ALGEBRA PH.D. QUALIFYING EXAM
FALL, 1999
PART I

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

Notation:
$\mathbb{Z}$: the ring of ordinary integers
$\mathbb{Q}$: the field of rational numbers
$\mathbb{R}$: the field of real numbers
$\mathbb{C}$: the field of complex numbers
$\mathbb{F}_q$: the finite field with q elements
$M_n(R)$: the ring of $n \times n$ matrices with entries in the ring R
$\mathrm{GL}_n(R)$: the group of invertible $n \times n$ matrices in $M_n(R)$
$R[t]$: the ring of polynomials with coefficients in the ring R
$\mathbb{Z}/n$: the ring of integers mod n. (Can also be thought of as the cyclic group of order n.)

1. If G is a simple group of order 60, determine, with proof, the number of elements of order 3 in G. (You may not assume there is only one such group.)

2. Let G be the group given by generators and relations $G = \{x, y \mid x^5 = xyx^{-1}y^{-2} = 1\}$.
(a) Prove G is finite.
(b) What is $|G|$?
(c) How many 5-Sylow subgroups are there in G?

3. Let G be the finite group of order 21 defined by generators and relations:

$$\langle x, y \mid x^3 = y^7 = 1, xyx^{-1} = y^2 \rangle.$$

Determine the conjugacy classes of G and construct its character table.

4. Let $\mathbb{K}$ be an arbitrary field and suppose that $T \in M_n(\mathbb{K})$. Prove that there exists a vector $v \in \mathbb{K}^n$ so that the vectors

$$\{v, Tv, T^2v, \ldots, T^{n-1}v\}$$

form a basis for $\mathbb{K}^n$ if and only if the only matrices in $M_n(\mathbb{K})$ which commute with T are expressible as polynomials in T (i.e., A commutes with T if and only if $A = a_0 I + a_1 T + \cdots + a_{n-1}T^{n-1}$ where $I \in M_n(\mathbb{K})$ is the identity matrix).

5.

(a) Determine the Galois group of $x^3 - x + 3$ over $\mathbb{Q}$.
(b) Determine the Galois group of $x^3 - x + 3$ over $\mathbb{F}_5$.
(c) Determine the Galois group of $x^4 + t$ over $\mathbb{R}[t]$.

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1. Suppose that $\mathbb{K}$ is a non-Galois extension of $\mathbb{Q}$ of degree 5. Let $\mathbb{L}$ be the Galois closure of $\mathbb{K}$ (the smallest Galois extension of $\mathbb{Q}$ containing $\mathbb{K}$), and suppose $\mathbb{L}$ does not contain any quadratic extensions of $\mathbb{Q}$. Prove $\mathrm{Gal}(\mathbb{L}/\mathbb{Q}) = \mathcal{A}_5$, the alternating group on 5 letters.

2. Determine all prime ideals of the ring $\mathbb{Z}[t]/(t^2)$.

3. Suppose that A, B are elements of $M_2(\mathbb{C})$ such that $A^2 = B^3 = I$, $ABA = B^{-1}$ with $A \neq I$, $B \neq I$. If $D \in M_2(\mathbb{C})$ commutes with A and B, show that D is a scalar matrix, i.e., a scalar multiple of I.

4. Let V be a valuation ring, i.e. a commutative ring (with unit) such that for all $a, b \in V$ either $a|b$ or $b|a$. (Here, $a|b$ means that $b = ac$ for some $c \in V$.)

(i) Prove that if I and J are two ideals in V then $I \subset J$ or $J \subset I$.

(ii) Prove that any finitely generated ideal of V is principal, that is, generated by a single element.

(iii) Prove that if V is a Noetherian valuation ring, then there exists an element $t \in V$ such that any proper nonzero ideal of V is (t^n) for some whole number $n \geq 1$.

5. Let G be a finite simple group, and let $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation, where $n > 1$. Let χ be its character. If $|\chi(g)| = n$, prove that g is the identity element of G.