Solutions to the 65th William Lowell Putnam Mathematical Competition
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A1 Yes. Suppose otherwise. Then there would be an N such that $S(N) < 80\%$ and $S(N+1) > 80\%$; that is, O'Keal's free throw percentage is under 80% at some point, and after one subsequent free throw (necessarily made), her percentage is over 80%. If she makes m of her first N free throws, then $m/N < 4/5$ and $(m+1)/(N+1) > 4/5$. This means that $5m < 4n < 5m+1$, which is impossible since then $4n$ is an integer between the consecutive integers $5m$ and $5m+1$.

Remark: This same argument works for any fraction of the form $(n-1)/n$ for some integer $n > 1$, but not for any other real number between 0 and 1.

A2 **First solution:** (partly due to Ravi Vakil) Yes, it does follow. For $i = 1, 2$, let P_i, Q_i, R_i be the vertices of T_i opposide the sides of length a_i, b_i, c_i, respectively.

We first check the case where $a_1 = a_2$ (or $b_1 = b_2$ or $c_1 = c_2$, by the same argument after relabeling). Imagine T_2 as being drawn with the base Q_2R_2 horizontal and the point P_2 above the line Q_2R_2. We may then position T_1 so that $Q_1 = Q_2$, $R_1 = R_2$, and P_1 lies above the line $Q_1R_1 = Q_2R_2$. Then P_1 also lies inside the region bounded by the circles through P_2 centered at Q_2 and R_2. Since $\angle Q_2$ and $\angle R_2$ are acute, the part of this region above the line Q_2R_2 lies within T_2. In particular, the distance from P_1 to the line Q_2R_2 is less than or equal to the distance from P_2 to the line Q_2R_2; hence $A_1 \leq A_2$.

To deduce the general case, put

$$r = \max\{a_1/a_2, b_1/b_2, c_1/c_2\}.$$

Let T_3 be the triangle with sides ra_2, rb_2, rc_2, which has area r^2A_2. Applying the special case to T_1 and T_3, we deduce that $A_1 \leq r^2A_2$; since $r \leq 1$ by hypothesis, we have $A_1 \leq A_2$ as desired.

Remark: Another geometric argument in the case $a_1 = a_2$ is that since angles $\angle Q_2$ and $\angle R_2$ are acute, the perpendicular to Q_2R_2 through P_2 separates Q_2 from R_2. If $A_1 > A_2$, then P_1 lies above the parallel to Q_2R_2 through P_2; if then it lies on or to the left of the vertical line through P_2, we have $c_1 > c_2$ because the inequality holds for both horizontal and vertical components (possibly with equality for one, but not both). Similarly, if P_1 lies to the right of the vertical, then $b_1 > b_2$.

Second solution: (attribution unknown) Retain notation as in the first paragraph of the first solution. Since the angle measures in any triangle add up to π, some angle of T_1 must have measure less than or equal to its counterpart in T_2. Without loss of generality assume that $\angle P_1 \leq \angle P_2$. Since the latter is acute (because T_2 is acute), we have $\sin \angle P_1 \leq \sin \angle P_2$. By the Law of Sines,

$$A_1 = \frac{1}{2}b_1c_1 \sin \angle P_1 \leq \frac{1}{2}b_2c_2 \sin \angle P_2 = A_2.$$

Remark: Many other solutions are possible; for instance, one uses Heron's formula for the area of a triangle in terms of its side lengths.

A3 Define a sequence v_n by $v_n = (n-1)(n-3)\cdots(4)(2)$ if n is odd and $v_n = (n-1)(n-3)\cdots(3)(1)$ if n is even; it suffices to prove that $u_n = v_n$ for all $n \geq 2$. Now $v_{n+3}v_n = (n+2)(n)(n-1)!$ and $v_{n+2}v_{n+1} = (n+1)!$, and so $v_{n+3}v_n - v_{n+2}v_{n+1} = n!$. Since we can check that $u_n = v_n$ for $n = 2, 3, 4$, and u_n and v_n satisfy the same recurrence, it follows by induction that $u_n = v_n$ for all $n \geq 2$, as desired.

A4 It suffices to verify that

$$x_1 \cdots x_n = \frac{1}{2^n n!} \sum_{e_i \in \{-1,1\}} (e_1 \cdots e_n)(e_1x_1 + \cdots + e_nx_n)^n.$$

To check this, first note that the right side vanishes identically for $x_1 = 0$, because each term cancels the corresponding term with e_1 flipped. Hence the right side, as a polynomial, is divisible by x_1; similarly it is divisible by $x_2, \ldots, x_n$. Thus the right side is equal to $x_1 \cdots x_n$ times a scalar. (Another way to see this: the right side is clearly odd as a polynomial in each individual variable, but the only degree n monomial in $x_1, \ldots, x_n$ with that property is $x_1 \cdots x_n$.) Since each summand contributes $\frac{1}{2^n}x_1 \cdots x_n$ to the sum, the scalar factor is 1 and we are done.

Remark: Several variants on the above construction are possible; for instance,

$$x_1 \cdots x_n = \frac{1}{n!} \sum_{e_i \in \{0,1\}} (-1)^{n-e_1-\cdots-e_n}(e_1x_1 + \cdots + e_nx_n)^n$$

by the same argument as above.

Remark: These construction work over any field of characteristic greater than n (at least for $n > 1$). On the other hand, no construction is possible over a field of characteristic $p \leq n$, since the coefficient of $x_1 \cdots x_n$ in the expansion of $(e_1x_1 + \cdots + e_nx_n)^n$ is zero for any e_i.

Remark: Richard Stanley asks whether one can use fewer than 2^n terms, and what the smallest possible number is.

A5 **First solution:** First recall that any graph with n vertices and e edges has at least $n - e$ connected components (add each edge one at a time, and note that it reduces the number of components by at most 1). Now imagine the squares of the checkerboard as a graph, whose vertices are connected if the corresponding squares share a side and are the same color. Let A be the number of edges in the graph, and let B be the number of 4-cycles (formed by monochromatic 2×2 squares). If we remove the bottom edge of each 4-cycle, the resulting graph has the same number of connected components as the original one; hence this number is at least

$$mn - A + B.$$

By the linearity of expectation, the expected number of connected components is at least

$$mn - E(A) + E(B).$$

Moreover, we may compute $E(A)$ by summing over the individual pairs of adjacent squares, and we may compute $E(B)$ by summing over the individual 2×2 squares. Thus

$$E(A) = \frac{1}{2}(m(n-1) + (m-1)n),$$
$$E(B) = \frac{1}{8}(m-1)(n-1),$$

and so the expected number of components is at least

$$mn - \frac{1}{2}(m(n-1) + (m-1)n) + \frac{1}{8}(m-1)(n-1)$$
$$= \frac{mn + 3m + 3n + 1}{8} > \frac{mn}{8}.$$

Remark: A "dual" approach is to consider the graph whose vertices are the corners of the squares of the checkerboard, with two vertices joined if they are adjacent and the edge between then does not separate two squares of the same color. In this approach, the 4-cycles become isolated vertices, and the bound on components is replaced by a call to Euler's formula relating the vertices, edges and faces of a planar figure. (One must be careful, however, to correctly handle faces which are not simply connected.)

Second solution: (by Noam Elkies) Number the squares of the checkerboard $1, \dots, mn$ by numbering the first row from left to right, then the second row, and so on. We prove by induction on i that if we just consider the figure formed by the first i squares, its expected number of monochromatic components is at least $i/8$. For $i = 1$, this is clear.

Suppose the i-th square does not abut the left edge or the top row of the board. Then we may divide into three cases.

- With probability $1/4$, the i-th square is opposite in color from the adjacent squares directly above and to the left of it. In this case adding the i-th square adds one component.
- With probability $1/8$, the i-th square is the same in color as the adjacent squares directly above and to the left of it, but opposite in color from its diagonal neighbor above and to the left. In this case, adding the i-th square either removes a component or leaves the number unchanged.
- In all other cases, the number of components remains unchanged upon adding the i-th square.

Hence adding the i-th square increases the expected number of components by $1/4 - 1/8 = 1/8$.

If the i-th square does abut the left edge of the board, the situation is even simpler: if the i-th square differs in color from the square above it, one component is added, otherwise the number does not change. Hence adding the i-th square increases the expected number of components by $1/2$; likewise if the i-th square abuts the top edge of the board. Thus the expected number of components is at least $i/8$ by induction, as desired.

Remark: Some solvers attempted to consider adding one row at a time, rather than one square; this must be handled with great care, as it is possible that the number of components can drop rather precipitously upon adding an entire row.

A6 By approximating each integral with a Riemann sum, we may reduce to proving the discrete analogue: for $x_{ij} \in \mathbb{R}$ for $i, j = 1, \dots, n$,

$$n \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x_{ij} \right)^2 + n \sum_{j=1}^{n} \left(\sum_{i=1}^{n} x_{ij} \right)^2$$
$$\leq \left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \right)^2 + n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}^2.$$

The difference between the right side and the left side is

$$\frac{1}{4} \sum_{i,j,k,l=1}^{n} (x_{ij} + x_{kl} - x_{il} - x_{kj})^2,$$

which is evidently nonnegative. If you prefer not to discretize, you may rewrite the original inequality as

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 F(x,y,z,w)^2 \, dx \, dy \, dz \, dw \geq 0$$

for

$$F(x,y,z,w) = f(x,y) + f(z,w) - f(x,w) - f(z,y).$$

Remark: (by Po-Ning Chen) The discrete inequality can be arrived at more systematically by repeatedly applying the following identity: for any real $a_1, \ldots, a_n$,

$$\sum_{1 \le i < j \le n} (x_i - x_j)^2 = n \sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i \right)^2.$$

Remark: (by David Savitt) The discrete inequality can also be interpreted as follows. For $c, d \in \{1, \ldots, n-1\}$ and $\zeta_n = e^{2\pi i/n}$, put

$$z_{c,d} = \sum_{i,j} \zeta_n^{ci+dj} x_{ij}.$$

Then the given inequality is equivalent to

$$\sum_{c,d=1}^{n-1} |z_{c,d}|^2 \ge 0.$$

B1 Let k be an integer, $0 \le k \le n-1$. Since $P(r)/r^k = 0$, we have

$$\begin{aligned} &c_n r^{n-k} + c_{n-1} r^{n-k+1} + \cdots + c_{k+1} r \\ &\quad = -(c_k + c_{k-1} r^{-1} + \cdots + c_0 r^{-k}). \end{aligned}$$

Write $r = p/q$ where p and q are relatively prime. Then the left hand side of the above equation can be written as a fraction with denominator q^{n-k}, while the right hand side is a fraction with denominator p^k. Since p and q are relatively prime, both sides of the equation must be an integer, and the result follows.

Remark: If we write $r = a/b$ in lowest terms, then $P(x)$ factors as $(bx - a)Q(x)$, where the polynomial Q has integer coefficients because you can either do the long division from the left and get denominators divisible only by primes dividing b, or do it from the right and get denominators divisible only by primes dividing a. The numbers given in the problem are none other than a times the coefficients of Q. More generally, if $P(x)$ is divisible, as a polynomial over the rationals, by a polynomial $R(x)$ with integer coefficients, then P/R also has integer coefficients; this is known as "Gauss's lemma" and holds in any unique factorization domain.

B2 **First solution:** We have

$$(m+n)^{m+n} > \binom{m+n}{m} m^m n^n$$

because the binomial expansion of $(m+n)^{m+n}$ includes the term on the right as well as some others. Rearranging this inequality yields the claim.

Remark: One can also interpret this argument combinatorially. Suppose that we choose $m+n$ times (with replacement) uniformly randomly from a set of $m+n$ balls, of which m are red and n are blue. Then the probability of picking each ball exactly once is $(m+n)!/(m+n)^{m+n}$. On the other hand, if p is the probability of picking exactly m red balls, then $p < 1$ and the probability of picking each ball exactly once is $p(m^m/m!)(n^n/n!)$.

Second solution: (by David Savitt) Define

$$S_k = \{i/k : i = 1, \ldots, k\}$$

and rewrite the desired inequality as

$$\prod_{x \in S_m} x \prod_{y \in S_n} y > \prod_{z \in S_{m+n}} z.$$

To prove this, it suffices to check that if we sort the multiplicands on both sides into increasing order, the i-th term on the left side is greater than or equal to the i-th term on the right side. (The equality is strict already for $i = 1$, so you do get a strict inequality above.)

Another way to say this is that for any i, the number of factors on the left side which are less than $i/(m+n)$ is less than i. But since $j/m < i/(m+n)$ is equivalent to $j < im/(m+n)$, that number is

$$\begin{aligned} &\left\lceil \frac{im}{m+n} \right\rceil - 1 + \left\lceil \frac{in}{m+n} \right\rceil - 1 \\ &\le \frac{im}{m+n} + \frac{in}{m+n} - 1 = i - 1. \end{aligned}$$

Third solution: Put $f(x) = x(\log(x+1) - \log x)$; then for $x > 0$,

$$\begin{aligned} f'(x) &= \log(1 + 1/x) - \frac{1}{x+1} \\ f''(x) &= -\frac{1}{x(x+1)^2}. \end{aligned}$$

Hence $f''(x) < 0$ for all x; since $f'(x) \to 0$ as $x \to \infty$, we have $f'(x) > 0$ for $x > 0$, so f is strictly increasing.

Put $g(m) = m \log m - \log(m!)$; then $g(m+1) - g(m) = f(m)$, so $g(m+1) - g(m)$ increases with m. By induction, $g(m+n) - g(m)$ increases with n for any positive integer n, so in particular

$$\begin{aligned} g(m+n) - g(m) &> g(n) - g(1) + f(m) \\ &\ge g(n) \end{aligned}$$

since $g(1) = 0$. Exponentiating yields the desired inequality.

B3 The answer is $\{a \mid a > 2\}$. If $a > 2$, then the function $f(x) = 2a/(a-2)$ has the desired property; both perimeter and area of R in this case are $2a^2/(a-2)$. Now suppose that $a \le 2$, and let $f(x)$ be a nonnegative continuous function on $[0, a]$. Let $P = (x_0, y_0)$ be a point on the graph of $f(x)$ with maximal y-coordinate; then the area of R is at most ay_0 since it lies below the line $y = y_0$. On the other hand, the points $(0,0)$, $(a,0)$, and P divide the boundary of R into three sections. The

length of the section between $(0,0)$ and P is at least the distance between $(0,0)$ and P, which is at least y_0; the length of the section between P and $(a,0)$ is similarly at least y_0; and the length of the section between $(0,0)$ and $(a,0)$ is a. Since $a \leq 2$, we have $2y_0 + a > ay_0$ and hence the perimeter of R is strictly greater than the area of R.

B4 **First solution:** Identify the xy-plane with the complex plane $\mathbb{C}$, so that P_k is the real number k. If z is sent to z' by a counterclockwise rotation by θ about P_k, then $z' - k = e^{i\theta}(z-k)$; hence the rotation R_k sends z to $\zeta z + k(1-\zeta)$, where $\zeta = e^{2\pi i/n}$. It follows that R_1 followed by R_2 sends z to $\zeta(\zeta z + (1-\zeta)) + 2(1-\zeta) = \zeta^2 z + (1-\zeta)(\zeta+2)$, and so forth; an easy induction shows that R sends z to

$$\zeta^n z + (1-\zeta)(\zeta^{n-1} + 2\zeta^{n-2} + \cdots + (n-1)\zeta + n).$$

Expanding the product $(1-\zeta)(\zeta^{n-1} + 2\zeta^{n-2} + \cdots + (n-1)\zeta + n)$ yields $-\zeta^n - \zeta^{n-1} - \cdots - \zeta + n = n$. Thus R sends z to $z+n$; in cartesian coordinates, $R(x,y) = (x+n, y)$.

Second solution: (by Andy Lutomirski, via Ravi Vakil) Imagine a regular n-gon of side length 1 placed with its top edge on the x-axis and the left endpoint of that edge at the origin. Then the rotations correspond to rolling this n-gon along the x-axis; after the n rotations, it clearly ends up in its original rotation and translated n units to the right. Hence the whole plane must do so as well.

Third solution: (attribution unknown) Viewing each R_k as a function of a complex number z as in the first solution, the function $R_n \circ R_{n-1} \circ \cdots \circ R_1(z)$ is linear in z with slope $\zeta^n = 1$. It thus equals $z+T$ for some $T \in \mathbb{C}$. Since $f_1(1) = 1$, we can write $1 + T = R_n \circ \cdots \circ R_2(1)$. However, we also have

$$R_n \circ \cdots \circ R_2(1) = R_{n-1} \circ R_1(0) + 1$$

by the symmetry in how the R_i are defined. Hence

$$R_n(1+T) = R_n \circ R_1(0) + R_n(1) = T + R_n(1);$$

that is, $R_n(T) = T$. Hence $T = n$, as desired.

B5 **First solution:** By taking logarithms, we see that the desired limit is $\exp(L)$, where $L = \lim_{x\to 1^-} \sum_{n=0}^{\infty} x^n \left(\ln(1+x^{n+1}) - \ln(1+x^n)\right)$. Now

$$\sum_{n=0}^{N} x^n \left(\ln(1+x^{n+1}) - \ln(1+x^n)\right)$$
$$= 1/x \sum_{n=0}^{N} x^{n+1}\ln(1+x^{n+1}) - \sum_{n=0}^{N} x^n \ln(1+x^n)$$
$$= x^N \ln(1+x^{N+1}) - \ln 2 + (1/x - 1)\sum_{n=1}^{N} x^n \ln(1+x^n);$$

since $\lim_{N\to\infty}(x^N \ln(1+x^{N+1})) = 0$ for $0 < x < 1$, we conclude that $L = -\ln 2 + \lim_{x\to 1^-} f(x)$, where

$$f(x) = (1/x-1)\sum_{n=1}^{\infty} x^n \ln(1+x^n)$$
$$= (1/x-1)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+1}x^{n+mn}/m.$$

This final double sum converges absolutely when $0 < x < 1$, since

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x^{n+mn}/m = \sum_{n=1}^{\infty} x^n(-\ln(1-x^n))$$
$$< \sum_{n=1}^{\infty} x^n(-\ln(1-x)),$$

which converges. (Note that $-\ln(1-x)$ and $-\ln(1-x^n)$ are positive.) Hence we may interchange the summations in $f(x)$ to obtain

$$f(x) = (1/x-1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+1}x^{(m+1)n}}{m}$$
$$= (1/x-1)\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left(\frac{x^m(1-x)}{1-x^{m+1}}\right).$$

This last sum converges absolutely uniformly in x, so it is legitimate to take limits term by term. Since $\lim_{x\to 1^-}\frac{x^m 1-x}{1-x^{m+1}} = \frac{1}{m+1}$ for fixed m, we have

$$\lim_{x\to 1^-} f(x) = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m(m+1)}$$
$$= \sum_{m=1}^{\infty}(-1)^{m+1}\left(\frac{1}{m} - \frac{1}{m+1}\right)$$
$$= 2\left(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\right) - 1$$
$$= 2\ln 2 - 1,$$

and hence $L = \ln 2 - 1$ and the desired limit is $2/e$.

Remark: Note that the last series is not absolutely convergent, so the recombination must be done without rearranging terms.

Second solution: (by Greg Price, via Tony Zhang and Anders Kaseorg) Put $t_n(x) = \ln(1+x^n)$; we can then write $x^n = \exp(t_n(x)) - 1$, and

$$L = \lim_{x\to 1^-}\sum_{n=0}^{\infty}(t_n(x) - t_{n+1}(x))(1-\exp(t_n(x))).$$

The expression on the right is a Riemann sum approximating the integral $\int_0^{\ln 2}(1-e^t)\,dt$, over the subdivision of $[0,\ln(2))$ given by the $t_n(x)$. As $x \to 1^-$, the maximum difference between consecutive $t_n(x)$ tends to 0, so the Riemann sum tends to the value of the integral. Hence $L = \int_0^{\ln 2}(1-e^t)\,dt = \ln 2 - 1$, as desired.

B6 **First solution:** (based on a solution of Dan Bernstein) Note that for any b, the condition that $b \notin \mathcal{B}$ already forces $\limsup N(x)/x$ to be at most 1/2: pair off $2mb+n$ with $(2m+1)b+n$ for $n = 1, \ldots, b$, and note that at most one member of each pair may belong to $\mathcal{A}$. The idea of the proof is to do something similar with pairs replaced by larger clumps, using long runs of excluded elements of B.

Suppose we have positive integers $b_0 = 1, b_1, \ldots, b_n$ with the following properties:

(a) For $i = 1, \ldots, n$, $c_i = b_i/(2b_{i-1})$ is an integer.

(b) For $e_i \in \{-1, 0, 1\}$, $|e_1 b_1 + \cdots + e_n b_n| \notin \mathcal{B}$.

Each nonnegative integer a has a unique "base expansion"

$$a = a_0 b_0 + \cdots + a_{n-1} b_{n-1} + m b_n \qquad (0 \leq a_i < 2c_i);$$

if two integers have expansions with the same value of m, and values of a_i differing by at most 1 for $i = 0, \ldots, n-1$, then their difference is not in $\mathcal{B}$, so at most one of them lies in $\mathcal{A}$. In particular, for any $d_i \in \{0, \ldots, c_i - 1\}$, any $m_0 \in \{0, 2c_0 - 1\}$ and any m_n, the set

$$\{m_0 b_0 + (2d_1 + e_1) b_0 + \cdots + (2d_{n-1} + e_{n-1}) b_{n-1} + (2m_n + e_n) b_n\},$$

where each e_i runs over $\{0, 1\}$, contains at most one element of $\mathcal{A}$; consequently, $\limsup N(x)/x \leq 1/2^n$.

We now produce such b_i recursively, starting with $b_0 = 1$ (and both (a) and (b) holding vacuously). Given $b_0, \ldots, b_n$ satisfying (a) and (b), note that $b_0 + \cdots + b_{n-1} < b_n$ by induction on n. By the hypotheses of the problem, we can find a set S_n of $6b_n$ consecutive integers, none of which belongs to $\mathcal{B}$. Let b_{n+1} be the second-smallest multiple of $2b_n$ in S_n; then $b_{n+1} + x \in S_n$ for $-2b_n \leq x \leq 0$ clearly, and also for $0 \leq x \leq 2b_n$ because there are most $4b_n - 1$ elements of S_n preceding b_{n+1}. In particular, the analogue of (b) with n replaced by $n+1$ holds for $e_{n+1} \neq 0$; of course it holds for $e_{n+1} = 0$ because (b) was already known. Since the analogue of (a) holds by construction, we have completed this step of the construction and the recursion may continue.

Since we can construct $b_0, \ldots, b_n$ satisfying (a) and (b) for any n, we have $\limsup N(x)/x \leq 1/2^n$ for any n, yielding $\lim N(x)/x = 0$ as desired.

Second solution: (by Paul Pollack) Let S be the set of possible values of $\limsup N(x)/x$; since $S \subseteq [0, 1]$ is bounded, it has a least upper bound L. Suppose by way of contradiction that $L > 0$; we can then choose $\mathcal{A}, \mathcal{B}$ satisfying the conditions of the problem such that $\limsup N(x)/x > 3L/4$.

To begin with, we can certainly find some positive integer $m \notin \mathcal{B}$, so that $\mathcal{A}$ is disjoint from $\mathcal{A}+m = \{a+m : a \in \mathcal{A}\}$. Put $\mathcal{A}' = \mathcal{A} \cup (\mathcal{A}+m)$ and let $N'(x)$ be the size of $\mathcal{A}' \cap \{1, \ldots, x\}$; then $\limsup N'(x)/x = 3L/2 > L$, so $\mathcal{A}'$ cannot obey the conditions of the problem statement. That is, if we let $\mathcal{B}'$ be the set of positive integers that occur as differences between elements of $\mathcal{A}'$, then there exists an integer n such that among any n consecutive integers, at least one lies in $\mathcal{B}'$. But

$$\mathcal{B}' \subseteq \{b + em : b \in \mathcal{B}, e \in \{-1, 0, 1\}\},$$

so among any $n + 2m$ consecutive integers, at least one lies in $\mathcal{B}$. This contradicts the condition of the problem statement.

We conclude that it is impossible to have $L > 0$, so $L = 0$ and $\lim N(x)/x = 0$ as desired.

Remark: A hybrid between these two arguments is to note that if we can produce $c_1, \ldots, c_n$ such that $|c_i - c_j| \notin \mathcal{B}$ for $i, j = 1, \ldots, n$, then the translates $\mathcal{A} + c_1, \ldots, \mathcal{A} + c_n$ are disjoint and so $\limsup N(x)/x \leq 1/n$. Given $c_1 \leq \cdots \leq c_n$ as above, we can then choose c_{n+1} to be the largest element of a run of $c_n + 1$ consecutive integers, none of which lie in $\mathcal{B}$.