ALGEBRA QUALIFYING EXAM, SPRING 2000: PART I

Directions: Work each problem in a separate bluebook. Give reasons for your assertions and state precisely any theorems that you quote.

Notation:
$\mathbb{Z}$: Integers
$\mathbb{Q}$: Rational Field
$\mathbb{R}$: Real Field
$\mathbb{C}$: Complex Field
$\mathrm{GL}_n(R)$: Group of invertible $n \times n$ matrices with entries in the ring R
$\mathbb{F}_q$: Finite field with q elements
$\mathbb{Z}/n$: Ring of integers mod n (can also be regarded as the cyclic group of order n)
S_n: Symmetric group of degree n

1. How many distinct isomorphism types are there for groups of order 5555?

2. Find the Galois group of $x^4 - 2$ over the fields $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{F}_3$ and $\mathbb{F}_{27}$.

3. Prove the following generalization of Nakayama's Lemma to noncommutative rings. Let R be a ring with 1 (not necessarily commutative) and suppose that $J \subset R$ is an ideal contained in every maximal left ideal of R. If M is a finitely generated left R-module such that $JM = M$, prove that $M = 0$.

4. Let S be a set of $n \times n$ nilpotent matrices over a field K that pairwise commute. Show that there is an invertible matrix M such that every matrix MAM^{-1} with $A \in S$ is strictly upper triangular, that is, all entries on or below the main diagonal are zero.

5.(a) Compute $|\,\mathrm{GL}_3(\mathbb{F}_2)|$, the number of invertible 3×3 matrices over the field $\mathbb{F}_2$. If $\mu \in \mathrm{GL}_3(\mathbb{F}_2)$ has order 7 explain why μ must act *transitively* on the non-zero elements of $\mathbb{F}_2^3 = (\mathbb{Z}/2)^3$.

(b) Using (a), show that there is a non-abelian group G of order $56 = 8 \cdot 7$ with a normal 2-Sylow subgroup isomorphic to $(\mathbb{Z}/2)^3$. Find the number of irreducible complex representations of G and their dimensions.

(c) Find the conjugacy classes of G and compute the character values for at least one irreducible complex representation of G of dimension greater than one.

Algebra Qualifying Exam, Spring 2000: Part II

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1. Suppose that A is a Noetherian local ring with maximal ideal $\mathfrak{m}$. If $\mathfrak{a} \subset A$ is an ideal such that the only prime ideal of A containing $\mathfrak{a}$ is $\mathfrak{m}$, show that $\mathfrak{m}^k \subset \mathfrak{a}$ for some $k \geq 1$.

2. A subgroup $H \subseteq S_n$ is *transitive* if for all i, j with $1 \leq i, j \leq n$, there exists some $\sigma \in H$ with $\sigma(i) = j$. An automorphism of a group G is called *inner* if it is of the form $x \to axa^{-1}$ for some $a \in G$.

   (a) Show that S_5 has six 5-Sylow subgroups.
   (b) Show that S_6 contains a transitive subgroup isomorphic to S_5.
   (c) The subgroup $H \subset S_6$ from part b has six cosets. Show that there is an isomorphism $\alpha : S_6 \to S_6$ such that $\alpha(H) \subset S_6$ is *not* a transitive subgroup of S_6.
   (d) Explain why the automorphism in part (c) is not inner.

3. How many similarity classes are there of 10×10 matrices with minimal polynomial $(x^2 + 1)(x^3 - 2)$ over the field $\mathbb{Q}$? Over the field $\mathbb{F}_5$?

4. Let k be a field of characteristic zero.

   (a) Suppose K and L are two finite extensions of k, in some fixed algebraic closure of k, such that K is *normal* over k. Prove that $|KL : L|$ divides $|K : k|$.
   (b) Suppose that E is a Galois extension of k with $\mathrm{Gal}(E/k) = S_n$, the symmetric group. Show that for any integer j with $1 < j < n$ there are subfields K, $L \subset E$ with $K \cap L = k$, $|K : k| = n$, $|KL : L| = j$ and $|L : k| = n!/j!$. [**Hint:** Galois correspondence.]

5. Let G be a group of odd order.

   (a) Show that the only irreducible complex character of G which is real valued is the trivial character χ_1. [**Hints:** Assume χ_V is a counterexample and get a contradiction from $0 = \langle \chi_1, \chi_V \rangle$. Make use of algebraic integers and the fact that $g \neq g^{-1}$ for $g \neq 1$.]

(b) Using (a), explain why the real group ring $\mathbb{R}[G]$ has structure

$$\mathbb{R} \times \prod_{i=1}^{\frac{s-1}{2}} \mathrm{Mat}_{r_i}(\mathbb{C}),$$

where s is the number of conjugacy classes of G and $\mathrm{Mat}_{r_i}(\mathbb{C})$ is the ring of $r_i \times r_i$ matrices with entries in $\mathbb{C}$.