HANOI MATHEMATICAL SOCIETY

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NGUYEN VAN MAU

HANOI OPEN MATHEMATICAL OLYMPIAD

PROBLEMS AND SOLUTIONS

Hanoi, 2009

Contents

Questions of Hanoi Open Mathematical Olympiad

1.1 Hanoi Open Mathematical Olympiad 2006

1.1.1 Junior Section, Sunday, 9 April 2006

Q1. What is the last two digits of the number

$$(11 + 12 + 13 + \cdots + 2006)^2?$$

Q2. Find the last two digits of the sum

$$2005^{11} + 2005^{12} + \cdots + 2005^{2006}.$$

Q3. Find the number of different positive integer triples (x, y, z) satisfying the equations

$$x^2 + y - z = 100 \quad \text{and} \quad x + y^2 - z = 124.$$

Q4. Suppose x and y are two real numbers such that

$$x + y - xy = 155 \quad \text{and} \quad x^2 + y^2 = 325.$$

Find the value of $|x^3 - y^3|$.

Q5. Suppose n is a positive integer and 3 arbitrary numbers are choosen from the set $\{1, 2, 3, \ldots, 3n + 1\}$ with their sum equal to $3n + 1$.

What is the largest possible product of those 3 numbers?

Q6. The figure $ABCDEF$ is a regular hexagon. Find all points M belonging to the hexagon such that

$$\text{Area of triangle } MAC = \text{Area of triangle } MCD.$$

Q7. On the circle (O) of radius 15cm are given 2 points A, B. The altitude OH of the triangle OAB intersect (O) at C. What is AC if $AB = 16cm$?

Q8. In ΔABC, $PQ//BC$ where P and Q are points on AB and AC respectively. The lines PC and QB intersect at G. It is also given $EF//BC$, where $G \in EF$, $E \in AB$ and $F \in AC$ with $PQ = a$ and $EF = b$. Find value of BC.

Q9. What is the smallest possible value of

$$x^2 + y^2 - x - y - xy?$$

1.1.2 Senior Section, Sunday, 9 April 2006

Q1. What is the last three digits of the sum

$$11! + 12! + 13! + \cdots + 2006!$$

Q2. Find the last three digits of the sum

$$2005^{11} + 2005^{12} + \cdots + 2005^{2006}.$$

Q3. Suppose that

$$a^{\log_b c} + b^{\log_c a} = m.$$

Find the value of

$$c^{\log_b a} + a^{\log_c b}?$$

Q4. Which is larger

$$2^{\sqrt{2}}, \quad 2^{1+\frac{1}{\sqrt{2}}} \quad \text{and} \quad 3.$$

Q5. The figure $ABCDEF$ is a regular hexagon. Find all points M belonging to the hexagon such that

$$\text{Area of triangle } MAC = \text{Area of triangle } MCD.$$

Q6. On the circle of radius 30cm are given 2 points A, B with $AB =$ 16cm and C is a midpoint of AB. What is the perpendicular distance from C to the circle?

Q7. In ΔABC, $PQ//BC$ where P and Q are points on AB and AC respectively. The lines PC and QB intersect at G. It is also given $EF//BC$, where $G \in EF$, $E \in AB$ and $F \in AC$ with $PQ = a$ and $EF = b$. Find value of BC.

Q8. Find all polynomials $P(x)$ such that

$$P(x) + P\Big(\frac{1}{x}\Big) = x + \frac{1}{x}, \quad \forall x \neq 0.$$

Q9. Let x, y, z be real numbers such that $x^2 + y^2 + z^2 = 1$. Find the largest possible value of

$$|x^3 + y^3 + z^3 - xyz|?$$

1.2 Hanoi Open Mathematical Olympiad 2007

1.2.1 Junior Section, Sunday, 15 April 2007

Q1. What is the last two digits of the number

$$(3 + 7 + 11 + \cdots + 2007)^2?$$

(A) 01; (B) 11; (C) 23; (D) 37; (E) None of the above.

Q2. What is largest positive integer n satisfying the following inequality:

$n^{2006} < 7^{2007}$?

(A) 7; (B) 8; (C) 9; (D) 10; (E) 11.

Q3. Which of the following is a possible number of diagonals of a convex polygon?

(A) 02; (B) 21; (C) 32; (D) 54; (E) 63.

Q4. Let m and n denote the number of digits in 2^{2007} and 5^{2007} when expressed in base 10. What is the sum $m+n$?

(A) 2004; (B) 2005; (C) 2006; (D) 2007; (E) 2008.

Q5. Let be given an open interval $(\alpha;\beta)$ with $\beta-\alpha=\frac{1}{2007}$. Determine the

maximum number of irreducible fractions $\frac{a}{b}$ in $(\alpha;\beta)$ with $1 \leq b \leq$ 2007?

(A) 1002; (B) 1003; (C) 1004; (D) 1005; (E) 1006.

Q6. In triangle ABC, $\angle BAC = 60^0$, $\angle ACB = 90^0$ and D is on BC. If AD

bisects $\angle BAC$ and $CD = 3cm$. Then DB is

(A) 3; (B) 4; (C) 5; (D) 6; (E) 7.

Q7. Nine points, no three of which lie on the same straight line, are located

inside an equilateral triangle of side 4. Prove that some three of these

points are vertices of a triangle whose area is not greater than $\sqrt{3}$.

Q8. Let a, b, c be positive integers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

Q9. A triangle is said to be the Heron triangle if it has integer sides and integer area. In a Heron triangle, the sides a, b, c satisfy the equation $b = a(a - c)$. Prove that the triangle is isosceles.

Q10. Let a, b, c be positive real numbers such that $\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} \geq 1$. Prove

that $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \geq 1$.

Q11. How many possible values are there for the sum $a + b + c + d$ if a, b, c, d

are positive integers and $abcd = 2007$.

Q12. Calculate the sum

$$\frac{5}{2.7} + \frac{5}{7.12} + \cdots + \frac{5}{2002.2007}.$$

Q13. Let be given triangle ABC. Find all points M such that

area of ΔMAB= area of ΔMAC.

Q14. How many ordered pairs of integers (x, y) satisfy the equation

$$2x^2 + y^2 + xy = 2(x + y)?$$

Q15. Let $p = \overline{abc}$ be the 3-digit prime number. Prove that the equation

$$ax^2 + bx + c = 0$$

has no rational roots.

1.2.2 Senior Section, Sunday, 15 April 2007

Q1. What is the last two digits of the number

$$\left(11^2 + 15^2 + 19^2 + \cdots + 2007^2\right)^2?$$

(A) 01; (B) 21; (C) 31; (D) 41; (E) None of the above.

Q2. Which is largest positive integer n satisfying the following inequality:

$$n^{2007} > (2007)^n.$$

(A) 1; (B) 2; (C) 3; (D) 4; (E) None of the above.

Q3. Find the number of different positive integer triples (x, y, z) satsfying

the equations

$$x + y - z = 1 \quad \text{and} \quad x^2 + y^2 - z^2 = 1.$$

(A) 1; (B) 2; (C) 3; (D) 4; (E) None of the above.

Q4. List the numbers $\sqrt{2}$, $\sqrt[3]{3}$, , $\sqrt[4]{4}$, $\sqrt[5]{5}$ and $\sqrt[6]{6}$ in order from greatest to

least.

Q5. Suppose that A, B, C, D are points on a circle, AB is the diameter, CD

is perpendicular to AB and meets AB at E, AB and CD are integers and $AE - EB = \sqrt{3}$. Find AE?

Q6. Let $P(x) = x^3 + ax^2 + bx + 1$ and $|P(x)| \leq 1$ for all x such that $|x| \leq 1$.

Prove that $|a| + |b| \leq 5$.

Q7. Find all sequences of integers $x_1, x_2, \dots, x_n, \dots$ such that ij divides $x_i + x_j$ for any two distinct positive integers i and j.

Q8. Let ABC be an equilateral triangle. For a point M inside ΔABC, let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB, respectively. Find the locus of all such points M for which $\angle FDE$ is a right angle.

Q9. Let $a_1, a_2, \ldots, a_{2007}$ be real numbers such that

$$a_1+a_2+\cdots+a_{2007} \geq (2007)^2 \text{ and } a_1^2+a_2^2+\cdots+a_{2007}^2 \leq (2007)^3-1.$$

Prove that $a_k \in [2006; 2008]$ for all $k \in \{1, 2, \ldots, 2007\}$.

Q10. What is the smallest possible value of

$$x^2 + 2y^2 - x - 2y - xy?$$

Q11. Find all polynomials $P(x)$ satisfying the equation

$$(2x-1)P(x) = (x-1)P(2x), \ \forall x.$$

Q12. Calculate the sum

$$\frac{1}{2.7.12} + \frac{1}{7.12.17} + \cdots + \frac{1}{1997.2002.2007}.$$

Q13. Let ABC be an acute-angle triangle with $BC > CA$. Let O, H and F

be the circumcenter, orthocentre and the foot of its altitude CH, respectively. Suppose that the perpendicular to OF at F meet the side

CA at P. Prove $\angle FHP = \angle BAC$.

Q14. How many ordered pairs of integers (x, y) satisfy the equation

$$x^2 + y^2 + xy = 4(x + y)?$$

Q15. Let $p = \overline{abcd}$ be the 4-digit prime number. Prove that the equation

$$ax^3 + bx^2 + cx + d = 0$$

has no rational roots.

1.3 Hanoi Open Mathematical Olympiad 2008

1.3.1 Junior Section, Sunday, 30 March 2008

Q1. How many integers from 1 to 2008 have the sum of their digits divisible
by 5 ?

Q2. How many integers belong to $(a, 2008a)$, where a $(a > 0)$ is given.

Q3. Find the coefficient of x in the expansion of

$$(1 + x)(1 - 2x)(1 + 3x)(1 - 4x)\cdots(1 - 2008x).$$

Q4. Find all pairs (m, n) of positive integers such that

$$m^2 + n^2 = 3(m + n).$$

Q5. Suppose x, y, z, t are real numbers such that

$$\begin{cases} |x + y + z - t| \leqslant 1 \\ |y + z + t - x| \leqslant 1 \\ |z + t + x - y| \leqslant 1 \\ |t + x + y - z| \leqslant 1 \end{cases}$$

Prove that $x^2 + y^2 + z^2 + t^2 \leqslant 1$.

Q6. Let $P(x)$ be a polynomial such that

$$P(x^2 - 1) = x^4 - 3x^2 + 3.$$

Find $P(x^2 + 1)$?

Q7. The figure $ABCDE$ is a convex pentagon. Find the sum

$$\angle DAC + \angle EBD + \angle ACE + \angle BDA + \angle CEB?$$

Q8. The sides of a rhombus have length a and the area is S. What is the length of the shorter diagonal?

Q9. Let be given a right-angled triangle ABC with $\angle A = 90^0$, $AB = c$, $AC = b$. Let $E \in AC$ and $F \in AB$ such that $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Denote by $P \in BC$ and $Q \in BC$ such that $EP \perp BC$ and $FQ \perp BC$. Determine $EP + EF + PQ$?

Q10. Let $a, b, c \in [1, 3]$ and satisfy the following conditions

$$\max\{a, b, c\} \geqslant 2, \ \ a + b + c = 5.$$

What is the smallest possible value of

$$a^2 + b^2 + c^2?$$

1.3.2 Senior Section, Sunday, 30 March 2008

Q1. How many integers are there in $(b, 2008b]$, where b $(b > 0)$ is given.

Q2. Find all pairs (m, n) of positive integers such that

$$m^2 + 2n^2 = 3(m + 2n).$$

Q3. Show that the equation

$$x^2 + 8z = 3 + 2y^2$$

has no solutions of positive integers x, y and z.

Q4. Prove that there exists an infinite number of relatively prime pairs (m, n) of positive integers such that the equation

$$x^3 - nx + mn = 0$$

has three distint integer roots.

Q5. Find all polynomials $P(x)$ of degree 1 such that

$$\max_{a \leq x \leq b} P(x) - \min_{a \leq x \leq b} P(x) = b - a, \ \forall a, b \in \mathbb{R} \ \text{ where } \ a < b.$$

Q6. Let $a, b, c \in [1, 3]$ and satisfy the following conditions

$$\max\{a, b, c\} \geqslant 2, \ a + b + c = 5.$$

What is the smallest possible value of

$$a^2 + b^2 + c^2?$$

Q7. Find all triples (a, b, c) of consecutive odd positive integers such that $a < b < c$ and $a^2 + b^2 + c^2$ is a four digit number with all digits equal.

Q8. Consider a convex quadrilateral $ABCD$. Let O be the intersection of AC and BD; M, N be the centroid of $\triangle AOB$ and $\triangle COD$ and P, Q be orthocenter of $\triangle BOC$ and $\triangle DOA$, respectively. Prove that $MN \perp PQ$.

Q9. Consider a triangle ABC. For every point $M \in BC$ we difine $N \in CA$ and $P \in AB$ such that $APMN$ is a parallelogram. Let O be the intersection of BN and CP. Find $M \in BC$ such that $\angle PMO = \angle OMN$.

Q10. Let be given a right-angled triangle ABC with $\angle A = 90^0$, $AB = c$, $AC = b$. Let $E \in AC$ and $F \in AB$ such that $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Denote by $P \in BC$ and $Q \in BC$ such that $EP \perp BC$ and $FQ \perp BC$. Determine $EP + EF + FQ$?

1.4 Hanoi Open Mathematical Olympiad 2009

1.4.1 Junior Section, Sunday, 29 March 2009

Q1. What is the last two digits of the number

$$1000.1001 + 1001.1002 + 1002.1003 + \cdots + 2008.2009?$$

(A) 25; (B) 41; (C) 36; (D) 54; (E) None of the above.

Q2. Which is largest positive integer n satisfying the inequality

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} < \frac{6}{7}.$$

(A) 3; (B) 4; (C) 5; (D) 6; (E) None of the above.

Q3. How many positive integer roots of the inequality

$$-1 < \frac{x-1}{x+1} < 2$$

are there in $(-10, 10)$.

(A) 15; (B) 16; (C) 17; (D) 18; (E) None of the above.

Q4. How many triples (a, b, c) where $a, b, c \in \{1, 2, 3, 4, 5, 6\}$ and $a < b < c$ such that the number $abc + (7 - a)(7 - b)(7 - c)$ is divisible by 7.

(A) 15; (B) 17; (C) 19; (D) 21; (E) None of the above.

Q5. Show that there is a natural number n such that the number $a = n!$ ends exacly in 2009 zeros.

Q6. Let a, b, c be positive integers with no common factor and satisfy the conditions

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Prove that $a + b$ is a square.

Q7. Suppose that $a = 2^b + 19$, where $b = 2^{10n+1}$. Prove that a is divisible by 23 for any positive integer n.

Q8. Prove that $m^7 - m$ is divisible by 42 for any positive integer m.

Q9. Suppose that 4 real numbers a, b, c, d satisfy the conditions

$$\begin{cases} a^2 + b^2 = c^2 + d^2 = 4 \\ ac + bd = 2 \end{cases}$$

Find the set of all possible values the number $M = ab + cd$ can take.

Q10. Let a, b be positive integers such that $a+b = 99$. Find the smallest and the greatest values of the following product $P = ab$.

Q11. Find all integers x, y such that $x^2 + y^2 = (2xy + 1)^2$.

Q12. Find all the pairs of the positive integers such that the product of the numbers of any pair plus the half of one of the numbers plus one third of the other number is three times less than 15.

Q13. Let be given ΔABC with area $(\Delta ABC) = 60\text{cm}^2$. Let R, S lie in BC such that $BR = RS = SC$ and P, Q be midpoints of AB and AC, respectively. Suppose that PS intersects QR at T. Evaluate area (ΔPQT).

Q14. Let ABC be an acute-angled triangle with $AB = 4$ and CD be the altitude through C with $CD = 3$. Find the distance between the midpoints of AD and BC.

1.4.2 Senior Section, Sunday, 29 March 2009

Q1. What is the last two digits of the number

$$1000.1001 + 1001.1002 + 1002.1003 + \cdots + 2008.2009?$$

(A) 25; (B) 41; (C) 36; (D) 54; (E) None of the above.

Q2. Which is largest positive integer n satisfying the inequality

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} < \frac{6}{7}.$$

(A) 3; (B) 4; (C) 5; (D) 6; (E) None of the above.

Q3. How many integral roots of the inequality

$$-1 < \frac{x-1}{x+1} < 2$$

are there in $(-10, 10)$.

(A) 15; (B) 16; (C) 17; (D) 18; (E) None of the above.

Q4. How many triples (a, b, c) where $a, b, c \in \{1, 2, 3, 4, 5, 6\}$ and $a < b < c$ such that the number $abc + (7 - a)(7 - b)(7 - c)$ is divisible by 7.

(A) 15; (B) 17; (C) 19; (D) 21; (E) None of the above.

Q5. Suppose that $a = 2^b + 19$, where $b = 2^{10n+1}$. Prove that a is divisible by 23 for any positive integer n.

Q6. Determine all positive integral pairs (u, v) for which

$$5u^2 + 6uv + 7v^2 = 2009.$$

Q7. Prove that for every positive integer n there exists a positive integer m such that the last n digists in decimal representation of m^3 are equal to 8.

Q8. Give an example of a triangle whose all sides and altitudes are positive integers.

Q9. Given a triangle ABC with $BC = 5,\ CA = 4,\ AB = 3$ and the points E, F, G lie on the sides $BC,\ CA,\ AB$, respectively, so that EF is parallel to AB and area $(\Delta EFG) = 1$. Find the minimum value of the perimeter of triangle EFG.

Q10. Find all integers x, y, z satisfying the system

$$\begin{cases} x + y + z = 8 \\ x^3 + y^3 + z^3 = 8 \end{cases}$$

Q11. Let be given three positive numbers α, β and γ. Suppose that 4 real numbers a, b, c, d satisfy the conditions

$$\begin{cases} a^2 + b^2 = \alpha \\ c^2 + d^2 = \beta \\ ac + bd = \gamma \end{cases}$$

Find the set of all possible values the number $M = ab + cd$ can take.

Q12. Let a, b, c, d be positive integers such that $a+b+c+d = 99$. Find the smallest and the greatest values of the following product $P = abcd$.

Q13.Given an acute-angled triangle ABC with area S, let points A', B', C' be located as follows: A' is the point where altitude from A on BC meets the outwards facing semicirle drawn on BC as diameter. Points B', C' are located similarly. Evaluate the sum

$$T = (\text{area } \Delta BCA')^2 + (\text{area } \Delta CAB')^2 + (\text{area } \Delta ABC')^2.$$

Q14. Find all the pairs of the positive integers such that the product of the numbers of any pair plus the half of one of the numbers plus one third of the other number is 7 times less than 2009.