PH. D QUALIFYING EXAMINATION
COMPLEX ANALYSIS–FALL 1998

Work all problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

1. Let $f(z)$ be an entire function that is not a polynomial. Assume f has only finitely many zeros. Let $m(r) = \min_{|z|=r} |f(z)|$. Show $m(r) \to 0$ as $r \to \infty$.

2(a). Let $f(z)$ be analytic in a bounded region Ω and be continuous up to the boundary B of Ω. Let $E = f(B)$. If a and b are in the same component of $\mathbb{C} \setminus E$, show that a and b are taken the same number of times by f.

(b). Prove that $z^4 + z^2 + 2$ has a zero in each quadrant.

3. Show:

$$\int_0^\infty \frac{x^\alpha}{(x+2)^2}\,dx = \frac{\alpha\pi 2^{\alpha-1}}{\sin \pi\alpha} \qquad \text{for } -1 < \alpha < 1.$$

4. Show that the annulus can be mapped conformally onto the Riemann sphere with two segments of the real axis removed.

Hint: You may use the Riemann mapping theorem, including boundary behavior.

5(a). Let $f(z)$ be an entire function such that $|f(z)| < A \exp |z|^\alpha$. Show that the number of zeros of f in the disk $|z| < r$ is $\leq Cr^\alpha$ for some constant C and for $r > 1$.

(b). Let $f(z) = \sum z^n n^{-an}$. Show that f satisfies

$$|f(z)| \leq C_1 \exp(C_2 |z|^{1/\alpha}) \qquad \text{for some } C_1, C_2 > 0.$$

6. Let Ω be a bounded connected region. Show that there is an analytic function $f(z)$ defined in Ω such that f cannot be extended to be analytic in any larger connected region.