

Solutions to the 61st William Lowell Putnam Mathematical Competition

Saturday, December 2, 2000

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A-1 The possible values comprise the interval $(0, A^2)$.

To see that the values must lie in this interval, note that

$$\left(\sum_{j=0}^m x_j \right)^2 = \sum_{j=0}^m x_j^2 + \sum_{0 \leq j < k \leq m} 2x_j x_k,$$

so $\sum_{j=0}^m x_j^2 \leq A^2 - 2x_0 x_1$. Letting $m \rightarrow \infty$, we have $\sum_{j=0}^{\infty} x_j^2 \leq A^2 - 2x_0 x_1 < A^2$.

To show that all values in $(0, A^2)$ can be obtained, we use geometric progressions with $x_1/x_0 = x_2/x_1 = \dots = d$ for variable d . Then $\sum_{j=0}^{\infty} x_j = x_0/(1-d)$ and

$$\sum_{j=0}^{\infty} x_j^2 = \frac{x_0^2}{1-d^2} = \frac{1-d}{1+d} \left(\sum_{j=0}^{\infty} x_j \right)^2.$$

As d increases from 0 to 1, $(1-d)/(1+d)$ decreases from 1 to 0. Thus if we take geometric progressions with $\sum_{j=0}^{\infty} x_j = A$, $\sum_{j=0}^{\infty} x_j^2$ ranges from 0 to A^2 . Thus the possible values are indeed those in the interval $(0, A^2)$, as claimed.

A-2 First solution: Let a be an even integer such that $a^2 + 1$ is not prime. (For example, choose $a \equiv 2 \pmod{5}$, so that $a^2 + 1$ is divisible by 5.) Then we can write $a^2 + 1$ as a difference of squares $x^2 - b^2$, by factoring $a^2 + 1$ as rs with $r \geq s > 1$, and setting $x = (r+s)/2$, $b = (r-s)/2$. Finally, put $n = x^2 - 1$, so that $n = a^2 + b^2$, $n+1 = x^2$, $n+2 = x^2 + 1$.

Second solution: It is well-known that the equation $x^2 - 2y^2 = 1$ has infinitely many solutions (the so-called ‘‘Pell’’ equation). Thus setting $n = 2y^2$ (so that $n = y^2 + y^2$, $n+1 = x^2 + 0^2$, $n+2 = x^2 + 1^2$) yields infinitely many n with the desired property.

Third solution: As in the first solution, it suffices to exhibit x such that $x^2 - 1$ is the sum of two squares. We will take $x = 3^{2^n}$, and show that $x^2 - 1$ is the sum of two squares by induction on n : if $3^{2^n} - 1 = a^2 + b^2$, then

$$\begin{aligned} (3^{2^{n+1}} - 1) &= (3^{2^n} - 1)(3^{2^n} + 1) \\ &= (3^{2^{n-1}} a + b)^2 + (a - 3^{2^{n-1}} b)^2. \end{aligned}$$

A-3 The maximum area is $3\sqrt{5}$.

We deduce from the area of $P_1 P_3 P_5 P_7$ that the radius of the circle is $\sqrt{5}/2$. An easy calculation using the Pythagorean Theorem then shows that the rectangle $P_2 P_4 P_6 P_8$ has sides $\sqrt{2}$ and $2\sqrt{2}$. For notational ease, denote the area of a polygon by putting brackets around the name of the polygon.

By symmetry, the area of the octagon can be expressed as

$$[P_2 P_4 P_6 P_8] + 2[P_2 P_3 P_4] + 2[P_4 P_5 P_6].$$

Note that $[P_2 P_3 P_4]$ is $\sqrt{2}$ times the distance from P_3 to $P_2 P_4$, which is maximized when P_3 lies on the midpoint of arc $P_2 P_4$; similarly, $[P_4 P_5 P_6]$ is $\sqrt{2}/2$ times the distance from P_5 to $P_4 P_6$, which is maximized when P_5 lies on the midpoint of arc $P_4 P_6$. Thus the area of the octagon is maximized when P_3 is the midpoint of arc $P_2 P_4$ and P_5 is the midpoint of arc $P_4 P_6$. In this case, it is easy to calculate that $[P_2 P_3 P_4] = \sqrt{5} - 1$ and $[P_4 P_5 P_6] = \sqrt{5}/2 - 1$, and so the area of the octagon is $3\sqrt{5}$.

A-4 We use integration by parts:

$$\begin{aligned} \int_0^B \sin x \sin x^2 dx &= \int_0^B \frac{\sin x}{2x} \sin x^2 (2x dx) \\ &= -\frac{\sin x}{2x} \cos x^2 \Big|_0^B \\ &\quad + \int_0^B \left(\frac{\cos x}{2x} - \frac{\sin x}{2x^2} \right) \cos x^2 dx. \end{aligned}$$

Now $\frac{\sin x}{2x} \cos x^2$ tends to 0 as $B \rightarrow \infty$, and the integral of $\frac{\sin x}{2x^2} \cos x^2$ converges absolutely by comparison with $1/x^2$. Thus it suffices to note that

$$\begin{aligned} \int_0^B \frac{\cos x}{2x} \cos x^2 dx &= \frac{\cos x}{4x^2} \cos x^2 (2x dx) \\ &= \frac{\cos x}{4x^2} \sin x^2 \Big|_0^B \\ &\quad - \int_0^B \frac{2x \cos x - \sin x}{4x^3} \sin x^2 dx, \end{aligned}$$

and that the final integral converges absolutely by comparison to $1/x^3$.

An alternate approach is to first rewrite $\sin x \sin x^2$ as $\frac{1}{2}(\cos(x^2 - x) - \cos(x^2 + x))$. Then

$$\int_0^B \cos(x^2 + x) dx = -\frac{2x+1}{\sin(x^2+x)} \Big|_0^B - \int_0^B \frac{2\sin(x^2+x)}{(2x+1)^2} dx$$

converges absolutely, and $\int_0^B \cos(x^2 - x)$ can be treated similarly.

A-5 Let a, b, c be the distances between the points. Then the area of the triangle with the three points as vertices is $abc/4r$. On the other hand, the area of a triangle whose vertices have integer coordinates is at least $1/2$ (for example, by Pick's Theorem). Thus $abc/4r \geq 1/2$, and so

$$\max\{a, b, c\} \geq (abc)^{1/3} \geq (2r)^{1/3} > r^{1/3}.$$

A-6 Recall that if $f(x)$ is a polynomial with integer coefficients, then $m-n$ divides $f(m) - f(n)$ for any integers m and n . In particular, if we put $b_n = a_{n+1} - a_n$, then b_n divides b_{n+1} for all n . On the other hand, we are given that $a_0 = a_m = 0$, which implies that $a_1 = a_{m+1}$ and so $b_0 = b_m$. If $b_0 = 0$, then $a_0 = a_1 = \dots = a_m$ and we are done. Otherwise, $|b_0| = |b_1| = |b_2| = \dots$, so $b_n = \pm b_0$ for all n .

Now $b_0 + \dots + b_{m-1} = a_m - a_0 = 0$, so half of the integers b_0, \dots, b_{m-1} are positive and half are negative. In particular, there exists an integer $0 < k < m$ such that $b_{k-1} = -b_k$, which is to say, $a_{k-1} = a_{k+1}$. From this it follows that $a_n = a_{n+2}$ for all $n \geq k-1$; in particular, for $m = n$, we have

$$a_0 = a_m = a_{m+2} = f(f(a_0)) = a_2.$$

B-1 Consider the seven triples (a, b, c) with $a, b, c \in \{0, 1\}$ not all zero. Notice that if r_j, s_j, t_j are not all even, then four of the sums $ar_j + bs_j + ct_j$ with $a, b, c \in \{0, 1\}$ are even and four are odd. Of course the sum with $a = b = c = 0$ is even, so at least four of the seven triples with a, b, c not all zero yield an odd sum. In other words, at least $4N$ of the tuples (a, b, c, j) yield odd sums. By the pigeonhole principle, there is a triple (a, b, c) for which at least $4N/7$ of the sums are odd.

B-2 Since $\gcd(m, n)$ is an integer linear combination of m and n , it follows that

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer linear combination of the integers

$$\frac{m}{n} \binom{n}{m} = \binom{n-1}{m-1} \text{ and } \frac{n}{n} \binom{n}{m} = \binom{n}{m}$$

and hence is itself an integer.

B-3 Put $f_k(t) = \frac{df^k}{dt^k}$. Recall Rolle's theorem: if $f(t)$ is differentiable, then between any two zeroes of $f(t)$ there exists a zero of $f'(t)$. This also applies when the zeroes are not all distinct: if f has a zero of multiplicity m at $t = x$, then f' has a zero of multiplicity at least $m - 1$ there.

Therefore, if $0 \leq a_0 \leq a_1 \leq \dots \leq a_r < 1$ are the roots of f_k in $[0, 1]$, then f_{k+1} has a root in each of the intervals $(a_0, a_1), (a_1, a_2), \dots, (a_{r-1}, a_r)$, so long as we adopt the convention that the empty interval (t, t) actually contains the point t itself. There is also a root in the "wraparound" interval (a_r, a_0) . Thus $N_{k+1} \geq N_k$.

Next, note that if we set $z = e^{2\pi it}$, then

$$f_{4k}(t) = \frac{1}{2i} \sum_{j=1}^N j^{4k} a_j (z^j - z^{-j})$$

is equal to z^{-N} times a polynomial of degree $2N$. Hence as a function of z , it has at most $2N$ roots; therefore $f_k(t)$ has at most $2N$ roots in $[0, 1]$. That is, $N_k \leq 2N$ for all N .

To establish that $N_k \rightarrow 2N$, we make precise the observation that

$$f_k(t) = \sum_{j=1}^N j^{4k} a_j \sin(2\pi jt)$$

is dominated by the term with $j = N$. At the points $t = (2i+1)/(2N)$ for $i = 0, 1, \dots, N-1$, we have $N^{4k} a_N \sin(2\pi Nt) = \pm N^{4k} a_N$. If k is chosen large enough so that

$$|a_N| N^{4k} > |a_1| 1^{4k} + \dots + |a_{N-1}| (N-1)^{4k},$$

then $f_k((2i+1)/(2N))$ has the same sign as $a_N \sin(2\pi Nt)$, which is to say, the sequence $f_k(1/2N), f_k(3/2N), \dots$ alternates in sign. Thus between these points (again including the "wraparound" interval) we find $2N$ sign changes of f_k . Therefore $\lim_{k \rightarrow \infty} N_k = 2N$.

B-4 For t real and not a multiple of π , write $g(t) = \frac{f(\cos t)}{\sin t}$. Then $g(t + \pi) = g(t)$; furthermore, the given equation implies that

$$g(2t) = \frac{f(2\cos^2 t - 1)}{\sin(2t)} = \frac{2(\cos t)f(\cos t)}{\sin(2t)} = g(t).$$

In particular, for any integer n and k , we have

$$g(1 + n\pi/2^k) = g(2^k + n\pi) = g(2^k) = g(1).$$

Since f is continuous, g is continuous where it is defined; but the set $\{1 + n\pi/2^k | n, k \in \mathbb{Z}\}$ is dense in the reals, and so g must be constant on its domain. Since $g(-t) = -g(t)$ for all t , we must have $g(t) = 0$ when t is not a multiple of π . Hence $f(x) = 0$ for $x \in (-1, 1)$. Finally, setting $x = 0$ and $x = 1$ in the given equation yields $f(-1) = f(1) = 0$.

B-5 We claim that all integers N of the form 2^k , with k a positive integer and $N > \max\{S_0\}$, satisfy the desired conditions.

It follows from the definition of S_n , and induction on n , that

$$\begin{aligned}\sum_{j \in S_n} x^j &\equiv (1+x) \sum_{j \in S_{n-1}} x^j \\ &\equiv (1+x)^n \sum_{j \in S_0} x^j \pmod{2}.\end{aligned}$$

From the identity $(x+y)^2 \equiv x^2 + y^2 \pmod{2}$ and induction on n , we have $(x+y)^{2^n} \equiv x^{2^n} + y^{2^n} \pmod{2}$. Hence if we choose N to be a power of 2 greater than

$\max\{S_0\}$, then

$$\sum_{j \in S_n} x^j \equiv (1+x^N) \sum_{j \in S_0} x^j$$

and $S_N = S_0 \cup \{N+a : a \in S_0\}$, as desired.

B-6 For each point P in B , let S_P be the set of points with all coordinates equal to ± 1 which differ from P in exactly one coordinate. Since there are more than $2^{n+1}/n$ points in B , and each S_P has n elements, the cardinalities of the sets S_P add up to more than 2^{n+1} , which is to say, more than twice the total number of points. By the pigeonhole principle, there must be a point in three of the sets, say S_P, S_Q, S_R . But then any two of P, Q, R differ in exactly two coordinates, so PQR is an equilateral triangle, as desired.