PH. D QUALIFYING EXAMINATION
COMPLEX ANALYSIS–SPRING 2003

1. Let $\mathcal{H}$ be the class of analytic functions $f(z)$ on $|z| \leq 1$ satisfying $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq 100$ for all $|z| < 1$. Prove that there is a constant $c > 0$ so that for any $f \in \mathcal{H}$ the image of the unit disk under f contains the disk $|z| < c$.

2. Let H be the upper half plane and let $F : H \to \mathbb{C}$ be defined by

$$F(z) = \int_0^z \frac{dw}{(4 - w^2)\sqrt{w - 1}}.$$

Prove, using the argument principle but not quoting the Christoffel-Schwarz Lemma directly, that F maps H one-one and onto a domain Ω in $\mathbb{C}$. Identify this domain Ω.

3. Factor the function

$$\cos\left(\frac{\pi z}{4}\right) - \sin\left(\frac{\pi z}{4}\right)$$

into an infinite product.

4. Prove that

$$\int_0^\pi \ln \sin\theta \, d\theta = -\pi \ln 2.$$

5. Let $f(z)$ be an analytic function defined on the unit disk $|z| < 1$ so that $f(0) = 0$ and $-1 < \operatorname{Re} f(z) < 1$ for all $|z| < 1$. Prove that

$$|\operatorname{Im} f(z)| \leq \frac{2}{\pi} \log \frac{1 + |z|}{1 - |z|}, \quad \text{for} 0 < |z| < 1.$$

Find an explicit form of $f(z)$ when equality holds for some $0 < |z| < 1$.

6. Let $\mathfrak{P}(z)$ be the Weierstrass $\mathfrak{P}$ function of periods 1 and τ. Prove that there is a single value branch of the meromorphic function

$$F(z) = \sqrt{\mathfrak{P}(z) - \mathfrak{P}(\frac{1}{2})}$$

with $F(\frac{1}{2}) = 0$. What are the periods of this function? Verify your assertion.