Chứng minh rằng **:**

1. $\frac{\pi}{4} \leqslant \int_{\pi/4}^{3\pi/4} \frac{1}{3-2\sin^2 x}dx \leqslant \frac{\pi}{2}$

2. $\frac{\sqrt{3}}{12} \leqslant \int_{\pi/4}^{\pi/3} \frac{\cot g}{x}dx \leqslant \frac{1}{3}$

3. $\frac{1}{2} \leqslant \int_0^{1/2} \frac{1}{\sqrt{1-x^6}}dx \leqslant \frac{\pi}{6}$

4. $\ln 2 < \int_0^1 \frac{1}{1+x\sqrt{x}}dx < \frac{\pi}{4}$

5. $\int_0^1 \frac{1}{x^2+x+1}dx \leqslant \frac{\pi}{8}$

6. $\frac{\pi}{18} \leqslant \int_0^1 \frac{\sqrt{x}}{x^5+x^4+x^3+3}dx \leqslant \frac{\pi}{9\sqrt{3}}$

Bài giải **:**

1. $\frac{\pi}{4} \leqslant x \leqslant \frac{3\pi}{4} \Rightarrow \frac{1}{\sqrt{2}} \leqslant \sin x \leqslant 1 \Rightarrow \frac{1}{2} \leqslant \sin^2 x \leqslant 1 \Rightarrow 1 \leqslant 2\sin^2 x \leqslant 2 \Rightarrow 1 \leqslant 3-2\sin^2 x \leqslant 2 \Rightarrow \frac{1}{2} \leqslant \frac{1}{3-2\sin^2 x} \leqslant 1$

$\Rightarrow \frac{1}{2}\int_{\pi/4}^{3\pi/4} dx \leqslant \int_{\pi/4}^{3\pi/4} \frac{1}{3-2\sin^2 x}dx \leqslant \int_{\pi/4}^{3\pi/4} dx \Rightarrow \frac{\pi}{4} \leqslant \int_{\pi/4}^{3\pi/4} \frac{1}{3-2\sin^2 x}dx \leqslant \frac{\pi}{2}$

2. $\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{3} \Rightarrow \begin{cases} \frac{1}{\sqrt{3}} \leqslant \cot gx \leqslant 1 \\ \frac{3}{\pi} \leqslant \frac{1}{x} \leqslant \frac{4}{\pi} \end{cases} \Rightarrow \frac{\sqrt{3}}{\pi} \leqslant \frac{\cot gx}{x} \leqslant \frac{4}{\pi} \Rightarrow \frac{\sqrt{3}}{\pi}\int_{\pi/4}^{\pi/3} dx \leqslant \int_{\pi/4}^{\pi/3} \frac{\cot gx}{x}dx \leqslant \frac{4}{\pi}\int_{\pi/4}^{\pi/3} dx$

$\Rightarrow \frac{\sqrt{3}}{12} \leqslant \int_{\pi/4}^{\pi/3} \frac{\cot gx}{x}dx \leqslant \frac{1}{3}$

Bài toán này có thể giải theo phương pháp đạo hàm.

3. $0 \leqslant x \leqslant \frac{1}{2} < 1 \Rightarrow 0 \leqslant x^6 \leqslant .... \leqslant x^2 < 1 \Rightarrow -1 \leqslant -x^2 \leqslant -x^6 \leqslant 0 \Rightarrow 0 \leqslant 1-x^2 \leqslant 1-x^6 \leqslant 1 \Rightarrow \sqrt{1-x^2} \leqslant \sqrt{1-x^6} \leqslant 1$

$\Rightarrow 1 \leqslant \frac{1}{\sqrt{1-x^6}} \leqslant \frac{1}{\sqrt{1-x^2}} \Rightarrow \int_0^{1/2} dx \leqslant \int_0^{1/2} \frac{1}{\sqrt{1-x^6}}dx \leqslant I$

Với $I = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}dx$ **Đặt** $x = \sin t\,;\, t \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right] \Rightarrow dx = \cos t dt$

x	0	$1/2$
t	0	$\pi/6$

$\Rightarrow I = \int_0^{1/2} \frac{\cos t dt}{\sqrt{1-\sin^2 t}} = \int_0^{1/2} dt = \frac{\pi}{6}$ **Vậy** $\frac{1}{2} \leqslant \int_0^{1/2} \frac{1}{\sqrt{1-x^6}}dx \leqslant \frac{\pi}{6}$

4. $0 \leqslant x \leqslant 1 \Rightarrow x \leqslant \sqrt{x} \leqslant 1 \Rightarrow x^2 \leqslant x\sqrt{x} \leqslant x \Rightarrow 1+x^2 \leqslant 1+x\sqrt{x} \leqslant 1+x$

$\Rightarrow \frac{1}{x+1} \leqslant \frac{1}{1+x\sqrt{x}} \leqslant \frac{1}{1+x^2}(1);\forall x \in [0,1]$

Dấu đẳng thức trong (1) xảy ra khi :

$\begin{cases} x = 0 & VT_{(1)} \leqslant VG_{(1)} \\ x = 1 & VG_{(1)} \leqslant VP_{(1)} \end{cases} \Rightarrow x \in \varnothing$

Do đó : $\int_0^1 \frac{1}{1+x}dx < \int_0^1 \frac{1}{1+x\sqrt{x}}dx < \int_0^1 \frac{dx}{x^2+1} \Rightarrow \ln 2 < \int_0^1 \frac{1}{1+x\sqrt{x}}dx < \frac{\pi}{4}$

Chú ý : $\int_0^1 \frac{1}{1+x^2}dx = \frac{\pi}{4}$ **Xem bài tập 5 .**

5. $0 \leqslant x \leqslant 1 \Rightarrow x^2 \leqslant x \Rightarrow x^2 + x^2 \leqslant x^2 + x \Rightarrow 2 + 2x^2 \leqslant x^2 + x + 2 \Rightarrow \dfrac{1}{x^2+x+2} \leqslant \dfrac{1}{2(x^2+1)}$

$\Rightarrow \int_0^1 \dfrac{1}{x^2+x+2}dx \leqslant \dfrac{1}{2}\int_0^1 \dfrac{1}{x^2+1}dx$; $I = \int_0^1 \dfrac{1}{1+x^2}dx$

Đặt $x = tgt \Rightarrow dx = \dfrac{1}{\cos^2 t}dt = (1+tg^2 t)dt$

x	0	1
t	0	$\pi/4$

$\Rightarrow I = \int_0^{\pi/4} \dfrac{1+tg^2 t}{1+tg^2 t}dt = \int_0^{\pi/4} dt = \dfrac{\pi}{4} \Rightarrow I = \dfrac{\pi}{4}$ **Vậy** $\int_0^1 \dfrac{1}{x^2+x+2}dx \leqslant \dfrac{\pi}{8}$

6. $0 \leqslant x \leqslant 1 \Rightarrow \begin{cases} 0 \leqslant x^5 \leqslant x^3 \\ 0 \leqslant x^4 \leqslant x^3 \end{cases} \Rightarrow 0 \leqslant x^5 + x^4 \leqslant 2x^3 \Rightarrow x^3 + 3 \leqslant x^5 + x^4 + x^3 + 3 \leqslant 3x^3 + 3$

$\Rightarrow \dfrac{1}{3x^3+3} \leqslant \dfrac{1}{x^5+x^4+x^3+3} \leqslant \dfrac{1}{x^3+3} \Rightarrow \dfrac{\sqrt{x}}{3x^3+3} \leqslant \dfrac{\sqrt{x}}{x^5+x^4+x^3+3} \leqslant \dfrac{\sqrt{x}}{x^3+3}$

$\Rightarrow \int_0^1 \dfrac{\sqrt{x}}{3x^3+3}dx \leqslant \int_0^1 \dfrac{\sqrt{x}}{x^5+x^4+x^3+3}dx \leqslant \int_0^1 \dfrac{\sqrt{x}}{x^3+3}dx \quad (1)$

• $I_1 = \int_0^1 \dfrac{\sqrt{x}}{3x^3+3}dx = \dfrac{1}{3}\int_0^1 \dfrac{\sqrt{x}}{x^3+1}dx$; **Đặt** $x = t^2; (t \geqslant 0) \Rightarrow dx = 2tdt$

x	0	1
t	0	1

$I_1 = \dfrac{1}{3}\int_0^1 \dfrac{2t}{t^6+1}dt = \dfrac{2}{9}\int_0^1 \dfrac{3t^2.dt}{(t^3)^2+1}$ **Đặt** $u = t^3 \Rightarrow du = 3t^2dt$

t	0	1
u	0	1

$\Rightarrow I_1 = \dfrac{2}{9}\int_0^1 \dfrac{du}{u^2+1} = \dfrac{\pi}{18}$

Kết quả : $I = \dfrac{\pi}{4}$ **(bài tập 5)**

• $I_2 = \int_0^1 \dfrac{\sqrt{x}}{x^3+3} = \dfrac{\pi}{9\sqrt{3}}$ **(tương tự) Vậy** $(1) \Leftrightarrow I_1 \leqslant \int_0^1 \dfrac{\sqrt{x}}{x^5+x^4+x^3+3}dx \leqslant I_2$

$\dfrac{\pi}{18} \leqslant \int_0^1 \dfrac{\sqrt{x}}{x^5+x^4+x^3+3}dx \leqslant \dfrac{\pi}{9\sqrt{3}}$

1,Chứng minh rằng **:** $\int_0^{\pi/2} \dfrac{\sin x.\cos x}{\left(1+\sin^4 x\right)\left(1+\cos^4 x\right)}dx \geqslant \dfrac{\pi}{12}$

2.Nếu **:** $I_{(t)} = \int_0^t \dfrac{tg^4 x}{\cos 2x}dx > 0, \forall t \in \left(0, \dfrac{\pi}{4}\right)$; **thì :** $tg\left(t + \dfrac{\pi}{4}\right) > e^{\frac{2}{3}\left(tg^3 t + 3tgt\right)}$

Bài giải **:**

1. Ta có : $\dfrac{3}{(1+\sin^4 x)(1+\cos^4 x)} = \dfrac{2+\cos^2 x+\sin^2 x}{(1+\sin^4 x)(1+\cos^4 x)} \geqslant \dfrac{2+\sin^4 x+\cos^4 x}{(1+\sin^4 x)(1+\cos^4 x)}$

$\Rightarrow \dfrac{3}{(1+\sin^4 x)(1+\cos^4 x)} \geqslant \dfrac{1+\sin^4 x+1+\cos^4 x}{(1+\sin^4 x)(1+\cos^4 x)} = \dfrac{1}{1+\sin^4 x} + \dfrac{1}{1+\cos^4 x}$

$$\Rightarrow \frac{3\sin x.\cos x}{(1+\sin^4 x)(1+\cos^4 x)} \geqslant \frac{\sin x.\cos x}{1+\sin^4 x}+\frac{\sin x.\cos x}{1+\cos^4 x} \Rightarrow \frac{\sin x.\cos x}{(1+\sin^4 x)(1+\cos^4 x)} \geqslant \frac{1}{6}\left(\frac{\sin 2x}{1+\sin^4 x}+\frac{\sin 2x}{1+\cos^4 x}\right)$$

$$\Rightarrow \int_0^{\pi/2} \frac{3\sin x.\cos x}{(1+\sin^4 x)(1+\cos^4 x)}dx \geqslant \frac{1}{6}\left(\int_0^{\pi/2}\frac{\sin 2x}{1+\sin^4 x}dx+\int_0^{\pi/2}\frac{\sin 2x}{1+\cos^4 x}dx\right)$$

• $J_1=\int_0^{\pi/2}\frac{\sin 2x}{1+\sin^4 x}dx$ Đặt $t=\sin^2 x \Rightarrow dt=\sin 2x dx$

x	0	$\pi/2$
t	0	1

$\Rightarrow J_1=\int_0^1\frac{dt}{t^2+1}=\frac{\pi}{4}$ (kết quả I= $\frac{\pi}{4}$ bài tập 5)

• $J_2=\int_0^{\pi/2}\frac{\sin 2x}{1+\cos^4 x}dx$ Đặt $u=\cos^2 x \Rightarrow du=-\sin 2x dx$

x	0	$\pi/2$
u	1	0

$\Rightarrow J_2=\int_0^1\frac{du}{u^2+1}=\frac{\pi}{4}$ (kết quả I= $\frac{\pi}{4}$ bài tập 5)

$\Rightarrow \int_0^{\pi/2}\frac{\sin x.\cos x}{(1+\sin^4 x)(1+\cos^4 x)}dx \geqslant \frac{1}{6}(I+J)$ Vậy $\int_0^{\frac{\pi}{2}}\frac{\sin x.\cos x}{(1+\sin^4 x)(1+\cos^4 x)}dx \geqslant \frac{\pi}{12}$

2. Đặt $t=tgx \Rightarrow dt=(1+tg^2x)dx \Rightarrow dx=\frac{dt}{1+t^2}$

$$I_t=\int_0^{tgt}\frac{t^4}{\frac{1-t^2}{1+t^2}}.\frac{dt}{1+t^2}=\int_0^{tgt}\frac{t^4dt}{1-t^2}=\int_0^{tgt}\left(-t^2-1+\frac{1}{1-t^2}\right)dt=\left(-\frac{1}{3}t^3-t-\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right)\Bigg|_0^{tgt}=-\frac{1}{3}tg^3t-tgt-\frac{1}{2}\ln\left|\frac{tgt-1}{tgt+1}\right|$$

Vì

$I_{(t)}>0$ nên : $-\frac{1}{3}tg^3t-tgt-\frac{1}{2}\ln\left|\frac{tgt-1}{tgt+1}\right|>0$

$$\Leftrightarrow \frac{1}{2}\ln\left|\frac{tgt-1}{tgt+1}\right|=\frac{1}{2}\ln\left|tg\left(t+\frac{\pi}{4}\right)\right|>\frac{1}{3}tg^3t+tgt \Rightarrow tg\left(t+\frac{\pi}{4}\right)>e^{\frac{2}{3}\left(tg^3t+3tgt\right)}$$

1. $I_n=\frac{x^2}{x+1}$ ***Chứng minh*** : $\frac{1}{2(n+1)} \le \int_0^1 I_n dx \le \frac{1}{n+1}$ và $\lim_{n\to+\infty} I_n dx=0$

2. $J_n=x^n\left(1+e^{-x}\right)$ ***Chứng minh*** : $0<\int_0^1 J_n dx \leqslant \frac{2}{n+1}$ và $\lim_{n\to+\infty} J_n dx=0$

Bài giải :

1. $0\leqslant x\leqslant 1 \Rightarrow 1\leqslant x+1\leqslant 2 \Rightarrow \frac{1}{2}\leqslant\frac{1}{x+1}\leqslant 1$; $\frac{x^n}{2}\leqslant\frac{x^n}{x+1}\leqslant x^n \Rightarrow \frac{1}{2}\int_0^1 x^n dx\leqslant\int_0^1\frac{x^n}{x+1}dx\leqslant\int_0^1 x^n dx$

$$\Rightarrow \frac{x^{n+1}}{2(n+1)}\Bigg|_0^1 \leqslant \int_0^1\frac{x^n}{x+1}dx \leqslant \frac{x^{n+1}}{n+1}\Bigg|_0^1 \Rightarrow \frac{1}{2(n+1)}\leqslant\int_0^1\frac{x^n}{x+1}dx\leqslant\frac{1}{n+1}$$

Ta có : $\begin{cases} \lim\limits_{n\to\infty} \dfrac{1}{2(n+1)} = 0 \\ \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0 \end{cases} \Rightarrow \lim\limits_{n\to\infty} \dfrac{x^n}{x+1} = 0$

2. $0 \leqslant x \leqslant 1 \Rightarrow 0 \leqslant e^{-x} \leqslant e^0 = 1 \Rightarrow 1 \leqslant 1 + e^{-x} \leqslant 2 \Rightarrow x^n \leqslant x^n\left(1+e^{-x}\right) \leqslant 2.x^n$ *hay* $0 \leqslant x^n\left(1+e^{-x}\right) \leqslant 2x^n$

$\Rightarrow 0 \leqslant \int_0^1 x^n\left(1+e^{-x}\right)dx \leqslant 2\int_0^1 x^n dx \Rightarrow 0 \leqslant \int_0^1 x^n\left(1+e^{-x}\right)dx \leqslant \dfrac{2}{n+1}$

Ta có : $\lim\limits_{n\to\infty} \dfrac{2}{n+1} = 0 \quad \Rightarrow \lim\limits_{n\to\infty} x^n\left(1+e^{-x}\right)dx = 0$

Chứng minh rằng **:**

1. $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x}(4-3\sqrt{\cos x})(2\sqrt{\cos x}+2)dx \leq 8\pi$ **2.** $\int_1^2 \sqrt{\ln x}(9-3\sqrt{\ln x}-2\ln x)dx \leq 8(e-1)$

3. $\int_{\pi/4}^{\pi/3} \sqrt{\sin x}(1+2\sqrt{\sin x})(5-3\sqrt{\sin x})dx < \dfrac{2\pi}{3}$ **4.** $\int_0^{\pi/4} \sqrt{tgx}(7-4\sqrt{tgx})dx \leq \dfrac{49\pi}{64}$

5. $\int_0^{\pi} \sin^4 x.\cos^6 x dx \leq \dfrac{243\pi}{6250}$

Bài giải **:**

Đặt $f(x) = \sqrt{\cos x}(4-3\sqrt{\cos x})(2\sqrt{\cos x}+2)$

$f(x) \overset{cauchy}{\leqslant} \left(\dfrac{\sqrt{\cos x}+4-3\sqrt{\cos x}+2\sqrt{\cos x}+2}{3}\right)^3 = 8$

$\Rightarrow \int_{-\pi/2}^{\pi/2} f(x)dx \leqslant 8\int_{-\pi/2}^{\pi/2} dx \Rightarrow \int_{-\pi/2}^{\pi/2} \sqrt{\cos x}(4-3\sqrt{\cos x})(2\sqrt{\cos x}+2)dx \leqslant 8\pi$

2. Đặt $f(x) = \sqrt{\ln x}(9-3\sqrt{\ln x}-2\ln x) = \sqrt{\ln x}(3+\sqrt{\ln x})(3-2\sqrt{\ln x})$

$f(x) \leqslant \left(\dfrac{\sqrt{\ln x}+3+\sqrt{\ln x}+3-2\sqrt{\ln x}}{3}\right)^3 = 8$

$\Rightarrow \int_1^e f(x)dx \leqslant 8\int_1^e dx \Rightarrow \int_1^e \sqrt{\ln x}(9-3\sqrt{\ln x}-2\ln x)dx \leqslant 8(e-1)$

3. Đặt $f(x) = \sqrt{\sin x}(1+2\sqrt{\sin x})(5-3\sqrt{\sin x})$; $f(x) \leqslant \left(\dfrac{\sqrt{\sin x}+1+2\sqrt{\sin x}+5-3\sqrt{\sin x}}{3}\right)^3 \leqslant 8$

Đẳng thức $\Leftrightarrow \begin{cases} \sqrt{\sin x} = 1+2\sqrt{\sin x} \\ \sqrt{\sin x} = 5-3\sqrt{\sin x} \end{cases} \Leftrightarrow \begin{cases} \sqrt{\sin x} = -1 \\ 4\sqrt{\sin x} = 5 \end{cases} \Leftrightarrow x \in \varnothing$

$\Rightarrow f(x) < 8 \Rightarrow \int_{\pi/4}^{\pi/3} f(x)dx < 8\int_{\pi/4}^{\pi/3} dx \quad \Rightarrow \int_{\pi/4}^{\pi/3} \sqrt{\sin x}(1+2\sqrt{\sin x})(5-3\sqrt{\sin x})dx < \dfrac{2\pi}{3}$

4. Đặt $f(x) = \sqrt{tgx}(7-4\sqrt{tgx}) = \dfrac{1}{4}.4\sqrt{tgx}(7-4\sqrt{tgx})$

$$f(x) \le \frac{1}{4}\left(\frac{4\sqrt{tgx}+7-4\sqrt{tgx}}{2}\right)^2 = \frac{49}{16}$$

$$\Rightarrow \int_0^{\Pi/4} f_{(x)}dx \leqslant \frac{49}{16}\int_0^{\Pi/4} dx \qquad \Rightarrow \int_0^{\Pi/4} \sqrt{tgx}\left(7-4\sqrt{tgx}\right)dx \leqslant \frac{49\Pi}{16}$$

5. $\sin^4 x.\cos^6 x = (1-\cos^2 x).(1-\cos^2 x).\cos^2 x.\cos^2 x.\cos^2 x$

$$= \frac{1}{2}(2-2\cos^2 x)(1-\cos^2 x).\cos^2 x.\cos^2 x.\cos^2 x$$

$$\le \frac{1}{2}\left(\frac{2-2\cos^2 x+1-\cos^2 x+\cos^2 x+\cos^2 x+\cos^2 x}{5}\right)^5$$

$$\Rightarrow \sin^4 x.\cos^6 x \le \frac{243}{6250} \Rightarrow \int_0^{\Pi} \sin^4 x.\cos^6 x dx \le \frac{243\Pi}{6250}$$

Chứng minh rằng **:**

1. $\int_{-\Pi/3}^{\Pi/2}\left(\sqrt{\cos^2 x+3\sin^2 x}+\sqrt{\sin^2 x+3\cos^2 x}\right)dx \leqslant \frac{5\Pi\sqrt{2}}{3}$
2. $\int_1^e\left(\sqrt{3+2\ln^2 x}+\sqrt{5-2\ln^2 x}\right)dx \leqslant 4(e-1)$
3. $-\frac{\Pi}{4} \leqslant \int \frac{\sqrt{3}\cos x+\sin x}{x^2+4}dx \leqslant \frac{\Pi}{4}$

Bài giải **:**

1. Đặt $f_{(x)} = 1\sqrt{\cos^2 x+3\sin^2 x}+1.\sqrt{\sin^2 x+3\cos^2 x}$

$f^2_{(x)} \leqslant 2\left(\cos^2 x+3\sin^2 x+3\cos^2 x+\sin^2 x\right) \Rightarrow f_{(x)} \leqslant 2\sqrt{2}$

$$\Rightarrow \int_{-\Pi/3}^{\Pi/2} f_{(x)}dx \leqslant 2\sqrt{2}\int_{-\Pi/3}^{\Pi/2} dx \Rightarrow \int_{-\Pi/3}^{\Pi/2}\left(\sqrt{\cos^2 x+3\sin^2 x}+\sqrt{\sin^2 x+3\cos^2 x}\right)dx \leqslant \frac{5\Pi\sqrt{2}}{3}$$

2. Đặt $f_{(x)} = 1\sqrt{3+2\ln^2 x}+1\sqrt{5-2\ln^2 x}$

$f_{(x)}^2 \le 2\left(3+2\ln^2 x+5-2\ln^2 x\right) \Rightarrow f_{(x)} \le 4$

$$\Rightarrow \int_1^e f_{(x)}dx \leqslant 4\int_1^e dx \Rightarrow \int_1^e\left(\sqrt{3+2\ln^2 x}+\sqrt{5-2\ln^2 x}\right)dx \le 4(e-1)$$

3. $\left|\sqrt{3}\cos x+\sin x\right| \le \sqrt{\left[(\sqrt{3})^2+1\right]\left(\cos^2 x+\sin^2 x\right)}$

$$\Rightarrow \frac{\left|\sqrt{3}\cos x+\sin x\right|}{x^2+4} \le \frac{2}{x^2+4} \Rightarrow \int_0^2 \frac{\left|\sqrt{3}\cos x+\sin x\right|}{x^2+4} \le 2\int_0^2 \frac{dx}{x^2+4}$$

Đặt $x = 2tgt \Rightarrow dx = 2\left(1+tg^2t\right)dt$

x	0	1
t	0	$\Pi/4$

$$\Rightarrow \int_0^2 \frac{dx}{x^2+4} = \int_0^{\Pi/4} \frac{2\left(1+tg^2t\right)}{4\left(1+tg^2t\right)}dt = \frac{1}{2}\int_0^{\Pi/4} dt = \frac{\Pi}{8}$$

$$\Rightarrow \int_0^2 \frac{\left|\sqrt{3}\cos x + \sin x\right|}{x^2+4}dx \leqslant \frac{\Pi}{4} \Rightarrow -\frac{\Pi}{4} \leqslant \int_0^2 \frac{\sqrt{3}\cos x + \sin x}{x^2+4}dx \leqslant \frac{\Pi}{4}$$

ĐÁNH GIÁ TÍCH PHÂN DỰA VÀO TẬP GIÁ TRỊ CỦA HÀM DƯỚI DẤU TÍCH PHÂN

Chứng minh rằng **:**

1. $\int_0^{\Pi/4} \sin 2x dx \le 2\int_0^{\Pi/4} \cos x dx$
2. $\int_0^{\Pi/2} \sin 2x dx \leqslant 2\int_0^{\Pi/2} \sin x dx$
3. $\int_1^2 \frac{x-1}{x} dx < \int_1^2 \frac{2x-1}{x+1} dx$
4. $\int_0^{\Pi/2} \frac{\sin x}{x} dx > \int_{\Pi/2}^{\Pi} \frac{\sin x}{x} dx$
5. $\int_1^2 (\ln x)^2 dx < \int_1^2 \ln x dx$
6. $\int_0^{\Pi/4} \sin x dx < \int_0^{\Pi/4} \cos x dx$

Bài giải **:**

$$1. \forall x \in \left[0;\frac{\Pi}{4}\right] \Rightarrow \begin{cases} 0 \le \sin x \le 1 \\ 0 \le \cos x \le 1 \end{cases} \Rightarrow 2\sin x.\cos x \le 2\cos x$$

$$\Leftrightarrow \sin 2x \le 2\cos x \qquad \Rightarrow \int_0^{\Pi/4} \sin 2x dx \le 2\int_0^{\Pi/4} \cos x dx$$

2. $\forall x \in \left[0;\frac{\Pi}{2}\right] \Rightarrow \begin{cases}\cos x \le 1\\ 0 \le \sin x\end{cases} \Rightarrow 2\sin 2x.\cos x \le 2\sin x$

$\Leftrightarrow \sin 2x \le 2\sin x \Rightarrow \int_0^{\Pi/2} \sin 2xdx \le 2\int_0^{\Pi/2} \sin xdx$

3. $\forall x \in [1;2]$ **Xét hiệu :** $\frac{x-1}{x} - \frac{2x-1}{x+1} = \frac{-x^2+x-1}{x(x+1)} < 0$

$\Rightarrow \frac{x-1}{x} < \frac{2x-1}{x+1} \Rightarrow \int_1^2 \frac{x-1}{x}dx < \int_1^2 \frac{2x-1}{x+1}dx$

4. Đặt $x = \Pi\text{-}u \quad \Rightarrow dx = \text{-}du$

x	$\Pi/2$	Π
u	$\Pi/2$	0

$\Rightarrow \int_{\Pi/2}^{\Pi} \frac{\sin x}{x}dx = \int_{\Pi/2}^{0} \frac{\sin(\Pi-u)}{\Pi-u}(-du) = \int_0^{\Pi/2} \frac{\sin x}{\Pi-x}dx$

$0 < x < \frac{\Pi}{2} \Rightarrow 0 < x < \Pi - x \Rightarrow \frac{1}{\Pi-x} < \frac{1}{x}$

Vì : $\sin x > 0 \Rightarrow \frac{\sin x}{\Pi-x} < \frac{\sin x}{x} \Rightarrow \int_0^{\Pi/2} \frac{\sin x}{\Pi-x}dx < \int_{\Pi}^{\Pi/2} \frac{\sin x}{x}dx$

$\Rightarrow \int_0^{\Pi/2} \frac{\sin x}{x}dx > \int_{\Pi/2}^{\Pi} \frac{\sin x}{x}dx$

5. Hàm số y = f(x) = lnx liên tục trên [1,2] nên y = g(x) = $(\ln x)^2$ cũng liên tục trên [1,2]

$1 \leqslant x \leqslant 2 \Rightarrow 0 \leqslant \ln x \leqslant \ln 2 < 1\ (*) \Rightarrow 0 \leqslant (\ln x)^2 < \ln x$

$\forall x \in [1,2] \Rightarrow \int_1^2 (\ln x)^2 dx < \int_1^2 \ln xdx$

Chú ý : dấu đẳng thức (*) xảy ra tại $x_0 = 1 \subset [1,2]$

6. $0 < x < \frac{\Pi}{4} \Rightarrow 0 < tgx < tg\frac{\Pi}{4} = 1 \Leftrightarrow \frac{\sin x}{\cos x} < 1$

$\Leftrightarrow \sin x < \cos x \Leftrightarrow \int_0^{\Pi/4} \sin xdx < \int_0^{\Pi/4} \cos xdx$

Chứng minh rằng **:**

1. $2 \leqslant \int_0^1 \sqrt{x^2+4}\,dx \leqslant \sqrt{5}$
2. $\frac{1}{\sqrt{2}} \leqslant \int_0^1 \frac{1}{\sqrt{x^8+1}}dx \leqslant 1$
3. $\frac{1}{26\sqrt[3]{2}} \leqslant \int_0^1 \frac{x^{25}}{\sqrt[3]{x^{10}+1}}dx \leqslant \frac{1}{26}$
4. $\int_0^1 \frac{x.\sin x}{1+x.\sin x}dx \leqslant 1 - \ln 2$
5. $0 < \int_1^{\sqrt{3}} \frac{e^{-x}.\sin x}{x^2+1}dx \leqslant \frac{\Pi}{12e}$
6. $\frac{\Pi}{6} \leqslant \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leqslant \frac{\Pi\sqrt{2}}{8}$

***Bài Giải*:**

1. $0 \le x \le 1 \Rightarrow 0 \le x^2 \le 1 \Rightarrow 4 \le x^2 + 4 \le 5 \Rightarrow 2 \leqslant \sqrt{x^2+4} \le \sqrt{5}$

$\Rightarrow 2\int_0^1 dx \le \int_0^1 \sqrt{x^2+4}\,dx \le \sqrt{5}\int_0^1 dx \Rightarrow 2 \le \int_0^1 \sqrt{x^2+4}\,dx \le \sqrt{5}$

2. $0 \le x \le 1 \Rightarrow 0 \le x^8 \le 1 \Rightarrow 1 \le x^8 + 1 \le 2$

$\Rightarrow 0 \le \sqrt{x^8+1} \le \sqrt{2} \Rightarrow \frac{1}{\sqrt{2}} \le \frac{1}{\sqrt{x^8+1}} \le 1$

$\Rightarrow \frac{1}{\sqrt{2}}\int_0^1 dx \le \int_0^1 \frac{dx}{\sqrt{x^8+1}} \le \int_0^1 dx \Rightarrow \frac{1}{\sqrt{2}} \le \int_0^1 \frac{dx}{\sqrt{x^8+1}} \le 1$

3. $0 \le x \le 1 \Rightarrow 1 \leqslant x^{10} + 1 \leqslant 2 \Rightarrow 1 \leqslant \sqrt[3]{x^{10}+1} \leqslant \sqrt[3]{2}$

$\Rightarrow \frac{1}{\sqrt[3]{2}} \leqslant \frac{1}{\sqrt[3]{x^{10}+1}} \leqslant 1 \Leftrightarrow \frac{x^{25}}{\sqrt[3]{2}} \leqslant \frac{x^{25}}{\sqrt[3]{x^{10}+1}} \leqslant x^{25}$

$\Rightarrow \frac{1}{\sqrt[3]{2}}\int_0^1 x^{25}dx \leqslant \int_0^1 \frac{x^{25}}{\sqrt[3]{x^{10}+1}}dx \leqslant \int_0^1 x^{25}dx \Rightarrow \frac{1}{26\sqrt[3]{2}} \leqslant \int_0^1 \frac{x^{25}}{\sqrt[3]{x^{10}+1}}dx \leqslant \frac{1}{26}$

4. Trước hết ta chứng minh : $\frac{x\sin x}{1+x\sin x} \leqslant \frac{x}{1+x};(1)\ \forall x \in [0,1].$

Giả sử ta có : (1).

$(1) \Leftrightarrow 1 - \frac{1}{1+x\sin x} \leqslant 1 - \frac{1}{1+x}; \forall x[0.1] \Leftrightarrow \frac{1}{1+x\sin x} \geqslant \frac{1}{1+x}$

$\Leftrightarrow 1 + x \geqslant 1 + x.\sin x \Leftrightarrow x(1-\sin x) \geqslant 0$ **đúng** $\forall x \in [0,1]$

$$(1) \Leftrightarrow \int_0^1 \frac{x\sin x}{x + x\sin x}dx \leqslant \int_0^1 \frac{x}{1+x}dx = \int_0^1 \left(1 - \frac{1}{1+x}\right)dx$$

Vậy (1) đẳng thức đúng , khi đó: $\Leftrightarrow \int_0^1 \frac{x.\sin x}{1+x\sin x}dx \leqslant \left(x - \ln|1+x|\right)\Big|_0^1 = 1 - \ln 2$

$$\Rightarrow \int_0^1 \frac{x.\sin x}{1+x.\sin x}dx \leqslant 1 - \ln 2.$$

5. $x \in \left[1,\sqrt{3}\right] \subset (0,\Pi) \Rightarrow \begin{cases} 0 < e^{-x} = \frac{1}{e^x} \leqslant \frac{1}{e} \\ 0 < \sin x < 1 \end{cases} \Rightarrow 0 < \frac{e^{-x}\sin x}{x^2+1} < \frac{1}{e\left(x^2+1\right)}$

$\Rightarrow 0 < \int_1^{\sqrt{3}} \frac{e^{-x}\sin x}{x^2+1}dx < \frac{1}{e}\int_1^{\sqrt{3}} \frac{dx}{x^2+1} = \frac{1}{e}I \quad ; I = \int_1^{\sqrt{3}} \frac{dx}{x^2+1}$

Đặt $x = tgt \Rightarrow dx = \frac{1}{\cos^2 t}dt = (1+tg^2 t)dt$

x	1	$\sqrt{3}$
t	$\Pi/4$	$\Pi/4$

$$\Rightarrow \mathrm{I} = \int_{\Pi/4}^{\Pi/3} \frac{\left(1+tg^2 t\right)}{1+tg^2 t}dt = \int_{\Pi/4}^{\Pi/3} dt = t\Big|_{\Pi/4}^{\Pi/3} = \frac{\Pi}{12}$$

Vậy $0 < \int_1^{\sqrt{3}} \frac{e^{-x}\sin x}{x^2+1}dx < \frac{\Pi}{12e}$

6. $0 \leqslant x \leqslant 1 \Rightarrow 0 \leqslant x^3 \leqslant x^2 \Rightarrow -x^2 \leqslant -x^3 \leqslant 0$

$\Rightarrow 4-2x^2 \leqslant 4-x^2-x^3 \leqslant 4-x^2$

$\Rightarrow \sqrt{4-2x^2} \leqslant \sqrt{4-x^2-x^3} \leqslant \sqrt{4-x^2}$

$\Rightarrow \frac{1}{\sqrt{4-2x^2}} \geqslant \frac{1}{\sqrt{4-x^2-x^3}} \geqslant \frac{1}{\sqrt{4-x^2}}$

$\Rightarrow I = \int_0^1 \frac{1}{\sqrt{4-x^2}}dx \leqslant \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}dx \leqslant \int_0^1 \frac{1}{\sqrt{4-2x^2}}dx = J$

Đặt $x = 2\sin t \Rightarrow dx = 2\cos tdt$

x	0	1
t	0	$\Pi/6$

$$\Rightarrow I = \int_0^{\Pi/6} \frac{2\cos tdt}{\sqrt{4-\left(2\sin t\right)^2}} = \int_0^{\Pi/6} dt = \frac{\Pi}{6}$$

Đặt $x = \sqrt{2}\sin t \Rightarrow dx = \sqrt{2}\cos tdt$

x	0	1
t	0	$\Pi/4$

$$\Rightarrow J = \int_0^{\Pi/4} \frac{\sqrt{2}\cos tdt}{\sqrt{4-2\left(\sqrt{2}\sin t\right)^2}} = \frac{\sqrt{2}}{2}\Big|_0^{\Pi/4} = \frac{\Pi\sqrt{2}}{8}$$

$$\Rightarrow \frac{\Pi}{6} \le \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \le \frac{\Pi\sqrt{2}}{8}$$

Chứng minh rằng **:**

1. $\frac{e-1}{e} \leqslant \int_0^1 e^{-x^2}dx \leqslant 1$

2. $\frac{\Pi}{2} \leqslant \int_0^{\Pi/2} e^{\sin^2 x}dx \leqslant \frac{\Pi}{2}e$

3. $\frac{\Pi}{2} \le \int_0^{\Pi/2} \sqrt{1+\frac{1}{2}\sin^2 x}\,.dx \le \frac{\Pi\sqrt{6}}{4}$

4. $0.88 < \int_0^1 \frac{1}{\sqrt{1+x^4}}dx < 1$

Bài giải **:**

1. • $0 \leqslant x \leqslant 1 \Rightarrow 0 \leqslant x^2 \leqslant x \leqslant 1 \Rightarrow 0 < e^{x^2} \leqslant e^x$

$\Rightarrow \frac{1}{e^{x^2}} \geqslant \frac{1}{e^x} \Leftrightarrow e^{-x^2} \geqslant e^{-x} \ (1)$

• $x^2 \geqslant 0 \Rightarrow e^{x^2} \geqslant e^0 = 1 \Rightarrow e^{-x^2} \leqslant 1 \ (2)$

Từ (1) và (2) suy ra $: e^{-x} \leqslant e^{-x^2} \leqslant 1$

$\Rightarrow \int_0^1 e^{-x^2} dx \leqslant \int_0^1 e^{-x^2} dx \leqslant \int_0^1 dx \Rightarrow \frac{e-1}{e} \leqslant \int_0^1 e^{-x^2} dx \leqslant 1$

2. $0 \leqslant \sin^2 x \leqslant 1 \Rightarrow 1 \leqslant e^{\sin^2 x} \leqslant e$

$\Rightarrow \int_0^{\Pi/2} dx \leqslant \int_0^{\Pi/2} e^{\sin^2 x} dx \leqslant e.\int_0^{\Pi/2} dx \Rightarrow \frac{\Pi}{2} \leqslant \int_0^{\Pi/2} e^{\sin^2 x} dx \leqslant \frac{\Pi}{2} e$

3. $0 \leqslant \sin^2 x \leqslant 1 \Rightarrow 0 \leqslant \frac{1}{2}\sin^2 x \leqslant \frac{1}{2} \Rightarrow 1 \leqslant \sqrt{1 + \frac{1}{2}\sin^2 x} \leqslant \sqrt{\frac{3}{2}}$

$\Rightarrow \int_0^{\Pi/2} dx \leqslant \int_0^{\Pi/2} \sqrt{1 + \frac{1}{2}\sin^2 x}\, dx \leqslant \sqrt{\frac{3}{2}} \int_0^{\Pi/2} dx \Rightarrow \frac{\Pi}{2} \leqslant \int_0^{\Pi/2} \sqrt{1 + \frac{1}{2}\sin^2 x}\, .dx \leqslant \frac{\Pi\sqrt{6}}{4}$

4. ***Cách 1*:**

$\forall x \in (0,1)$ **thì** $x^4 < x^2 \Rightarrow 1 + x^4 < 1 + x^2 \Rightarrow \frac{1}{\sqrt{1+x^4}} > \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \int_0^1 \frac{1}{\sqrt{1+x^4}} dx > \int_0^1 \frac{1}{\sqrt{1+x^2}} dx = \ln\left| x + \sqrt{1+x^2} \right| \Big|_0^1 = \ln\left(1 + \sqrt{2}\right) > 0,88$

Mặt khác : $1 + x^4 > 1 \Rightarrow \frac{1}{\sqrt{1+x^4}} < 1 \Rightarrow \int_0^1 \frac{1}{\sqrt{1+x^4}} dx < 1$

Vậy : $0,88 < \int_0^1 \frac{1}{\sqrt{1+x^4}} dx < 1$

***Chú ý* : học sinh tự chứng minh** $\int \frac{1}{\sqrt{a^2+x^2}} dx = \ln\left| x + \sqrt{x^2+a^2} \right| + C$ **bằng phương pháp tích phân từng phần .**

***Cách 2* :**

$x \in (0,1) \Rightarrow x^4 < x^2 \Rightarrow 1 + x^4 < 1 + x^2$

$\Rightarrow \frac{1}{\sqrt{1+x^4}} > \frac{1}{\sqrt{1+x^2}} \Rightarrow \int_0^1 \frac{1}{\sqrt{1+x^4}} dx > I$

Với : $I = \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$

Đặt $x = tgt \Rightarrow dx = \frac{1}{\cos^2} dt = \left(1 + tg^2 t\right) dt$

x	0	1
t	0	$\Pi/4$

$$I=\int_0^{\Pi/4}\frac{\left(1+tg^2t\right)}{\sqrt{\left(1+tg^2t\right)}}dt=\int_0^{\Pi/4}\frac{1}{\cos t}dt$$

$$I=\int_0^{\Pi/4}\frac{\cos t}{1-\sin^2 t}dt$$

Đặt $u=\sin t\Rightarrow du=\cos tdt$

t	0	$\Pi/4$
u	0	$1/\sqrt{2}$

$$I=\int_0^{1/\sqrt{2}}\frac{du}{1-u^2}=\frac{1}{2}\int_0^{1/\sqrt{2}}\frac{1-u+u+1}{(1-u)(1+u)}du=\frac{1}{2}\int_0^{1/\sqrt{2}}\left(\frac{1}{1+u}+\frac{1}{1-u}\right)du$$

$$=\frac{1}{2}\int_0^{1/\sqrt{2}}\frac{1}{1+u}du+\frac{1}{2}\int_0^{1/\sqrt{2}}\frac{1}{1-u}du=\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|\Bigg|_0^{1/\sqrt{2}}$$

$$I=\frac{1}{2}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}}>0,88\Rightarrow\int_0^1\frac{1}{\sqrt{1+x^4}}dx>0,88$$

Mặt khác $:1+x^4>1\Rightarrow\frac{1}{\sqrt{1+x^4}}<1$

$$\Rightarrow\int_0^1\frac{1}{\sqrt{1+x^4}}dx<\int_0^1dx=1\quad(2)$$

Từ (1) và (2) suy ra : $0.88<\int_0^1\frac{1}{\sqrt{1+x^4}}dx<1$

<u>Chứng minh rằng</u> **:**

1. $0<\int_0^{\Pi/4}x\sqrt{tgx}\,dx<\frac{\Pi^2}{32}$

2. $\left|\int_0^1\frac{\cos nx}{1+x}dx\right|\leqslant\ln 2$

3. $\left|\int_1^{\sqrt{3}}\frac{e^{-x}.\sin x}{1+x^2}dx\right|<\frac{\Pi}{12e}$

4. $\left|\int_1^{\sqrt{3}}\frac{e^{-x}\cos x}{1+x^2}dx\right|<\frac{\Pi}{12e}$

5. $\int_{100\Pi}^{200\Pi}\frac{\cos x}{x}dx\leqslant\frac{1}{200\Pi}$

6. $\frac{1}{n-1}\left(1-\frac{1}{2^{n-1}}\right)\leqslant\int_0^1\frac{e^x}{(1+x)^n}dx\leqslant\frac{e}{n-1}\left(1-\frac{1}{2^{n-1}}\right)$

<u>Bài giải</u> **:**

1. $0\leqslant x\leqslant\frac{\Pi}{4}\Rightarrow 0\leqslant tgx\leqslant 1\Rightarrow 0\leqslant\sqrt{tgx}\leqslant 1\Rightarrow 0\leqslant x\sqrt{tgx}\leqslant x$

Xét $:0<\alpha<x<\beta<\frac{\Pi}{4}$ **ta có :**

$$\left.\begin{array}{l}0<tgx<1\\0<x<\frac{\Pi}{4}\end{array}\right\}\Rightarrow 0<x\sqrt{tgx}\leqslant x$$

$$I=\int_0^{\Pi/4}x\sqrt{tgx}\,dx=\int_0^{\alpha}x\sqrt{tgx}\,dx+\int_{\alpha}^{\beta}x\sqrt{tgx}\,dx+\int_{\beta}^{\Pi/4}x\sqrt{tgx}\,dx$$

Ta có :

$$\left.\begin{array}{l} 0 \leqslant \int_0^{\alpha} x\sqrt{tgx}\,dx \leqslant \int_0^{\alpha} xdx \\ 0 < \int_{\alpha}^{\beta} x\sqrt{tgx}\,dx < \int_{\alpha}^{\beta} xdx \\ 0 \leqslant \int_{\beta}^{\Pi/4} x\sqrt{tgx}\,dx \leqslant \int_{\beta}^{\Pi/4} xdx \end{array}\right\} \Rightarrow 0 \leqslant \int_0^{\Pi/4} x\sqrt{tgx}\,dx < \int_0^{\Pi/4} xdx$$

$$\Rightarrow 0 < \int_0^{\Pi/4} x\sqrt{tgx}\,dx < \frac{\Pi^2}{32}$$

Chú ý : $(\alpha,\beta) \subset [a,b]$ **thì** $\int_a^b f_{(x)}dx = \int_b^{\alpha} f_{(x)}dx + \int_{\alpha}^{\beta} f_{(x)}dx + \int_{\beta}^{b} f_{(x)}dx$

Tuy nhiên nếu : $m \leqslant f_{(x)} \leqslant M$ **thì :**

$$m\int_a^b dx \leqslant \int_a^b f_{(x)}dx \leqslant M\int_a^b dx \Rightarrow m(b-a) \leqslant \int_a^b f_{(x)}dx \leqslant M(b-a)$$

Nhưng $(\alpha,\beta) \subset [a,b]$ **thì** $m\int_a^b dx < \int_a^b f_{(x)}dx < M\int_a^b f_{(x)}dx$

(Đây là phần mắc phải sai lầm phổ biến nhất)Do chưa hiểu hết ý nghĩa hàm số $f_{(x)}$ **chứa** (α,β) **liên tục** $[a,b]$ **mà** $(\alpha,\beta) \subset [a,b]$**)**

2. $\left|\int_0^1 \frac{\cos nx}{1+x}dx\right| \leqslant \int_0^1 \left|\frac{\cos nx}{1+x}\right| dx = \int_0^1 \frac{|\cos nx|}{1+x}dx \leqslant \int_0^1 \frac{1}{1+x} = \ln|1+x|\Big|_0^1 = \ln 2$

$$\Rightarrow \left|\int_0^1 \frac{\cos nx}{1+x}dx\right| \leqslant \ln 2$$

3. $1 \leqslant x \leqslant \sqrt{3} \Rightarrow \begin{cases} e^{-x} \leqslant e^{-1} = 1/e \\ |\sin x| \leqslant 1 \end{cases}$

$$\Rightarrow \left|\int_1^{\sqrt{3}} \frac{e^{-x}.\sin x}{1+x^2}dx\right| \leqslant \int^{\sqrt{3}} \left|\frac{e^{-x}.\sin x}{1+x^2}\right| dx \leqslant \int_1^{\sqrt{3}} \frac{1/e}{1+x^2}dx$$

$$\Rightarrow \left|\int_1^{\sqrt{3}} \frac{e^{-x}.\sin x}{1+x^2}dx\right| \leqslant \frac{1}{e}.I \qquad \textbf{với } I = \int_1^{\sqrt{3}} \frac{1}{1+x^2}dx$$

Đặt $x = tgt \Rightarrow dx = (1+tg^2t)dt$

x	1	$\sqrt{3}$
t	$\Pi/4$	$\Pi/3$

$$\Rightarrow \mathrm{I} = \int_{\Pi/4}^{\Pi/3} \frac{(1+tg^2t)}{1+tg^2t}dt = \int_{\Pi/4}^{\Pi/3} dt = \frac{\Pi}{12}$$

$\Rightarrow \left|\int_1^{\sqrt{3}} \frac{e^{-x}.\sin x}{1+x}dx\right| \leqslant \frac{\Pi}{12e}(*)$ **(Cách 2 xem bài 4 dưới đây)**

Đẳng thức xảy ra khi :

$$\begin{cases} e^{-x} = e^{-1} \\ \sin x = 1 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ \sin x = 1 \end{cases} \Rightarrow x \in \varnothing, \forall x \in \left[1, \sqrt{3}\right]$$

Vậy : $\left|\int_1^{\sqrt{3}} \frac{e^{-x}.\sin x}{1+x^2} dx\right| < \frac{\Pi}{12e}$

Xem lại chú ý trên , đây là phần sai lầm thường mắc phải không ít người đã vội kết luận đẳng thức (*) đúng . Thật vô lý

4. $\left|\int_1^{\sqrt{3}} \frac{e^{-x}\cos x}{1+x^2} dx\right| \leqslant \int_1^{\sqrt{3}} \left|\frac{e^{-x}\cos x}{1+x^2}\right| dx \leqslant \int_1^{\sqrt{3}} \frac{e^{-x}}{1+x^2} dx$

Do $y = e^{-x}$ **giảm** $\Rightarrow \max\left(e^{-x}\right) = e^{-1} = \frac{1}{e}$

$\Rightarrow \left|\int_1^{\sqrt{3}} \frac{e^{-x}\cos x}{1+x^2} dx\right| \leqslant \frac{1}{e}\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx = \frac{\Pi}{12e}$ **;do I bài 3**

Dấu đẳng thức :

$$\begin{cases} e^{-x} = e^{-1} \\ \cos x = 1 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ \cos x = 1 \end{cases} \Leftrightarrow x \in \varnothing, \forall x \in \left[1, \sqrt{3}\right]$$

Vậy $\left|\int_1^{\sqrt{3}} \frac{e^{-x}\cos x}{1+x^2} dx\right| < \frac{\Pi}{12e}$

5. Đặt $\begin{cases} u = 1/x \\ dv = \cos x dx \end{cases} \Rightarrow \begin{cases} du = -1/x^2\, dx \\ v = \sin x \end{cases}$

$\Rightarrow \int_{100\Pi}^{200\Pi} \frac{\cos x}{x} dx = \frac{1}{x}\sin x \Big|_{100\Pi}^{200\Pi} + \int_{100\Pi}^{200\Pi} \frac{\sin x}{x^2} dx$

$\Rightarrow \int_{100\Pi}^{200\Pi} \frac{\cos x}{x} dx \leqslant \int_{100\Pi}^{200\Pi} \frac{1}{x^2} dx = -\frac{1}{x}\Big|_{100\Pi}^{200\Pi} = \frac{1}{200\Pi}$

Vậy $\int_{100\Pi}^{200\Pi} \frac{\cos x}{x} dx \leqslant \frac{1}{200\Pi}$

Bài toán này có thể giải theo phương pháp đạo hàm .

6. $0 \leqslant x \leqslant 1 \Rightarrow 1 \leqslant e^x \leqslant e \Rightarrow \frac{1}{(1+x)^n} \leqslant \frac{e^x}{(1+x)^n} \leqslant \frac{e}{(1+x)^n}$

$\Rightarrow \int_0^1 \frac{1}{(1+x)^n}dx \leqslant \int_0^1 \frac{e^x}{(1+x)^n}dx \leqslant e\int_0^1 \frac{1}{(1+x)^n}dx$

$\Leftrightarrow \left.\frac{(x+1)^{1-n}}{1-n}\right|_0^1 \leqslant \int_0^1 \frac{e^x}{(1+x)^n}dx \leqslant e.\left.\frac{(x+1)^{1-n}}{1-n}\right|_0^1$

Vậy $: \frac{1}{n-1}\left(1-\frac{1}{2^{n-1}}\right) \leqslant \int_0^1 \frac{e^x}{(1+x)^n}dx \leqslant \frac{e}{n-1}\left(1-\frac{1}{2^{n-1}}\right); n>1$

Bài toán này có thể giải theo phương pháp nhị thức Newton .

<u>Chứng minh rằng</u> **: nếu f(x) và g(x) là 2 hàm số liên tục và x xác định trên [a,b] , thì ta có :**

$\left(\int_a^b f_{(x)}.g_{(x)}.dx\right)^2 \leqslant \int_a^b f^2{}_{(x)}dx \,.\, \int_a^b g^2{}_{(x)}dx$

***Cách 1* :**

Cho các số α_1 **, tuỳ ý** $\left(i \in \overline{1,n}\right)$ **ta có :**

$\left(\alpha^2{}_1+\alpha^2{}_2+...+\alpha^2{}_n\right)\left(\beta^2{}_1+\beta^2{}_2+...+\beta^2{}_n\right) \geqslant \left(\alpha_1\beta_1+\alpha_2\beta_2+...+\alpha_n\beta_n\right)$ (1)

Đẳng thức (1) xảy ra khi : $\frac{\alpha_1}{\beta_1}=\frac{\alpha_2}{\beta_2}=...\frac{\alpha_n}{\beta_n}$

Thật vậy : phân hoạch [a,b] thành n đoạn nhỏ bằng nhau bởi các điểm chia :

a = $x_0 < x_1 < x_2 < .... < x_n$ **= b và chọn :**

$\xi_1 \in \left[x_{i-1}, x_i\right] = \frac{b-a}{n} \quad \forall i \in \overline{i,n}$

Do f và g liên tục , ta có :

$$\begin{cases} \int_a^b f^2{}_{(x)}dx = \lim\limits_{n\to+\infty}\sum\limits_{i=1}^n f^2\left(\xi_i\right)\frac{b-a}{n} \quad (2) \\ \int_a^b g^2{}_{(x)}dx = \lim\limits_{n\to+\infty}\sum\limits_{i=1}^n g^2\left(\xi_i\right)\frac{b-a}{n} \quad (3) \\ \qquad\qquad n\to\infty \end{cases}$$

Khi đó (1)

$\Leftrightarrow \lim\limits_{n\to+\infty}\sum\limits_{i=1}^n f^2\left(\xi_i\right)\frac{b-a}{n} \;.\; \lim\limits_{n\to+\infty}\sum\limits_{i=1}^n g^2\left(\xi_i\right)\frac{b-a}{n}.$

$\geqslant \left[\lim\limits_{n\to+\infty}\sum\limits_{i=1}^n f\left(\xi_i\right).g\left(\xi_i\right)\frac{b-a}{n}.\right]^2$ (4)

Từ (4) ta cũng có :

$\sum\limits_{i=1}^n f^2\left(\xi_i\right)\sum\limits_{i=1}^n g^2\left(\xi_i\right). \geqslant \left[\sum\limits_{i=1}^n f\left(\xi_i\right)\sum\limits_{i=1}^n g\left(\xi_i\right).\right]^2$ (5)

Đẳng thức xảy ra khi : f(x):g(x) = k hay f(x) = k.g(x)

Từ (5) $\Rightarrow\left(\int_a^b f(x).g(x)dx\right)^2 \leqslant \int_a^b f^2(x)dx\,.\int_a^b g^2(x)dx$

Cách 2 **:** $\forall t\in R^+$ **ta có :**

$0\leqslant\left[tf(x)-g(x)\right]^2 = t^2f^2(x)-2.t.f(x).g(x)+g^2(x)$

$\Rightarrow h(t)=t^2\int_a^b f^2(x)dx-2t\int_a^b f(x).g(x)dx+\int_a^b g^2(x)dx\geqslant 0$

h(t) là 1 tam thức bậc 2 luôn không âm nên cần phải có điều kiện :

$\begin{cases} a_h = t^2>0 \\ \Delta_h\leqslant 0\end{cases} \Leftrightarrow \Delta'_h\leqslant 0$

$\Leftrightarrow\left[\int_a^b f(x).g(x)dx\right]^2-\int_a^b f^2(x)dx\,.\int_a^b g^2(x)dx\le 0$

$\Rightarrow\left(\int_a^b f(x).g(x)dx\right)^2\leqslant\int_a^b f^2(x)dx\,.\int_a^b g^2(x)dx$

<u>Chứng minh rằng</u> **:**

1. $\int_0^1\sqrt{1+x^3}dx<\frac{\sqrt5}{2}$

2. $\int_0^1 e^{\sin^2 x}\,dx>\frac{3\Pi}{2}$

3. $e^x-1<\int_0^x\sqrt{e^{2t}+e^{-t}}dt<\sqrt{\left(e^x-1\right)\left(e^x-\frac12\right)}$

4. $\left|\int_0^1\frac{3\cos x-4\sin x}{1+x^2}dx\right|\leqslant\frac{5\Pi}{4}$

<u>Bài giải</u> **:**

1. Ta có $:\left(\int_a^b f(x).g(x)dx\right)^2\leqslant\int_a^b f^2(x)dx\,.\int_a^b g^2(x)dx$ **(đã chứng minh bài trước)**

$\Rightarrow\left|\int_a^b f(x).g(x)dx\right|\leqslant\sqrt{\int_a^b f^2(x)dx}\,.\sqrt{\int_a^b g^2(x)dx}$

$\sqrt{1+x^3}=\sqrt{(1+x).(1-x+x^2)}=\sqrt{(1+x)}.\sqrt{(1-x+x^2)}$

$\Rightarrow\int_0^1\sqrt{1+x^3}dx=\int_0^1\sqrt{(1+x)}\sqrt{(1-x+x^2)}\,dx<\sqrt{\int_0^1(1+x)\,dx}\sqrt{\int_0^1(x^2-x+1)\,dx}$

$\int_0^1\sqrt{1+x^3}dx<\sqrt{\left(\frac{x^2}{2}+x\right)\Big|_0^1}\sqrt{\left(\frac{x^3}{3}-\frac{x^2}{2}+x\right)\Big|_0^1}=\frac{\sqrt5}{2}$

$\Rightarrow\int_0^1\sqrt{1+x^3}dx<\frac{\sqrt5}{2}$

2. $\int_0^\Pi e^{\sin^2 x}\,dx=\int_0^{\Pi/2}e^{\sin^2 x}\,dx+\int_0^{\Pi/2}e^{\sin^2 x}\,dx$

Đặt $t=\frac{x}{2}+t\Rightarrow dx=dt$

x	$\Pi/2$	Π
t	0	$\Pi/2$

$\Rightarrow \int_0^{\Pi} e^{\sin^2 x}\, dx = \int_0^{\Pi/2} e^{\sin^2 x}\, dx + \int_0^{\Pi/2} e^{\sin^2(\Pi/2+t)}\, dt$

$= \int_0^{\Pi/2} e^{\sin^2 x}\, dx + \int_0^{\Pi/2} e^{\cos^2 x}\, dx = 2\int_0^{\Pi/2} e^{\sin^2 x}\, dx$

Ta lại có $\left(\int_0^{\Pi/2} \sqrt{e}dx\right)^2 = \left(\int_0^{\Pi/2} e^{\sin^2 x/2}.e^{\cos^2 x/2} dx\right)^2$

$< \int_0^{\Pi/2} e^{\sin^2 x}dx \,.\, \int_0^{\Pi/2} e^{\cos^2 x}dx$

$hay \left(\int_0^{\Pi/2} \sqrt{e}\, dx\right)^2 < \left(\int_0^{\Pi/2} e^{\sin^2 x}dx\right)^2 \Rightarrow \int_0^{\Pi/2} \sqrt{e}\, dx < \int_0^{\Pi/2} e^{\sin^2 x}dx$

$\Rightarrow \int_0^{\Pi} e^{\sin^2 x}dx > \frac{1}{2}\sqrt{e}\Big|_0^{\Pi/2} = \Pi\sqrt{e}\,;\left(\sqrt{e} > \frac{3}{2}\right)$

$\Rightarrow \int_0^{\Pi} e^{\sin^2 x}dx > \frac{3}{2}$

Chú ý : bài này có thể giải theo phương pháp đạo hàm .

3. $\int_0^x \sqrt{e^{2t}+e^{-t}}\,dt = \int_0^x e^{t/2}\sqrt{e^t+e^{-2t}}\,dt$

$\left(\int_0^x e^{t/2}\sqrt{e^t+e^{-2t}}\,dt\right)^2 \leqslant \int_0^t e^t dt \int_0^t \left(e^t+e^{-2t}\right)dt$

$vi \left(\int_a^b f(x).g(x)dx\right)^2 \leqslant \int_a^b f^2(x)dx \,.\, \int_a^b g^2(x)dx$

$\Rightarrow \left(\int_o^x \sqrt{e^{2t}+e^{-t}}\,dt\right)^2 \leqslant \left(e^x-1\right)\left(e^x-\frac{1}{2}-\frac{1}{e^{2x}}\right) < \left(e^x-1\right)\left(e^x-\frac{1}{2}\right)$

$\Rightarrow \int_0^1 \sqrt{e^{2t}+e^{-t}}\,dt \leqslant \sqrt{\left(e^x-1\right)\left(e^x-\frac{1}{2}\right)}$ (1)

Mặt khác $:\sqrt{e^{2t}+e^{-t}} > e^t \;\; ;\forall 0 < t < x$

$\Rightarrow \int_0^x \sqrt{e^{2t}+e^{-t}}\; dt > \int_0^x e^t dt = e^x - 1$ (2)

Từ (1) và (2) suy ra $: e^x - 1 < \int_0^x \sqrt{e^{2t}+e^{-t}}\,dt < \sqrt{\left(e^x-1\right)\left(e^x-\frac{1}{2}\right)}$

4. $\left|\frac{3\cos x - 4\sin x}{1+x^2}\right| \leqslant \frac{1}{1+x^2}\sqrt{\left[3^2+(-4)^2\right]\left[\sin^2 x+\cos^2 x\right]} = \frac{5}{x^2+1}$

$\Rightarrow \left|\int_0^1 \frac{3\cos x-4\sin x}{1+x^2}dx\right| \leqslant \int_0^1 \left|\frac{3\cos x-4\sin x}{1+x^2}\right|dx \leqslant 5\int_0^1 \frac{1}{1+x^2}dx$

Đặt $x = tgt \Rightarrow dx = \left(1+tg^2 t\right)dt$

x	0	1
t	0	$\Pi/4$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx = \int_0^1 \frac{(1+tg^2t)}{1+tg^2t}dt = \int_0^1 dt = \frac{\Pi}{4}$$

$$\Rightarrow 4.\left|\int_0^1 \frac{3\cos x - 4\sin x}{1+x^2}dx\right| \leqslant \frac{5\Pi}{4}$$

Chứng minh bất đẳng thức tích phân bằng phương pháp đạo hàm.

Chứng minh rằng **:**

1. $54\sqrt{2} \leqslant \int_{-7}^{11}\left(\sqrt{x+7}\right)+\left(\sqrt{11-x}\right)dx \leqslant 108$ $\qquad \frac{\Pi}{4} \leqslant \int_0^{\Pi/4}(\sin x + \cos x)dx \leqslant \frac{\Pi\sqrt{2}}{4}$

2. $0 < \int_0^1 x(1-x^2)dx < \frac{4}{27}$ $\qquad$ 4. $\int_0^e e^{\sin^2 x}dx > \frac{3\Pi}{2}$

Bài giải **:**

1. Xét $f(x) = \left(\sqrt{x+7}\right)+\left(\sqrt{11-x}\right);\ x\in[-7,11]$

$$f'(x) = \frac{\sqrt{11-x}-\sqrt{x+7}}{2\sqrt{11-x}\sqrt{x+7}} \Rightarrow f'(x) = 0 \Leftrightarrow x = 2$$

x	**-7**	**2**	**11**
f'(x)	+	**0**	-
f(x)	$3\sqrt{2}$ ↗	6	↘ $3\sqrt{2}$

$$\Rightarrow 3\sqrt{2} \leqslant f(x) \leqslant 6 \Rightarrow 3\sqrt{2}\int_{-7}^{11}dx \leqslant \int_{-7}^{11} f(x)dx \leqslant 6\int_{-7}^{11}dx$$

$$\Rightarrow 54\sqrt{2} \leqslant \int_{-7}^{11}\left(\sqrt{x+7}+\sqrt{11-x}\right)dx \leqslant 108$$

2. Xét hàm số : f(x) = x(1-x²) ; $\forall x\in[0,1]$ $\qquad \Rightarrow f'(x) = 3x^2 - 4x + 1$

$\Rightarrow$ **f'(x)=0** $\Leftrightarrow x = \frac{1}{3} \vee x = 1$

x	**-∞ 0**	**1/3**	**1 +∞**
f'(x)	+	**0**	-
f(x)	0 ↗	4/27	↘ 0

$\Rightarrow 0 \leqslant f(x) \leqslant \dfrac{4}{27}$

$va \begin{cases} \exists x \in \left(0, \frac{1}{3}\right); \left(\frac{1}{3}, 0\right) \Rightarrow 0 < f_{(x)} < \frac{4}{27} \\ f_{(0)} = f_{(1)} = 0 \end{cases}$

$\Rightarrow 0 < \int_0^1 f(x)dx < \dfrac{4}{27}\int_0^1 dx \Rightarrow 0 < \int_0^1 f(x)dx < \dfrac{4}{27}$

3. Xét hàm số :

$f(x) = \sin x + \cos x = \sqrt{2}\sin\left(x + \dfrac{\Pi}{4}\right); x \in \left[0, \dfrac{\Pi}{4}\right]$

$f^{'}(x) = \sqrt{2}\cos\left(x + \dfrac{\Pi}{4}\right) \geqslant 0 \quad, \forall x \in \left(0, \dfrac{\Pi}{4}\right)$

$\Rightarrow$ **f(x) là hàm số tăng** $\forall x \in \left[0, \dfrac{\Pi}{4}\right] \Rightarrow f_{(0)} \leqslant f_{(x)} \leqslant f_{\left(\frac{\Pi}{4}\right)}$

$\Rightarrow 1 \leqslant \sin x + \cos x \leqslant \sqrt{2} \Rightarrow \dfrac{\Pi}{4} \leqslant \int_0^{\Pi/4}(\sin x + \cos x)dx \leqslant \dfrac{\Pi\sqrt{2}}{4}$

4. Nhận xét $\forall x > 0$ **thì** $e^x > 1 + x$ **(đây là bài tập Sgk phần chứng minh bất đẳng thức bằng pp đạo hàm)**

Xét $f_{(t)} = e^t - 1 - t \quad ; t \geqslant 0 \Rightarrow f^{'}_{(t)} = e^t - 1 > 0 ; \forall t > 0$

$\Rightarrow$ **hàm số f(t) đồng biến** $\forall t \geqslant 0$

Vì x > 0 nên f(x) > f(0) = 0 $\Rightarrow e^x - 1 - x > 0 \Leftrightarrow e^x > 1 + x\ (1)$

Do vậy : $\forall x \in (0, \Pi)\ thi\ e^{\sin^2 x} > 1 + \sin^2 x \quad (do(1))$

$\Rightarrow \int_0^{\Pi} e^{\sin^2 x}dx > \int_0^{\Pi}(1 + \sin^2 x)dx = \Pi + \int_0^{\Pi}\dfrac{1 - \cos 2x}{2}dx$

$\Rightarrow \int_0^{\Pi} e^{\sin^2 x}dx > \dfrac{3\Pi}{2}$

<u>Chứng minh rằng</u> **:**

1. $\dfrac{2}{5} \leqslant \int_1^2 \dfrac{x}{x^2+1}dx \leqslant \dfrac{1}{2}$
2. $\dfrac{\sqrt{3}}{4} \leqslant \int_{\Pi/4}^{\Pi/3} \dfrac{\sin x}{x}dx \leqslant \dfrac{1}{2}$
3. $\dfrac{\Pi\sqrt{3}}{3} \leqslant \int_0^{\Pi} \dfrac{1}{\sqrt{\cos^2 x + \cos x + 1}}dx \leqslant \dfrac{2\Pi\sqrt{3}}{3}$
4. $\dfrac{\sqrt{3}}{12} \leqslant \int_{\Pi/6}^{\Pi/3} \dfrac{\cot gx}{x}dx \leqslant \dfrac{1}{3}$
5. $\dfrac{2}{3} < \int_0^1 \dfrac{1}{\sqrt{2 + x - x^2}}dx < \dfrac{1}{\sqrt{2}}$
6. $2\sqrt[4]{2} < \int_{-1}^1\left(\sqrt[4]{1+x} + \sqrt[4]{1-x}\right)dx < 4$

<u>Bài giải</u> **:**

1. Xét : $f_{(x)} = \dfrac{x}{x^2+1}$ $;x \in [1,2]$. **có** $f'_{(x)} = \dfrac{1-x^2}{\left(1+x^2\right)^2} \leqslant 0\,;\forall x \in [1,2]$

$\Rightarrow$ **hàm số nghịch biến** $\forall x \in [1,2] \Rightarrow f_{(2)} \leqslant f_{(x)} \leqslant f_{(1)}$

$$\Rightarrow \frac{2}{5} \leqslant \frac{x}{x^2+1} \leqslant \frac{1}{2} \Rightarrow \frac{2}{5}\int_1^2 dx \leqslant \int_1^2 \frac{x}{x^2+1}dx \leqslant \frac{1}{2}\int_1^2 dx$$

$$\Rightarrow \frac{2}{5} \leqslant \int_1^2 \frac{x}{x^2+1} \leqslant \frac{1}{2}$$

2. Xét $f_{(x)} = \dfrac{\sin x}{x}\,;\forall x \in \left[\dfrac{\Pi}{6};\dfrac{\Pi}{3}\right] \Rightarrow f'_{(x)} = \dfrac{x.\cos x - \sin x}{x^2}$

Đặt $Z = x.\cos x - \sin x \Rightarrow Z' = -x$ $\quad x < 0\,;\forall x \in \left[\dfrac{\Pi}{6};\dfrac{\Pi}{3}\right]$

$\Rightarrow$ **Z đồng biến trên** $\forall x \in \left[\dfrac{\Pi}{6};\dfrac{\Pi}{3}\right]$ **và :**

$$Z \leqslant Z_{\left(\Pi/3\right)} = \frac{\Pi - 3\sqrt{3}}{6} < 0\,;\forall x \in \left[\frac{\Pi}{6};\frac{\Pi}{3}\right]$$

$$\Rightarrow f'_{(x)} < 0\,;\forall x \in \left[\frac{\Pi}{6};\frac{\Pi}{3}\right]$$

x	$-\infty$ $\quad \Pi/6$		$\Pi/3$ $\quad +\infty$
f'(x)		−	
f(x)		$\Pi/3$ ↘ $3\sqrt{3}/2\Pi$	

$$\Rightarrow \frac{3\sqrt{3}}{2\Pi} \leqslant f_{(x)} \leqslant \frac{3}{\Pi}$$

$$hay: \frac{3\sqrt{3}}{2\Pi} \leqslant \frac{\sin x}{x} \leqslant \frac{3}{\Pi}$$

$$\Rightarrow \frac{3\sqrt{3}}{2\Pi}\int_{\Pi/6}^{\Pi/3} dx \leqslant \int_{\Pi/6}^{\Pi/3} \frac{\sin x}{x}dx \leqslant \frac{3}{\Pi}\int_{\Pi/6}^{\Pi/3} dx \Rightarrow \frac{\sqrt{3}}{4} \leqslant \int_{\Pi/6}^{\Pi/3} \frac{\sin x}{x}dx \leqslant \frac{1}{2}$$

3. Đặt $t = \cos x\,; x \in [0,\Pi] \Rightarrow t \in [-1,1]$

và $f_{(t)} = t^2 + t + 1\,; t \in [-1,1]$

$f'_{(t)} = 2t + 1$; $f'_{(t)} = 0 \Leftrightarrow t = -\dfrac{1}{2}$

t	**-∞**	**-1** $\quad$ $-\frac{1}{2}$ $\quad$ **1**	**+∞**
f'(t)		$-$ $\quad$ **0** $\quad$ **+**	
f(t)		1 ↘ $\frac{3}{4}$ ↗ 3	

$\Rightarrow \dfrac{3}{4} \leqslant f_{(t)} \leqslant 3$; $\forall t \in [-1,1]$

$\Rightarrow \dfrac{3}{4} \leqslant \cos^2 x + \cos x + 1 \leqslant 3$; $\forall x \in [0,\Pi]$

hay $\dfrac{\sqrt{3}}{2} \leqslant \sqrt{\cos^2 x + \cos x + 1} \leqslant \sqrt{3}$ $\quad \Rightarrow \dfrac{1}{\sqrt{3}} \leqslant \dfrac{1}{\sqrt{\cos^2 x + \cos x + 1}} \leqslant \dfrac{2}{\sqrt{3}}$

$\Rightarrow \dfrac{1}{\sqrt{3}} \int_0^{\Pi} dx \leqslant \int_0^{\Pi} \dfrac{1}{\cos^2 x + \cos x + 1} dx \leqslant \dfrac{2}{\sqrt{3}} \int_0^{\Pi} dx$

$\Rightarrow \dfrac{\Pi\sqrt{3}}{3} \leqslant \int_0^{\Pi} \dfrac{1}{\sqrt{\cos^2 x + \cos x + 1}} dx \leqslant \dfrac{2\Pi\sqrt{3}}{3}$

Chú ý : thực chất bất đẳng thức trên phải là :

$\dfrac{\Pi\sqrt{3}}{3} < \int_0^{\Pi} \dfrac{1}{\sqrt{\cos^2 x + \cos x + 1}} dx < \dfrac{2\Pi\sqrt{3}}{3}$ **(học sinh tự giải thích vì sao)**

4. $f_{(x)} = \dfrac{\cot gx}{x}$; **liên tục** $\forall x \in \left[\dfrac{\Pi}{4};\dfrac{\Pi}{3}\right]$

có $f'_{(x)} = \dfrac{-(2x + \sin 2x)}{2x^2 \sin^2 x} < 0$; $\forall x \in \left(\dfrac{\Pi}{4};\dfrac{\Pi}{3}\right)$ $\quad \Rightarrow$ **f(x) :nghịch biến trên** $\left[\dfrac{\Pi}{4};\dfrac{\Pi}{3}\right]$

$\Rightarrow f_{\left(\Pi/3\right)} \leqslant f_{(x)} \leqslant f_{\left(\Pi/4\right)}$

$\Rightarrow \dfrac{\sqrt{3}}{\Pi} \leqslant \dfrac{\cot gx}{x} \leqslant \dfrac{4}{\Pi} \Rightarrow \dfrac{\sqrt{3}}{\Pi} \int_{\Pi/4}^{\Pi/3} dx \leqslant \int_{\Pi/4}^{\Pi/3} \dfrac{\cot gx}{x} dx \leqslant \dfrac{4}{\Pi} \int_{\Pi/4}^{\Pi/3} dx$

$\Rightarrow \dfrac{\sqrt{3}}{12} \leqslant \int_{\Pi/4}^{\Pi/3} \dfrac{\cot gx}{x} dx \leqslant \dfrac{1}{3}$

5. $f_{(x)} = 2 + x - x^2$; $\forall x \in [0,1]$ **có f'(x)=1- 2x**

$\Rightarrow f'_{(x)} = 0 \Leftrightarrow x = \dfrac{1}{2}$

x	**-∞** **0** $\quad$ $\frac{1}{2}$ $\quad$ **1** **+∞**

f'(x)		+	**0**	−		
f(x)			$\frac{9}{4}$			
		↗		↘		
		2		2		

$\Rightarrow 2 \leqslant f_{(x)} \leqslant \frac{9}{4}$

và $\begin{cases} \exists x \in \left(0, \frac{1}{2}\right); \left(\frac{1}{2}, 1\right) \\ f_{(0)} = f_{(1)} = 2 \end{cases} \Rightarrow 2 < f_{(x)} < \frac{9}{4}$

$\Rightarrow 2 < 2 + x - x^2 < \frac{9}{4} \quad \Rightarrow \frac{2}{3} < \frac{1}{\sqrt{2 + x - x^2}} < \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{2}{3}\int_0^1 dx < \int_0^1 \frac{1}{\sqrt{2 + x - x^2}}\, dx < \frac{1}{\sqrt{2}}\int_0^1 dx$

$\Rightarrow \frac{2}{3} < \int_0^1 \frac{1}{\sqrt{2 + x - x^2}} dx < \frac{1}{\sqrt{2}}$

6. Xét :

$f_{(x)} = \sqrt[4]{1+x} + \sqrt[4]{1-x} \quad ; x \in [-1, 1]$

$f'_{(x)} = \frac{1}{4}\left(\frac{1}{\sqrt[4]{(1+x)^3}} - \frac{1}{\sqrt[4]{(1-x)^3}}\right)$

$f'_{(x)} = 0 \Leftrightarrow \sqrt[4]{(1-x)^3} = \sqrt[4]{(1+x)^3} \Leftrightarrow x = 0$

Mặt khác : $f'_{(x)} > 0 \Leftrightarrow \frac{1}{\sqrt[4]{(1+x)^3}} > \frac{1}{\sqrt[4]{(1-x)^3}} \Leftrightarrow -1 < x < 0$

x	**-∞ -1**		**0**		**1 +∞**	
f'(x)		+	**0**	−		
f(x)			2			
		↗		↘		
		$\sqrt[4]{2}$		$\sqrt[4]{2}$		

$\Rightarrow \sqrt[4]{2} \le f_{(x)} \le 2$

$va \begin{cases} \exists x \in (-1, 0); (0, 1) \\ f_{(-1)} = f_{(1)} = \sqrt[4]{2} \end{cases} \Rightarrow \sqrt[4]{2} < f_{(x)} < 2$

$\Rightarrow \sqrt[4]{2}\int_{-1}^1 dx < \int_{-1}^1 \left(\sqrt[4]{1+x} + \sqrt[4]{1-x}\right) dx < 2\int_{-1}^1 dx \Rightarrow 2\sqrt[4]{2} < \int_{-1}^1 \left(\sqrt[4]{1+x} + \sqrt[4]{1-x}\right) dx < 4$

Chứng minh rằng **:**

1. $2.e^{-2} \le \int_0^2 e^{x-x^2}dx \le 2\sqrt[4]{e}$
2. $\int_{100}^{200} e^{-x^2}dx < 0,005$
3. $90 - \ln 10 \le \int_{10}^{100} e^{1/x}dx < 90 + \frac{9}{200} + \ln 10$
4. $9 \leqslant \int_0^{\Pi/3}\left(\frac{3}{\cos^4} - 2tg^4x\right)dx \le 90$
5. $\int_0^1 e^{1/x^2+1}dx \ge 1 + \frac{\Pi}{4}$
6. $\int_0^{\Pi/2} \frac{tg^{x/2}}{x}dx < 1$

Bài giải **:**

1. Đặt $f_{(x)} = x - x^2 \quad ; x \in [0,2]$ **có** $f'_{(x)} = 1 - 2x$

có $f'_{(x)} = 0 \Leftrightarrow x = \frac{1}{2}$

x	**-∞**	**0**		$\frac{1}{2}$		**2**	**+∞**
f'(x)			+	**0**	−		
f(x)		0	↗	$\frac{1}{4}$	↘	−2	

$\Rightarrow -2 \leqslant f_{(x)} \leqslant \frac{1}{4}$

$hay\ -2 \leqslant x - x^2 \leqslant \frac{1}{4}$

$\Rightarrow e^{-2} \leqslant e^{x-x^2} \leqslant e^{1/4} = \sqrt[4]{e} \Rightarrow e^{-2} \le \int_0^2 dx \le \int_0^2 e^{x-x^2}dx \leqslant \sqrt[4]{e}\int_0^2 dx$

$2.e^{-2} \leqslant \int_0^2 e^{x-x^2}dx \leqslant 2.\sqrt[4]{e}$

Chú ý : thực chất bất đẳng thức trên là : $2.e^{-2} < \int_0^2 e^{x-x^2}dx < 2.\sqrt[4]{e}$

2. Trước hết ta chứng minh : $e^{-x^2} \le \frac{1}{x^2}; (1)\ x \ne 0$

Đặt $t = x^2 \quad ; x \ne 0 \Rightarrow t > 0$

Giả sử ta có (1) và (1) $\Leftrightarrow e^{-t} \leqslant \frac{1}{t} \quad ; t > 0 \Leftrightarrow e^t \geqslant t\, ; t > 0$

$\Leftrightarrow e^t - t \geqslant 0\,(2)\, ; t > 0$

Đặt $f_{(x)} = e^t - t\ \ co\ f'_{(t)} = e^t - 1 > 0\, , t > 0$

$\Rightarrow f_{(t)}$ **luôn đồng biến** $\forall t > 0$ **và** $f_{(t)} \geqslant f_{(0)} = 1 > 0$

$\Rightarrow f_{(t)} \geqslant 0\,, t > 0 \Rightarrow e^{-x^2} \leq \dfrac{1}{x^2} \Rightarrow \int_{100}^{200} e^{-x^2}dx \leq \int_{100}^{200} \dfrac{1}{x^2}dx$

$\Rightarrow \int_{100}^{200} e^{-x^2}dx < 0,005$

3. Trước hết ta chứng minh : $1 - \dfrac{1}{x} \leqslant e^{-1/x} \leqslant 1 - \dfrac{1}{x} + \dfrac{1}{2x^2}\,;(1)\ \forall x > 0$

Đặt $t = -\dfrac{1}{x}\ ; x > 0 \Rightarrow t < 0$

$(1) \Leftrightarrow 1 + t \leqslant e^t \leqslant 1 + t + \dfrac{1}{2}t^2\ ;(2)\, t < 0$

Xét hàm số $f_{(t)} = e^t - t - 1\ \ ; h_{(t)} = e^t - 1 - t - \dfrac{1}{2}t^2\ ; t < 0$

● $f'_{(t)} = e^t - 1$

t	-∞	0	+∞
f'(t)	–		
f(t)	+∞ ↘ 0		

$\Rightarrow f_{(t)} > 0 \quad ;\forall \tau < 0$

$hay\ e^t - 1 - t > 0 \quad ;\forall t < 0$

$\Rightarrow 1 + t < e^t \quad ;\forall t < 0\,(3)$

•$h'_{(t)} = e^t - 1 - t$

x	-∞	0	+∞
h't		+	
ht		↗ 0	

$\Rightarrow h_{(t)} < 0\ ;\forall t < 0$

$hay\ e^t < 1 + t + \dfrac{1}{2} > 0\,;\forall t < 0\,(4)$

Từ (3) và (4) suy ra :

$1+t\leqslant e^t\leqslant 1+t+\frac{1}{2}t^2 \quad ;\forall t<0$

$hay\ 1-\frac{1}{x}\leqslant e^{-1/x}\leqslant 1-\frac{1}{x}+\frac{1}{2x^2}\ ;x>0$

$\Rightarrow \int_{10}^{100}\left(1-\frac{1}{x}\right)dx\leqslant \int_{10}^{100}e^{-1/x}dx\leqslant \int_{10}^{100}\left(1-\frac{1}{x}+\frac{1}{2x^2}\right)dx$

$90-\ln 10\leq \int_{10}^{100}e^{1/x}dx<90+\frac{9}{200}+\ln 10$

*** Là bài toán khó , hi vọng các em tìm điều thú vị trong bài toán trên – chúc thành công .**

4. Xét $f_{(x)}=\frac{3}{\cos^4 x}-2tg^4x \quad ;x\in\left[0,\frac{\Pi}{3}\right]$

Đặt $t=\frac{1}{\cos^2 x}=1+tg^2x \quad ;x\in x\in\left[0,\frac{\Pi}{3}\right] \qquad \Rightarrow t\in[1;4]$

$\Rightarrow f_{(t)}=t^2+4t-2\Rightarrow f'_{(t)}=4t^3+4>0\ ;\forall t\in[1,4]$

$\Rightarrow f_{(1)}\leqslant f_{(t)}\leqslant f_{(4)} \quad \Rightarrow 3\leqslant f_{(t)}\leqslant 30$

$\Rightarrow 3\int_1^4 dt\leqslant \int_1^4 f_{(t)}dt\leq 30\int_1^4 dt$

$\Rightarrow 9\leqslant \int_0^{\Pi/3}\left(\frac{3}{\cos^4}-2tg^4x\right)dx\leqslant 90$

5. Xét hàm số $f_{(x)}=e^x-1-x\ ;\forall x\geqslant 0$

có $f'_{(x)}=e^x-1>0\ ,\forall x\geqslant 0\Rightarrow f_{(x)}$ **đồng biến** $\forall x\in[0,+\infty)$

$\Rightarrow f_{(x)}\geqslant f_{(0)}=0\Rightarrow e^x-1-x\geqslant 0\Rightarrow e^x\geqslant 1+x\ ;\forall x\geqslant 0$

$\Rightarrow e^{1/1+x^2}\geqslant 1+\frac{1}{1+x^2} \quad ;\forall x\geqslant 0$

$\Rightarrow \int_0^1 e^{1/1+x^2}dx\geqslant \int_0^1\left(1+\frac{1}{1+x^2}\right)dx=1+\int_0^1\frac{1}{1+x^2}dx \ (*)$

Đặt $x=tgt\Rightarrow dx=\left(1+tg^2t\right)dt$

$\begin{cases}x=0\\x=1\end{cases}\Rightarrow\begin{cases}t=0\\t=\Pi/4\end{cases} \qquad \Rightarrow \int_0^1\frac{1}{1+x^2}dx=\int_0^1\frac{\left(1+tg^2t\right)dt}{1+tg^2t}=\frac{\Pi}{4}$

Từ (*) suy ra : $\int e^{1/x^2+1}dx\geqslant 1+\frac{\Pi}{4}$

6. Trước hết ta chứng minh : $\frac{tg\,x/2}{x}<\frac{2}{\Pi} \qquad ;x\in\left(0,\frac{\Pi}{2}\right)$

Xét hàm số $f_{(x)} = \frac{1}{x}.tg\frac{x}{2} \quad ; x \in \left(0, \frac{\Pi}{2}\right)$

$f'_{(x)} = \frac{x - \sin x}{2x^2.\cos^2 x/2}$

Đặt $Z = x - \sin x \Rightarrow Z' = 1 - \cos x > 0 \, , \forall x \in \left(0, \frac{\Pi}{2}\right)$

$\Rightarrow Z > Z_{(0)} = 0 \Rightarrow f'_{(x)} > 0 \, , \forall x \in \left(0, \frac{\Pi}{2}\right)$

x	**-∞**	**0**	$\Pi/2$	**+∞**
f'(x)		**+**		
f(x)		$-\infty \nearrow 2/\Pi$		

$\Rightarrow f_{(x)} < \frac{2}{\Pi} \quad \Rightarrow \frac{tg\, x/2}{x} < \frac{2}{\Pi}$

$\Rightarrow \int_0^{\Pi/2} \frac{tg\, x/2}{x} dx < \int_0^{\Pi/2} \frac{2}{\Pi} dx \quad \Rightarrow \int_0^{\Pi/2} \frac{tg\, x/2}{x} dx < 1$

Chứng minh rằng **:**

1. $\frac{1}{2}\int_0^{\Pi} x^{1999}.e^{2x}.dx > \frac{\Pi^{2001}}{2001} + \frac{\Pi^{2001}}{2002}$
2. $\int_0^1 x \ln\left(x + \sqrt{1 + x^2}\right) dx \geqslant \frac{1}{2}\ln\left(1 + \sqrt{2}\right) + \frac{\sqrt{2}}{2} - 1$
3. $\int_0^{\Pi/4} x tg^n x dx \geqslant \frac{1}{n+2}\left(\frac{\Pi}{4}\right)^{n+2}$

Bài giải **:**

1. Trước hết ta chứng minh : $e^{2x} > 2\left(x^2 + x\right) \; ; \forall x > 0$

Xét hàm số:

$f_{(x)} = e^{2x} - 2\left(x^2 + x\right) \quad ; \forall x > 0$

$f'_{(x)} = 2.e^{2x} - 4x - 2 \quad ; f''_{(x)} = 4.e^{2x} - 4 > 0 \quad ; \forall x > 0$

$\Rightarrow f'_{(x)}$ **là hàm tăng** $; \forall x > 0 \Rightarrow f'_{(x)} > f_{(0)} = 0$

$\Rightarrow f_{(x)}$ **là hàm tăng** $; \forall x > 0 \Rightarrow f_{(x)} > f_{(0)}$

$$\Rightarrow e^{2x} > 2\left(x^2 + x\right) \Rightarrow x^{1999}.e^{2x} > 2.x^{1999}\left(x^2 + x\right)$$

$$\Rightarrow \frac{1}{2}\int_0^{\Pi} x^{1999}.e^{2x}dx > \int_0^{\Pi} x^{1999}\left(x^2 + x\right)dx$$

$$\Rightarrow \frac{1}{2}\int_0^{\Pi} x^{1999}.e^{2x}.dx > \frac{\Pi^{2001}}{2001} + \frac{\Pi^{2001}}{2002}$$

2. Trước hết ta chứng minh : $1 + x\ln\left(x + \sqrt{1 + x^2}\right) \geqslant \sqrt{1 + x^2} \quad ;\forall x \in R$

Xét hàm số :

$$f_{(x)} = 1 + x\ln\left(x + \sqrt{1 + x^2}\right) - \sqrt{1 + x^2}$$

$$f'_{(x)} = \ln\left(x + \sqrt{1 + x^2}\right) \Rightarrow f'_{(x)} = 0 \Leftrightarrow x + \sqrt{1 + x^2} = 1$$

$$\Leftrightarrow \begin{cases} 1 - x \geq 0 \\ 1 + x^2 = (1 - x)^2 \end{cases} \Leftrightarrow x = 0$$

và $f'_{(x)} < 0 \Leftrightarrow \ln\left(x + \sqrt{1 + x^2}\right) < 0 \Leftrightarrow x < 0$

x	**-∞**	**0**	**+∞**
f'(x)	**-**	**0**	**+**
f(x)	↘		↗
		0	

$$\Rightarrow f_{(x)} \geqslant f_{(0)} = 0 \quad ;\forall x \in R$$

$$\Rightarrow 1 + x\ln\left(x + \sqrt{1 + x^2}\right) \geqslant \sqrt{1 + x^2}$$

$$\Rightarrow x\ln\left(x + \sqrt{1 + x^2}\right) \geqslant \sqrt{1 + x^2} - 1$$

$$\Rightarrow \int_0^1 x\ln\left(x + \sqrt{1 + x^2}\right)dx \geqslant \int_0^1\left(\sqrt{1 + x^2} - 1\right)dx = \frac{1}{2}\left[x\sqrt{x^2 + 1} + \frac{1}{2}\ln\left(x + \sqrt{1 + x^2}\right) - x\right]\Bigg|_0^1$$

$$\Rightarrow \int_0^1 x\ln\left(x + \sqrt{1 + x^2}\right)dx \geqslant \frac{1}{2}\ln\left(1 + \sqrt{2}\right) + \frac{\sqrt{2}}{2} - 1$$

3. Đặt $f_{(x)} = tgx - x\,;\forall x \in \left[0, \frac{\Pi}{4}\right]$

$$f'_{(x)} = \frac{1}{\cos^2 x} - 1 = tg^2x > 0 \quad ;\forall x \in \left(0, \frac{\Pi}{4}\right)$$

$\Rightarrow f_{(x)}$ **đồng biến trên** $\left[0, \frac{\Pi}{4}\right] \Rightarrow f_{(x)} \geqslant f_{(0)} = 0$

$$\Rightarrow tgx \geqslant x \quad ; \forall x \in \left[0, \frac{\Pi}{4}\right] \Rightarrow tg^n x \geqslant x^n$$

$$\Rightarrow xtg^n x \geqslant x^{n+1} \Rightarrow \int_0^{\Pi/4} xtg^n x dx \geqslant \int_0^{\Pi/4} x^{n+1} dx$$

$$\Rightarrow \int_0^{\Pi/4} xtg^n x dx \geqslant \frac{1}{n+2}\left(\frac{\Pi}{4}\right)^{n+2}$$

Giả sử f(x) có đạo hàm liên tục trên [0,1] và f(1) – f(0) = 1

***<u>Chứng minh rằng</u>* :** $\int_0^1 \left(f'_{(x)}\right)^2 dx \geqslant 1$

Ta có : $\int_0^1 \left(f'_{(x)} - 1\right)^2 dx \geqslant 1 \; ; \forall x \in [0,1]$

$$\Rightarrow \int_0^1 \left(f'_{(x)}\right)^2 dx - 2\int_0^1 f'_{(x)} dx \geqslant 1 + \int_0^1 dx \geqslant 0 \Leftrightarrow \int_0^1 \left(f'_{(x)}\right)^2 dx - 2\left[f_{(1)} - f_{(0)}\right] + 1 \geqslant 0$$

$$\Leftrightarrow \int_0^1 \left(f'_{(x)}\right)^2 dx - 2 + 1 \geqslant 0 \Rightarrow \int_0^1 \left(f'_{(x)}\right)^2 dx \geqslant 1$$

Cho f là 1 hàm liên tục trên [0;1] đồng thời thoả mãn

$$\begin{cases} 1 \leqslant f_{(x)} \leqslant 2 \; ; \; ; \forall x \in [0,1] \, (a) \\ \int_0^1 f_{(x)} dx = \frac{3}{2} \, (b) \end{cases}$$

<u>Chứng minh</u> $\frac{2}{3} \leqslant \int_0^1 \frac{1}{f_{(x)}} dx < \frac{3}{4}$

Theo BĐT Bunhiacosky

$$1 \leqslant \left(\int_0^1 1.dx\right)^2 = \left(\int_0^1 \sqrt{f_{(x)}} \cdot \frac{1}{\sqrt{f_{(x)}}} dx\right)^2 \leqslant \int_0^1 f_{(x)} dx \cdot \int_0^1 \frac{dx}{f_{(x)}}$$

$$= \frac{3}{2}\int_0^1 \frac{dx}{f_{(x)}} \Rightarrow \int_0^1 \frac{dx}{f_{(x)}} \geqslant \frac{2}{3} \, (1)$$

Dấu "=" không xảy ra :

$$\frac{\sqrt{f_{(x)}}}{1/\sqrt{f_{(x)}}} = k \Leftrightarrow f_{(x)} = k = \frac{3}{2}$$

$$do \int_0^1 f_{(x)}.dx = \frac{3}{2}$$

Từ (a) : $1 \leqslant f_{(x)} \leqslant 2 \; ; \forall x \in [0,1]$ **thì** $\begin{cases} 2 - f_{(x)} \geqslant 0 \\ f_{(x)} - 1 \geqslant 0 \end{cases}$

$$\Leftrightarrow \left(2 - f_{(x)}\right)\left(f_{(x)} - 1\right) \geqslant 0 \Leftrightarrow f^2_{(x)} - 3f_{(x)} + 2 \leqslant 0$$

$f_{(x)} - 3 + \frac{2}{f_{(x)}} \leqslant 0 \; (2)$ **Đặt** $t = f_{(x)}$

$\Rightarrow 1 \le t \le 2$ **thì (2)** $\Leftrightarrow t - 3 - \frac{2}{t} = f_{(t)} \leqslant 0$

t	**1**	$\sqrt{2}$	**2**
f'(t)	–	**0**	**+**
f(t)	↘	$2\sqrt{2} - 3$	↗

$\Rightarrow \int_0^1 f_{(x)} dx - 3\int_0^1 dx + 2\int_0^1 \frac{dx}{f_{(x)}} < 0$

$\Rightarrow 2\int_0^1 \frac{dx}{f_{(x)}} < 3\int_0^1 dx - \int_0^1 f_{(x)} dx = \frac{3}{2} \Rightarrow \int_0^1 \frac{dx}{f_{(x)}} < \frac{3}{4}$

Từ (1) và (2) suy ra :

$\frac{2}{3} \leqslant \int \frac{1}{f_{(x)}} dx < \frac{3}{4}$

BÀI TẬP TỰ LUYỆN

Chứng minh rằng :

1. $\dfrac{\Pi}{28} \leqslant \int_0^{\Pi/4} \dfrac{1}{5+2\cos^2 x}\,dx \leqslant \dfrac{\Pi}{24}$

2. $\dfrac{\Pi}{24} \leqslant \int_0^{\Pi/4} \dfrac{1}{3+4\sin^2 x}\,dx \leqslant \dfrac{\Pi}{18}$

3. $\dfrac{2}{9} \leqslant \int_{-1}^{1} \dfrac{1}{x^3+8}dx \leqslant \dfrac{2}{7}$

4. $\int_0^{10} \dfrac{x}{x^3+16}dx < \dfrac{5}{6}$

5. $\left|\int_0^{18} \dfrac{\cos x}{\sqrt{1+x^4}}\,dx\right| < \dfrac{5}{6}$

6. $\dfrac{\Pi}{16} \leqslant \int_0^{\Pi/2} \dfrac{1}{5+3\cos^2 x}dx \leqslant \dfrac{\Pi}{10}$

7. $\int_1^{\sqrt{3}} \dfrac{e^{-x}.\sin x}{x^2+1}dx < \dfrac{\Pi}{12e}$

8. $\int_0^1 \sqrt{3+e^{-x}}\,dx \leqslant 2$

9. $\int_1^{\Pi} x^{2001}.\ln x.dx < \left(\sqrt{\Pi}\right)^{4003}$

10. $\dfrac{\Pi}{6} \leqslant \int_0^1 \dfrac{1}{\sqrt{4-x^2-x^3}}dx \leqslant \dfrac{\Pi\sqrt{2}}{8}$

11. $\dfrac{1}{2} \leqslant \int_0^{1/2} \dfrac{1}{\sqrt{1-x^{2\Pi}}}dx < \dfrac{\Pi}{6}$ $\quad (n = 2, 3...)$

12. $\dfrac{2}{\sqrt[4]{e}} \leqslant \int_0^2 e^{x^2-x}\,dx \leqslant 2e^2$

13. $0 < \int_0^1 \dfrac{x^7}{\sqrt[3]{1+x^8}}dx < \dfrac{1}{8}$

14. $1 < \int_0^1 e^{x^2}\,dx < e$

15. $\dfrac{1}{20\sqrt[3]{2}} < \int_0^1 \dfrac{x^{19}}{\sqrt{1+x^6}}dx < \dfrac{1}{20}$

16. $\dfrac{\Pi}{10} \leqslant \int_0^{\Pi/2} \dfrac{1}{5-3\cos^2 x}dx \leqslant \dfrac{\Pi}{4}$

17. $0 \leqslant \int_0^1 \dfrac{x^n}{1+x}dx \leqslant \dfrac{1}{n+1}$

18. $1 \leqslant \int_0^3 \dfrac{1}{\sqrt{-x^2+2x+8}}dx \leqslant \dfrac{3}{\sqrt{5}}$

19. $1 \leqslant \int_0^1 \sqrt{1+x^2}\,dx \leqslant \sqrt{2}$

20. $\sqrt{3} \leqslant \int_0^1 \sqrt{3+x^2}\,dx \leqslant 2$

21. $\dfrac{\Pi}{8} \leqslant \int_{\Pi/4}^{3\Pi/4} \dfrac{1}{3+\sin^2 x}dx \leqslant \dfrac{\Pi}{7}$

22. $2 \leqslant \int_{-1}^{1} \sqrt{5-4x}dx \leqslant 6$

23. $2 \leqslant \int_0^1 \sqrt{4+x^2}\,dx \leqslant \sqrt{5}$

24. $\dfrac{\Pi\sqrt{3}}{18} < \int_0^1 \dfrac{1}{x^2+x+2}dx < \dfrac{\Pi}{8}$

25. $\dfrac{1}{\sqrt{2}} \leqslant \int_0^{1/\sqrt{2}} \dfrac{1}{\sqrt{1-x^{2004}}}dx \leqslant \dfrac{\Pi}{4}$

26. $\dfrac{\Pi}{18} \leqslant \int_0^1 \dfrac{\sqrt{x}}{x^7+x^5+x^3+3}dx < \dfrac{\Pi\sqrt{3}}{27}$

27. $0 \leqslant \int_0^e x\ln x dx \leqslant e^2$

28. $9 < \int_0^3 \sqrt{81+x^2}\,dx < 10$

29. $\dfrac{2\Pi}{3} < \int_0^{2\Pi} \dfrac{dx}{10+3\cos x} < \dfrac{2\Pi}{7}$

30. $\dfrac{\Pi}{2} < \int_0^{\Pi/2} \sqrt{1+\dfrac{1}{2}\sin^2 x}\;dx < \dfrac{\Pi\sqrt{6}}{4}$

31. $0 < \int_{-1}^{1} tgx^2 dx < 2\sqrt{3}$

32. $\Pi/4 \leqslant \int_{\Pi/4}^{3\Pi/4} \dfrac{1}{3-2\sin^2 x}dx \leqslant \dfrac{\Pi}{2}$

33. $\dfrac{1}{e}\left(e-1\right) < \int_0^1 e^{-x^2}\,dx < 1$

34. $\left|\int_0^1 \dfrac{\sin(nx)}{x+1}\,dx\right| \leqslant \ln 2$

35. $\int_0^1 \dfrac{\cos(nx)}{x+1}dx \leqslant \ln 2$

36. $\int_1^{1/\sqrt{2}} \dfrac{x}{\sqrt{1-x^{2n}}}dx < \dfrac{\Pi}{12}$; $n = 3, 4$

37. $\int_0^{\sqrt{2\Pi}} \sin\left(x^2\right)dx > 0$

<u>Chứng minh rằng :</u>

1. $\int_0^{\Pi}\left(\sqrt{2+\cos^2 x}+\sqrt{2-\cos^2 x}\right)dx \leqslant 2\sqrt{2}\,\Pi$

2. $0 < \int_1^{\sqrt{3}}\frac{3\cos x+4\sin x}{x^2+1}dx \leqslant \frac{5\Pi}{12}$

3. $\int_{\Pi/6}^{\Pi/3}\left(3-2\sqrt{\sin x}\right)\left(\sin x+6\sqrt{\sin x}+5\right)dx \leqslant \frac{9\Pi}{2}$

4. $\int_0^{\Pi/4}\sqrt{tgx}\left(2+3\sqrt{tgx}\right)\left(7-4\sqrt{tgx}\right)dx \leqslant \frac{27\Pi}{4}$

5. $\int_0^{\Pi/4}\sin^2 x\left(2+3\cos^2 x\right)dx < \frac{25\Pi}{48}$

6. $\int_0^{\Pi/2}\cos^4 x\left(2\sin^2 x+3\right)dx < \frac{125\Pi}{54}$

7. $\int_{-\Pi/4}^{\Pi/4}\left(\sqrt{5-2\cos^2 x}+\sqrt{3-2\sin^2 x}\right)dx \leqslant \frac{3\Pi\sqrt{3}}{2}$

8. $\int_0^{\Pi/2}\sqrt{\sin x}\left(2+3\sqrt{\sin x}\right)\left(7-4\sqrt{\sin x}\right)dx \leqslant \frac{27\Pi}{2}$

9. $\int_{\Pi/6}^{\Pi/3}\left(3-2\sqrt{\sin x}\right)\left(5+\sqrt{\sin x}\right)\left(1+\sqrt{\sin x}\right)dx \leqslant \frac{9\Pi}{2}$

10. $\int_0^{\Pi}\left(2\sin x+tgx\right)dx > 0$

11. $0 \leqslant \int_0^1\left(e^{-x}+x-1\right)dx \leqslant e^{-1}$

12. $\frac{1}{2} \leqslant \int_0^1\frac{e^{-x^2}}{x^2+1}dx \leqslant \frac{5}{24}+\frac{1}{2}$

Chứng minh rằng :

1. $\frac{2\Pi}{13} \leqslant \int_0^{2\Pi}\frac{1}{10+3\cos x}dx \leqslant \frac{2\Pi}{7}$

2. $\frac{\Pi}{14} \leqslant \int_0^{\Pi/2}\frac{1}{4+3\cos^2 x} \leqslant \frac{\Pi}{8}$

3. $\left|\int_0^{18}\frac{\cos x}{\sqrt{1+x^4}}dx\right| < 0,1$

4. $\int_0^1\sqrt{1+x^2}dx > \int_0^1 xdx$

5. $\int_0^1 x^2\sin^2 xdx < \int_0^1 x\sin^2 xdx$

6. $\int_1^2 e^{x^2}dx > \int_1^2 e^x dx$

11. $-1 \leqslant \int_0^1\frac{x\sin a+\sqrt{a+1}\cos a}{x+1}dx \leqslant 1$ $(a \in R)$

12. $\frac{\Pi}{4\sqrt{2}} < \int_0^1\frac{\sqrt{1-x^2}}{1+x^2}dx < \sqrt{\frac{\Pi}{6}}$

13. $1 \leqslant \int_{-1}^1 2^{x^3}dx \leqslant 4$

14. $1 \leqslant \int_0^1 e^{x^2}dx \leqslant e$

15. $\Pi \leqslant \int_0^{\Pi}\frac{1}{\sin^4 x+\cos^4 x}dx \leqslant 2\Pi$

16. $\int_0^1\frac{1}{x^2+x+2}dx < \frac{\Pi}{8}$

7. $\left|\int_{-1}^{1}\frac{x^2\cos\alpha-2x+\cos\alpha}{x^2-2x\cos\alpha+1}dx\right|\leqslant 2$

$\alpha\in(0,\Pi)$

8. $\int_0^{\Pi/2}\sin^{10}xdx\leqslant\int_0^{\Pi/2}\sin^2 xdx$

9. $\int_{-\Pi/6}^{\Pi/6}\left(\sqrt{\cos^2 x+2\sin^2 x}+\sqrt{\sin^2 x+2\cos^2 x}\right)dx\leqslant\Pi\sqrt{6}$

10. $\int_0^{\Pi/2}\left(\sqrt{3\cos^2 x+\sin^2 x}+\sqrt{3\sin^2 x+\cos^2 x}\right)dx\leqslant\Pi\sqrt{2}$

17. $\int_{\Pi/4}^{\Pi/2}\sin xdx>\int_{\Pi/4}^{\Pi/2}\cos xdx$

18. $3\sqrt{6}\leqslant\int_{-2}^{1}\left(\sqrt{2+x}+\sqrt{4-x}\right)dx\leqslant 6\sqrt{3}$

19. $25\leqslant\int_3^5\frac{x^3}{x^2-4x+5}dx\leqslant 27$

Chứng minh rằng :

1. $\frac{\sqrt{3}}{4}<\int_{\Pi/6}^{\Pi/3}\frac{\sin x}{x}dx<\frac{1}{2}$

2. $\frac{\sqrt{3}}{8}<\int_{\Pi/4}^{\Pi/3}\frac{\sin x}{x}<\frac{\sqrt{2}}{6}$

3. $0<\int_0^3 x\left(1-x^2\right)dx<\frac{4}{27}$

4. $\frac{\Pi\sqrt{3}}{3}<\int_0^{\Pi}\frac{1}{\sqrt{\cos^2 x+\cos x+1}}dx<\frac{2\Pi\sqrt{3}}{3}$

5. $\frac{2}{5}<\int_1^2\frac{x}{x^2+1}dx<\frac{1}{2}$

6. $0<\int_0^1 x\left(1-x^2\right)dx<\frac{2\sqrt{3}}{9}$

7. $54\sqrt{2}\leqslant\int_{-7}^{11}\left(\sqrt{11-x}+\sqrt{7+x}\right)dx\leqslant 108$

8. $\int_0^{\Pi/2}\left(\frac{8}{3\cos x+\cos 3x}+3\cos^2 x\right)dx\geqslant 5$

9. $-\frac{49}{8}\leqslant\int_{\frac{\Pi}{6}}^{\frac{5\Pi}{2}}\left(\sin x-\frac{1}{\sin x}-\sin^2 x-\frac{1}{\sin^2 x}\right)dx\leqslant -3$

10. $0,65\leqslant\int_0^1\frac{dx}{x^2+1}\leqslant 0,9$

11. $\frac{2}{3}\leqslant\int_0^1\frac{1}{\sqrt{2+x-x^2}}dx\leqslant\frac{1}{\sqrt{2}}$

28. $\int_0^{2\Pi}\frac{\cos x}{x}dx<\frac{1}{2\Pi}$

29. $0\leqslant\int_{-2}^{0}x^2e^xdx\leqslant 8e^{-2}$

30. $-24\leqslant\int_{-1}^{2}\left(-x^5-5x^3+20x+2\right)dx\leqslant 32$

31. $-2\sqrt[3]{4}\leqslant\int_{-1}^{1}\sqrt{-x^3+3x-2}dx\leqslant 0$

32. $0<\int_0^{e^2}x^{1/\sqrt{x}}dx<e^{2(e+1)}$

33. $\frac{1}{2}<\int_0^{2\Pi/3}\left(2\cos^2 x+2\cos x+1\right)dx<5$

34. $\left|\int_{-\Pi}^{\Pi}\sin x\left(1+\cos x\right)dx\right|<\frac{\Pi\sqrt{3}}{2}$

35. $\left|\int_{-\Pi}^{\Pi}\left(\sin x+\cos x\right)dx\right|<2\Pi\sqrt{2}$

36. $\frac{7\Pi\sqrt{3}}{3}<\int_0^{2\Pi}\left(x\sqrt{3}+2\sin x\right)dx<\frac{5\Pi\sqrt{3}+6}{3}$

37. $\int_0^{2\Pi}\left(\cos x-x\right)dx<2\Pi$

38. $\left|\int_0^{\Pi}\left(\frac{\cos x}{\sin^3 x}-2\cot gx\right)dx\right|<\frac{\Pi 2\sqrt{3}}{9}$

39. $-2\leqslant\int_2^4\frac{x^2-7x+5}{x^2-5x+7}dx\leqslant 6$

40. $0\leqslant\int_{-1}^{1}\frac{x^2+3x+1}{x^2-x+1}dx\leqslant 10$

12. $\frac{5}{2} < \int_2^3 \frac{x^2}{\sqrt{x^2-1}}dx < \frac{9\sqrt{2}}{4}$

13. $\int_{100\Pi}^{200\Pi} \frac{\cos x}{x}dx < \frac{1}{200\Pi}$

14. $\frac{1}{2} < \int_{\frac{\Pi}{4}}^{\frac{\Pi}{2}} \frac{\sin x}{x}dx < \frac{\sqrt{2}}{2}$

15. $2\left(e^2 - e\right) < \int_2^{e^2}\left(3\ln x - \frac{1}{\ln x}\right)dx$

16. $\frac{2}{5} < \int_1^2 \frac{x}{x^2+1}dx < \frac{1}{2}$

17. $\int_0^1 x(1-x)dx < \frac{1}{2}$

18. $\Pi \leqslant \int_{\Pi/2}^{\Pi} \sqrt{\cos 2x - \cos x + 1}\, dx \leqslant 2\Pi$

19. $5\sqrt{2} \leqslant \int_0^{\sqrt{2}}\left(-x^4 + 4x^2 + 5\right)dx \leqslant 9\sqrt{2}$

20. $-141 \leqslant \int_{-1}^{2}\left(-3x^4 - 8x^3 + 30x^2 + 72x - 20\right)dx \leqslant 369$

21. $\frac{5}{2} \leqslant \int_{-2}^{1/2} \frac{2x^2+4x+5}{x^2+1}dx \leqslant 15$

22. $0 \leqslant \int_0^e x\ln x \leqslant e^2$

23. $e^2(e-1) < \int_e^{e^2} \frac{x}{\ln x}dx < e^3(e-1)$

24. $\frac{5}{4} < \int_1^2 \frac{2x}{x^2+1}dx < 1$

25. $\ln 2 < \int_1^3 \ln\frac{x+1}{x}dx < \ln 3$

26. $\frac{2}{\sqrt[4]{e}} < \int_0^2 e^{x^2-x}dx < 2e^2$

27. $e < \int_1^2 \frac{e^x}{x}dx < \frac{1}{2}e^2$