Ph. D Qualifying Examination
Complex analysis–Autumn 2002

Work all six problems. All problems have equal weight. Write the solution to each problem in a separate bluebook.

1. Let $P(w, z) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z)$ be a polynomial in two variables. Suppose $z_0 \in \mathbb{C}$ is a point such that $a_0(z_0) \neq 0$ and $P(w, z_0)$ has n-distinct zeros, say $w_1, \cdots, w_n$. Show that there exists an open disc $\Delta \subset \mathbb{C}$ containing z_0 and n-holomorphic functions $f_i(z) : \Delta \to \mathbb{C}$, $i = 1, \cdots, n$, such that: (1) $P(f_i(z), z) = 0$ on Δ; (2) $f_i(z_0) = w_i$ and (3) whenever $P(w, z) = 0$ and $z \in \Delta$, then $w = f_i(z)$ for some i.

2. Find the Green's function for the region consisting of the complement in the $\mathbb{C}$-plane of the intervals $(-\infty, -1]$ and $[1, \infty)$ on the real axis.

3. Let H be the upper half plane and $\bar{H}$ be its closure in $\mathbb{C}$. Let $f : \bar{H} \to \mathbb{C}$ be a continuous function that is analytic in H. Suppose f is bounded on H and

$$\lim_{t \to \pm\infty} f(t) = 0, \quad t \in \mathbb{R}.$$

Show that

$$\lim_{|z| \to \infty} f(z) = 0, \; t \in H.$$

(Hint 1: You can use the following result if you like: *Let D be the open unit disk and $g : \bar{D} - \{1\} \to \mathbb{C}$ be a continuous map that is analytic in D. Suppose $|g| \leq M$ on D and $|g(e^{i\theta})| \leq 1$ for $0 < \theta < 2\pi$. Then $|g(z)| \leq 1$ on D.*)

(Hint 2: Consider functions of the form $\frac{\log z}{A + B \log z} f(z)$.)

4. Use Argument Principle to show that the function $f(z) = e^{\pi z} - e^{-\pi z}$ assumes any value w with positive real part once and only once in the half strip $\mathrm{Re}\, z > 0$, $-\frac{1}{2} < \mathrm{Im}\; z < \frac{1}{2}$.

5. Show that

$$\pi^2 \frac{\cos \pi z}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(z - n)^2}.$$

(Hint: Study the principal parts at the poles and use periodicity.)

6(a). Describe the Riemann surface of $w = \sqrt{z(z-1)(z-\lambda)}, \quad \lambda \neq 0, 1.$

(b). Show that dz/w is a holomorphic differential on the Riemann surface and describe the mapping defined by

$$f(\zeta) = \int_{\zeta_0}^{\zeta} \frac{dz}{w},$$

where ζ is a general point on the Riemann surface and ζ_0 is a chosen basepoint.