Stanford Math PhD Qualifying Exam, Part I
Spring, 2004

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Classify the finite groups of order $333 = 3^2 \cdot 37$.

2. (a) If $\mathbb{F}_q$ is the finite field with q elements, show that $X^{q^r} - X \in \mathbb{F}_q[X]$ is exactly the product of all irreducible polynomials $f(X) \in \mathbb{F}_q[X]$ whose degree divides r.

(b) Prove that the nmber of irreducible polynomials of degree r in $\mathbb{F}_q[X]$ is

$$\frac{1}{r} \sum_{d|r} \mu\left(\frac{r}{d}\right) q^d,$$

where μ is the Moebius function:

$$\mu(d) = \begin{cases} (-1)^k & \text{if } d \text{ is a product of } k \text{ distinct primes;} \\ 0 & \text{otherwise}. \end{cases}$$

3. Let A be an integral domain with field of fractions F. Assume that for every prime ideal $\mathfrak{p} \subset A$ the localization $A_{\mathfrak{p}}$ is integrally closed (in F). Prove that A is integrally closed (in F).

4. Let A and B be nilpotent complex $n \times n$ matrices. Suppose that $\text{rank}(A^k) = \text{rank}(B^k)$ for all k. Prove that $A = MBM^{-1}$ for some $M \in \text{GL}(n, \mathbb{C})$.

5. Here is a partial character table of A_5.

	1	(123)	(12)(34)	(12345)	(13524)
χ_1	1	1	1	1	1
χ_2	4	1	0	-1	-1
χ_3	5	-1	1	0	0
χ_4	3				
χ_5	3				

Complete this character table by constructing χ_4 and χ_5.

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General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. An abelian group G (written additively) is called *divisible* if the homomorphism $x \longmapsto nx = x + \ldots + x$ (n terms) is surjective for all $n \geqslant 1$. The abelian group G is called *injective* if whenever A and B are abelian groups with $A \subset B$, a homomorphism $\varphi : A \longrightarrow G$ can be extended to a homomorphism $\Phi : B \longrightarrow G$. Assume that G is divisible. Prove that G is injective. [**Hint:** Use Zorn's Lemma.]

2. Show that if G is a finite abelian group, then there exists a finite extension F of $\mathbb{Q}$ such that $\mathrm{Gal}(F/\mathbb{Q}) \cong G$. [**Hint:** Think about roots of unity.]

3. Suppose that A is a commutative Noetherian ring.

(a) Prove that every ideal $I \subset A$ contains a finite product of prime ideals.

(b) Prove that A has only finitely many minimal prime ideals. [**Hint:** Think about the zero ideal.]

(c) Prove that if A has no nilpotent elements then the set of zero divisors in A is exactly the union of the minimal prime ideals of A.

4. Let (π, V) be a nontrivial irreducible complex representation of the finite group G with character χ. Suppose that $1 \neq g \in G$ is such that $|\chi(g)| = \chi(1)$. Show that $\pi(g)$ is a scalar endomorphism of V and deduce that G is not a nonabelian simple group.

5. Determine the number of conjugacy classes of elements of orders 3, 5 and 11 in $\mathrm{GL}(2, \mathbb{F}_{11})$.