,c are non-negative. Prove that

$$\frac{a^4}{b+c}+\frac{b^4}{a+c}+\frac{c^4}{a+b}\geq\frac{1}{2}(a^3+b^3+c^3)$$

Solution:
Suppose $a\geq b\geq c>0$. Then:

$$a^4\geq b^4\geq c^4 and\frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}$$

Use Chebushev's inequality with 2 pairs of number $(a^4,b^4,c^4$ and $(\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b})$ We have:

$$\frac{a^4}{b+c}+\frac{b^4}{c+a}+\frac{c^4}{a+b}\geq\frac{1}{3}(a^4+b^4+c^4)(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b})(1)$$

Use inequality

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{9}{x+y+z}$$

We have:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{9}{2(a+b+c)}(2)$$

From (1),(2):

$$\frac{a^4}{b+c}+\frac{b^4}{a+c}+\frac{c^4}{a+b}\geq\frac{3}{2}.\frac{a^4+b^4+c^4}{a+b+c}$$

$$\geq \frac{a^3+b^3+c^3}{2}$$

Because

$$a^4+b^4+c^4 \geq \frac{1}{3}(a^3+b^3+c^3)(a+b+c)$$

Use

$$\frac{a^4}{b+c}+\frac{a^2(b+c)}{4} \geq a^3$$
$$6a^3+b^3 \geq ab(a+b)$$

491.

Let $a,b,c>0$. Prove that:

$$\frac{a^2}{b+c}+\frac{b^2}{a+c}+\frac{c^2}{a+b} \geq \frac{3}{2}.\frac{a^3+b^3+c^3}{a^2+b^2+c^2}.$$

Solution:

Use Cauchy:

$$\left(\sum_{\text{cyc}} \frac{a^2}{b+c}\right)\left(\sum_{\text{cyc}} a^4(b+c)\right) \geq (a^3+b^3+c^3)^2$$

The remaining inequality is

$$2(a^3+b^3+c^3)(a^2+b^2+c^2) \geq 3\left(\sum_{\text{cyc}} a^4(b+c)\right)$$

which is very nice. We have

$$\sum_{\text{cyc}} a^4(b+c) = (a^3+b^3+c^3)(ab+bc+ca) - abc(a^2+b^2+c^2)$$

and

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

so it factors! Mr. Green it's now equivalent to

$$(a^2+b^2+c^2-ab-bc-ca)\left(3(a^3+b^3+c^3)-(a^2+b^2+c^2)(a+b+c)\right) \geq 0$$

which is easy enough.

492,

For any three positive reals a,b,c We have

$$3\frac{a^2+b^2+c^2}{a+b+c} \geq \sum \frac{a^2+b^2}{a+b}$$

Solution:

$$3\frac{a^2+b^2+c^2}{a+b+c} - \sum \frac{a^2+b^2}{a+b} = \sum \left(\frac{a^2+b^2+c^2}{a+b+c} - \frac{a^2+b^2}{a+b}\right)$$
$$= \sum \left(\frac{1}{a+b+c} \cdot \left((a-b)\, ab \frac{1}{b+c} - (c-a)\, ca \frac{1}{b+c}\right)\right)$$
$$= \frac{1}{a+b+c} \cdot \sum \left((a-b)\, ab \frac{1}{b+c} - (c-a)\, ca \frac{1}{b+c}\right)$$
$$= \frac{1}{a+b+c} \cdot \left(\sum \left((a-b)\, ab \frac{1}{b+c}\right) - \sum \left((c-a)\, ca \frac{1}{b+c}\right)\right)$$

$$= \frac{1}{a+b+c} \cdot \left(\sum \left((a-b)\, ab \frac{1}{b+c} \right) - \sum \left((a-b)\, ab \frac{1}{c+a} \right) \right)$$

$$= \frac{1}{a+b+c} \cdot \sum \left((a-b)\, ab \frac{1}{b+c} - (a-b)\, ab \frac{1}{c+a} \right)$$

$$= \frac{1}{a+b+c} \cdot \sum \frac{(a-b)^2\, ab}{(b+c)\,(c+a)} \geq 0$$

and thus

$$3 \frac{a^2+b^2+c^2}{a+b+c} \geq \sum \frac{a^2+b^2}{a+b}$$

Solution complete.

ill a little over two years later

why harazis stronger?

The first is

$$\iff \frac{a^2+b^2+c^2}{a+b+c} \geq \sqrt[3]{\frac{\prod(a^2+b^2)}{\prod(a+b)}}$$

So the second is stronger because:

$$\frac{a^2+b^2+c^2}{a+b+c} \geq \frac{\sum \frac{a^2+b^2}{a+b}}{3} \geq \sqrt[3]{\frac{\prod(a^2+b^2)}{\prod(a+b)}}$$

$$3 \frac{a^2+b^2+c^2}{a+b+c} - \sum \frac{a^2+b^2}{a+b} = \sum \left(\frac{a^2+b^2+c^2}{a+b+c} - \frac{a^2+b^2}{a+b} \right)$$

$$= \sum \left(\frac{1}{a+b+c} \cdot \left((a-b)\, ab \frac{1}{b+c} - (c-a)\, ca \frac{1}{b+c} \right) \right)$$

Q.E.D

493,

Let a, b, c be non-negative real numbers with sum 2. Prove that

$$\sqrt{a+b-2ab} + \sqrt{b+c-2bc} + \sqrt{c+a-2ca} \geq 2.$$

Solution:

We may write the inequality in the form

$$\sqrt{\frac{a+b}{2} - ab} + \sqrt{\frac{b+c}{2} - bc} + \sqrt{\frac{c+a}{2} - ca} \geq \sqrt{2}.$$

Squaring both sides of the inequality gives

$$a+b+c-(ab+bc+ca) + 2\sqrt{\left(\frac{a+b}{2} - ab\right)\left(\frac{b+c}{2} - bc\right)}$$

$$+2\sqrt{\left(\frac{c+a}{2} - ca\right)\left(\frac{a+b}{2} - ab\right)} + 2\sqrt{\left(\frac{b+c}{2} - bc\right)\left(\frac{c+a}{2} - ca\right)} \geq 2,$$

which reads as follows after some simple manupilations

$$\sqrt{\left(\frac{a+b}{2} - ab\right)\left(\frac{b+c}{2} - bc\right)} + \sqrt{\left(\frac{c+a}{2} - ca\right)\left(\frac{a+b}{2} - ab\right)} + \sqrt{\left(\frac{b+c}{2} - bc\right)\left(\frac{c+a}{2} - ca\right)} \geq \frac{1}{2}(ab+bc+ca).$$

Put

$$t = \sqrt{\left(\frac{a+b}{2} - ab\right)\left(\frac{b+c}{2} - bc\right)}.$$

Notice that
$$\frac{a+b}{2}-ab=\frac{1}{4}(a-b)^2+\frac{1}{4}c(a+b).$$
By Cauchy Schwarz inequality, We have
$$t\geq\frac{1}{2}|(a-b)(b-c)|+\frac{1}{2}\sqrt{ca(a+b)(b+c)}.$$
Now the rest is no problem We think because
$$|(a-b)(b-c)|+|(b-c)(c-a)|+|(c-a)(a-b)|+\sqrt{ca(a+b)(c+b)}+\sqrt{ab(a+c)(b+c)}+\sqrt{bc(b+a)(c+a)}\geq 2(ab+bc+c$$
So the art of problem solving is to get a problem no problem.
By Cauchy Schwarz,
$$(a+c)(a+b)\geq(a+\sqrt{bc})^2.$$
Thus, it is sufficient to prove that
$$\sum_{\text{cyclic}}|(a-b)(b-c)|+ab+bc+ca+\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})\geq 2(ab+bc+ca).$$
Note that
$$a^2+b^2+c^2-ab-bc-ca\leq\sum_{\text{cyclic}}|(a-b)(b-c)|.$$
To finish the **Solution**, We show that
$$a^2+b^2+c^2+\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})\geq 2(ab+bc+ca),$$

494.
, For any positive real numbers a,b and c,
$$\frac{a(a+c)}{b(b+c)}+\frac{b(b+a)}{c(c+a)}+\frac{c(c+b)}{a(a+b)}\geq\frac{3(a^2+b^2+c^2)}{ab+bc+ca}$$
Your way is very complicated.our **Solution** is very simple.
Hence,We have:
$$\frac{a(a+c)}{b(b+c)}+\frac{b(b+a)}{c(c+a)}+\frac{c(c+b)}{a(a+b)}\geq\frac{(a(a+c)+b(b+a)+c(c+b))^2}{ab(a+c)(b+a)+bc(b+a)(c+a)+ca(c+b)(a+b)}=A$$
Now,We will prove:
$$A\geq 3\frac{a^2+b^2+c^2}{ab+bc+ca}$$
We have;
$$\frac{(\sum a^2+\sum ab)^2}{(\sum ab)^2+abc\sum a}\geq 4\frac{\sum a^2\sum ab}{(\sum ab)^2+\frac{1}{3}(\sum ab)^2}$$
this inequality is proved!
$$\frac{a(a+c)}{b(b+c)}+\frac{b(b+a)}{c(c+a)}+\frac{c(c+b)}{a(a+b)}\geq_{\text{(Cauchy)}}\frac{(a(a+c)+b(b+a)+c(c+b))^2}{ab(a+c)(b+a)+bc(b+a)(c+a)+ca(c+b)(a+b)}$$
$$=\frac{(\sum a^2+\sum ab)^2}{(\sum ab)^2+abc\sum a}\geq_{\text{(Am-Gm)}}4\frac{\sum a^2\sum ab}{(\sum ab)^2+abc\sum a}\geq_{\text{(Muirhead)}}4\frac{\sum a^2\sum ab}{(\sum ab)^2+\frac{1}{3}(\sum ab)^2}$$
$$=\frac{3(a^2+b^2+c^2)}{ab+bc+ca};\qquad\text{Thus }\frac{a(a+c)}{b(b+c)}+\frac{b(b+a)}{c(c+a)}+\frac{c(c+b)}{a(a+b)}\geq\frac{3(a^2+b^2+c^2)}{ab+bc+ca}$$

495.
, if a, b, c are non-negative numbers, no two of which are zero, then

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{2(a+b+c)}.$$

Solution:
this follows trivially from

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{a+b+c}$$

We think the ineq Vacs post is true because it is a problem of Vu Dinh Quy our friend.You can solve it by using

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}=\frac{b^2}{a+b}+\frac{c^2}{b+c}+\frac{a^2}{c+a}$$

496.
Given some nonnegative numbers. Prove the following inequality:

$$\frac{a^2+b^2}{2a+2b+c}+\frac{b^2+c^2}{2b+2c+a}+\frac{c^2+a^2}{2c+2a+b} \ge \frac{2\sqrt{3(a^2+b^2+c^2)}}{5}.$$

Solution:
We have

$$\sum_{cycl}\left(\frac{5(b^2+c^2)}{a+2b+2c}-(b+c)\right)$$

$$=\sum_{cycl}\frac{b(c-a)+c(b-a)+3(b-c)^2}{a+2b+2c}$$

$$=\sum_{cycl}(a-b)^2\left(\frac{3}{c+2a+2b}-\frac{c}{(a+2b+2c)(b+2c+2a)}\right)$$

and

$$2\sqrt{3(a^2+b^2+c^2)}-2(a+b+c)=2\frac{(a-b)^2+(b-c)^2+(c-a)^2}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}$$

The given inequality is thus equivant to $x(b-c)^2+y(c-a)^2+z(a-b)^2 \ge 0$ where

$$z=\frac{3}{c+2a+2b}-\frac{c}{(a+2b+2c)(b+2c+2a)}-\frac{2}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}$$

(and similarly for x and y). But $z \ge 0$ since

$$\frac{1}{c+2a+2b} \ge \frac{c}{(a+2b+2c)(b+2c+2a)}$$

and

$$\frac{2}{c+2a+2b} \ge \frac{1}{a+b+c} \ge \frac{2}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}$$

(and similarly for x and y).

497.
For positive real numbers a, b and c,

$$\sum_{cycl}\frac{ab(a^2+bc)}{b+c} \ge \sqrt{3abc(ab^2+bc^2+ca^2)}$$

Solution:
By the ill-known inequality $(x+y+z)^2 \geq 3(xy+yz+zx)$ it suffices to prove that

$$\sum_{cycl}\left(ab\frac{a^2+bc}{b+c}\cdot bc\frac{b^2+ca}{c+a}\right) \geq abc(ab^2+bc^2+ca^2)$$

or equivalently

$$\sum_{cycl}\frac{b(a^2+bc)(b^2+ca)}{(b+c)(c+a)} \geq ab^2+bc^2+ca^2$$

Note that $(a^2+bc)(b^2+ca)-ab(a+c)(b+c) = c(a+b)(a-b)^2 \geq 0$, that is,

$$\frac{(a^2+bc)(b^2+ca)}{(a+c)(b+c)} \geq ab(*)$$

Hence

$$\frac{b(a^2+bc)(b^2+ca)}{(a+c)(b+c)} \geq ab^2$$

Adding this and similar inequalities, the conclusion follows.
Apropos: (*) implies trivially

$$\sqrt{\frac{(a^3+abc)(b^3+abc)}{(a+c)(b+c)}} \geq ab$$

and so

$$\sum_{cycl}\sqrt{\frac{(a^3+abc)(b^3+abc)}{(a+c)(b+c)}} \geq ab+bc+ca$$

This together with a ill-known inequality by Cezar Lupu yields

$$\sum_{cycl}\sqrt{\frac{a^3+abc}{b+c}} \geq a+b+c$$

in similar manner, We also have

$$\sum_{cycl}\sqrt{\frac{b+c}{a^3+abc}} \leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

$$\sum_{cycl}\sqrt{\frac{a^2+bc}{b+c}} \geq \sqrt{a}+\sqrt{b}+\sqrt{c}$$

$$\sum_{cycl}\sqrt{\frac{b+c}{a^2+bc}} \leq \frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}$$

498.
, Given some nonnegative numbers. Prove the following inequality:

$$\frac{a^2+b^2}{2a+2b+c}+\frac{b^2+c^2}{2b+2c+a}+\frac{c^2+a^2}{2c+2a+b} \geq \frac{2\sqrt{3(a^2+b^2+c^2)}}{5}.$$

Solution:
We have

$$\sum_{cycl}\left(\frac{5(b^2+c^2)}{a+2b+2c}-(b+c)\right)$$

$$=\sum_{cycl}\frac{b(c-a)+c(b-a)+3(b-c)^2}{a+2b+2c}$$

$$= \sum_{cycl} (a-b)^2 \left(\frac{3}{c+2a+2b} - \frac{c}{(a+2b+2c)(b+2c+2a)} \right)$$

and

$$2\sqrt{3(a^2+b^2+c^2)} - 2(a+b+c) = 2\frac{(a-b)^2+(b-c)^2+(c-a)^2}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}$$

The given inequality is thus equivant to $x(b-c)^2 + y(c-a)^2 + z(a-b)^2 \geq 0$ where

$$z = \frac{3}{c+2a+2b} - \frac{c}{(a+2b+2c)(b+2c+2a)} - \frac{2}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}$$

(and similarly for x and y). But $z \geq 0$ since

$$\frac{1}{c+2a+2b} \geq \frac{c}{(a+2b+2c)(b+2c+2a)}$$

and

$$\frac{2}{c+2a+2b} \geq \frac{1}{a+b+c} \geq \frac{2}{\sqrt{3(a^2+b^2+c^2)}+a+b+c}$$

(and similarly for x and y).

499.

For any positive real numbers a, b and c, prove that

$$\sqrt[6]{\frac{a+b}{a+c}} + \sqrt[6]{\frac{b+c}{b+a}} + \sqrt[6]{\frac{c+a}{c+b}} \leq \frac{a+b+c}{\sqrt[3]{abc}}$$

Solution:

By Holder's:

$$LHS^3 \leq 3\sum_{cyc} \sqrt{a+b} \sum_{cyc} \frac{1}{\sqrt{a+b}} = 9 + 3\sum_{sym} \sqrt{\frac{a+b}{a+c}}.$$

We are now to prove that:

$$\frac{(a+b+c)^3}{abc} \geq 9 + 3\sum_{sym} \sqrt{\frac{a+b}{a+c}}.$$

And We also have:

$$\frac{2(a+b+c)^3}{9abc} \geq \sum_{sym} \sqrt{\frac{a+b}{a+c}} \iff \frac{(a+b+c)\sum_{cyc}(b-c)^2 + 6\sum_{cyc} a(b-c)^2}{9abc} \geq$$

$$\sum_{sym} \frac{(b-c)^2}{\sqrt{a+b}\sqrt{a+c}\left(\sqrt{a+b}+\sqrt{a+c}\right)^2}.$$

But:

$$(7a+b+c)\sqrt{a+b}\sqrt{a+c}\left(\sqrt{a+b}+\sqrt{a+c}\right)^2 \geq 9abc.$$

$$\sqrt[3]{3\sum_{cyc} \sqrt{a+b} \sum_{cyc} \frac{1}{\sqrt{a+b}}} =$$

$$(1^3+1^3+1^3)^{\frac{1}{3}} \left(\left(\sqrt[6]{a+b}\right)^3 + \left(\sqrt[6]{b+c}\right)^3 + \left(\sqrt[6]{c+a}\right)^3 \right)^{\frac{1}{3}} \left(\left(\frac{1}{\sqrt[6]{a+c}}\right)^3 + \left(\frac{1}{\sqrt[6]{b+a}}\right)^3 + \left(\frac{1}{\sqrt[6]{c+b}}\right)^3 \right)^{\frac{1}{3}} \geq$$

$$\geq \sum_{cyc} 1 \cdot \sqrt[6]{a+b} \cdot \frac{1}{\sqrt[6]{a+c}} = LHS.$$

500.

, Actually, the following stronger result holds

$$\sqrt[3]{\frac{a+b}{a+c}}+\sqrt[3]{\frac{b+c}{b+a}}+\sqrt[3]{\frac{c+a}{c+b}}\le\frac{a+b+c}{\sqrt[3]{abc}}$$

Solution:

Let's Holder it agian!

$$\left(\sqrt[3]{\frac{a+b}{a+c}}+\sqrt[3]{\frac{b+c}{b+a}}+\sqrt[3]{\frac{c+a}{c+b}}\right)^3$$

$$\le 6(a+b+c)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

$$\le 3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

The rest is:

$$\frac{(a+b+c)^3}{abc}\ge 3(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\iff (a+b+c)^2\ge 3(ab+bc+ca).$$

501.

Let a,b,c be real positive numbers. Prove that

$$(a^2-ab+b^2)(b^2-bc+c^2)+(b^2-bc+c^2)(c^2-ca+a^2)+(c^2-ca+a^2)(a^2-ab+b^2)\ge a^2b^2+b^2c^2+c^2a^2.$$

it's true for all reals a, b and c.

Solution:

this ineq is equivalent to

$$\sum_{cyc}((a-b)^2+ab)((a-c)^2+ac)\ge\sum_{cyc}a^2b^2$$

$$\Leftrightarrow\sum_{cyc}2(a-b)^2(a-c)^2+2ac(a-b)^2+2ab(a-c)^2\ge\sum_{cyc}c^2(a-b)^2$$

$$\Leftrightarrow\sum_{cyc}(a-b)^2(a-c)^2+(a-b)^2(b-c)^2+2ac(a-b)^2+2bc(a-b)^2-c^2(a-b)^2\ge 0$$

$$\Leftrightarrow\sum_{cyc}(a-b)^2((a-c)^2+(b-c)^2+2ac+2bc-c^2)\ge 0$$

$$\Leftrightarrow\sum_{cyc}(a-b)^2(a^2+b^2+c^2)\ge 0$$

502.

Prove that for all positive reals a,b,c We have:

$$1)\frac{ab+b^2}{ac+c^2}+\frac{bc+c^2}{ba+a^2}+\frac{ca+a^2}{cb+b^2}\ge 2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

$$2)\frac{a^4}{a^2+ab+b^2}+\frac{b^4}{b^2+bc+c^2}+\frac{c^4}{c^2+ca+a^2}\ge\frac{a^3+b^3+c^3}{a+b+c}$$

Solution:

1)Applying AM-GM, We obtain:

$$\left[\frac{ab+b^2}{ac+c^2}\equiv\frac{b(a+b)}{c(a+c)}\right]+\frac{4bc}{(a+b)(a+c)}\ge 4\frac{b}{c+a},$$

Take sum of three inequalities,

$$LHS + \sum_{cyc} \frac{4bc}{(a+b)(a+c)} \geq 4 \sum_{cyc} \frac{a}{b+c}$$

Moreover,

$$LHS \geq 4 \sum_{cyc} \frac{a}{b+c} - 4 \sum_{cyc} \frac{bc}{(a+b)(a+c)} \geq 2 \sum_{cyc} \frac{a}{b+c}$$

$$\iff 2 \sum_{cyc} \frac{bc}{(a+b)(a+c)} \leq \sum_{cyc} \frac{a}{b+c} \iff a^3 + b^3 + c^3 + 3abc \geq \sum_{sym} a^2 b$$

Reducing to Schur's.

2)We have

$$\frac{a^4}{a^2+ab+b^2} + \frac{b^4}{b^2+bc+c^2} + \frac{c^4}{c^2+ca+a^2} \geq \frac{a^3+b^3+c^3}{a+b+c}$$

$$<=> \frac{a^4}{a^2+ab+b^2} + \frac{b^4}{b^2+bc+c^2} + \frac{c^4}{c^2+ca+a^2} \geq \frac{3abc}{a+b+c} + (a^2 - ab + (b^2 - bc) + (c^2 - ca)$$

$$<=> \sum_{cyc} \left(\frac{a^4}{a^2+ab+b^2} - a^2 + ab) \right) \geq \frac{3abc}{a+b+c} (*)$$

But

$$\sum (\frac{a^4}{a^2+ab+b^2} - a^2 + ab) = \sum \frac{ab^3}{a^2+ab+b^2} = \sum \frac{b^2}{1 + \frac{a}{b} + \frac{b}{a}}$$

By Cauchy-Schwarz's inequality we have

$$\sum \frac{b^2}{1 + \frac{a}{b} + \frac{b}{a}} \geq \frac{(a+b+c)^2}{3 + \sum (\frac{a}{b} + \frac{b}{a})} = \frac{(a+b+c)abc}{ab+bc+ca} \geq \frac{3abc}{a+b+c}$$

Hence $(*)$ is true inequality.

Q.E,D

503., Prove that, for any positive real numbers a, b and c,

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + \frac{12(a+b)(b+c)(c+a)}{(a+b+c)^3} \geq \frac{41}{9}$$

Solution:

$a+b+c=3$. Set $w = ab+bc+ca, r = abc$, then We have $a^2+b^2+c^2 = 9-2w$, and

$$(a+b)(b+c)(c+a) = (3-a)(3-b)(3-c) = (9-3(a+b)+ab)(3-c) = 27-9(a+b+c)-abc+3(ab+bc+ca) = 3w-r.$$

The desired inequality reads

$$\frac{9-2w}{w} + \frac{12(3w-r)}{27} \geq \frac{41}{9}.$$

Equivalently,

$$27(9-2w) + 12w(3w-r) \geq 123w.$$

504.

, $ab+ac+bc \geq 0$. Prove that

$$(a+b+c)^6 \geq 27(a^2+b^2+c^2)(ab+ac+bc)^2.$$

Solution:
in fact,

$$(a+b+c)^6-27(a^2+b^2+c^2)(ab+bc+ca)^2 = (a^2+b^2+c^2+8(ab+bc+ca))(a^2+b^2+c^2-ab-bc-ca)^2$$

The latter expresion is positive whenever $ab+bc+ca \geq 0$.

Q.E.D
505.
, Prove that if $x, y, z \geq 0$ then

$$(x+y+z)^3 \geq 3(xy+yz+zx)\sqrt{3(x^2+y^2+z^2)}.$$

Solution: Since the desired is homogenous, We can suppose WLOG that $x+y+z=3$. Set $w = xy+yz+zx$, now all We need to prove is

$$27 \geq w^2(9-2w).$$

This is true since $w \leq 3$.

it's just am-gm... Mr. Green

$$(a+b+c)^2 = 3\frac{(a^2+b^2+c^2)+(ab+bc+ca)+(ab+bc+ca)}{3} \geq 3\sqrt[3]{a^2+b^2+c^2}\sqrt[3]{(ab+bc+ca)^2}$$

506.
, Let a, b, c three non-negative real numbers such that $a+b+c=3$. Prove the following inequality

$$\frac{1}{a^2}+\frac{1}{a^2}+\frac{1}{a^2} \geq a^2+b^2+c^2.$$

Solution.
We will use Thuan's lemma:

$$(a+b+c)^3 \geq 3(ab+bc+ca)\sqrt{3(a^2+b^2+c^2)}.$$

Using $a+b+c=3$, We get

$$27 \geq (ab+bc+ca)^2(a^2+b^2+c^2).$$

Now, if We prove that

$$\frac{1}{a^2}+\frac{1}{a^2}+\frac{1}{a^2} \geq \frac{27}{(ab+bc+ca)^2}$$

We are done. This last inequality reduces to

$$(ab+bc+ca)^2\left(\frac{1}{a^2}+\frac{1}{a^2}+\frac{1}{a^2}\right) \geq 27$$

But, this follows immediately from these two trivial inequalities:

$$ab+bc+ca \geq \sqrt{3abc(a+b+c)} and \frac{1}{a^2}+\frac{1}{a^2}+\frac{1}{a^2} \geq \frac{a+b+c}{abc}.$$

507.
x, y, z are positive real numbers such, that $x+y+z=3$. Prove that:

$$\frac{1}{x+yz}+\frac{1}{y+zx}+\frac{1}{z+xy} \leq \frac{9}{2(xy+xz+yz)}.$$

Solution:
it is equivalent to

$$\frac{3}{2(xy+yz+zx)} \geq \sum_{cycl} \frac{1}{x(x+y+z)+3yz}$$

We have

$$\frac{1}{2(xy+yz+zx)} - \frac{1}{x(x+y+z)+3yz} = \frac{(x-y)(x-z)}{2(xy+yz+zx)(x(x+y+z)+3yz)}$$

it suffices to prove

$$\sum_{cycl} \frac{(x-y)(x-z)}{x(x+y+z)+3yz} \geq 0$$

Assume $x \geq y \geq z$. Then

$$\frac{(z-x)(z-y)}{z(x+y+z)+3xy} \geq 0$$

Further,

$$\frac{(x-y)(x-z)}{x(x+y+z)+3yz} + \frac{(y-z)(y-x)}{y(x+y+z)+3zx} = \frac{z(2x+2y-z)(x-y)^2}{(x(x+y+z)+3yz)(y(x+y+z)+3zx)} \geq 0$$

and the **Solution** is completed.

Q.E.D.
508.
, Prove that, for any positive real numbers a, b and c,

$$\sum_{cycl} \frac{a^2+bc}{a^2+(b+c)^2} \leq \frac{18}{5} \cdot \frac{a^2+b^2+c^2}{(a+b+c)^2}$$

Solution:
The inequality is equivalent to

$$\sum \frac{(b+c)^2-bc}{a^2+(b+c)^2} + \frac{18}{5} \cdot \frac{a^2+b^2+c^2}{(a+b+c)^2} \geq 3$$

Since $(b+c)^2 \geq 4bc$ hence it suffices to prove

$$\sum \frac{(b+c)^2}{4(a^2+(b+c)^2)} + \frac{6}{5} \cdot \frac{a^2+b^2+c^2}{(a+b+c)^2} \geq 1$$

By Cauchy inequality We have

$$\sum \frac{(b+c)^2}{4(a^2+(b+c)^2)} \geq \frac{(a+b+c)^2}{\sum(a^2+(b+c)^2)} = \frac{(a+b+c)^2}{2(a^2+b^2+c^2)+(a+b+c)^2}$$

WLOG, We may assume $a+b+c=1$. Setting $x = a^2+b^2+c^2$ then $3x \geq 1$, it remains to prove that

$$\frac{1}{2x+1} + \frac{6x}{5} \geq 1$$
$$\Leftrightarrow x(3x-1) \geq 0$$

Which is true.

WLOG,assume that $a+b+c=3$. We have to prove that

$$\frac{2}{5}(a^2+b^2+c^2) \geq \sum \frac{4a^2+(3-a)^2}{4(a^2+(3-a)^2)}$$

This is an easy problem because:

$$\frac{2}{5}a^2 - \frac{4a^2+(3-a)^2}{4(a^2+(3-a)^2)} \geq \frac{11}{25}a - \frac{11}{25} \Leftrightarrow (a-1)^2(80a^2-168a+171) \geq 0 \ldots$$

509.

Let a, b, c be three positive real numbers. Prove that :

$$\frac{1}{2} + \frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 2 - \frac{ab+bc+ca}{2(a^2+b^2+c^2}$$

Solution:

$$\frac{1}{2} + \frac{a^2+b^2+c^2}{ab+bc+ca} - (\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b})$$
$$= \frac{\sum(a^3+abc)(b-c)^2}{(a+b)(b+c)(c+a)(2(ab+bc+ca))} \geq 0$$
$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} + \frac{ab+bc+ca}{2(a^2+b^2+c^2)} - 2$$
$$= \frac{\sum a^3(a-b)(a-c) + (a^3+b^3+c^3)(a^2+b^2+c^2-ab-bc-ca)}{(a+b)(b+c)(c+a)(2(a^2+b^2+c^2))} \geq 0$$

510.

, Let a, b, c be three positive real numbers. Prove that :

$$\frac{1}{2} + \frac{a^2+b^2+c^2}{ab+bc+ca} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$$

Solution:

Using the AM-GM:

$$\sum_{cycl} ab \sum_{cycl} \frac{a}{b+c} = \sum_{cycl} a^2 + \sum_{cycl} \frac{abc}{b+c} \leq \sum_{cycl} a^2 + \sum_{cycl} \frac{a(b+c)^2}{4(b+c)}$$

it seems two **Solution** are hard to chew.

$$\frac{1}{2} + \frac{a^2+b^2+c^2}{ab+bc+ca} \geq \sum_{cyc} \frac{a}{b+c} \iff \sum_{cyc} (a-b)^2 \cdot \left(\frac{c^2}{(ab+bc+ca)(c+a)(c+b)} \right) \geq 0$$

$$\sum_{cyc} \frac{a}{b+c} \geq 2 - \frac{ab+bc+ca}{2(a^2+b^2+c^2)} \iff \sum_{cyc} (a-b)^2 \cdot \underbrace{\frac{ab+bc+ca-a^2}{(a+b)(b+c)(ab+bc+ca)}}_{S_c} \geq 0$$

Wlog assume $c \geq b \geq a, then S_c \geq 0 and S_a + S_b \geq 0.$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 2 - \frac{ab+bc+ca}{2(a^2+b^2+c^2)}$$

By Cauchy,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)}$$

it remains to prove that

$$B^2 + A^2C \geq 4BC(*)$$

where $A = a+b+c, B = ab+bc+ca$ and $C = a^2+b^2+c^2$. But

$$A^2C = C^2 + 2BC$$

So (*) becomes

$$B^2 + C^2 \geq 2BC$$

which is obvious.

2)

The following stronger inequality holds:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{13}{6} - \frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)}.$$

Rewrite it as

$$\sum_{cycl} \frac{a}{b+c} - \frac{3}{2} \geq \frac{2}{3}\left(1 - \frac{ab+bc+ca}{a^2+b^2+c^2}\right)$$

$$\sum_{cycl} \frac{(a-b)^2}{2(a+c)(b+c)} \geq \sum_{cycl} \frac{(a-b)^2}{3(a^2+b^2+c^2)}$$

$$\sum_{cycl} (a-b)^2 \left(\frac{1}{2(a+c)(b+c)} - \frac{1}{3(a^2+b^2+c^2)}\right) \geq 0$$

and note that

$$3(a^2+b^2+c^2) - 2(a+c)(b+c) = (a+b-c)^2 + 2(a-b)^2$$

The best inequality of this type is:

$$\sum_{cycl} \frac{a}{b+c} - \frac{3}{2} \geq (\sqrt{3}-1)\left(1 - \frac{ab+bc+ca}{a^2+b^2+c^2}\right)$$

Hmm... inequality:

$$\Longleftrightarrow \sum_{cyc} \left(\frac{1}{(a+c)(b+c)} - \frac{1-\sqrt{3}}{a^2+b^2+c^2}\right) \geq 0.$$

511.

, Let a, b, c be real nonnegative numbers, prove that

$$\frac{(a+b)(a+c)}{a^2+bc} + \frac{(b+c)(b+a)}{b^2+ca} + \frac{(c+a)(c+b)}{c^2+ab} \geq 5.$$

Solution:

$$A \geq \frac{(a+b)(a+c)}{a^2+ab} + \frac{(b+c)(b+a)}{b^2+ab} + \frac{(c+a)(c+b)}{ca+ab}$$

$$A \geq \frac{a+c}{a} + \frac{b+c}{b} + \frac{c+a}{a}$$

$$A \geq 5$$

Another one is

$$\frac{x}{(x+y)(x+z)} + \frac{y}{(y+z)(y+x)} + \frac{z}{(z+x)(z+y)} \geq \frac{2}{x+y+z}.$$

We don't know, maybe We misunderstanding something, but your inequality is equivalent with

$$x(y+z) + y(z+x) + z(x+y) \geq \frac{2(x+y)(y+z)(z+x)}{x+y+z} \Leftrightarrow$$

$$2(x+y+z)(xy+yz+zx) \geq 2(x+y)(y+z)(z+x) \Leftrightarrow$$

$$(x+y+z)(xy+yz+zx) \geq (x+y)(y+z)(z+x) \Leftrightarrow$$

$$(x+y+z)(xy+yz+zx) \geq (x+y+z)(xy+yz+zx) - xyz \Leftrightarrow$$

$$xyz \geq 0.$$

512.
Let a and b be positive real numbers, prove that

$$\frac{a+\sqrt{ab}+b}{3} \leq \sqrt{(\frac{a^{2/3}+b^{2/3}}{2})^3}$$

Solution:
Suppose $a^{\frac{2}{3}}+b^{\frac{2}{3}}=2$ And We put $a=x^6 b=y^6$ So We will have a new puzzle : $x^4+y^4=2$.
Prove that:

$$x^6+y^6+x^3y^3 \leq 3$$

We have :

$$x^6+y^6+x^3y^3 = x^2(2-y^4)+x^3y^3+y^2(2-x^4) = (x^2+y^2)(2-x^2y^2)+x^3y^3 =$$

$$\sqrt{2(x^2y^2+1)}(2-x^2y^2)+x^3y^3 \leq \frac{1}{2}(2+x^2y^2+1)(2-x^2y^2)+x^3y^3 =$$

$$\frac{1}{2}(6-x^2y^2-x^4y^4)+x^3y^3 = 3+x^3y^3-\frac{1}{2}(x^2y^2+x^4y^4) \leq 3+x^3y^3-\frac{1}{2}(2\sqrt{x^6y^6}) = 3$$

Q.E.D

513.
Prove, for $a,b,c>0$

$$\frac{a^2}{(a+b)(a+c)}+\frac{b^2}{(b+c)(b+a)}+\frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}$$

Solution:
is it the same as

$$\frac{(a-b)(a-c)}{(a+b)(a+c)}+\frac{(b-c)(b-a)}{(b+c)(b+a)}+\frac{(c-a)(c-b)}{(c+a)(c+b)} \geq 0$$

$$LHS = \frac{a^2(b+c)+b^2(a+c)+c^2(a+b)}{(a+b)(a+c)(b+c)} = \frac{(a+b)(a+c)(b+c)-2abc}{(a+b)(a+c)(b+c)} =$$

$$1-\frac{2abc}{(a+b)(a+c)(b+c)} \geq 1-\frac{2abc}{8abc} = \frac{3}{4}$$

514.
Let $a,b,c>0$ and $a+b+c=3$. Prove that:

$$\sum \frac{a}{b^{\frac{8}{3}}+1} \geq \frac{3}{2}$$

Solution:
Using a ill-known approach:

$$3-\sum_{cycl}\frac{a}{b^{8/3}+1} = \sum_{cycl}\left(a-\frac{a}{b^{8/3}+1}\right) = \sum_{cycl}\frac{ab^{8/3}}{1+b^{8/3}} \leq \sum_{cycl}\frac{ab^{8/3}}{2b^{4/3}} = \frac{1}{2}\sum_{cycl}ab^{4/3} \leq \frac{3}{2}$$

since

$$3\sum_{cycl} ab^{4/3} \le \sum_{cycl}(ab^2+2ab) \le \frac{1}{3}(a+b+c)(a^2+b^2+c^2)+2(ab+bc+ca) = a^2+b^2+c^2+2(ab+bc+ca) = (a+b+c)^2 = 9$$

where We use $3(ab^2+bc^2+ca^2) \le (a+b+c)(a^2+b^2+c^2)$ which is not difficult to prove.

515.
, Prove that $\forall a,b,c>0$ We have

$$\sum \frac{a^2-bc}{b^2+c^2+2a^2} \ge 0$$

Yet another **Solution**: By Cauchy-Schwarz,

$$3-2\sum_{cyc}\frac{c^2-ab}{2c^2+a^2+b^2} = \sum_{cyc}\frac{(a+b)^2}{a^2+b^2+2c^2} \le \sum_{cycl} \le \frac{a^2}{a^2+c^2}+\frac{b^2}{b^2+c^2} = 3$$

516.
Let x,y,z be positive real numbers such that $x^2+y^2+z^2 \le 3$. Prove that

$$\frac{1+xy}{z^2+xy}+\frac{1+yz}{x^2+yz}+\frac{1+zx}{y^2+zx} \ge 3.$$

Solution:
by Cauchy-Schwarz, We get:

$$\left(\sum_{cyc}(z^2+xy)(1+xy)\right)\left(\frac{1+xy}{z^2+xy}+\frac{1+yz}{x^2+yz}+\frac{1+zx}{y^2+zx}\right) \ge (3+xy+yz+zx)^2$$

so We should only to prove:

$$(3+xy+yz+zx)^2 \ge 3\left(\sum_{cyc}(z^2+xy)(1+xy)\right)$$

$$\iff 9+6\sum_{cyc}xy+(\sum_{cyc}xy)^2 \ge 3\sum_{cyc}x^2y^2+3\sum_{cyc}xyz^2+9+3\sum_{cyc}xy$$

$$\iff 3\sum_{cyc}xy \ge 2\sum_{cyc}x^2y^2+\sum_{cyc}xyz^2$$

by

$$x^2+y^2+z^2 \le 3$$

We should only to prove:

$$(\sum_{cyc}xy)(\sum_{cyc}x^2) \ge 2\sum_{cyc}x^2y^2+\sum_{cyc}xyz^2$$

$$\iff \sum_{sym}x^3y \ge 2\sum_{cyc}x^2y^2$$

obvious true

There is a shorter **Solution** but it uses a stronger result

$$\sum_{cycl}\frac{3+3xy}{z^2+xy} \ge \sum_{cycl}\frac{x^2+y^2+z^2+3xy}{z^2+xy} = 3+\sum_{cycl}\frac{(x+y)^2}{z^2+xy} \ge 9$$

where the last inequality follows from

For any three positive real numbers a, b, c, prove the inequality

$$\frac{(b+c)^2}{a^2+bc}+\frac{(c+a)^2}{b^2+ca}+\frac{(a+b)^2}{c^2+ab}\geq 6.$$

517.
Let a, b, c three nonnegative real numbers. Prove that the following inequality holds true:

$$\frac{b^2+c^2}{a(b+c)}+\frac{c^2+a^2}{b(c+a)}+\frac{a^2+b^2}{c(a+b)}\geq 3\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Solution:
By Chebyshev and Cauchy,

$$\sum\frac{b^2+c^2}{a(b+c)}\geq\frac{2(a^2+b^2+c^2)}{3}\sum\frac{1}{a(b+c)}\geq\frac{2(a^2+b^2+c^2)}{3}\cdot\frac{9}{2(ab+bc+ca)}=3\cdot\frac{a^2+b^2+c^2}{ab+bc+ca}$$

411, Let x, y, z be positive real numbers such that $x+y+z=1$. Prove that

$$\frac{1}{\sqrt{x+yz}}+\frac{1}{\sqrt{y+zx}}+\frac{1}{\sqrt{z+xy}}\geq\frac{9}{2}$$

$$\frac{1}{\sqrt{x+yz}}+\frac{1}{\sqrt{y+zx}}+\frac{1}{\sqrt{z+xy}}\geq\frac{9}{\sqrt{3\sum x+3\sum yz}}\geq\frac{9}{2}$$

$$\sum_{cycl}\frac{1}{\sqrt{a+bc}}\geq\sum_{cycl}\frac{1}{\sqrt{a+\left(\frac{b+c}{2}\right)^2}}=2\sum_{cycl}\frac{1}{\sqrt{4a+(1-a)^2}}=2\sum_{cycl}\frac{1}{\sqrt{(1+a)^2}}=2\sum_{cycl}\frac{1}{1+a}\geq 2\frac{9}{3+a+b+c}=\frac{9}{2}$$

Since $x+y+z=1$, We have $x+yz=x(x+y+z)+yz=y(x+z)+x(x+z)=(x+z)(y+x)$.
Applying AM-GM inequality to get

$$\sqrt{(x+z)(y+x)}\leq\frac{1}{2}(2x+y+z)$$

Two other similar inequalities and AM-HM inequality solve the desired inequality.

518.
Prove that $\forall a, b, c>0$ We have

$$\sum\frac{a^2-bc}{\sqrt{b^2+c^2+2a^2}}\geq 0$$

Solution:
Observe that

$$\sum_{cycl}\frac{a^2-bc}{\sqrt{2a^2+b^2+c^2}}=\frac{1}{2}\sum_{cycl}\left(\sqrt{2a^2+b^2+c^2}-\frac{(b+c)^2}{\sqrt{2a^2+b^2+c^2}}\right)$$

By Cauchy, $\sqrt{2(x^2+y^2)}\geq x+y$. Using this,

$$\sum_{cycl}\sqrt{2a^2+b^2+c^2}\geq\frac{1}{\sqrt{2}}\sum_{cycl}(\sqrt{a^2+b^2}+\sqrt{a^2+c^2})=\sqrt{2}\sum_{cycl}\sqrt{a^2+b^2}$$

On the other hand, also by Cauchy

$$\sum_{cycl}\frac{(b+c)^2}{\sqrt{2a^2+b^2+c^2}}\leq\sqrt{2}\sum_{cycl}\frac{(b+c)^2}{\sqrt{a^2+b^2}+\sqrt{a^2+c^2}}$$

$$\leq \sqrt{2}\sum_{cycl}\left(\frac{b^2}{\sqrt{a^2+b^2}}+\frac{c^2}{\sqrt{a^2+c^2}}\right)=\sqrt{2}\sum_{cycl}\frac{a^2+b^2}{\sqrt{a^2+b^2}}=\sqrt{2}\sum_{cycl}\sqrt{a^2+b^2}$$

Hence

$$\sum_{cycl}\sqrt{2a^2+b^2+c^2}\geq\sqrt{2}\sum_{cycl}\sqrt{a^2+b^2}\geq\sum_{cycl}\frac{(b+c)^2}{\sqrt{2a^2+b^2+c^2}}$$

from which the result follows.

519.

Given that $a, b, c > 0$ and $abc = 1$.Prove that:

$$\sum_{cyc}\frac{1}{a+b+1}\leq 1$$

Solution:

Let $a = x^3, b = y^3, c = z^3$ then $xyz = 1$ and our inequality becomes

$$\sum\frac{1}{x^3+y^3+xyz}\leq\frac{1}{xyz}$$

Using the fact that $x^3 + y^3 \geq\ xy(x + y)$ and i've done.

can be bashed as follows: AM-GM yields

$$\frac{a}{b}+\frac{a}{c}+1\geq 3\sqrt[3]{\frac{a}{b}\cdot\frac{a}{c}\cdot 1}=3\sqrt[3]{\frac{a^2}{bc}}=3\sqrt[3]{\frac{a^3}{abc}}=3\sqrt[3]{a^3}=3a$$

and similarly

$$\frac{b}{c}+\frac{b}{a}+1\geq 3b;$$
$$\frac{c}{a}+\frac{c}{b}+1\geq 3c.$$

Adding these three inequalities together, We get

$$\frac{a}{b}+\frac{a}{c}+\frac{b}{c}+\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+3\geq 3a+3b+3c=(2a+2b+2c)+(a+b+c)$$

But AM-GM again gives

$$a+b+c\geq 3\sqrt[3]{abc}=3;$$

hence,

$$\frac{a}{b}+\frac{a}{c}+\frac{b}{c}+\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+3\geq(2a+2b+2c)+(a+b+c)\geq(2a+2b+2c)+3$$

in other words,

$$\frac{a}{b}+\frac{a}{c}+\frac{b}{c}+\frac{b}{a}+\frac{c}{a}+\frac{c}{b}\geq 2a+2b+2c.$$

Now, algebraic computation, at first without the condition abc $= 1$, yields

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}-1=\frac{2a+2b+2c-\left(a^2c+a^2b+b^2a+b^2c+c^2b+c^2a\right)+2-2abc}{(a+b+1)(b+c+1)(c+a+1)}$$

Now, using $abc = 1$, We can simplify this to

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}-1=\frac{2a+2b+2c-\left(a^2c+a^2b+b^2a+b^2c+c^2b+c^2a\right)}{(a+b+1)(b+c+1)(c+a+1)}$$

$$=\frac{2a+2b+2c-\left(\frac{a}{b}+\frac{a}{c}+\frac{b}{c}+\frac{b}{a}+\frac{c}{a}+\frac{c}{b}\right)}{(a+b+1)(b+c+1)(c+a+1)}$$

But this is negative, since We have already seen that

$$\frac{a}{b}+\frac{a}{c}+\frac{b}{c}+\frac{b}{a}+\frac{c}{a}+\frac{c}{b}\geq 2a+2b+2c.$$

Hence, We have

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\leq 1,$$

Here is our **Solution** for 1. By Cauchy-Schwarz We have

$$(a+b+1)(1+1+c)\geq(\sqrt{a}+\sqrt{b}+\sqrt{c})^2, so$$

$$\frac{1}{a+b+1}\leq\frac{c+2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}$$

Thus

$$\sum\frac{1}{a+b+1}\leq\sum\frac{c+2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}=\frac{a+b+c+6}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\leq 1$$

520.

With the above conditions, prove that:

$$\sum_{cyc}\frac{c}{a+b+1}\geq 1$$

Solution:

$$\sum_{cyc}\frac{c}{a+b+1}=\sum_{cyc}\frac{c^2}{ac+bc+c}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)+a+b+c}=\frac{a^2+b^2+c^2+2(ab+bc+ca)}{(a+b+c)+2(ab+bc+ca)}$$

it is easy by AM-GM to prove

$$a^2+b^2+c^2\geq a+b+c$$

521.

Let a,b,c be positive reals. Then

$$\sum_{cycl}\frac{a+b-c}{a^2+ab+b^2}\leq\frac{3}{a+b+c}$$

Solution:

it's equivlent to

$$\sum\frac{(a+b)^2-c^2}{a^2+b^2+ab}\leq 3$$

Since

$$a^2+b^2+ab\geq\frac{3(a+b)^2}{4}$$

the inequality becomes

$$\sum\frac{c^2}{(a+b)^2}\geq\frac{3}{4}$$

which is ill known. 522.

Let $a,b,c>0$ such that their sum is 3. Prove that the following inequality holds:

$$\frac{a^3+abc}{(b+c)^2}+\frac{b^3+abc}{(c+a)^2}+\frac{c^3+abc}{(a+b)^2}\geq\frac{3}{2}.$$

Solution:
Assume $a \geq b \geq c$. By Schur's inequality,

$$\sum \left(\frac{a}{(b+c)^2}\right)(a-b)(a-c) \geq 0$$

it follows that

$$LHS \geq \sum \frac{a^2(b+c)}{(b+c)^2} = \sum \frac{a^2}{b+c} \geq \frac{a+b+c}{2} = \frac{3}{2}.$$

By Chebyshev, then iran 1996, then Schur

$$LHS \geq \frac{1}{3}\left(\sum x^3 + xyz\right)\left(\sum \frac{1}{(x+y)^2}\right)$$

$$\geq \frac{3}{4(xy+yz+zx)}\left(\sum x^3 + xyz\right) \geq \frac{1}{2}\frac{(xy+yz+zx)(x+y+z)}{xy+yz+zx}$$

$$\frac{x+y+z}{2} = frac32$$

which yields the desired result.

We can do without iran 1996 as follows: by Chebyshev and CBS,

$$\sum_{cycl} \frac{a^3+abc}{(b+c)^2} \geq \frac{a^3+b^3+c^3+3abc}{3}\sum_{cycl}\frac{1}{(b+c)^2} \geq \frac{3(a^3+b^3+c^3+3abc)}{2(a^2+b^2+c^2+ab+bc+ca)}$$

and note that using Schur We have

$$(ab+bc+ca)(a+b+c) \leq a^3+b^3+c^3+6abc$$

and

$$(a^2+b^2+c^2)(a+b+c) = a^3+b^3+c^3+a^2(b+c)+b^2(c+a)+c^2(a+b) \leq 2(a^3+b^3+c^3)+3abc$$

so that

$$(a+b+c)(a^2+b^2+c^2+ab+bc+ca) \leq 3(a^3+b^3+c^3+3abc)$$

523.
, Let a, b, c be positive real numbers and $a+b+c=1$. Prove that this inequality holds:

$$\sqrt{\frac{ab}{ab+c}} + \sqrt{\frac{ac}{ac+b}} + \sqrt{\frac{bc}{bc+a}} \geq \frac{3}{2}$$

Solution:

$$\sqrt{\frac{bc}{bc+a}} = \sqrt{\frac{bc}{bc+1-b-c}} = \sqrt{\frac{bc}{(a+b)(a+c)}} \leq \frac{1}{2}.\left(\frac{b}{a+b}+\frac{c}{a+c}\right)$$

$$\frac{a+b+c}{1+a+b+c} \geq \frac{a}{1+3a} + \frac{b}{1+3b} + \frac{c}{1+3c}$$

Solution:
Rewrite as

$$f\left(\frac{a+b+c}{3}\right) \geq \frac{f(a)+f(b)+f(c)}{3}$$

with

$$f(x) = \frac{1}{1+1/x}$$

Another **Solution**: the given ineq is a special case of the following

$$\sum \frac{a_1}{1+ka_1} \leq \frac{nS}{n+kS}$$

where $k \geq 0andS = \sum a_i$, which, in turn, can be deduced from

$$\sum x_We \sum y_We \geq \sum (x_We + y_i) \sum \frac{x_i y_i}{x_i + y_i}$$

by letting $x_i = ka_i$ and $y_i = 1$.

525.

Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+d)} + \frac{1}{d(1+a)} \geq 2$$

Solution:

Put

$$a = \frac{x}{w}, b = \frac{y}{x}, c = \frac{z}{y}, d = \frac{w}{z}.$$

The given ineq becomes

$$\sum_{cycl} \frac{w}{x+y} \geq 2$$

which can be easily proved using Cauchy-Schwarz.

526.

1) if a, b,c are positive reals such that $a^2 + b^2 + c^2 = 1$ prove that

$$2abc(a+b+c) \leq 2(a+b+c)^2 + 1.$$

Solution:

$$1 = (a^2 + b^2 + c^2)^2 \geq (ab + bc + ca)^2 \geq 3abc(a+b+c)$$

$$2(a+b+c)^2 + 1 = (a^2+b^2+c^2)(3a^2+3b^2+3c^2+4ab+4ac+4bc) =$$

$$\sum_{cyc} 3a^4 + 3a^2b^2 + 3a^2c^2 + 4a^3b + 4a^3c + 4a^2bc \geq 2abc(a+b+c)$$

$$\Leftrightarrow \sum_{cyc} 3a^4 + 3a^2b^2 + 3a^2c^2 + 4a^3b + 4a^3c + 2a^2bc \geq 0$$

$$2(a+b+c)^2 + 1 \geq 21abc(a+b+c)$$

2) Let $a_1, \ldots, a_n (n \geq 2)$ be positive reals, and let $S = a_1 + \cdots + a_n$. The for any $k \in \mathbf{Z}$,

$$\sum_{i=1}^{n} \frac{a_i^k}{S - a_i} \geq \frac{S^{k-1}}{n^{k-2}(n-1)}$$

by hölder's inequality We have that

$$\left(\sum_{i=1}^{n} \frac{a_i^k}{S-a_i}\right)\left(\sum_{i=1}^{n} S - a_i\right)\left(\sum_{i=1}^{n} 1\right)^{k-2} \geq \left(\sum_{i=1}^{n} a_i\right)^k$$

$$\Rightarrow \sum_{i=1}^{n} \frac{a_i^k}{S-a_i} \geq \frac{S^k}{n^{k-2}(n-1)S} = \frac{S^{k-1}}{n^{k-2}(n-1)}$$

527.
if $a, b, c > 0$ then
$$\frac{a^5}{a^3+b^3}+\frac{b^5}{b^3+c^3}+\frac{c^5}{c^3+a^3} \geq \frac{a^2+b^2+c^2}{2}$$
Solution:
The inequality is equivalent to:
$$\frac{a^2(a^3-b^3)}{a^3+b^3}+\frac{b^2(b^3-c^3)}{b^3+c^3}+\frac{c^2(c^3-a^3)}{c^3+a^3} \geq 0$$
since
$$\frac{a^2+ab+b^2}{a^2-ab+b^2} \geq 1,$$
it suffices to prove that:
$$\sum \frac{a^2(a-b)}{a+b} \geq 0,$$
which is equivalent to:
$$\frac{a(a^2+b^2)}{a+b}+\frac{b(b^2+c^2)}{b+c}+\frac{c(a^2+c^2)}{a+c} \geq ab+ac+bc$$
but We know that:
$$a^2+b^2 \geq \frac{1}{2}(a+b)^2,$$
similarly for others then it suffices to prove that:
$$a^2+b^2+c^2 \geq ab+ac+bc$$
which is true.

2)
$$a^2+b^2+c^2 \geq \sum \frac{2a^2b^3}{a^3+b^3}$$
is equivalent to
$$\Leftrightarrow a^2+b^2+c^2 \geq \sum \frac{2ab^2}{a+b}$$
We think that it follows from
$$\frac{2ab^2}{a+b} \geq \frac{2a^2b^3}{a^3+b^3}$$
which is
$$(a+b)(a-b)^2 \geq 0$$
$$\sum_{cyc} \frac{a^5}{a^3+b^3} \geq \frac{a^2+b^2+c^2}{2} \Leftrightarrow \sum_{cyc}\left(\frac{a^5}{a^3+b^3}-\frac{a^2+b^2}{4}\right) \Leftrightarrow$$
$$\Leftrightarrow \sum_{cyc}\left(\frac{(a-b)(3a^4+3a^3b+2a^2b^2+ab^3+b^4)}{a^3+b^3}-\frac{5(a^2-b^2)}{2}\right) \geq 0 \Leftrightarrow$$
$$\Leftrightarrow \sum_{cyc} \frac{(a-b)^2(a^3+2a^2b+6ab^2+3b^3)}{a^3+b^3} \geq 0.$$
528.
Given $a, b, c \geq 0$. Prove that:
$$(a^4+b^4+c^4)(a^3+b^3+c^3)(a+b+c) \geq \ 9abc(a^5+b^5+c^5)$$

Solution:
ineq became:

$$\Leftrightarrow \frac{a^3+b^3+c^3}{3abc} \geq \frac{3(a^5+b^5+c^5)}{(a+b+c)(a^4+b^4+c^4)}$$

$$Sa = \frac{a+b+c}{2abc} + \frac{a^3+2bc(b+c)-(b^2+bc+c^2)(a+b+c)}{(a+b+c)(a^4+b^4+c^4)}$$

We easily to see $Sa, Sb \geq 0$
We only need to prove that: $Sb+Sc \geq 0$
We have:

$$Sb+Sc \geq \frac{a+b+c}{abc} - \frac{2abc+2(a^2+b^2+c^2)(a+b+c)}{(a^4+b^4+c^4)(a+b+c)} \geq \frac{a+b+c}{abc} - \frac{20(a^2+b^2+c^2)(a+b+c)}{9(a^4+b^4+c^4)} > 0$$

Because

$$(a+b+c)^5 \geq 81abc(a^2+b^2+c^2)$$

$$\Rightarrow Q.E.D$$

529..
if a, b, c are non-negative numbers, no two of which are zero, then

$$\sum \frac{2a^2+bc}{b+c} \geq \frac{9(a^2+b^2+c^2)}{2(a+b+c)}.$$

Solution:
The inequality is equivalent to:

$$\sum_{cyc} \left(\frac{2a^3+abc}{b+c} + 2a^2+bc \right) \geq \frac{9}{2}(a^2+b^2+c^2).$$

Recalling the known inequality:

$$\sum_{cyc} \frac{a^3+abc}{b+c} \geq a^2+b^2+c^2.$$

it suffices to prove that:

$$\sum_{cyc} \frac{a^3}{b+c} + bc \geq \frac{3}{2}(a^2+b^2+c^2)$$

$$\Leftrightarrow \sum_{cyc} \frac{a^2+b^2-c^2}{(a+c)(b+c)}(a-b)^2 \geq 0$$

$$\Leftrightarrow \sum_{cyc} S_c(a-b)^2 \geq 0.$$

Assuming $a \geq b \geq c$, hence:

$$S_c \geq 0,$$

$$S_b \geq 0,$$

$$S_b + S_a = \frac{c^2(a+b+2c)+(a+b)(a-b)^2}{(a+b)(b+c)(c+a)} \geq 0.$$

530.
, Given $a, b, c \geq 0$. Prove that:

$$\sqrt{a^2+4bc} + \sqrt{b^2+4ca} + \sqrt{c^2+4ab} \geq \sqrt{15(ab+bc+ca)}$$

Solution:
Used AM-GM, We only need to prve that:

$$216(a^2+4bc)(b^2+4ca)(c^2+4ab) \geq \ (11\sum\ ab - \sum a^2)^3$$

With $a+b+c=3$, it's becames:

$$216(125r^2+4q^3-180qr+432r) \geq \ (13q-9)^3$$

$$f'(r) = 216(432+250r-180q) \geq 0$$

$$\Rightarrow f(r) \geq \ f(0) \geq \ 0 always trues with q \leq 2.59$$

$$With q \geq \ 2.59, We have : f(r) \geq \ f(\frac{4q-9}{3}) = (3-q)(q+4.48)(q-2.02) \geq 0$$

$$\Rightarrow Q.E.D$$

To allnames :
First, squaring .Next, use am-gm

Because of Schur:

$$9+3abc \geq \ 4(ab+bc+ca)$$

if $k=5$, ineq is not true. if $k<3.59$, We can prove easily by:

$$(a^2+kbc)(b^2+kca)(c^2+kab) \geq \ \frac{(1+k)^3}{27}(ab+bc+ca)^3$$

So, $k=4$ is nice

531..
Let a,b and c be non-negative numbers, no two of them are zero. Prove that:

$$\frac{ab}{(a+b)^2}+\frac{ac}{(a+c)^2}+\frac{bc}{(b+c)^2} \geq \frac{(a+b+c)(ab+ac+bc)}{4(a^3+b^3+c^3)}$$

Solution:

$$\sum\ \frac{ab}{a^2+b^2} \geq \ \frac{(ab+bc+ca)^2}{\sum ab(a^2+b^2)} \geq \frac{(a+b+c)(ab+bc+ca)}{2(a^3+b^3+c^3)}$$

$$\Leftrightarrow 2(ab+bc+ca)(a^3+b^3+c^3) \geq \ (a+b+c)\sum ab(a^2+b^2)$$

$$\Leftrightarrow \sum\ ab(a^3+b^3) \geq \sum\ a^2b^2(a+b)$$

$$\Leftrightarrow \sum\ ab(a+b)(a-b)^2 \geq 0$$

532.
Given $a,b,c>0$. Prove that:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \ \frac{9}{a+b+c+\sqrt{3(ab+bc+ca)}}$$

Solution:
The ineq is equivalent to :

$$\frac{a^2+b^2+c^2+3(ab+bc+ca)}{(a+b)(b+c)(c+a)} \geq \frac{9}{a+b+c+\sqrt{3(ab+bc+ca)}}$$

Let

$$a+b+c=3u, ab+bc+ca=3v^2, abc=w^3$$

The ineq becomes:

$$\frac{3u^2+v^2}{9uv^2-w^3} \geq \frac{1}{u+v}$$

Therefore We only have to prove the ineq when a=b and c=0,which is trivial.

We can continue without considering 2 cases(a=b and c=0).

$$<=> w^3+v^3+3u^2v+3u^3 \geq 8uv^2$$

$$<=> (w^3+3u^3-4uv^2)+v(u-v)(3u-v) \geq 0$$

which is true since

$$(w^3+3u^3-4uv^2) \geq 0$$

(Schur ineq) and $u \geq v$

$$\Longleftrightarrow 0 \leq \sum a^3 - 5\sum a^2(b+c) - 9abc + \left(\sum a^2 + 3\sum bc\right)\sqrt{3\sum bc}$$

$$= \sum a^3 - \sum a^2(b+c) + 3abc - 4\sum bc \sum a + \left(\sum a^2 + 3\sum bc\right)\sqrt{3\sum bc}$$

$$= \sum a(a-b)(a-c) + \frac{\left(\sum a^2 - \sum bc\right)\left(3\sum a^2 + 5\sum bc\right)\sqrt{\sum bc}}{\left(\sum a^2 + 3\sum bc\right)\sqrt{3} + 4\sum a\sqrt{\sum bc}}.$$

Q.E.D

533.
Let a, b, c be nonnegative numbers and $ab+bc+ca=1$. Prove:

$$\frac{1}{(1+a^2)^2} + \frac{1}{(1+b^2)^2} + \frac{1}{(1+c^2)^2} \leq 2$$

Solution:
$\sum_{cyc} \frac{1}{(1+a^2)^2} \leq 2 \Leftrightarrow \sum_{cyc} \frac{1}{(a+b)^2(a+c)^2} \leq \frac{2}{(ab+ac+bc)^2} \Leftrightarrow$

$\Leftrightarrow (ab+ac+bc)^2 \sum_{cyc}(a^2+ab) \leq \prod_{cyc}(a+b)^2$.

Let $a+b+c=3u$, $ab+bc+ca=3v^2$ and $abc=w^3$.

Hence, $(ab+ac+bc)^2 \sum_{cyc}(a^2+ab) \leq \prod_{cyc}(a+b)^2 \Leftrightarrow w^3 \leq 9uv^2 - \sqrt{27(3u^2-v^2)v^4}$.

But w^3 gets a maximal value when two numbers from $\{a, b, c\}$ are equal.

id est, it remains to prove that

$(ab+ac+bc)^2 \sum_{cyc}(a^2+ab) \leq \prod_{cyc}(a+b)^2$ for $b=c=1$,

which gives $4a^3+3a^2+2a+1 \geq 0$.

it is true because $a \geq 0$

534.

Let $a, b, c > 0$. Prove that:

$$\frac{3a^4+a^2b^2}{a^3+b^3}+\frac{3b^4+b^2c^2}{b^3+c^3}+\frac{3c^4+c^2a^2}{c^3+a^3} \geq 2\left(a+b+c\right)$$

Solution:

$$\frac{3a^4+a^2b^2}{a^3+b^3}+\frac{3b^4+b^2c^2}{b^3+c^3}+\frac{3c^4+c^2a^2}{c^3+a^3} \geq 2\left(a+b+c\right)$$

$$\Leftrightarrow \sum\left(\frac{3a^4+a^2b^2}{a^3+b^3}-2a\right) \geq 0$$

$$\Leftrightarrow \sum\left(\frac{\left(a-b\right)\left(a^3+a^2b+2ab^2\right)}{a^3+b^3}\right) \geq 0$$

$$\Leftrightarrow \sum\left(\frac{\left(a-b\right)\left(a^3+a^2b+2ab^2\right)}{a^3+b^3}-2\left(a-b\right)\right) \geq 0$$

$$\Leftrightarrow \frac{\left(2c^2-b^2\right)\left(b-c\right)^2}{b^3+c^3}+\frac{\left(2a^2-c^2\right)\left(c-a\right)^2}{c^3+a^3}+\frac{\left(2b^2-a^2\right)\left(a-b\right)^2}{a^3+b^3} \geq 0$$

Setting

$$S_a = \frac{\left(2c^2-b^2\right)}{b^3+c^3}$$

$$S_b = \frac{\left(2a^2-c^2\right)}{c^3+a^3}$$

$$S_c = \frac{\left(2b^2-a^2\right)}{a^3+b^3}$$

$$\Rightarrow S_b \geq 0$$

Then We will show that

$$S_b + 2S_c \geq 0$$

$$a^2S_b^2 + 2b^2S_a \geq 0$$

First We will prove $S_b + 2S_c \geq 0$.

$$S_b + 2S_c = \frac{\left(2a^2-c^2\right)\left(a^3+b^3\right)+\left(a^3+c^3\right)\left(4b^2-2a^2\right)}{\left(a^3+b^3\right)\left(a^3+c^3\right)}$$

$$= \frac{\left(2a^2b^3-2a^2c^3\right)+\left(4a^3b^2-a^3c^2-b^3c^2\right)+4b^2c^3}{\left(a^3+b^3\right)\left(a^3+c^3\right)} \geq 0$$

Now We will prove $a^2S_b^2 + 2b^2S_a \geq 0$

$$a^2S_b^2 + 2b^2S_a \geq 0$$

$$\Leftrightarrow \frac{a^2\left(2a^2-c^2\right)}{a^3+c^3}+\frac{2b^2\left(2c^2-b^2\right)}{b^3+c^3} \geq 0$$

Setting $f\left(a\right) = \frac{a^2\left(2a^2-c^2\right)}{a^3+c^3}+\frac{2b^2\left(2c^2-b^2\right)}{b^3+c^3}$

$$\Rightarrow f'\left(a\right) = \frac{2a^6+8a^3b^3+a^4c^2-2ac^5}{\left(a^3+c^3\right)^2} \geq 0$$

$$\Rightarrow f(a) \geq f(b) = \frac{3b^2c^2}{b^3+c^3} \geq 0$$

Applying the two inequality We have proved We get

$$\sum S_a (b-c)^2 \geq \left(S_a (b-c)^2 + \frac{S_b (c-a)^2}{2} \right) + \frac{S_b (b-c)^2}{2} + \left(\frac{S_b (a-b)^2}{2} + S_c (a-b)^2 \right) \geq 0$$

We have done in the case $a \geq b \geq c$

535.

if a, b, c, d, e are positive real numbers such that $a+b+c+d+e = 5$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{20}{a^2+b^2+c^2+d^2+e^2} \geq 9$$

Solution:

in this **Solution**, $\sum_{sym} f(a,b)$ means $f(a,b)+f(a,c)+f(a,d)+f(a,e)+f(b,c)+f(b,d)$ $+f(b,e)+f(c,d)+f(c,e)+f(d,e)$. We will firstly rewrite the inequality as

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} - \frac{25}{a+b+c+d+e} \geq 4 - \frac{4(a+b+c+d+e)^2}{5(a^2+b^2+c^2+d^2+e^2)}.$$

Using the identities

$$(a+b+c+d+e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) - 25 = \sum_{sym} \frac{(a-b)^2}{ab}$$

and $5(a^2+b^2+c^2+d^2+e^2) - (a+b+c+d+e)^2 = \sum_{sym}(a-b)^2$

We can rewrite again the inequality as

$$\frac{1}{a+b+c+d+e} \sum_{sym} \frac{(a-b)^2}{ab} \geq \frac{4}{5} \times \frac{\sum_{sym}(a-b)^2}{a^2+b^2+c^2+d^2+e^2}$$

or $\sum_{sym} S_{ab}(a-b)^2 \geq 0$ where $S_{xy} = \frac{1}{xy} - \frac{4}{a^2+b^2+c^2+d^2+e^2}$ for all $x, y \in \{a, b, c, d, e\}$.

Assume that $a \geq b \geq c \geq d \geq e > 0$. We will show that $S_{bc} + S_{bd} \geq 0$ and $S_{ab} + S_{ac} + S_{ad} + S_{ae} \geq 0$. indeed, We have

$$\begin{aligned} S_{bc} + S_{bd} &= \frac{1}{bc} + \frac{1}{bd} - \frac{8}{a^2+b^2+c^2+d^2+e^2} \\ &> \frac{1}{bc} + \frac{1}{bd} - \frac{8}{b^2+b^2+c^2+d^2} \\ &\geq \frac{1}{bc} + \frac{1}{bd} - \frac{8}{2bc+2bd} \geq 0 \end{aligned}$$

and

$$S_{ab}+S_{ac}+S_{ad}+S_{ae} = \frac{1}{ab}+\frac{1}{ac}+\frac{1}{ad}+\frac{1}{ae}-\frac{16}{a^2+b^2+c^2+d^2+e^2} \geq \frac{16}{a(b+c+d+e)}-\frac{16}{a^2+\frac{1}{4}(b+c+d+e)^2} \geq 0.$$

Hence, with notice that $S_{bd} \geq S_{bc}$ and $S_{ae} \geq S_{ad} \geq S_{ac} \geq S_{ab}$ We have $S_{bd} \geq 0$ and $S_{ae} \geq 0, S_{ae} + S_{ad} \geq 0, S_{ae} + S_{ad} + S_{ac} \geq 0$. Thus,

$S_{bd}(b-d)^2 + S_{bc}(b-c)^2 \geq (S_{bd} + S_{bc})(b-c)^2 \geq 0 \quad (1)$

and

$$S_{ae}(a-e)^2+S_{ad}(a-d)^2+S_{ac}(a-c)^2+S_{ab}(a-b)^2 \geq (S_{ae}+S_{ad})(a-d)^2+S_{ac}(a-c)^2+S_{ab}(a-b)^2$$

$$\geq (S_{ae} + S_{ad} + S_{ac})(a-c)^2 + S_{ab}(a-b)^2 \geq (S_{ae} + S_{ad} + S_{ac} + S_{ab})(a-b)^2 \geq 0 \qquad (2)$$

On the other hand, $S_{be} \geq S_{bd} \geq 0$ and $S_{de} \geq S_{ce} \geq S_{cd} \geq S_{bd} \geq 0 \quad (3)$. Therefore, from (1), (2) and (3) We get $\sum_{sym} S_{ab}(a-b)^2 \geq 0$. Equality occurs when $a = b = c = d = e$ or $a = 2b = 2c = 2d = 2e$

536.
Problem.For three positive real numbers a, b, c, prove that

$$\frac{1}{(a-b)^2} + \frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} \geq \frac{11+5\sqrt{5}}{2(a^2+b^2+c^2)}$$

When does the equality hold ?
Solution:
Since the inequality is symmetric We can assume that
$a \geq b \geq c$. if We fix $a-b, b-c, c-a$, then the maximum of RHS is when $c = 0$.
So, $x = a-b, y = b-c$ then the inequality is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{(x+y)^2} \geq \frac{11+5\sqrt{5}}{2((x+y)^2+y^2)}$$

Multiply x^2 to this inequality and

$$p = \frac{y}{x}$$

then,the inequality is

$$1 + \frac{1}{p^2} + \frac{1}{(1+p)^2} \geq \frac{11+5\sqrt{5}}{2((1+p)^2+p^2)}$$

and when We multiply $2(p^2+(p+1)^2)$ to this inequality,

it is equivalent to

$$\frac{4}{p(p+1)} + \frac{2(2p^2+2p+1)}{p^2(p+1)^2} + 4p(p+1) \geq 1 + \sqrt{5}$$

. $x = p(p+1)$
then
$\frac{8}{x} + \frac{2}{x^2} + 4x \geq 1 + 5\sqrt{5}$.
if We differentiate $\frac{8}{x} + \frac{2}{x^2} + 4x$ to evaluate minimum,
$x^3 - 2x + 1 = 0$ is the condition which is equal to $(x^2 - x - 1)(x+1) = 0$. Since $x > 0$, $x^2 = x + 1$ and $x = \frac{1+\sqrt{5}}{2}$.
if We evaluate the value, minimum is when
$x = \frac{1+\sqrt{5}}{2}$.
The condition of equality is $c = 0$, and

$$\frac{b}{a-b}(\frac{b}{a-b}+1) = \frac{ab}{(a-b)^2} = \frac{1+\sqrt{5}}{2} \iff a^2 - 2ab + b^2$$

$$= \frac{\sqrt{5}-1}{2}ab \iff a^2 - \frac{\sqrt{5}+3}{2}ab + b^2 = 0 \iff \frac{a}{b}$$

or

$$\frac{b}{a} = \frac{\frac{\sqrt{5}+3}{2} + \sqrt{\frac{3\sqrt{5}-1}{2}}}{2}$$

537.

if a, b, c are positive numbers such that $a+b+c=3$, then

$$\frac{a}{3a+b^2} + \frac{b}{3b+c^2} + \frac{c}{3c+a^2} \leq \frac{3}{4}$$

Solution:

The inequality is equivalent to

$$\sum \frac{b^2}{b^2+3a} \geq \frac{3}{4}$$

By Cauchy Schwarz inequality, We have

$$LHS \geq \frac{(a^2+b^2+c^2)^2}{\sum a^4 + (a+b+c)\sum ab^2}$$

it suffices to prove

$$4(a^2+b^2+c^2)^2 \geq 3\sum a^4 + 3\sum a^2b^2 + 3\sum ab^3 + 3\sum a^2bc$$

$$\Leftrightarrow (a^2+b^2+c^2)^2 - 3\sum ab^3 + 3(\sum a^2b^2 - \sum a^2bc) \geq 0$$

By VasC's inequality, We have

$$(a^2+b^2+c^2)^2 - 3\sum ab^3 \geq 0$$

By Am -GM inequality,

$$\sum a^2b^2 - \sum a^2bc \geq 0$$

We are done.

538.

Let $a, b, c > 0$.Prove that:

$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{a+b+c}{2(ab+bc+ca)} + \frac{3}{a+b+c}$. **Solution**:

Let put $p = a+b+c, q = ab+bc+ca, r = abc$, This inequality is equivalent to:

$$\frac{p^2+q}{pq-r} \geq \frac{p}{2q} + \frac{3}{p}$$

$$\iff \frac{p^2+3}{3p-r} \geq \frac{p}{6} + \frac{3}{p}$$

By expanding expression We have:

$$(p^2+3)6p - p^2(3p-r) - 18(3p-r) \geq 0$$

$$\iff 3p^3 + p^2r - 36p + 18r \geq 0$$

From the ill-known inequality, the third degree Schur's inequality states:

$$p^3 - 4pq + 9r \geq 0 \iff p^3 - 12p + 9r \geq 0$$

We have:

$$\iff 3p^3 + p^2r - 36p + 18r \geq 0$$

$$\iff 3(p^3 - 12p + 9r) + r(p^2 - 9) \geq 0$$

On the other hand, We have:

$$r(p^2 - 9) \geq 0 \iff (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$$

539// Let a,b,c be nonnegative real numbers, not all are zero. Prove that:

$$\sqrt{\frac{7a}{a+3b+3c}} + \sqrt{\frac{7b}{b+3c+3a}} + \sqrt{\frac{7c}{c+3a+3b}} \leq 3$$

Solution:
By Cauchy Schwarz inequality, We have:

$$(\sum_{cyc} \sqrt{\frac{7a}{a+3b+3c}})^2 \leq [\sum_{cyc}(17a+2b+2c)][\sum_{cyc} \frac{7a}{(17a+2b+2c)(a+3b+3c)}]$$

$$= 3\sum_{cyc} \frac{49a(a+b+c)}{(17a+2b+2c)(a+3b+3c)}$$

We need to prove:

$$\sum_{cyc} \frac{49a(a+b+c)}{(17a+2b+2c)(a+3b+3c)} \leq 3$$

$$\leftrightarrows \sum_{cyc}[1 - \frac{49a(a+b+c)}{(17a+2b+2c)(a+3b+3c)}] \geq 0 \leftrightarrows \sum_{cyc} \frac{(b+c-2a)(8a+3b+3c)}{(17a+2b+2c)(a+3b+3c)}] \geq 0$$

Normalize that

$$a+b+c = 1$$

, then the inequality becomes

$$\sum_{cyc} \frac{(b+c-2a)(5a+3)}{(15a+2)(3-2a)} \geq 0$$

$$\leftrightarrows \sum_{cyc}(a-b)[\frac{5b+3}{(15b+2)(3-2b)} - \frac{5a+3}{(15a+2)(3-2a)}] \geq 0$$

$$\leftrightarrows \sum_{cyc}(a-b)^2(1+30c-50ab)(15c+2)(3-2c) \geq 0$$

Without loss of generality, We may assume

$$a \geq b \geq c \to a \geq \frac{1}{3},$$

then We have

$$1+30a-50bc \geq 1+30a-50(\frac{b+c}{2})^2 = 1+30a-50(\frac{1-a}{2})^2$$

$$= \frac{1}{2}(110a-25a^2-23) \geq \frac{1}{2}(110-25a^2-23.3a) = 4a > 0$$

$$1+30b-50ca = 1+30b-50b(a+c-b)+50(a-b)(b-c)$$

$$= 1+30b-50b(1-2b)+50(a-b)(b-c)$$

$$= (10b-1)^2+50(a-b)(b-c) \geq 0$$

$$1+30b-50ca+1+30c-50ab = 2+30(b+c)-50a(b+c)$$

$$= 2+30(1-a)-50a(1-a) = 2(5a-4)^2 \geq 0$$

And

$$(15b+2)(3-2b)-(15c+2)(3-2c) = (b-c)(41-30b-30c) \geq 0$$

Therefore

$$LSH \geq (a-c)^2(1+30b-50ca)(15b+2)(3-2b)+(a-b)^2(1+30c-50ab)(15c+2)(3-2c)$$

$$\geq (a-b)^2(1+30b-50ca)(15c+2)(3-2c)+(a-b)^2(1+30c-50ab)(15c+2)(3-2c)$$

$$= (a-b)^2((15c+2)(3-2c)(1+30b-50ca+1+30c-50ab) \geq 0$$

Our inequality is proved. Equality holds if and only if

$$a = b = c = 1$$

or

$$a = 8b = 8c$$

or any cyclic permutations.
540.
if a, b, c are nonnegative real numbers, then

$$\sum a^2(a-b)(a-c)(3a-5b)(3a-5c) \geq 0$$

Solution:
Let $a+b+c=3u$, $ab+ac+bc=3v^2$, $abc=w^3$ and $u^2=tv^2$.
Hence,

$$\sum_{cyc} a^2(a-b)(a-c)(3a-5b)(3a-5c) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{cyc}(4.5a^6-24a^5b+15a^4b^2+32a^4bc-40a^3b^2c+12.5a^2b^2c^2) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow 49w^6+112(3u^2-4v^2)uw^3+9(9u^2-2v^2)(3u^2-4v^2)^2 \geq 0.$$

$\frac{\Delta}{14} = (3u^2 - 4v^2)^2(18v^2 - 17u^2)$. Thus, for $t \geq \frac{18}{17}$ the inequality is true.
While for $1 \leq t \leq \frac{18}{17}$ it's enough to prove that

$$w^3 \leq \frac{8u(4v^2 - 3u^2) - (4v^2 - 3u^2)\sqrt{18v^2 - 17u^2}}{7}.$$

$(a-b)^2(a-c)^2(b-c)^2 \geq 0$ gives $w^3 \leq 3uv^2 - 2u^3 + 2\sqrt{(u^2 - v^2)^3}$.
Hence, it remains to prove that

$$3uv^2 - 2u^3 + 2\sqrt{(u^2 - v^2)^3} \leq \frac{8u(4v^2 - 3u^2) - (4v^2 - 3u^2)\sqrt{18v^2 - 17u^2}}{7},$$

which is equivalent to

$$(4v^2 - 3u^2)\sqrt{18v^2 - 17u^2} \leq 11uv^2 - 10u^3 - 14\sqrt{(u^2 - v^2)^3}.$$

$11uv^2 - 10u^3 \geq 14\sqrt{(u^2 - v^2)^3}$ is true for $1 \leq t \leq \frac{18}{17}$.
Hence,

$$(4v^2 - 3u^2)\sqrt{18v^2 - 17u^2} \leq 11uv^2 - 10u^3 - 14\sqrt{(u^2 - v^2)^3} \Leftrightarrow$$

$$\Leftrightarrow \left((4v^2 - 3u^2)\sqrt{18v^2 - 17u^2}\right)^2 \leq \left(11uv^2 - 10u^3 - 14\sqrt{(u^2 - v^2)^3}\right)^2 \Leftrightarrow$$

$$\Leftrightarrow 449t^3 - 1378t^2 + 1413t - 484 \geq 28(11 - 10t)\sqrt{t(t-1)^3} \Leftrightarrow$$

$$\Leftrightarrow 449t^2 - 929t + 484 \geq 28(11 - 10t)\sqrt{t(t-1)} \Leftrightarrow$$

$$\Leftrightarrow (3t - 4)^2(117t - 121)^2 \geq 0,$$

which is true.
541.
Let$a, b, c, d, e \geq 0$

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)(e^2 + 1) \geq (a + b + c + d + e - 1)^2$$

Solution:

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)(e^2 + 1) - (a + b + c + d + e - 1)^2 \geq$$

$$\geq \frac{1}{12}\sum_{sym} a^2b^2c^2 + \frac{1}{12}\sum_{sym} a^2b^2 - \frac{1}{6}\sum_{sym} ab + 2(a + b + c + d + e) =$$

$$= \frac{1}{36}\sum_{sym}(3a^2b^2c^2 + a^2b^2 + a^2c^2 + b^2c^2 - 2(ab + ac + bc) + a + b + c)$$

Let $f(a,b,c) = 3a^2b^2c^2 + a^2b^2 + a^2c^2 + b^2c^2 - 2(ab + ac + bc) + a + b + c$, where $a = min\{a, b, c\}$.
Hence,

$$f(a,b,c) - f\left(a, \sqrt{bc}, \sqrt{bc}\right) = a^2(b-c)^2 - 2a(\sqrt{b} - \sqrt{c})^2 + (\sqrt{b} - \sqrt{c})^2 =$$

$$= (\sqrt{b} - \sqrt{c})^2\left((\sqrt{b} + \sqrt{c})^2a^2 - 2a + 1\right) \geq (\sqrt{b} - \sqrt{c})^2(4a^3 - 2a + 1) =$$

$$= (\sqrt{b} - \sqrt{c})^2\left(4a^3 + 0.5 + 0.5 - 2a\right) \geq (\sqrt{b} - \sqrt{c})^2 a \geq 0$$

Hence, it remains to prove that $f(a, b, b) \geq 0$.
But

$$f(a,b,b) \geq 0 \Leftrightarrow (3b^4 + 2b^2)a^2 - (4b - 1)a + b^4 - 2b^2 + 2b \geq 0$$

which is true for
$$b \le \frac{1}{4}$$
because
$$b^4 - 2b^2 + 2b = b(b^3 + 1 + 1 - 3b + b) \ge 0$$
Thus, it remains to prove that
$$(4b-1)^2 - 4(3b^4 + 2b^2)(b^4 - 2b^2 + 2b) \le 0$$

Which is true for $b > \frac{1}{4}$.
$=> Q.E.D$

542.
Let $a, b, c > 0 : ab + bc + ca = 2$. Prove that :
$$\sqrt{\frac{2a^2+bc}{a^2+bc}} + \sqrt{\frac{2b^2+ca}{b^2+ca}} + \sqrt{\frac{2c^2+ab}{c^2+ab}} \le \frac{2}{abc}$$
Solution:
By Cauchy-Schwars inequality,
$$\left(\sum \sqrt{\frac{2a^2+bc}{a^2+bc}}\right)^2 \le \frac{1}{2}\left(\sum \frac{1}{a}\right)\left(\sum \frac{(2a^2+bc)(b+c)}{a^2+bc}\right) = \frac{1}{abc}\sum \frac{(2a^2+bc)(b+c)}{a^2+bc}$$
We have
$$\sum \frac{(2a^2+bc)(b+c)}{a^2+bc} = \sum \frac{(2b+c)(a^2+bc)+c(a^2-b^2)}{a^2+bc}$$
$$= 3\sum a + \sum \frac{c(a^2-b^2)}{a^2+bc} = 3\sum a + \frac{abc(\sum a^2b^2 - \sum a^4)}{\prod(a^2+bc)} \le 3\sum a \le \frac{(\sum ab)^2}{abc} = \frac{4}{abc}$$
thus,
$$\left(\sum \sqrt{\frac{2a^2+bc}{a^2+bc}}\right)^2 \le \frac{4}{a^2b^2c^2}$$
and that is the desired result.
543.
Suppose a,b,c,d are positive integers with $ab + cd = 1$.
Then, For $We = 1, 2, 3, 4$,let $(x_i)^2 + (y_i)^2 = 1$, where x_i and y_i are real numbers.
Show that
$$(ay_1 + by_2 + cy_3 + dy_4)^2 + (ax_4 + bx_3 + cx_2 + dx_1)^2 \le 2(\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c}).$$
Solition:
Use Cauchy-Schwartz , We have
$$(ay_1 + by_2 + cy_3 + dy_4)^2 \le$$
$$(ab+cd)(\frac{(ay_1+by_2)^2}{ab} + \frac{(cy_3+dy_4)^2}{cd}) =$$
$$\frac{(ay_1+by_2)^2}{ab} + \frac{(cy_3+dy_4)^2}{cd}$$

Similar:

$$(ax_4 + bx_3 + cx_2 + dx_1)^2 \leq$$
$$(ab + cd)(\frac{(ax_4 + bx_3)^2}{ab} + \frac{(cx_2 + dx_1)^2}{cd})$$
$$= \frac{(ax_4 + bx_3)^2}{ab} + \frac{(cx_2 + dx_1)^2}{cd}$$

But:

$$(ay_1 + by_2)^2 \leq (ay_1 + by_2)^2 + (ax_1 - bx_2)^2 = a^2 + b^2 + 2ab(y_1y_2 - x_1x_2)$$

Similar.

$$(cx_2 + dx_1)^2 \leq c^2 + d^2 + 2cd(x_1x_2 - y_1y_2)$$

then We get:

$$\frac{(ay_1 + by_2)^2}{ab} + \frac{(cx_2 + dx_1)^2}{cd} \leq$$
$$\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c}$$

(1)
The same argument show that:

$$\frac{(cy_3 + dy_4)^2}{cd} + \frac{(ax_4 + bx_3)^2}{ab} \leq$$
$$\frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c}$$

(2)
Combining (1);(2) We get . Q.E.D

] 544.
Let $a, b, c > 0$.Prove that:

$$\frac{a^2 - bc}{\sqrt{8a^2 + (b + c)^2}} + \frac{b^2 - ca}{\sqrt{8b^2 + (c + a)^2}} + \frac{c^2 - ab}{\sqrt{8c^2 + (a + b)^2}} \geq 0$$

Solution:

$$\sum_{cyc} \frac{a^2 - bc}{\sqrt{8a^2 + (b + c)^2}} \geq 0 \Leftrightarrow \sum_{cyc} \frac{(a - b)(a + c) - (c - a)(a + b)}{\sqrt{8a^2 + (b + c)^2}} \geq 0.$$

But

$$\sum_{cyc} \frac{(a - b)(a + c) - (c - a)(a + b)}{\sqrt{8a^2 + (b + c)^2}} =$$
$$= \sum_{cyc} (a - b) \left(\frac{a + c}{\sqrt{8a^2 + (b + c)^2}} - \frac{b + c}{\sqrt{8b^2 + (a + c)^2}} \right) =$$
$$= \sum_{cyc} \frac{(a - b)((a + c)^2(8b^2 + (a + c)^2) - (b + c)^2(8a^2 + (b + c)^2)}{\sqrt{(8a^2 + (b + c)^2)(8b^2 + (a + c)^2)} \left((a + c)\sqrt{8b^2 + (a + c)^2} + (b + c)\sqrt{(8a^2 + (b + c)^2} \right)} =$$
$$= \sum_{cyc} \frac{(a - b)^2(4c^3 - 2(a + b)c^2 + 4(a^2 - 3ab + b^2)c + (a + b)(a^2 + b^2))}{\sqrt{(8a^2 + (b + c)^2)(8b^2 + (a + c)^2)} \left((a + c)\sqrt{8b^2 + (a + c)^2} + (b + c)\sqrt{(8a^2 + (b + c)^2} \right)} \geq$$

$$\geq \sum_{cyc} \frac{(a-b)^2(4c^3-2(a+b)c^2-(a+b)^2c+\frac{(a+b)^3}{2})}{\sqrt{(8a^2+(b+c)^2)(8b^2+(a+c)^2)}\left((a+c)\sqrt{8b^2+(a+c)^2}+(b+c)\sqrt{(8a^2+(b+c)^2}\right)} =$$

$$= \sum_{cyc} \frac{(a-b)^2(2c+a+b)(2c-a-b)^2}{2\sqrt{(8a^2+(b+c)^2)(8b^2+(a+c)^2)}\left((a+c)\sqrt{8b^2+(a+c)^2}+(b+c)\sqrt{(8a^2+(b+c)^2}\right)}.$$

Q.E.D

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545.

Let $a, b, c \geq 0$, s.t. $a+b+c=3$. Prove that:

$$(a^2b+b^2c+c^2a)(ab+bc+ca) \leq 9$$

Solution:

1)Let $a+b+c=3u$, $ab+ac+bc=3v^2$, $abc=w^3$ and $u^2=tv^2$.

Hence, $t \geq 1$ and $(a^2b+b^2c+c^2a)(ab+bc+ca) \leq 9 \Leftrightarrow$

$$\Leftrightarrow 3u^5 \geq (a^2b+b^2c+c^2a)v^2 \Leftrightarrow$$

$$\Leftrightarrow 6u^5 - v^2\sum_{cyc}(a^2b+a^2c) \geq v^2\sum_{cyc}(a^2b-a^2c) \Leftrightarrow$$

$$\Leftrightarrow 6u^5-9uv^4+3v^2w^3 \geq (a-b)(a-c)(b-c)v^2.$$

$(a-b)^2(a-c)^2(b-c)^2 \geq 0$ gives $w^3 \geq 3uv^2-2u^3-2\sqrt{(u^2-v^2)^3}$. Hence,

$$2u^5-3uv^4+v^2w^3 \geq 2u^5-3uv^4+v^2\left(3uv^2-2u^3-2\sqrt{(u^2-v^2)^3}\right) \geq 0$$

because

$$2u^5-3uv^4+v^2\left(3uv^2-2u^3-2\sqrt{(u^2-v^2)^3}\right) \geq 0 \Leftrightarrow$$

$\Leftrightarrow u^5-u^3v^2 \geq v^2\sqrt{(u^2-v^2)^3} \Leftrightarrow t^3-t+1 \geq 0$, which is true. Hence,

$$6u^5-9uv^4+3v^2w^3 \geq 0$$

and enough to prove that

$$(6u^5-9uv^4+3v^2w^3)^2 \geq v^4(a-b)^2(a-c)^2(b-c)^2.$$

Since,

$$(a-b)^2(a-c)^2(b-c)^2 = 27(3u^2v^4-4v^6+6uv^2w^3-4u^3w^3-w^6)$$

We obtain

$$(6u^5-9uv^4+3v^2w^3)^2 \geq v^4(a-b)^2(a-c)^2(b-c)^2 \Leftrightarrow$$

$$\Leftrightarrow v^4w^6+uv^2(u^4+3u^2-6v^4)w^3+u^{10}-3u^6v^4+3v^{10} \geq 0.$$

id est, it remains to prove that $u^2v^4(u^4+3u^2-6v^4)^2-4v^4(u^{10}-3u^6v^4+3v^{10}) \leq 0$. But

$$u^2v^4(u^4+3u^2-6v^4)^2-4v^4(u^{10}-3u^6v^4+3v^{10}) \leq 0 \Leftrightarrow$$

$$\Leftrightarrow t(t^2+3t-6)^2-4(t^5-3t^3+3) \leq 0 \Leftrightarrow (t-1)^2(t^3-4t+4) \geq 0,$$

which is true.

Q.E.D

.

2)Let $\{x,y,z\}=\{a,b,c\}$ such that $x\geq y\geq z$. By the Rearrangement inequality We have

$$a^2b+b^2c+c^2a=a(ab)+b(bc)+c(ca)\leq x(xy)+y(zx)+z(yz)=y(x^2+xz+z^2)$$

Using AM-GM inequality We get

$$(xy+yz+zx)y(x^2+xz+z^2)\leq y\frac{(xy+yz+zx+x^2+xz+z^2)^2}{4}=$$
$$=\frac{1}{4}y(x+z)^2(x+y+z)^2=\frac{9}{8}.2y.(z+x).(z+x)\leq\frac{9}{8}.[\frac{2(x+y+z)}{3}]^3=9$$

Q.E.D

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546.

Leta,b,c,d are non-negative reals. Prove that

$$\sum_{cyc}a^4+\sum_{cyc}abc(a+b+c)\geq 2\sum_{sym}a^2b^2+4abcd$$

Solution:

The following stronger inequality is also true

$$\frac{1}{6}\sum_{sym}a^4+\frac{1}{2}\sum_{sym}a^2bc\geq\frac{1}{3}\sum_{sym}a^3b+\frac{1}{6}\sum_{sym}a^2b^2+\frac{1}{6}\sum_{sym}abcd$$

First

$$(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}+\frac{\partial}{\partial c}+\frac{\partial}{\partial d})(\frac{1}{6}\sum_{sym}a^4+\frac{1}{2}\sum_{sym}a^2bc-\frac{1}{3}\sum_{sym}a^3b-\frac{1}{6}\sum_{sym}a^2b^2-\frac{1}{6}\sum_{sym}abcd)$$

$$=\frac{1}{6}\sum_{sym}4a^3+\frac{1}{2}\sum_{sym}(2abc+a^2b+a^2c)-\frac{1}{3}\sum_{sym}(a^3+3a^2b)-\frac{1}{6}\sum_{sym}(2a^2b+2ab^2)-\frac{1}{6}\sum_{sym}(abc+bcd+cda+dab)$$

$$=\frac{1}{3}\sum_{sym}a^3+\frac{1}{3}\sum_{sym}abc-\frac{2}{3}\sum_{sym}a^2b$$

$$=\frac{1}{9}\sum_{sym}(a^3+b^3+c^3+3abc-a^2b-ab^2-b^2c-bc^2-c^2a-ac^2)\geq 0$$

So We can assume $d=0$

We only have to prove

$$\sum_{cyc}a^4+\sum_{cyc}a^2bc\geq\frac{2}{3}\sum_{sym}a^3b+\frac{2}{3}\sum_{sym}a^2b^2$$

(these sums are symmetric for a,b,c.) it comes from Schur.

547.

★ Let $a,b,c\geq 0$, find the best k constant such that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\sqrt{9-k+\frac{k(a^2+b^2+c^2)}{ab+bc+ca}}$$

Solution:

Let $a+b+c=3u$, $ab+ac+bc=3v^2$, where $v>0$, $abc=w^3$, $k>0$

and $\sqrt{9-k+\frac{k(a^2+b^2+c^2)}{ab+bc+ca}}=3p$. Then $p\geq 1$, $\frac{u^2}{v^2}=\frac{3p^2+k-3}{k}$ and

$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\sqrt{9-k+\frac{k(a^2+b^2+c^2)}{ab+bc+ca}}\Leftrightarrow\sum_{cyc}a^2c\geq 3pw^3\Leftrightarrow$

$\Leftrightarrow\sum_{cyc}(a^2b+a^2c)\geq 6pw^3+\sum_{cyc}(a^2b-a^2c)\Leftrightarrow$

$\Leftrightarrow 9uv^2-3w^3\geq 6pw^3+(a-b)(a-c)(b-c)$.

Now We will understand when $9uv^2-3w^3\geq 6pw^3$ is true.

$(a-b)^2(b-c)^2(c-a)^2\geq 0\Leftrightarrow w^6-2(3uv^2-2u^3)w^3+4v^6-3u^2v^4\leq 0$.

it gives $w^3\leq 3uv^2-2u^3+2\sqrt{(u^2-v^2)^3}$.

Hence, if $3uv^2\geq(1+2p)\left(3uv^2-2u^3+2\sqrt{(u^2-v^2)^3}\right)$ is true then $9uv^2-3w^3\geq 6pw^3$ is true.

But $3uv^2\geq(1+2p)\left(3uv^2-2u^3+2\sqrt{(u^2-v^2)^3}\right)\Leftrightarrow$

$\Leftrightarrow 3\geq(1+2p)\left(3-2\cdot\frac{3p^2+k-3}{k}-2\sqrt{\frac{\left(\frac{3p^2+k-3}{k}-1\right)^3}{\frac{3p^2+k-3}{k}}}\right)\Leftrightarrow$

$\Leftrightarrow 3k\geq(1+2p)\left(k+6-6p^2+2\sqrt{\frac{27(p^2-1)^3}{3p^2+k-3}}\right)\Leftrightarrow$

$\Leftrightarrow 6p^3+3p^2-(k+6)p+k-3\geq(1+2p)\sqrt{\frac{27(p^2-1)^3}{3p^2+k-3}}$.

But $6p^3+3p^2-(k+6)p+k-3=(p-1)(6p^2+9p-k+3)\geq 0$ for all $k\leq 18$.

Hence, for $0<k\leq 18$ We obtain $6p^3+3p^2-(k+6)p+k-3\geq(1+2p)\sqrt{\frac{27(p^2-1)^3}{3p^2+k-3}}\Leftrightarrow$

$(6p^3+3p^2-(k+6)p+k-3)^2(3p^2+k-3)\geq 27(p^2-1)^3(1+2p)^2\Leftrightarrow$

$\Leftrightarrow(p-1)^2(54p^3-9(k-15)p^2-18(k-6)p+k^2-9k+27)\geq 0$.

Let $f(p)=54p^3-9(k-15)p^2-18(k-6)p+k^2-9k+27$.

Then $f'(p)=162p^2-18(k-15)p-18(k-6)=18(p+1)(9p-k+6)\geq 0$ for all $0<k\leq 15$.

Hence, for all $0<k\leq 15$ We obtain $f(p)\geq f(1)=k^2-36k+324=(k-18)^2\geq 0$.

if $15<k\leq 18$ then $f(p)\geq f\left(\frac{k-6}{9}\right)=\frac{-k^3+18k^2-27k-27}{27}$.

$-k^3+18k^2-27k-27\geq 0$ and $k>15$ gives $15<k\leq 6+6\sqrt{3}\cos 10^\circ=16.23...$

Thus, $9uv^2-3w^3\geq 6pw^3$ is true for all $0\leq k\leq 6+6\sqrt{3}\cos 10^\circ$.

Now it remains to understand for which k the following inequality is true.

$9uv^2-3w^3\geq 6pw^3+\sqrt{(a-b)^2(a-c)^2(b-c)^2}.$

But $9uv^2-3w^3\geq 6pw^3+\sqrt{(a-b)^2(a-c)^2(b-c)^2}\Leftrightarrow$

$\Leftrightarrow(3uv^2-(1+2p)w^3)^2\geq 3(3u^2v^4-4v^6+2(3uv^2-2u^3)w^3-w^6)\Leftrightarrow$

$\Leftrightarrow(1+p+p^2)w^6-3(2uv^2+puv^2-u^3)w^3+3v^6\geq 0\Leftrightarrow$

$\Leftrightarrow(1+p+p^2)w^6-3\sqrt{\frac{3p^2+k-3}{k}}\left(2+p-\frac{3p^2+k-3}{k}\right)v^3w^3+3v^6\geq 0\Leftrightarrow$

$\Leftrightarrow(1+p+p^2)w^6-\frac{3}{k}\cdot\sqrt{\frac{3p^2+k-3}{k}}(k+3+pk-3p^2)v^3w^3+3v^6\geq 0.$

if $p\geq\frac{k+3}{3}$ then $k+3+pk-3p^2\leq 0$ and our inequality holds.

if $1\leq p<\frac{k+3}{3}$ then We need understand for which k holds:

$\frac{9(3p^2+k-3)(k+3+pk-3p^2)^2}{k^3}-12(1+p+p^2)\leq 0.$

But $\frac{9(3p^2+k-3)(k+3+pk-3p^2)^2}{k^3}-12(1+p+p^2)\leq 0\Leftrightarrow$

$\Leftrightarrow(p-1)^2g(p)\leq 0$, where

$g(p)=81p^4-54(k-3)p^3+9(k^2-15k)p^2+18(k^2-6k-9)p-k^3+9k^2-27k-81.$

We see that $g(1)=-k(k-18)^2\leq 0$ and $g\left(\frac{k+3}{3}\right)=-4k(k^2+9k+27)\leq 0.$

$g'(p)=18(p+1)(18p^2-9(k-1)p+k^2-6k-9).$

Hence, $p_{max}=\frac{3(k-1)-\sqrt{k^2+30k+81}}{12}.$

id est, it remains to solve the following inequality:

$g\left(\frac{3(k-1)-\sqrt{k^2+30k+81}}{12}\right)\leq 0$, which indeed gives $k\leq 3\left(1+\sqrt[3]{2}\right)^2$ and

for $k=\left(1+\sqrt[3]{2}\right)^2$ We obtain $p_{max}=\sqrt[3]{2}.$

Q.E.D

548.
Let a, b, c be positive real numbers such that $a+b+c=1$. Prove inequality:

$$\frac{1}{bc+a+\frac{1}{a}}+\frac{1}{ac+b+\frac{1}{b}}+\frac{1}{ab+c+\frac{1}{c}}\leqslant\frac{27}{3}.$$

Solution:
1)

the inequality is equivalent to :

$$\sum \frac{1}{bc+a+\frac{1}{a}} - a \leq \frac{-4}{31}$$

(because $a+b+c=1$)

$$\sum \frac{a^2(bc+a)}{abc+a^2+1} \geq \frac{4}{31}$$

$$\sum \frac{(a^2(bc+a)^2)}{(abc+a^2+1)(bc+a)} \geq \frac{4}{31}$$

but by cauchy shwarz :

$$LHS(\sum (abc+a^2+1)(bc+a)) \geq (\sum a(bc+a))^2$$

setting : $r = abc, q = ab+ac+bc$ and since $a+b+c=1$ it's easy the check that:

$$\sum (abc+a^2+1)(bc+a) = 2-2q+5r+qr$$

$$\sum a(bc+a) = 3r+1-2q$$

so We need to prove that:

$$(3r+1-2q)^2 - \frac{4(2-2q+5r+qr)}{31} \geq 0$$

or:

$$9r^2 + \frac{166r}{31} - \frac{376qr}{31} + \frac{23}{31} - \frac{116q}{31} + 4q^2 \geq 0$$

now We put:

$$f(r) = 9r^2 + \frac{166r}{31} - \frac{376qr}{31} + \frac{23}{31} - \frac{116q}{31} + 4q^2$$

We have:

$$f(r)' = 18r + \frac{166}{31} - \frac{376q}{31}$$

it's easy to check that : $f(r)' \geq 0$ since $q \leq \frac{1}{3}$

so f is an increasing function , and by shur : $r \geq \frac{4q-1}{9}$

thus :

$$f(r) \geq f(\frac{4q-1}{9}) = \frac{12q^2}{31} - \frac{28q}{31} + \frac{8}{31}$$

hence it sufficies to prove , that : $\frac{12q^2}{31} - \frac{28q}{31} + \frac{8}{31} \geq 0$

wich is equivalent to : $\frac{4(3q-1)(q-2)}{31} \geq 0$, wich is true since $q \leq \frac{1}{3} \leq 2$.

2)

The ineq becomes to $\sum \frac{x}{xyz+x^2+1} \leq \frac{27}{31}$

$$\iff 31 \sum x(xyz+1+y^2)(xyz+1+z^2) \leq 27 \prod (xyz+x^2+1)$$

Let $x+y+z=3u=1$,$xy+yz+zx=3v^2$ and $xyz=w^3$

$$\sum x(xyz+1+y^2)(xyz+1+z^2)$$
$$=x^2y^2z^2\sum x+xyz\sum(xy^2+x^2y)+2xyz\sum x+\sum x+\sum(xy^2+x^2y)+xyz\sum xy$$
$$LHS=w^6+w^3(9uv^2-3w^3)+2w^3+1+9uv^2-3w^3+3w^3v^2$$
$$=-2w^6+6w^3v^2-w^3+A$$

$$\prod(xyz+x^2+1)$$
$$=x^2y^2z^2\sum x^2+xyz\sum x^2y^2+2xyz\sum x^2+\sum x^2y^2+\sum x^2+x^3y^3z^3+4x^2y^2z^2+3xyz+1$$
$$=w^9+2w^6-6w^6v^2-3w^3v^2+2w^3+B$$

the ineq becomes to

$$f(w^3)=27w^9+116w^6-162w^6v^2-267w^3v^2+85w^3\geq 0$$
$$f'(w^3)=243w^6+232w^3-324w^3v^2-267v^2+85$$

549.
Let $x,y,z>1$ and $x+y+z=xyz$. Find the minimum value of :

$$A=\frac{x-2}{y^2}+\frac{y-2}{z^2}+\frac{z-2}{x^2}$$

Solution:
This problem can be done by this way:
+) We have

$$A=\left(\frac{x-2}{y^2}+1\right)+\left(\frac{y-2}{z^2}+1\right)+\left(\frac{z-2}{x^2}+1\right)-3$$
$$=\frac{(x-1)+(y^2-1)}{y^2}+\frac{(y-1)+(z^2-1)}{z^2}+\frac{(z-1)+(x^2-1)}{x^2}-3$$
$$=\left(\frac{x-1}{y^2}+\frac{x^2-1)}{x^2}\right)+\left(\frac{y-1}{z^2}+\frac{y^2-1)}{y^2}\right)+\left(\frac{z-1}{x^2}+\frac{z^2-1)}{z^2}\right)-3$$

$$=(x-1)(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{x})+(y-1)(\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{y})+(z-1)(\frac{1}{x^2}+\frac{1}{z^2}+\frac{1}{z})-3$$

$$\geq(x-1)(\frac{2}{xy}+\frac{1}{x})+(y-1)(\frac{2}{yz}+\frac{1}{y})+(z-1)(\frac{2}{xz}+\frac{1}{z})-3$$
$$=(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})-2(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx})$$

+) But from $x+y+z=xyz$ We get

$$\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}=1$$

and

$$(\frac{1}{x}+\frac{1}{y}+\frac{1}{z})^2$$

$$=(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2})+2(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx})\geq 3(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx})=3$$

Hence $A\geq\sqrt{3}-2$ Q.E.D

550.

Let a,b,c be positive integers such that $a+b+c=3$. Prove that
$\sum_{cyc}\frac{a}{3a^2+abc+27}\leq\frac{3}{31}$.

Solution:

By Schur inequality, We get $3abc\geq 4(ab+bc+ca)-9$. it suffices to prove that

$$\sum\frac{3a}{9a^2+4(ab+bc+ca)+72}\leq\frac{3}{31}$$

$$\sum\left(1-\frac{31a(a+b+c)}{9a^2+4(ab+bc+ca)+72}\right)\geq 0$$

$$\sum\frac{(7a+8c+10b)(c-a)-(7a+8b+10c)(a-b)}{a^2+s}\geq 0$$

where $s=\frac{4(ab+bc+ca)+72}{9}$.

$$\sum(a-b)^2\frac{8a^2+8b^2+15ab+10c(a+b)+s}{(a^2+s)(b^2+s)}\geq 0$$

which is true.

551.

Let $x_1,x_2,\cdots,x_m,y_1,y_2,\cdots,y_n$ be positive real numbers.

Denote by $X=\sum_{We=1}^{m}x,Y=\sum_{j=1}^{n}y$.

Prove that

$$2XY\sum_{We=1}^{m}\sum_{j=1}^{n}|x_i-y_j|\geq X^2\sum_{j=1}^{n}\sum_{l=1}^{n}|y_i-y_l|+Y^2\sum_{We=1}^{m}\sum_{k=1}^{m}|x_i-x_k|$$

Solution:

ill assume $x_1\geq x_2\geq\cdots\geq x_n$ and $X\geq Y$, then make an induction :

if the inequality is true for $m-1$, then We can prove for $x=x_1+x_2$

the following statement, which will solve our problem in matter of fact:

$X\sum_{j=1}^{n}|x-y_j|-Y\sum_{We=3}^{m}|x-x_i|\leq X\sum_{j=1}^{n}(|x_1-y_j|+|x_2-y_j|)-Y\left(\sum_{We=3}^{m}(|x_1-x_i|+|x_2-x_i|)+|x_1-x_2|\right)$ (*).

First

$$LHS=X\sum_{j=1}^{n}|x-y_j|-Y\sum_{We=3}^{m}|x-x_i|=\sum_{j=1}^{n}\sum_{We=1}^{m}x_i|x-y_j|-\sum_{j=1}^{n}\sum_{We=3}^{m}y_j|x-x_j|$$

$RHS=X\sum_{j=1}^{n}(|x_1-y_j|+|x_2-y_j|)-Y\left(\sum_{We=3}^{m}(|x_1-x_i|+|x_2-x_i|)+|x_1-x_2|\right)=$
$\sum_{j=1}^{n}\sum_{We=1}^{m}(x_i|x_1-y_j|+x_i|x_2-y_j|)-\sum_{j=1}^{n}\sum_{We=3}^{m}\left(y_j|x_1-x_i|+y_j|x_2-x_i|\right)+y_j|x_1-x_2|)$

and now using the inequality

$$|xx_We-y_jx_i|-|y_jx-y_jx_i|\leq x_i|x_1-y_j|+x_i|x_2-y_j|-y_j|x_1-x_i|-y_j|x_2-x_i|$$

which follows by verifying all cases - $x_1 \geq x_2 \geq y_j \cup x \geq y_j$ etc. , We will take what We need.

$\Rightarrow 2XY \sum_{We=1}^{m} \sum_{j=1}^{n} |x_We - y_j| - X^2 \sum_{We=1}^{n} \sum_{j=1}^{n} |y_We - y_j| - Y^2 \sum_{We=1}^{m} \sum_{j=1}^{m} |x_We - x_j| \geq 2XY \sum_{We=2}^{m} \sum_{j=1}^{n} |x'_We - y_j| - X^2 \sum_{We=1}^{n} \sum_{j=1}^{n} |y_We - y_j| - Y^2 \sum_{We=2}^{m} \sum_{j=1}^{m} |x'_We - x'_j| \geq 0$ by (*), where $x'_2 = x_1 + x_2$ and $x'_We = x_i$ for $i \geq 3$.
Q.E.D

552.
Let$a, b, c > 0, a+b+c = 1$. Prove that

$$\frac{a^2+3b}{b+c} + \frac{b^2+3c}{c+a} + \frac{c^2+3a}{a+b} \geq 5$$

Solution:
We have :$a+b+c = 1$

$$\frac{a^2+3b}{b+c} + \frac{b^2+3c}{c+a} + \frac{c^2+3a}{a+b} \geq 5$$

First, We are regrouping LHS in the way:

$$LHS = \sum \frac{a^2+3b}{b+c} = \sum (\frac{a^2-1}{1-a} + \frac{3b+1}{b+c})$$
$$= -(a+b+c+3) + \sum \frac{b+c+3b+a}{b+c}$$
$$= -(a+b+c+3) + 3 + \sum \frac{a+3b}{b+c}$$
$$<=> \sum \frac{a^2+3b}{b+c} = -1 + \sum \frac{a+3b}{b+c}$$

Now We have to prove:

$$-1 + \sum \frac{a+3b}{b+c} \geq 5$$
$$<=> \sum \frac{a+3b}{b+c} \geq 6$$

After clearing denominators We have:

$$\sum (a+3b)(a+c)(b+c) \geq 6(a+b)(a+c)(b+c)$$
$$<=> \sum (a^3 + ac^2) \geq 2 \sum a^2c$$

Wich is true by Am-Gm inequality:

$$\frac{a^3+ac^2}{2} \geq a^2c$$

Q.E.D

553.
if a, b, c are REALS such that $a^2+b^2+c^2 = 1$ Prove that $a+b+c-2abc \leq \sqrt{2}$ **Solution**:
Use Cauchy-Schwartz:

$$LHS = a(1-2bc) + (b+c) \leq \sqrt{(a^2 + (b+c)^2)((1-2bc)^2 + 1)}$$

So it'll be enough to prove that :

$$(a^2 + (b+c)^2)((1-2bc)^2 + 1) \leq 2 \Leftrightarrow (1+2bc)(1-bc+2b^2c^2) \leq 1 \Leftrightarrow 4b^2c^2 \leq 1$$

which is true because

$$1 \geq b^2 + c^2 \geq 2bc$$

554.

Let a, b, c be positive reals satisfying $a^2 + b^2 + c^2 = 3$. Prove that

$$(abc)^2(a^3 + b^3 + c^3) \leq 3$$

Solution:

For the sake of convenience, let us introduce the new unknowns u, v, w as follows:

$$u = a + b + c, v = ab + bc + ca, w = abc$$

Now note that $u^2 - 2v = 3$ and $a^3 + b^3 + c^3 = u(u^2 - 3v) = u\left(\frac{9-u^2}{2}\right)$.

We are to prove that $w^2\left(u \cdot \frac{9-u^2}{2} + 3w\right) \leq 3$.

By AM-GM, We have

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3} \implies w \leq \frac{u^3}{3^3}$$

Hence, it suffices to prove that

$$u^7 \cdot \frac{9-u^2}{2} + \frac{u^9}{3^2} \leq 3^7$$

Hoiver, by QM-AM We have

$$\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3} \implies u \leq 3$$

which proves the above inequality.

555.

Let $a, b, c \geq 0$ and $a + b + c = 1$. Prove that :

$$\frac{a}{\sqrt{b^2+3c}} + \frac{b}{\sqrt{c^2+3a}} + \frac{c}{\sqrt{a^2+3b}} \geq \frac{1}{\sqrt{1+3abc}}$$

Solution:

1)

Using Holder's inequality

$$\left(\sum_{cyc} \frac{a}{\sqrt{b^2+3c}}\right)^2 \cdot \sum_{cyc} a(b^2+3c) \geq (a+b+c)^3 = 1$$

it is enough to prove that

$1 + 3abc \geq \sum_{cyc} a(b^2 + 3c)$

Homogenise $(a+b+c)^3 = 1$,

Also after Homogenising $\sum_{cyc} a(b^2+3c) = a^2b+b^2c+c^2a+9abc+3\sum_{sym} a^2b$

$(a+b+c)^3 = a^3+b^3+c^3+6abc+3\sum_{sym} a^2b$

it is enough to prove that

$a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$

By AM-GM

$a^3+a^3+b^3 \geq 3a^2b$

$b^3+b^3+c^3 \geq 3b^2c$

$c^3+c^3+a^3 \geq 3c^2a$

Then $a^3+b^3+c^3 \geq a^2b+b^2c+c^2a$,done 2)

$f(t) = \dfrac{1}{\sqrt{t}}; f'(t) < 0; f''(t) > 0$

Using Jensen with iights $a, b, c,$ We have

$af(b^2+3c)+bf(c^2+3a)+cf(a^2+3b) \geqslant f(ab^2+bc^2+ca^2+3ab+3bc+3ca)$

Now,

By Holder, $(a^3+b^3+c^3) = \sqrt[3]{(a^3+b^3+c^3)(b^3+c^3+a^3)(b^3+c^3+a^3)} \geqslant ab^2+bc^2+ca^2$

Again,

$3(a+b)(b+c)(c+a)+3abc = 9abc+3\sum_{sym} a^2b = 3(a+b+c)(ab+bc+ca) = 3(ab+bc+ca)$

$\therefore 1+3abc = (a+b+c)^3+3abc > ab^2+bc^2+ca^2+3ab+3bc+3ca$

$\therefore f(ab^2+bc^2+ca^2+3ab+3bc+3ca) > f(1+3abc)$

QED

556.

Let a,b,c,d be positive real numbers satisfying $a+b+c+d=4$.Prove that

$$\frac{1}{11+a^2}+\frac{1}{11+b^2}+\frac{1}{11+c^2}+\frac{1}{11+d^2} \leq \frac{1}{3}$$

$$f(x) = \frac{1}{11+x^2} \Leftrightarrow f''(x) = \frac{6(x^2 - \frac{11}{3})}{(11+x^2)^3}$$

$$(x^2 - \frac{11}{3}) = \left(x - \sqrt{\frac{11}{3}}\right)\left(x + \sqrt{\frac{11}{3}}\right)$$

if x $\in \left(-\sqrt{\frac{11}{3}}, \sqrt{\frac{11}{3}}\right), \ f''(x) < 0$

Thus within the interval $\left(-\sqrt{\frac{11}{3}}, \sqrt{\frac{11}{3}}\right)$, the quadratic polynomial is negative

thereby making $f''(x) < 0$, and thus $f(x)$ is concave within $\left(-\sqrt{\frac{11}{3}}, \sqrt{\frac{11}{3}}\right)$.

Solution: Let $a \leqslant b \leqslant c \leqslant d$. if all of $a, b, c, d \in \left(0, \sqrt{\frac{11}{3}}\right)$,

Then by Jensen, $f(a) + f(b) + f(c) + f(d) \leqslant 4f\left(\frac{a+b+c+d}{4}\right) = 4f(1) = \frac{4}{12} = \frac{1}{3}$

$$f'(x) = \frac{-2x}{(11+x^2)^3} < 0 \text{ (for all positive } x)$$

At most 2 of a, b, c, d(namely c & d) can be greater than $\sqrt{\frac{11}{3}}$

in that case,

$f(a) + f(b) + f(c) + f(d) < f(a-1) + f(b-1) + f(c-3) + f(d-3) < 4f\left(\frac{a+b+c+d-8}{4}\right) = 4f($

QED

557.

Let $0 < a < b$ and $x_We \in [a, b]$.Prove that

$$(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \leq \frac{n^2(a+b)^2}{4ab}$$

Solution:

1)We will prove that if $a_1, a_2, \ldots, a_n \in [a, b]$ $(0 < a < b)$ then

$$(a_1 + a_2 + \cdots + a_n)(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}) \leq \frac{(a+b)^2}{4ab}n^2$$

$$P = (a_1 + a_2 + \cdots + a_n)(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}) =$$
$$(\frac{a_1}{c} + \frac{a_2}{c} + \cdots + \frac{a_n}{c})(\frac{c}{a_1} + \frac{c}{a_2} + \cdots + \frac{c}{a_n}) \leq$$
$$\frac{1}{4}(\frac{a_1}{c} + \frac{c}{a_1} + \frac{a_2}{c} + \frac{c}{a_2} + \cdots + \frac{a_n}{c} + \frac{c}{a_n})^2$$

Function $f(t) = \frac{c}{t} + \frac{t}{c}$ have its maximum on $[a, b]$ in a or b. We will choose c such that $f(a) = f(b), c = \sqrt{ab}$. Then $f(t) \leq \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}$. Then

$$P \leq n^2(\sqrt{\frac{a}{b}} + \sqrt{b\frac{b}{a}})^2 \cdot \frac{1}{4} = n^2\frac{(a+b)^2}{4ab}$$

2)

$$(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \leq n\left(\frac{x_1}{x_n} + \frac{x_2}{x_n - 1} + ...\right)$$

using Chebyshev.

which is equal to

$$(1^2+1^2+...)\left(\frac{x_1}{x_n}+\frac{x_2}{x_n-1}+...\right)\leq\left(\sqrt{\frac{x_1}{x_n}}+...\right)^2$$

by Cauchy-Schwartz.

Now,

$$\sqrt{\frac{x_i}{x_j}}\leq\sqrt{\frac{b}{a}}\leq\frac{a+b}{2\sqrt{ab}}\implies\left(\sqrt{\frac{x_1}{x_n}}+...\right)^2\leq\frac{n^2(a+b)^2}{4ab}$$

Hence proved.

558.

Let x, y, z be positive real number such that $xy+yz+zx=1$. Prove that

$$\frac{27}{4}(x+y)(y+z)(z+x)\geq(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x})^2\geq 6\sqrt{3}$$

Solution:

From the constraint, We have

$$(x+y)(y+z)=y^2+1$$
$$(y+z)(z+x)=z^2+1$$
$$(z+x)(x+y)=x^2+1$$

so that the right inequality can be rewritten as

$$x+y+z+\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}\geq 3\sqrt{3}(1)$$

Now,

$$(x+y+z)^2=x^2+y^2+z^2+2\geq xy+yz+zx+2=3$$

hence

$$x+y+z\geq 3(2)$$

Also the function

$$f(t)=\sqrt{t^2+1}$$

is a convex function (its second derivative satisfies

$$f''(t)=(t^2+1)^{-3/2}>0$$

Thus,

$$\sqrt{x^2+1}+\sqrt{y^2+1}+\sqrt{z^2+1}\geq 3\sqrt{(\frac{x+y+z}{3})^2+1}$$

and using (2) We obitan

$$\sqrt{x^2+z}+\sqrt{y^2+1}+\sqrt{z^2+1}\geq 2\sqrt{3}$$

Adding (2) and (3) yields (1). Asfor the left inequality ,it is equivalent to

$$\frac{1}{x^2+1}+\frac{1}{y^2+1}+\frac{1}{z^2+1}\leq\frac{3\sqrt{3}}{2}(4)$$

The constraint allows us to write

$$x=tan\frac{a}{2},y=tan\frac{b}{2},z=tan\frac{c}{2}$$

where a,b,ca are the angles of a triangle . Then (4) can be rewritten as

$$cos\frac{a}{2}+cos\frac{b}{2}+cos\frac{c}{2}\leq\frac{3\sqrt{3}}{2},$$

which holds because from the concavity of cos on $(0,\frac{\pi}{2})$We have

$$cos\frac{a}{2}+cos\frac{b}{2}+cos\frac{c}{2}\leq 3cos\frac{a+b+c}{6}=\frac{3\sqrt{3}}{2}.$$

559.(Tack Garfulkel inequality)
Let triangle ABC.Prove that:

$$m_a+l_b+h_c\leq\frac{\sqrt{3}}{2}(a+b+c)$$

Proof:
Let

$$x=p-a>0,y=p-b>0,z=p-c>0.$$

$$=>a=y+z,b=z+x,c=x+y.$$

We have

$$m_a=\frac{1}{2}\sqrt{2(b^2+c^2)-a^2}=\frac{1}{2}\sqrt{2(z+x)^2+2(x+y)^2-(y+z)^2}$$

$$=\frac{1}{2}\sqrt{4x^2+4x(y+z)+y^2-2yz+z^2}=\sqrt{(x+\frac{y+z}{2})^2-yz}$$

$$=\frac{1}{\sqrt{3}}\sqrt{3(x+\frac{y+z}{2}-\sqrt{yz})(x+\frac{y+z}{2}+\sqrt{yz})}$$

$$\leq\frac{1}{\sqrt{3}}\frac{3(x+\frac{y+z}{2}-\sqrt{yz})(x+\frac{y+z}{2}+\sqrt{yz})}{2}\leq\frac{1}{\sqrt{3}(2x+y+z-\sqrt{yz})}$$

And We have too:

$$l_b=\frac{2\sqrt{ac}}{a+c}\sqrt{p(p-b)}\leq\sqrt{p(p-b)}=\sqrt{y(x+y+z)}$$

$$l_c=\frac{2\sqrt{ab}}{a+b}\sqrt{p(p-c)}\leq\sqrt{p(p-c)}=\sqrt{z(x+y+z)}$$

$$=>m_a+l_b+l_c\leq\frac{2x+y+z-\sqrt{yz}}{\sqrt{3}}+\sqrt{x+y+z}(\sqrt{y}+\sqrt{z})$$

$$\leq\frac{2x+y+z-\sqrt{yz}}{\sqrt{3}}+\frac{2}{3}\sqrt{x+y+z}\frac{\sqrt{3}}{2}(\sqrt{y}+\sqrt{z})$$

$$\leq\frac{1}{\sqrt{3}}2x+y+z-\sqrt{yz}+x+y+z+\frac{3}{4}(\sqrt{y}+\sqrt{z})^2$$

$$\leq\frac{1}{\sqrt{3}}[3(x+y+z)-(\sqrt{y}-\sqrt{z})^2]$$

$$\leq \frac{1}{\sqrt{3}}3(x+y+z) \leq \frac{\sqrt{3}}{2}(a+b+c)$$

$$=> m_a, l_b, l_c \leq \frac{\sqrt{3}}{2}(a+b+c)(1)$$

$$=> m_a + l_b + h_c \leq \frac{\sqrt{3}}{2}(a+b+c)$$

Equality ocur if $a=b=c$

560.

Let a,b,c be pove real number such that $abc=1$.Prove that:

$$\sum \sqrt[4]{2a^2+bc} \leq \frac{ab+bc+ca}{\sqrt[4]{3}}.\sqrt{\frac{a+b+c+3}{2}}.$$

Proof:

$$\sum \sqrt[4]{2a^2+bc} = \sum \frac{\sqrt[4]{2a^2bc+b^2c^2}}{\sqrt[4]{bc}} \leq \sqrt{\left(\sum \sqrt{2a^2bc+b^2c^2}\right)\left(\sum \frac{1}{\sqrt{bc}}\right)}$$

$$\leq \sqrt{\sqrt{3\sum(2a^2bc+b^2c^2)}.\sqrt{3\sum\left(\frac{1}{bc}\right)}}$$

$$= \sqrt{3(ab+bc+ca)\sqrt{a+b+c}} = \sqrt{3(ab+bc+ca)\sqrt{abc(a+b+c)}}$$

$$\leq \sqrt{3(ab+bc+ca)\frac{ab+bc+ca}{\sqrt{3}}} = \sqrt[4]{3}(ab+bc+ca) \leq \frac{ab+bc+ca}{\sqrt[4]{3}}.\sqrt{\frac{a+b+c+3}{2}}$$

561.

Let n is a positive integer, real numbers $a_1, a_2, ..., a_n$ and $r_1, r_2, ..., r_n$ satisfies $a_1 \leqslant a_2 \leqslant ... \leqslant a_n$ and $0 \leqslant r_1 \leqslant r_2 \leqslant ... \leqslant r_n$,

Prove that:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j min(r_i, r_j) \geq 0$$

Proof:

for n=1, it is trivial.

assume $n \geq 2$, and $a_We(i=1,2,...,n)$ are neither all positive nor all negative, otherwise LHS is obviously $>$=0.

WLOG, let

$$a_t \leq 0 \leq a_{t+1}$$

let$b_i = -a_We$ for $i=1,2,...,t$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j min(r_i, r_j) \geq 0$$

$$\Longleftrightarrow \sum_{i=t+1}^{n}\sum_{j=t+1}^{n} a_i a_j min(r_i, r_j) + \sum_{i=1}^{t}\sum_{j=1}^{t} a_i a_j min(r_i, r_j) + 2\sum_{i=t+1}^{n}\sum_{j=1}^{t} a_i a_j min(r_i, r_j) \geq 0$$

$$\Longleftrightarrow \sum_{i=t+1}^{n}\sum_{j=t+1}^{n} a_i a_j min(r_i, r_j) + \sum_{i=1}^{t}\sum_{j=1}^{t} b_i b_j min(r_i, r_j) - 2\sum_{i=t+1}^{n}\sum_{j=1}^{t} a_i b_j r_j \geq 0$$

$$\iff 2\sum_{t+1\le i<j\le n} a_ia_jr_i + \sum_{i=t+1}^{n} a_i^2r_i + 2\sum_{1\le i<j\le t} b_ib_jr_i + \sum_{i=1}^{t} b_i^2r_i - 2(\sum_{i=t+1}^{n} a_i)(\sum_{j=1}^{t} b_jr_j) \ge 0$$

if$r_{t+1} = 0$,it is trivial. If $r_{t+1} \neq 0$ We have :
$\frac{r_i}{r_{t+1}} \le 1$ $(i = 1, 2, ..., t)$,since $\{r_We\}$ is monotonously increasing.
Hence,

$$\begin{aligned}
&2\sum_{t+1\le i<j\le n} a_ia_jr_i + \sum_{i=t+1}^{n} a_i^2r_i + 2\sum_{1\le i<j\le t} b_ib_jr_i + \sum_{i=1}^{t} b_i^2r_i - 2(\sum_{i=t+1}^{n} a_i)(\sum_{j=1}^{t} b_jr_j)\\
&\ge r_{t+1}(\sum_{t+1}^{n} a_i)^2 + 2\sum_{1\le i<j\le t} b_ib_jr_i\frac{r_j}{r_{t+1}} + \sum_{i=1}^{t} b_i^2r_i\frac{r_i}{r_{t+1}} - 2(\sum_{i=t+1}^{n} a_i)(\sum_{j=1}^{t} b_jr_j)\\
&= r_{t+1}(\sum_{t+1}^{n} a_i)^2 + \frac{(\sum_{i=1}^{t} b_Wer_i)^2}{r_{t+1}} - 2(\sum_{i=t+1}^{n} a_i)(\sum_{j=1}^{t} b_jr_j) \ge 0
\end{aligned}$$

By AM-GM's inequalities.
equality holds if $r_1 = r_2 = ... = r_n$ and $\sum_{i=1}^{n} r_ia_i = 0$
562.
Let $a, b, c > 0, a + b + c = 3$. Prove that:

$$\frac{1}{\sqrt{a^2+1}} + \frac{1}{\sqrt{b^2+1}} + \frac{1}{\sqrt{c^2+1}} + \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}} \ge 3 + \frac{3}{\sqrt{2}}$$

Proof:
1)
Let

$$f(x) = \frac{1}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x}} = (\sqrt{x^2+1})^{-\frac{1}{2}} + (\sqrt{x})^{-\frac{1}{2}}$$

Inequality

$$<=> f(a) + f(b) + f(c) \ge 3 + \frac{3}{\sqrt{2}}$$

We have

$$f''(x) = (x^2+1)^{-\frac{5}{2}}.(2x^2-1) + \frac{3}{4}.x^{-\frac{5}{2}}$$

$$f''(x) = (x^2+1)^{-\frac{5}{2}}.(2x^2-1) + \frac{3}{4}.x^{-\frac{5}{2}} > 0$$

$$\Leftrightarrow \frac{2x^2-1}{\sqrt{(x^2+1)^5}} + \frac{3}{4\sqrt{x^5}} > 0$$

$$\Leftrightarrow 4\sqrt{x^5}.(2x^2-1) + 3\sqrt{(x^2+1)^5 > 0}$$

$$+)x > \frac{1}{\sqrt{2}}$$

$$+)0 < x < \frac{1}{\sqrt{2}}$$

$$3.\sqrt{(x^2+1)^5} > 4\sqrt{x^5(2x^2-1)^2} \Leftrightarrow 9.(x^2+1)^5 > 16x^5(2x^2-1)^2$$

We have

$$LHS \ge 9.(2x)^5 > 16x^5 > 16x^5(2x^2-1)^2 = RHS$$

Because $0 < x < \frac{1}{\sqrt{2}}$
Hence $f''(x) > 0 \forall x$.

By fensen's inequality, We have:

$$f(a)+f(b)+f(c)\geq 3f(\frac{a+b+c}{3})=3f(1)=3+\frac{3}{\sqrt{2}}$$

2)

$$LHS=\sum(\frac{1}{\sqrt{a^2+1}}+\frac{1}{\sqrt{2a}})+(1-\frac{1}{\sqrt{2}}).\sum(\frac{1}{\sqrt{a}})$$

$$\geq 2\sum(\frac{1}{\sqrt[4]{2a.(a^2+1)}})+(1-\frac{1}{\sqrt{2}}).\sum(\frac{1}{\sqrt{a}})$$

$$\geq 2\sum(\frac{\sqrt{2}}{a+1})+(1-\frac{1}{\sqrt{2}}).\frac{9}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$$

$$\geq 2\sqrt{2}.\frac{9}{a+b+c+3}+(1-\frac{1}{\sqrt{2}}).\frac{9}{\sqrt{3(a+b+c)}}$$

$$=2\sqrt{2}.\frac{9}{3+3}+(1-\frac{1}{\sqrt{2}}).\frac{9}{\sqrt{3.3}}=3+\frac{3}{\sqrt{2}}$$

563.
Let ABC be a triangle, and A, B, C its angles. Prove that

$$\sin\frac{3A}{2}+\sin\frac{3B}{2}+\sin\frac{3C}{2}\leq\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+\cos\frac{C-A}{2}.$$

Proof:
We have:

$$\sum\cos\frac{B-C}{2}\geq\sum\sin\frac{3A}{2}\Longleftrightarrow 2\sum\sin\frac{B}{2}\sin\frac{C}{2}+\sum\sin\frac{A}{2}\geq$$

$$3\sum\sin\frac{A}{2}-4\sum\sin^3\frac{A}{2}\iff 4\sum\sin^3\frac{A}{2}+2\sum\sin\frac{B}{2}\sin\frac{C}{2}\geq 2\sum\sin\frac{A}{2}\quad(*)$$

Now, We will prove that:

$$2\left(\sin^3\frac{A}{2}+\sin^3\frac{B}{2}\right)+\sin\frac{C}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)\geq\sin\frac{A}{2}+\sin\frac{B}{2}\quad(1)$$

Indeed, We have:

$$(1)\iff 2\left(\sin^2\frac{A}{2}-\sin\frac{A}{2}\sin\frac{B}{2}+\sin^2\frac{B}{2}\right)+\sin\frac{C}{2}\geq 1$$

$$\iff 1-\cos A+1-\cos B+\cos\frac{A+B}{2}-\cos\frac{A-B}{2}+\sin\frac{C}{2}\geq 1$$

$$\iff 1+2\sin\frac{C}{2}\geq\cos A+\cos B+\cos\frac{A-B}{2},\text{ what is truly.}$$

Similar, We have:

$$2\left(\sin^3\frac{B}{2}+\sin^3\frac{C}{2}\right)+\sin\frac{A}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)\geq\sin\frac{B}{2}+\sin\frac{C}{2}\quad(2)$$

$$2\left(\sin^3\frac{C}{2}+\sin^3\frac{A}{2}\right)+\sin\frac{B}{2}\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)\geq\sin\frac{C}{2}+\sin\frac{A}{2}\quad(3)$$

From (1), (2) and (3) We have

$$4\sum\sin^3\frac{A}{2}+2\sum\sin\frac{B}{2}\sin\frac{C}{2}\geq 2\sum\sin\frac{A}{2}\quad\mathbb{QED}$$

564.
If a, b, c are non-negative numbers,then

$$\sum(a^2-bc)\sqrt{a^2+4bc}\geq 0$$

Proof:

$$\sum_{cyc}(a^2-bc)\sqrt{a^2+4bc}\geq 0\Leftrightarrow\sum_{cyc}a^2\sqrt{a^2+4bc}\geq bc\sqrt{a^2+4bc}\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}\left(a^6+4a^4bc+2a^2b^2\sqrt{(a^2+4bc)(b^2+4ac)}\right)\geq$$

$$\geq\sum_{cyc}\left(a^2b^2c^2+4a^3b^3+2c^2ab\sqrt{(a^2+4bc)(b^2+4ac)}\right).$$

But $2a^2b^2\sqrt{(a^2+4bc)(b^2+4ac)}\geq 2a^3b^3+8a^2b^2c\sqrt{ab}$ and

$$2c^2ab\sqrt{(a^2+4bc)(b^2+4ac)}\leq a^3c^2b+b^3c^2a+4c^3a^2b+4c^3b^2a.$$

Id est, it remains to prove that

$$\sum_{cyc}\left(a^6-2a^3b^3+4a^4bc+8a^2b^2c\sqrt{ab}-5a^3b^2c-5a^3c^2b-a^2b^2c^2\right)\geq 0.$$

We obtain:

$$\sum_{cyc}(a^6-2a^3b^3+a^4bc)=\sum_{cyc}(a^6-a^5b-a^5c+a^4bc)+\sum_{cyc}(a^5b+a^5c-2a^3b^3)\geq 0$$

and

$$\sum_{cyc}(a^2b^2c\sqrt{ab}-a^2b^2c^2)\geq 0.$$

Let's assume $a=x^2$, $b=y^2$ and $c=z^2$, where x, y and z are non-negative numbers. Hence, it remains to prove that

$$\sum_{cyc}(3x^6-5x^4y^2-5x^4z^2+7x^3y^3)\geq 0\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}(3x^6-10x^4y^2+14x^3y^3-10x^2y^4+3y^6)\geq 0\Leftrightarrow$$

$$\Leftrightarrow\sum_{cyc}(x-y)^2(3x^4+6x^3y-x^2y^2+6xy^3+3y^4)\geq 0.$$

565.
Prove that if $k,\ n\in\mathbb{N}^*$ so that

$$\sum_{i=1}^{n}a_i^k=1, then:\sum_{i=1}^{n}a_i+\prod_{i=1}^{n}\frac{1}{a_i}\geq\sqrt[k]{n^{k-1}}+\left(\sqrt[k]{n}\right)^n$$

Proof:
Manifestly the statement have to specify $k\geq 1,\ a_We>0.$
Because of $AM-GM$ inequality We have

$$\prod_{i=1}^{n}\frac{1}{a_i}\geq(n/\sum_{i=1}^{n}a_i)^n$$

Because of x^k is convexe We have

$$n(\sum_{i=1}^{n} a_i/n)^k \leq \sum_{i=1}^{n} a_i^k = 1$$

so $\sum_{i=1}^{n} a_i \leq n^{\frac{k-1}{k}} Let'sdenote$f(x)=x+(n$_{\overline{x)^n}}$ so

$$f'(x) = 1 - (\frac{n}{x})^{n+1} \leq 0$$

so $f(x)$ is decreasing for $x \in]0, n]$.
We have

$$0 < A = \sum_{i=1}^{n} a_i \leq n^{\frac{k-1}{k}} < n$$

so

$$\sum_{i=1}^{n} a_i + \prod_{i=1}^{n} \frac{1}{a_i} \geq \sum_{i=1}^{n} a_i + (n/\sum_{i=1}^{n} a_i)^n = f(A) \geq f(n^{\frac{k-1}{k}}) = n^{\frac{k-1}{k}} + n^{\frac{n}{k}}$$

566.
Let x, y, z are non-zero numbers such that $x + y + z = 0$. Find the maximum value of

$$E = \frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2}$$

Proof:
1)

$$\frac{yz}{x^2} = a^3 \, , \, \frac{zx}{y^2} = b^3 \, , \, \frac{xy}{z^2} = c^3$$

where a, b, c are real numbers.

$$xyz = a^3x^3 = b^3y^3 = c^3z^3 \, , \, x + y + z = \sqrt[3]{xyz}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 0$$

$$\Longrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

since $xyz \neq 0 On the other hand abc = 1, ab + bc + ca = 0 \frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2} = a^3 + b^3 + c^3 = (a + b + c)^3 - 3[(a + b + c)(ab + bc + ca) - abc] = (a + b + c)^3 + 3$

Let u be a real number such that $u = a + b + c$

Then it is easy to see that a, b, c are roots of the polynomial $P(t) = t^3 - ut^2 - 1$
Let $f(t)$ be a function such that

$$f(t) = t - \frac{1}{t^2}$$

Then a, b, c satisfy the equation $f(t) = u$ Now We will prove that if

$$u > \frac{-3}{\sqrt[3]{4}},$$

then the equation $f(t) = u$ has no more than one root.

$$u > \frac{-3}{\sqrt[3]{4}} \Longrightarrow f(t) = t - \frac{1}{t^2} > \frac{-3}{\sqrt[3]{4}} \Longrightarrow$$

$$(t - \sqrt[3]{-2})^2 \left(t - \frac{1}{\sqrt[3]{4}}\right) > 0 \Longrightarrow t > \frac{1}{\sqrt[3]{4}}$$

$$\implies f'(t) = 1 + \frac{2}{t^3} > 0$$

This shows that if $u > \frac{-3}{\sqrt[3]{4}}$, then the equation $f(t) = u$ has no more than one root.

$$u_{max} = \frac{-3}{\sqrt[3]{4}}$$

$$E_{max} = (u^3 + 3)_{max} = -\frac{15}{4}$$

Equality holds when $(x, y, z) = (k, k, -2k)$

2)

By Dirichlet Principle , exits two number from x, y, z (assume that x, y) such that $xy \geq 0$

Then $z = -(x + y)$ And

$$E = -\frac{y(x+y)}{x^2} - \frac{x(x+y)}{y^2} + \frac{xy}{(x+y)^2}$$

$$= \frac{xy}{(x+y)^2} - \left(\frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x}\right)$$

$\leq \frac{1}{4} - 4 = -\frac{15}{4}$ So $MaxE = -\frac{15}{4}$, equality holds when $(x, y, z) = (k, k, -2k)$ or cyclic permutation.

567.

Let ABC be a triangle with altitudes h_a, h_b, h_c, angle bisectors l_a, l_b, l_c, exradiWe r_a, r_b, r_c, inradius r and circumradius R. Prove or disprove the inequality

$$8 < \frac{h_a r_a + h_b r_b + h_c r_c}{Rr} \leq \frac{l_a r_a + l_b r_b + l_c r_c}{Rr} \leq \frac{27}{2}$$

Solution:

$$\frac{\sum h_a r_a}{Rr} = \frac{2 r_a r_b r_c}{Rr} \left(\sum \frac{1}{r_b + r_c}\right).$$

This is because:

$$h_a = \frac{2 r_b r_c}{r_b + r_c}.$$

Then

$$\frac{2 r_a r_b r_c}{Rr} \left(\sum \frac{1}{r_b + r_c}\right) = \frac{2 r_a r_b r_c}{Rr} \left(\frac{\sum (r_a + r_b)(r_a + r_c)}{\prod (r_a + r_b)}\right).$$

Now we use the fact that:

$$r_a r_b r_c = p^2 r,$$

$$\prod (r_a + r_b) = 4p^2 R,$$

$$\sum (r_a + r_b)(r_a + r_c) = \sum r_a^2 + 3 \sum r_a r_b = (r_a + r_b + r_c)^2 + 2 \sum r_a r_b.$$

Also

$$\sum r_a = 4R + r$$

and

$$\sum r_a r_b = p^2.$$

We put all these relations head to head and it follows that

$$\frac{\sum h_a r_a}{Rr} = \frac{(4R + r)^2 + p^2}{2R^2}.$$

Thus we have $8Rr + r^2 + p^2 > 0$, which is obvious.