Solutions to the 64th William Lowell Putnam Mathematical Competition
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A1 There are n such sums. More precisely, there is exactly one such sum with k terms for each of $k = 1, \ldots, n$ (and clearly no others). To see this, note that if $n = a_1 + a_2 + \cdots + a_k$ with $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$, then

$$
\begin{aligned}
ka_1 &= a_1 + a_1 + \cdots + a_1 \\
&\leq n \leq a_1 + (a_1 + 1) + \cdots + (a_1 + 1) \\
&= ka_1 + k - 1.
\end{aligned}
$$

However, there is a unique integer a_1 satisfying these inequalities, namely $a_1 = \lfloor n/k \rfloor$. Moreover, once a_1 is fixed, there are k different possibilities for the sum $a_1 + a_2 + \cdots + a_k$: if i is the last integer such that $a_i = a_1$, then the sum equals $ka_1 + (i - 1)$. The possible values of i are $1, \ldots, k$, and exactly one of these sums comes out equal to n, proving our claim.

Note: In summary, there is a unique partition of n with k terms that is "as equally spaced as possible". One can also obtain essentially the same construction inductively: except for the all-ones sum, each partition of n is obtained by "augmenting" a unique partition of $n-1$.

A2 **First solution:** Assume without loss of generality that $a_i + b_i > 0$ for each i (otherwise both sides of the desired inequality are zero). Then the AM-GM inequality gives

$$
\begin{aligned}
&\left(\frac{a_1 \cdots a_n}{(a_1 + b_1) \cdots (a_n + b_n)} \right)^{1/n} \\
&\leq \frac{1}{n} \left(\frac{a_1}{a_1 + b_1} + \cdots + \frac{a_n}{a_n + b_n} \right),
\end{aligned}
$$

and likewise with the roles of a and b reversed. Adding these two inequalities and clearing denominators yields the desired result.

Second solution: Write the desired inequality in the form

$$(a_1 + b_1) \cdots (a_n + b_n) \geq [(a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n}]^n,$$

expand both sides, and compare the terms on both sides in which k of the terms are among the a_i. On the left, one has the product of each k-element subset of $\{1, \ldots, n\}$; on the right, one has $\binom{n}{k}(a_1 \cdots a_n)^{k/n} \cdots (b_1 \ldots b_n)^{(n-k)/n}$, which is precisely $\binom{n}{k}$ times the geometric mean of the terms on the left. Thus AM-GM shows that the terms under consideration on the left exceed those on the right; adding these inequalities over all k yields the desired result.

Third solution: Since both sides are continuous in each a_i, it is sufficient to prove the claim with $a_1, \ldots, a_n$ all positive (the general case follows by taking limits as some of the a_i tend to zero). Put $r_i = b_i/a_i$; then the given inequality is equivalent to

$$(1 + r_1)^{1/n} \cdots (1 + r_n)^{1/n} \geq 1 + (r_1 \cdots r_n)^{1/n}.$$

In terms of the function

$$f(x) = \log(1 + e^x)$$

and the quantities $s_i = \log r_i$, we can rewrite the desired inequality as

$$\frac{1}{n}(f(s_1) + \cdots + f(s_n)) \geq f\left(\frac{s_1 + \cdots + s_n}{n} \right).$$

This will follow from Jensen's inequality if we can verify that f is a convex function; it is enough to check that $f''(x) > 0$ for all x. In fact,

$$f'(x) = \frac{e^x}{1 + e^x} = 1 - \frac{1}{1 + e^x}$$

is an increasing function of x, so $f''(x) > 0$ and Jensen's inequality thus yields the desired result. (As long as the a_i are all positive, equality holds when $s_1 = \cdots = s_n$, i.e., when the vectors $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. Of course other equality cases crop up if some of the a_i vanish, i.e., if $a_1 = b_1 = 0$.)

Fourth solution: We apply induction on n, the case $n = 1$ being evident. First we verify the auxiliary inequality

$$(a^n + b^n)(c^n + d^n)^{n-1} \geq (ac^{n-1} + bd^{n-1})^n$$

for $a, b, c, d \geq 0$. The left side can be written as

$$
\begin{aligned}
& a^n c^{n(n-1)} + b^n d^{n(n-1)} \\
& + \sum_{i=1}^{n-1} \binom{n-1}{i} a^n c^{ni} d^{n(n-1-i)} \\
& + \sum_{i=1}^{n-1} \binom{n-1}{i-1} b^n c^{n(n-i)} d^{n(i-1)}.
\end{aligned}
$$

Applying the weighted AM-GM inequality between matching terms in the two sums yields

$$(a^n + b^n)(c^n + d^n)^{n-1}$$
$$\geq a^n c^{n(n-1)} + b^n d^{n(n-1)}$$
$$+ \sum_{i=1}^{n-1} \binom{n}{i} a^i b^{n-i} c^{(n-1)i} d^{(n-1)(n-i)},$$

proving the auxiliary inequality.

Now given the auxiliary inequality and the $n-1$ case of the desired inequality, we apply the auxiliary inequality with $a = a_1^{1/n}$, $b = b_1^{1/n}$, $c = (a_2 \cdots a_n)^{1/n(n-1)}$, $d = (b_2 \ldots b_n)^{1/n(n-1)}$. The right side will be the n-th power of the desired inequality. The left side comes out to

$$(a_1 + b_1)((a_2 \cdots a_n)^{1/(n-1)} + (b_2 \cdots b_n)^{1/(n-1)})^{n-1},$$

and by the induction hypothesis, the second factor is less than $(a_2 + b_2) \cdots (a_n + b_n)$. This yields the desired result.

Note: Equality holds if and only if $a_i = b_i = 0$ for some i or if the vectors $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are proportional. As pointed out by Naoki Sato, the problem also appeared on the 1992 Irish Mathematical Olympiad. It is also a special case of a classical inequality, known as Hölder's inequality, which generalizes the Cauchy-Schwarz inequality (this is visible from the $n = 2$ case); the first solution above is adapted from the standard proof of Hölder's inequality. We don't know whether the declaration "Apply Hölder's inequality" by itself is considered an acceptable solution to this problem.

A3 **First solution:** Write

$$f(x) = \sin x + \cos x + \tan x + \cot x + \sec x + \csc x$$
$$= \sin x + \cos x + \frac{1}{\sin x \cos x} + \frac{\sin x + \cos x}{\sin x \cos x}.$$

We can write $\sin x + \cos x = \sqrt{2}\cos(\pi/4 - x)$; this suggests making the substitution $y = \pi/4 - x$. In this new coordinate,

$$\sin x \cos x = \frac{1}{2}\sin 2x = \frac{1}{2}\cos 2y,$$

and writing $c = \sqrt{2}\cos y$, we have

$$f(y) = (1 + c)\left(1 + \frac{2}{c^2 - 1}\right) - 1$$
$$= c + \frac{2}{c - 1}.$$

We must analyze this function of c in the range $[-\sqrt{2}, \sqrt{2}]$. Its value at $c = -\sqrt{2}$ is $2 - 3\sqrt{2} < -2.24$, and at $c = \sqrt{2}$ is $2 + 3\sqrt{2} > 6.24$. Its derivative is

$1 - 2/(c-1)^2$, which vanishes when $(c-1)^2 = 2$, i.e., where $c = 1 \pm \sqrt{2}$. Only the value $c = 1 - \sqrt{2}$ is in bounds, at which the value of f is $1 - 2\sqrt{2} > -1.83$. As for the pole at $c = 1$, we observe that f decreases as c approaches from below (so takes negative values for all $c < 1$) and increases as c approaches from above (so takes positive values for all $c > 1$); from the data collected so far, we see that f has no sign crossings, so the minimum of $|f|$ is achieved at a critical point of f. We conclude that the minimum of $|f|$ is $2\sqrt{2} - 1$.

Alternate derivation (due to Zuming Feng): We can also minimize $|c + 2/(c - 1)|$ without calculus (or worrying about boundary conditions). For $c > 1$, we have

$$1 + (c - 1) + \frac{2}{c - 1} \geq 1 + 2\sqrt{2}$$

by AM-GM on the last two terms, with equality for $c - 1 = \sqrt{2}$ (which is out of range). For $c < 1$, we similarly have

$$-1 + 1 - c + \frac{2}{1 - c} \geq -1 + 2\sqrt{2},$$

here with equality for $1 - c = \sqrt{2}$.

Second solution: Write

$$f(a, b) = a + b + \frac{1}{ab} + \frac{a + b}{ab}.$$

Then the problem is to minimize $|f(a, b)|$ subject to the constraint $a^2 + b^2 - 1 = 0$. Since the constraint region has no boundary, it is enough to check the value at each critical point and each potential discontinuity (i.e., where $ab = 0$) and select the smallest value (after checking that f has no sign crossings).

We locate the critical points using the Lagrange multiplier condition: the gradient of f should be parallel to that of the constraint, which is to say, to the vector (a, b). Since

$$\frac{\partial f}{\partial a} = 1 - \frac{1}{a^2 b} - \frac{1}{a^2}$$

and similarly for b, the proportionality yields

$$a^2 b^3 - a^3 b^2 + a^3 - b^3 + a^2 - b^2 = 0.$$

The irreducible factors of the left side are $1 + a$, $1 + b$, $a - b$, and $ab - a - b$. So we must check what happens when any of those factors, or a or b, vanishes.

If $1 + a = 0$, then $b = 0$, and the singularity of f becomes removable when restricted to the circle. Namely, we have

$$f = a + b + \frac{1}{a} + \frac{b + 1}{ab}$$

and $a^2 + b^2 - 1 = 0$ implies $(1 + b)/a = a/(1 - b)$. Thus we have $f = -2$; the same occurs when $1 + b = 0$.

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If $a - b = 0$, then $a = b = \pm\sqrt{2}/2$ and either $f = 2 + 3\sqrt{2} > 6.24$, or $f = 2 - 3\sqrt{2} < -2.24$.

If $a = 0$, then either $b = -1$ as discussed above, or $b = 1$. In the latter case, f blows up as one approaches this point, so there cannot be a global minimum there.

Finally, if $ab - a - b = 0$, then

$$a^2 b^2 = (a + b)^2 = 2ab + 1$$

and so $ab = 1 \pm \sqrt{2}$. The plus sign is impossible since $|ab| \leq 1$, so $ab = 1 - \sqrt{2}$ and

$$\begin{aligned} f(a, b) &= ab + \frac{1}{ab} + 1 \\ &= 1 - 2\sqrt{2} > -1.83. \end{aligned}$$

This yields the smallest value of $|f|$ in the list (and indeed no sign crossings are possible), so $2\sqrt{2} - 1$ is the desired minimum of $|f|$.

Note: Instead of using the geometry of the graph of f to rule out sign crossings, one can verify explicitly that f cannot take the value 0. In the first solution, note that $c + 2/(c - 1) = 0$ implies $c^2 - c + 2 = 0$, which has no real roots. In the second solution, we would have

$$a^2 b + ab^2 + a + b = -1.$$

Squaring both sides and simplifying yields

$$2a^3 b^3 + 5a^2 b^2 + 4ab = 0,$$

whose only real root is $ab = 0$. But the cases with $ab = 0$ do not yield $f = 0$, as verified above.

A4 We split into three cases. Note first that $|A| \geq |a|$, by applying the condition for large x.

Case 1: $B^2 - 4AC > 0$. In this case $Ax^2 + Bx + C$ has two distinct real roots r_1 and r_2. The condition implies that $ax^2 + bx + c$ also vanishes at r_1 and r_2, so $b^2 - 4ac > 0$. Now

$$\begin{aligned} B^2 - 4AC &= A^2 (r_1 - r_2)^2 \\ &\geq a^2 (r_1 - r_2)^2 \\ &= b^2 - 4ac. \end{aligned}$$

Case 2: $B^2 - 4AC \leq 0$ **and** $b^2 - 4ac \leq 0$. Assume without loss of generality that $A \geq a > 0$, and that $B = 0$ (by shifting x). Then $Ax^2 + Bx + C \geq ax^2 + bx + c \geq 0$ for all x; in particular, $C \geq c \geq 0$. Thus

$$\begin{aligned} 4AC - B^2 &= 4AC \\ &\geq 4ac \\ &\geq 4ac - b^2. \end{aligned}$$

Alternate derivation (due to Robin Chapman): the ellipse $Ax^2 + Bxy + Cy^2 = 1$ is contained within the ellipse $ax^2 + bxy + cy^2 = 1$, and their respective enclosed areas are $\pi/(4AC - B^2)$ and $\pi/(4ac - b^2)$.

Case 3: $B^2 - 4AC \leq 0$ **and** $b^2 - 4ac > 0$. Since $Ax^2 + Bx + C$ has a graph not crossing the x-axis, so do $(Ax^2 + Bx + C) \pm (ax^2 + bx + c)$. Thus

$$\begin{aligned} (B - b)^2 - 4(A - a)(C - c) &\leq 0, \\ (B + b)^2 - 4(A + a)(C + c) &\leq 0 \end{aligned}$$

and adding these together yields

$$2(B^2 - 4AC) + 2(b^2 - 4ac) \leq 0.$$

Hence $b^2 - 4ac \leq 4AC - B^2$, as desired.

A5 **First solution:** We represent a Dyck n-path by a sequence $a_1 \cdots a_{2n}$, where each a_i is either $(1, 1)$ or $(1, -1)$.

Given an $(n - 1)$-path $P = a_1 \cdots a_{2n-2}$, we distinguish two cases. If P has no returns of even-length, then let $f(P)$ denote the n-path $(1, 1)(1, -1)P$. Otherwise, let $a_i a_{i+1} \cdots a_j$ denote the rightmost even-length return in P, and let $f(P) = (1, 1)a_1 a_2 \cdots a_j (1, -1)a_{j+1} \cdots a_{2n-2}$. Then f clearly maps the set of Dyck $(n - 1)$-paths to the set of Dyck n-paths having no even return.

We claim that f is bijective; to see this, we simply construct the inverse mapping. Given an n-path P, let $R = a_i a_{i+1} ... a_j$ denote the leftmost return in P, and let $g(P)$ denote the path obtained by removing a_1 and a_j from P. Then evidently $f \circ g$ and $g \circ f$ are identity maps, proving the claim.

Second solution: (by Dan Bernstein) Let C_n be the number of Dyck paths of length n, let O_n be the number of Dyck paths whose final return has odd length, and let X_n be the number of Dyck paths with no return of even length.

We first exhibit a recursion for O_n; note that $O_0 = 0$. Given a Dyck n-path whose final return has odd length, split it just after its next-to-last return. For some k (possibly zero), this yields a Dyck k-path, an upstep, a Dyck $(n - k - 1)$-path whose odd return has even length, and a downstep. Thus for $n \geq 1$,

$$O_n = \sum_{k=0}^{n-1} C_k (C_{n-k-1} - O_{n-k-1}).$$

We next exhibit a similar recursion for X_n; note that $X_0 = 1$. Given a Dyck n-path with no even return, splitting as above yields for some k a Dyck k-path with no even return, an upstep, a Dyck $(n-k-1)$-path whose final return has even length, then a downstep. Thus for $n \geq 1$,

$$X_n = \sum_{k=0}^{n-1} X_k (C_{n-k-1} - O_{n-k-1}).$$

To conclude, we verify that $X_n = C_{n-1}$ for $n \geq 1$, by induction on n. This is clear for $n = 1$ since $X_1 = C_0 = 1$. Given $X_k = C_{k-1}$ for $k < n$, we have

$$X_n = \sum_{k=0}^{n-1} X_k(C_{n-k-1} - O_{n-k-1})$$

$$= C_{n-1} - O_{n-1} + \sum_{k=1}^{n-1} C_{k-1}(C_{n-k-1} - O_{n-k-1})$$

$$= C_{n-1} - O_{n-1} + O_{n-1}$$

$$= C_{n-1},$$

as desired.

Note: Since the problem only asked about the *existence* of a one-to-one correspondence, we believe that any proof, bijective or not, that the two sets have the same cardinality is an acceptable solution. (Indeed, it would be highly unusual to insist on using or not using a specific proof technique!) The second solution above can also be phrased in terms of generating functions. Also, the C_n are well-known to equal the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$; the problem at hand is part of a famous exercise in Richard Stanley's *Enumerative Combinatorics, Volume 1* giving 66 combinatorial interpretations of the Catalan numbers.

A6 **First solution:** Yes, such a partition is possible. To achieve it, place each integer into A if it has an even number of 1s in its binary representation, and into B if it has an odd number. (One discovers this by simply attempting to place the first few numbers by hand and noticing the resulting pattern.)

To show that $r_A(n) = r_B(n)$, we exhibit a bijection between the pairs (a_1, a_2) of distinct elements of A with $a_1 + a_2 = n$ and the pairs (b_1, b_2) of distinct elements of B with $b_1 + b_2 = n$. Namely, given a pair (a_1, a_2) with $a_1 + a_2 = n$, write both numbers in binary and find the lowest-order place in which they differ (such a place exists because $a_1 \neq a_2$). Change both numbers in that place and call the resulting numbers b_1, b_2. Then $a_1 + a_2 = b_1 + b_2 = n$, but the parity of the number of 1s in b_1 is opposite that of a_1, and likewise between b_2 and a_2. This yields the desired bijection.

Second solution: (by Micah Smukler) Write $b(n)$ for the number of 1s in the base 2 expansion of n, and $f(n) = (-1)^{b(n)}$. Then the desired partition can be described as $A = f^{-1}(1)$ and $B = f^{-1}(-1)$. Since $f(2n) + f(2n+1) = 0$, we have

$$\sum_{i=0}^{n} f(n) = \begin{cases} 0 & n \text{ odd} \\ f(n) & n \text{ even.} \end{cases}$$

If p, q are both in A, then $f(p) + f(q) = 2$; if p, q are both in B, then $f(p) + f(q) = -2$; if p, q are in different sets, then $f(p) + f(q) = 0$. In other words,

$$2(r_A(n) - r_B(n)) = \sum_{p+q=n, p<q} (f(p) + f(q))$$

and it suffices to show that the sum on the right is always zero. If n is odd, that sum is visibly $\sum_{i=0}^{n} f(i) = 0$. If n is even, the sum equals

$$\left(\sum_{i=0}^{n} f(i) \right) - f(n/2) = f(n) - f(n/2) = 0.$$

This yields the desired result.

Third solution: (by Dan Bernstein) Put $f(x) = \sum_{n \in A} x^n$ and $g(x) = \sum_{n \in B} x^n$; then the value of $r_A(n)$ (resp. $r_B(n)$) is the coefficient of x^n in $f(x)^2 - f(x^2)$ (resp. $g(x)^2 - g(x^2)$). From the evident identities

$$\frac{1}{1-x} = f(x) + g(x)$$
$$f(x) = f(x^2) + xg(x^2)$$
$$g(x) = g(x^2) + xf(x^2),$$

we have

$$f(x) - g(x) = f(x^2) - g(x^2) + xg(x^2) - xf(x^2)$$
$$= (1-x)(f(x^2) - g(x^2))$$
$$= \frac{f(x^2) - g(x^2)}{f(x) + g(x)}.$$

We deduce that $f(x)^2 - g(x)^2 = f(x^2) - g(x^2)$, yielding the desired equality.

Note: This partition is actually unique, up to interchanging A and B. More precisely, the condition that $0 \in A$ and $r_A(n) = r_B(n)$ for $n = 1, \ldots, m$ uniquely determines the positions of $0, \ldots, m$. We see this by induction on m: given the result for $m - 1$, switching the location of m changes $r_A(m)$ by one and does not change $r_B(m)$, so it is not possible for both positions to work. Robin Chapman points out this problem is solved in D.J. Newman's *Analytic Number Theory* (Springer, 1998); in that solution, one uses generating functions to find the partition and establish its uniqueness, not just verify it.

B1 No, there do not.

First solution: Suppose the contrary. By setting $y = -1, 0, 1$ in succession, we see that the polynomials $1 - x + x^2, 1, 1 + x + x^2$ are linear combinations of $a(x)$ and $b(x)$. But these three polynomials are linearly independent, so cannot all be written as linear combinations of two other polynomials, contradiction.

Alternate formulation: the given equation expresses a diagonal matrix with $1, 1, 1$ and zeroes on the diagonal, which has rank 3, as the sum of two matrices of rank 1. But the rank of a sum of matrices is at most the sum of the ranks of the individual matrices.

Second solution: It is equivalent (by relabeling and rescaling) to show that $1 + xy + x^2y^2$ cannot be written as $a(x)d(y) - b(x)c(y)$. Write $a(x) = \sum a_i x^i$, $b(x) = \sum b_i x^i$, $c(y) = \sum c_j y^j$, $d(y) = \sum d_j y^j$. We now start comparing coefficients of $1 + xy + x^2y^2$. By comparing coefficients of $1 + xy + x^2y^2$ and $a(x)d(y) - b(x)c(y)$, we get

$$1 = a_i d_i - b_i c_i \qquad (i = 0, 1, 2)$$
$$0 = a_i d_j - b_i c_j \qquad (i \neq j).$$

The first equation says that a_i and b_i cannot both vanish, and c_i and d_i cannot both vanish. The second equation says that $a_i/b_i = c_j/d_j$ when $i \neq j$, where both sides should be viewed in $\mathbb{R} \cup \{\infty\}$ (and neither is undetermined if $i, j \in \{0, 1, 2\}$). But then

$$a_0/b_0 = c_1/d_1 = a_2/b_2 = c_0/d_0$$

contradicting the equation $a_0 d_0 - b_0 c_0 = 1$.

Third solution: We work over the complex numbers, in which we have a primitive cube root ω of 1. We also use without further comment unique factorization for polynomials in two variables over a field. And we keep the relabeling of the second solution.

Suppose the contrary. Since $1 + xy + x^2y^2 = (1 - xy/\omega)(1 - xy/\omega^2)$, the rational function $a(\omega/y)d(y) - b(\omega/y)c(y)$ must vanish identically (that is, coefficient by coefficient). If one of the polynomials, say a, vanished identically, then one of b or c would also, and the desired inequality could not hold. So none of them vanish identically, and we can write

$$\frac{c(y)}{d(y)} = \frac{a(\omega/y)}{b(\omega/y)}.$$

Likewise,

$$\frac{c(y)}{d(y)} = \frac{a(\omega^2/y)}{b(\omega^2/y)}.$$

Put $f(x) = a(x)/b(x)$; then we have $f(\omega x) = f(x)$ identically. That is, $a(x)b(\omega x) = b(x)a(\omega x)$. Since a and b have no common factor (otherwise $1 + xy + x^2y^2$ would have a factor divisible only by x, which it doesn't since it doesn't vanish identically for any particular x), $a(x)$ divides $a(\omega x)$. Since they have the same degree, they are equal up to scalars. It follows that one of $a(x), xa(x), x^2a(x)$ is a polynomial in x^3 alone, and likewise for b (with the same power of x).

If $xa(x)$ and $xb(x)$, or $x^2a(x)$ and $x^2b(x)$, are polynomials in x^3, then a and b are divisible by x, but we know a and b have no common factor. Hence $a(x)$ and $b(x)$ are polynomials in x^3. Likewise, $c(y)$ and $d(y)$ are polynomials in y^3. But then $1 + xy + x^2y^2 = a(x)d(y) - b(x)c(y)$ is a polynomial in x^3 and y^3, contradiction.

Note: The third solution only works over fields of characteristic not equal to 3, whereas the other two work over arbitrary fields. (In the first solution, one must replace -1 by another value if working in characteristic 2.)

B2 It is easy to see by induction that the j-th entry of the k-th sequence (where the original sequence is $k = 1$) is $\sum_{i=1}^{k} \binom{k-1}{i-1}/(2^{k-1}(i + j - 1))$, and so $x_n = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1}/i$. Now $\binom{n-1}{i-1}/i = \binom{n}{i}/n$; hence

$$x_n = \frac{1}{n2^{n-1}} \sum_{i=1}^{n} \binom{n}{i} = \frac{2^n - 1}{n2^{n-1}} < 2/n,$$

as desired.

B3 **First solution:** It is enough to show that for each prime p, the exponent of p in the prime factorization of both sides is the same. On the left side, it is well-known that the exponent of p in the prime factorization of $n!$ is

$$\sum_{i=1}^{n} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

(To see this, note that the i-th term counts the multiples of p^i among $1, \ldots, n$, so that a number divisible exactly by p^i gets counted exactly i times.) This number can be reinterpreted as the cardinality of the set S of points in the plane with positive integer coordinates lying on or under the curve $y = np^{-x}$: namely, each summand is the number of points of S with $x = i$.

On the right side, the exponent of p in the prime factorization of $\mathrm{lcm}(1, \ldots, \lfloor n/i \rfloor)$ is $\lfloor \log_p \lfloor n/i \rfloor \rfloor = \lfloor \log_p(n/i) \rfloor$. However, this is precisely the number of points of S with $y = i$. Thus

$$\sum_{i=1}^{n} \lfloor \log_p \lfloor n/i \rfloor \rfloor = \sum_{i=1}^{n} \left\lfloor \frac{n}{p^i} \right\rfloor,$$

and the desired result follows.

Second solution: We prove the result by induction on n, the case $n = 1$ being obvious. What we actually show is that going from $n - 1$ to n changes both sides by the same multiplicative factor, that is,

$$n = \prod_{i=1}^{n-1} \frac{\mathrm{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}}{\mathrm{lcm}\{1, 2, \ldots, \lfloor (n-1)/i \rfloor\}}.$$

Note that the i-th term in the product is equal to 1 if n/i is not an integer, i.e., if n/i is not a divisor of n. It is also equal to 1 if n/i is a divisor of n but not a prime power, since any composite number divides the lcm of all smaller numbers. However, if n/i is a power of p, then the i-th term is equal to p.

Since n/i runs over all proper divisors of n, the product on the right side includes one factor of the prime p for each factor of p in the prime factorization of n. Thus the whole product is indeed equal to n, completing the induction.

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B4 First solution: Put $g = r_1 + r_2$, $h = r_3 + r_4$, $u = r_1 r_2$, $v = r_3 r_4$. We are given that g is rational. The following are also rational:

$$\frac{-b}{a} = g + h$$
$$\frac{c}{a} = gh + u + v$$
$$\frac{-d}{a} = gv + hu$$

From the first line, h is rational. From the second line, $u + v$ is rational. From the third line, $g(u + v) - (gv + hu) = (g - h)u$ is rational. Since $g \neq h$, u is rational, as desired.

Second solution: This solution uses some basic Galois theory. We may assume $r_1 \neq r_2$, since otherwise they are both rational and so then is $r_1 r_2$.

Let τ be an automorphism of the field of algebraic numbers; then τ maps each r_i to another one, and fixes the rational number $r_1 + r_2$. If $\tau(r_1)$ equals one of r_1 or r_2, then $\tau(r_2)$ must equal the other one, and vice versa. Thus τ either fixes the set $\{r_1, r_2\}$ or moves it to $\{r_3, r_4\}$. But if the latter happened, we would have $r_1 + r_2 = r_3 + r_4$, contrary to hypothesis. Thus τ fixes the set $\{r_1, r_2\}$ and in particular the number $r_1 r_2$. Since this is true for any τ, $r_1 r_2$ must be rational.

Note: The conclusion fails if we allow $r_1 + r_2 = r_3 + r_4$. For instance, take the polynomial $x^4 - 2$ and label its roots so that $(x - r_1)(x - r_2) = x^2 - \sqrt{2}$ and $(x - r_3)(x - r_4) = x^2 + \sqrt{2}$.

B5 Place the unit circle on the complex plane so that A, B, C correspond to the complex numbers $1, \omega, \omega^2$, where $\omega = e^{2\pi i/3}$, and let P correspond to the complex number x. The distances a, b, c are then $|x - 1|, |x - \omega|, |x - \omega^2|$. Now the identity

$$(x - 1) + \omega(x - \omega) + \omega^2(x - \omega^2) = 0$$

implies that there is a triangle whose sides, as vectors, correspond to the complex numbers $x - 1, \omega(x - \omega), \omega^2(x - \omega^2)$; this triangle has sides of length a, b, c.

To calculate the area of this triangle, we first note a more general formula. If a triangle in the plane has vertices at 0, $v_1 = s_1 + it_1$, $v_2 = s_2 + it_2$, then it is well known that the area of the triangle is $|s_1 t_2 - s_2 t_1|/2 = |v_1 \overline{v_2} - v_2 \overline{v_1}|/4$. In our case, we have $v_1 = x - 1$ and $v_2 = \omega(x - \omega)$; then

$$v_1 \overline{v_2} - v_2 \overline{v_1} = (\omega^2 - \omega)(x\overline{x} - 1) = i\sqrt{3}(|x|^2 - 1).$$

Hence the area of the triangle is $\sqrt{3}(1 - |x|^2)/4$, which depends only on the distance $|x|$ from P to O.

B6 First solution: (composite of solutions by Feng Xie and David Pritchard) Let μ denote Lebesgue measure

on $[0, 1]$. Define

$$E_+ = \{x \in [0, 1] : f(x) \geq 0\}$$
$$E_- = \{x \in [0, 1] : f(x) < 0\};$$

then E_+, E_- are measurable and $\mu(E_+) + \mu(E_-) = 1$. Write μ_+ and μ_- for $\mu(E_+)$ and $\mu(E_-)$. Also define

$$I_+ = \int_{E_+} |f(x)|\, dx$$
$$I_- = \int_{E_-} |f(x)|\, dx,$$

so that $\int_0^1 |f(x)|\, dx = I_+ + I_-$.

From the triangle inequality $|a + b| \geq \pm(|a| - |b|)$, we have the inequality

$$\iint_{E_+ \times E_-} |f(x) + f(y)|\, dx\, dy$$
$$\geq \pm \iint_{E_+ \times E_-} (|f(x)| - |f(y)|)\, dx\, dy$$
$$= \pm(\mu_- I_+ - \mu_+ I_-),$$

and likewise with $+$ and $-$ switched. Adding these inequalities together and allowing all possible choices of the signs, we get

$$\iint_{(E_+ \times E_-) \cup (E_- \times E_+)} |f(x) + f(y)|\, dx\, dy$$
$$\geq \max\left\{0, 2(\mu_- I_+ - \mu_+ I_-), 2(\mu_+ I_- - \mu_- I_+)\right\}.$$

To this inequality, we add the equalities

$$\iint_{E_+ \times E_+} |f(x) + f(y)|\, dx\, dy = 2\mu_+ I_+$$
$$\iint_{E_- \times E_-} |f(x) + f(y)|\, dx\, dy = 2\mu_- I_-$$
$$- \int_0^1 |f(x)|\, dx = -(\mu_+ + \mu_-)(I_+ + I_-)$$

to obtain

$$\int_0^1 \int_0^1 |f(x) + f(y)|\, dx\, dy - \int_0^1 |f(x)|\, dx$$
$$\geq \max\{(\mu_+ - \mu_-)(I_+ + I_-) + 2\mu_-(I_- - I_+),$$
$$(\mu_+ - \mu_-)(I_+ - I_-),$$
$$(\mu_- - \mu_+)(I_+ + I_-) + 2\mu_+(I_+ - I_-)\}.$$

Now simply note that for each of the possible comparisons between μ_+ and μ_-, and between I_+ and I_-, one of the three terms above is manifestly nonnegative. This yields the desired result.

Second solution: We will show at the end that it is enough to prove a discrete analogue: if $x_1, \ldots, x_n$ are real numbers, then

$$\frac{1}{n^2} \sum_{i,j=1}^n |x_i + x_j| \geq \frac{1}{n} \sum_{i=1}^n |x_i|.$$

In the meantime, we concentrate on this assertion.

Let $f(x_1, \ldots, x_n)$ denote the difference between the two sides. We induct on the number of nonzero values of $|x_i|$. We leave for later the base case, where there is at most one such value. Suppose instead for now that there are two or more. Let s be the smallest, and suppose without loss of generality that $x_1 = \cdots = x_a = s$, $x_{a+1} = \cdots = x_{a+b} = -s$, and for $i > a + b$, either $x_i = 0$ or $|x_i| > s$. (One of a, b might be zero.)

Now consider

$$f(\overbrace{t, \cdots, t}^{a \text{ terms}}, \overbrace{-t, \cdots, -t}^{b \text{ terms}}, x_{a+b+1}, \cdots, x_n)$$

as a function of t. It is piecewise linear near s; in fact, it is linear between 0 and the smallest nonzero value among $|x_{a+b+1}|, \ldots, |x_n|$ (which exists by hypothesis). Thus its minimum is achieved by one (or both) of those two endpoints. In other words, we can reduce the number of distinct nonzero absolute values among the x_i without increasing f. This yields the induction, pending verification of the base case.

As for the base case, suppose that $x_1 = \cdots = x_a = s > 0$, $x_{a+1} = \cdots = x_{a+b} = -s$, and $x_{a+b+1} = \cdots = x_n = 0$. (Here one or even both of a, b could be zero, though the latter case is trivial.) Then

$$
\begin{aligned}
f(x_1, \ldots, x_n) &= \frac{s}{n^2}(2a^2 + 2b^2 + (a+b)(n-a-b)) \\
&\quad - \frac{s}{n}(a+b) \\
&= \frac{s}{n^2}(a^2 - 2ab + b^2) \\
&\geq 0.
\end{aligned}
$$

This proves the base case of the induction, completing the solution of the discrete analogue.

To deduce the original statement from the discrete analogue, approximate both integrals by equally-spaced Riemann sums and take limits. This works because given a continuous function on a product of closed intervals, any sequence of Riemann sums with mesh size tending to zero converges to the integral. (The domain is compact, so the function is uniformly continuous. Hence for any $\epsilon > 0$ there is a cutoff below which any mesh size forces the discrepancy between the Riemann sum and the integral to be less than ϵ.)

Alternate derivation (based on a solution by Dan Bernstein): from the discrete analogue, we have

$$\sum_{1 \leq i < j \leq n} |f(x_i) + f(x_j)| \geq \frac{n-2}{2} \sum_{i=1}^{n} |f(x_i)|,$$

for all $x_1, \ldots, x_n \in [0, 1]$. Integrating both sides as $(x_1, \ldots, x_n)$ runs over $[0, 1]^n$ yields

$$\frac{n(n-1)}{2} \int_0^1 \int_0^1 |f(x) + f(y)|\, dy\, dx$$

$$\geq \frac{n(n-2)}{2} \int_0^1 |f(x)|\, dx,$$

or

$$\int_0^1 \int_0^1 |f(x) + f(y)|\, dy\, dx \geq \frac{n-2}{n-1} \int_0^1 |f(x)|\, dx.$$

Taking the limit as $n \to \infty$ now yields the desired result.

Third solution: (by David Savitt) We give an argument which yields the following improved result. Let μ_p and μ_n be the measure of the sets $\{x : f(x) > 0\}$ and $\{x : f(x) < 0\}$ respectively, and let $\mu \leq 1/2$ be $\min(\mu_p, \mu_n)$. Then

$$\int_0^1 \int_0^1 |f(x) + f(y)|\, dx\, dy$$

$$\geq (1 + (1 - 2\mu)^2) \int_0^1 |f(x)|\, dx.$$

Note that the constant can be seen to be best possible by considering a sequence of functions tending towards the step function which is 1 on $[0, \mu]$ and -1 on $(\mu, 1]$.

Suppose without loss of generality that $\mu = \mu_p$. As in the second solution, it suffices to prove a strengthened discrete analogue, namely

$$\frac{1}{n^2} \sum_{i,j} |a_i + a_j| \geq \left(1 + \left(1 - \frac{2p}{n}\right)^2\right) \left(\frac{1}{n} \sum_{i=1}^{n} |a_i|\right),$$

where $p \leq n/2$ is the number of $a_1, \ldots, a_n$ which are positive. (We need only make sure to choose meshes so that $p/n \to \mu$ as $n \to \infty$.) An equivalent inequality is

$$\sum_{1 \leq i < j \leq n} |a_i + a_j| \geq \left(n - 1 - 2p + \frac{2p^2}{n}\right) \sum_{i=1}^{n} |a_i|.$$

Write $r_i = |a_i|$, and assume without loss of generality that $r_i \geq r_{i+1}$ for each i. Then for $i < j$, $|a_i + a_j| = r_i + r_j$ if a_i and a_j have the same sign, and is $r_i - r_j$ if they have opposite signs. The left-hand side is therefore equal to

$$\sum_{i=1}^{n} (n - i) r_i + \sum_{j=1}^{n} r_j C_j,$$

where

$$
\begin{aligned}
C_j = &\#\{i < j : \operatorname{sgn}(a_i) = \operatorname{sgn}(a_j)\} \\
&- \#\{i < j : \operatorname{sgn}(a_i) \neq \operatorname{sgn}(a_j)\}.
\end{aligned}
$$

Consider the partial sum $P_k = \sum_{j=1}^{k} C_j$. If exactly p_k of $a_1, \ldots, a_k$ are positive, then this sum is equal to

$$\binom{p_k}{2} + \binom{k - p_k}{2} - \left[\binom{k}{2} - \binom{p_k}{2} - \binom{k - p_k}{2}\right],$$

which expands and simplifies to

$$-2p_k(k - p_k) + \binom{k}{2}.$$

For $k \leq 2p$ even, this partial sum would be minimized with $p_k = \frac{k}{2}$, and would then equal $-\frac{k}{2}$; for $k < 2p$ odd, this partial sum would be minimized with $p_k = \frac{k \pm 1}{2}$, and would then equal $-\frac{k-1}{2}$. Either way, $P_k \geq -\lfloor \frac{k}{2} \rfloor$. On the other hand, if $k > 2p$, then

$$-2p_k(k - p_k) + \binom{k}{2} \geq -2p(k - p) + \binom{k}{2}$$

since p_k is at most p. Define Q_k to be $-\lfloor \frac{k}{2} \rfloor$ if $k \leq 2p$ and $-2p(k-p) + \binom{k}{2}$ if $k \geq 2p$, so that $P_k \geq Q_k$. Note that $Q_1 = 0$.

Partial summation gives

$$\begin{aligned}
\sum_{j=1}^{n} r_j C_j &= r_n P_n + \sum_{j=2}^{n} (r_{j-1} - r_j) P_{j-1} \\
&\geq r_n Q_n + \sum_{j=2}^{n} (r_{j-1} - r_j) Q_{j-1} \\
&= \sum_{j=2}^{n} r_j (Q_j - Q_{j-1}) \\
&= -r_2 - r_4 - \cdots - r_{2p} \\
&\quad + \sum_{j=2p+1}^{n} (j - 1 - 2p) r_j.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{1 \leq i < j \leq n} |a_i + a_j| &= \sum_{i=1}^{n} (n - i) r_i + \sum_{j=1}^{n} r_j C_j \\
&\geq \sum_{i=1}^{2p} (n - i - [i \text{ even}]) r_i \\
&\quad + \sum_{i=2p+1}^{n} (n - 1 - 2p) r_i \\
&= (n - 1 - 2p) \sum_{i=1}^{n} r_i + \\
&\quad \sum_{i=1}^{2p} (2p + 1 - i - [i \text{ even}]) r_i \\
&\geq (n - 1 - 2p) \sum_{i=1}^{n} r_i + p \sum_{i=1}^{2p} r_i \\
&\geq (n - 1 - 2p) \sum_{i=1}^{n} r_i + p \frac{2p}{n} \sum_{i=1}^{n} r_i,
\end{aligned}$$

as desired. The next-to-last and last inequalities each follow from the monotonicity of the r_i's, the former by pairing the ith term with the $(2p + 1 - i)^{\text{th}}$.

Note: Compare the closely related Problem 6 from the 2000 USA Mathematical Olympiad: prove that for any nonnegative real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$, one has

$$\sum_{i,j=1}^{n} \min\{a_i a_j, b_i b_j\} \leq \sum_{i,j=1}^{n} \min\{a_i b_j, a_j b_i\}.$$