1996 UIUC UNDERGRAD MATH CONTEST

Problem 1. Let $a_1 < a_2 < \cdots < a_{43} < a_{44}$ be positive integers not exceeding 125. Prove that among the 43 differences $d_i = a_{i+1} - a_i$ $(i = 1, 2, \ldots, 43)$ some value must occur at least 10 times.

Solution. The sum of the 43 differences d_i is $a_{44} - a_1 \le 124$. If no value among the d_i's occurred more than 9 times, this sum would be at least $9 \cdot 1 + 9 \cdot 2 + 9 \cdot 3 + 9 \cdot 4 + 7 \cdot 5 = 125$, contradicting the above upper bound. Hence some value must occur at least 10 times.

Problem 2. Suppose f is a real positive continuous function on $\mathbf{R}$ with $\int_{-\infty}^{\infty} f(x)dx = 1$. Let $0 < \alpha < 1$, and suppose $[a, b]$ is an interval of *minimal length* with $\int_a^b f(x)dx = \alpha$. Show that $f(a) = f(b)$.

Solution. Let $0 < \alpha < 1$, and suppose $[a, b]$ is an interval *of minimal length* with $\int_a^b f(x)dx = \alpha$. Show that $f(a) = f(b)$. Let $F(y) = \int_y^{y+b-a} f(x)dx$. By hypothesis, $F(a) = \alpha$, and by the fundamental theorem of calculus we have $F'(a) = f(b) - f(a)$. Thus, it suffices to show that $F'(a) = 0$. Since f is continuous, nonnegative and integrable over $(-\infty, \infty)$, $F(y)$ is continuous, nonnegative and tends to zero as $x \to \pm\infty$. The function $F(y)$ therefore attains a finite maximum value M at some point. If we can show that the maximum M is attained at the point $y = a$ it follows that $F'(a) = 0$ as desired. To prove this, suppose that $F(a)$ is not the maximum value of F. Then there exists a number y_0 such that $F(y_0) > F(a)$. Setting $G(y) = \int_{y_0}^{y} f(x)dx$, we have $G(y_0) = 0$ and $G(y_0 + b - a) = F(y_0) > F(a)$. Since f is continuous, so is G, and the intermediate value theorem implies that, for some $y_1 \in (y_0, y_0 + b - a)$, $G(y_1) = F(a) = \alpha$. Since the interval $[y_0, y_1]$ has length $< (b - a)$ this contradicts the assumption that $[a, b]$ is an interval of minimal length with $\int_a^b f(x)dx = \alpha$. Hence the claim is proved.

Problem 3. Evaluate the infinite product $\prod_{k=1}^{\infty} \cos(x2^{-k})$. (Hint: $\sin 2\alpha = 2 \sin\alpha \cos\alpha$.)

Solution. The infinite product is, by definition, the limit of the partial products $P_n = \prod_{k=1}^{n} \cos(x2^{-k})$, as $n \to \infty$. If $x = 0$ this limit is 1. If $x \neq 0$, then the given identity yields

$$P_n = \prod_{k=1}^{n} \frac{\sin(x2^{-k+1})}{2\sin(x2^{-k})} = \frac{\sin x}{2^n \sin(x2^{-n})}$$

(provided n is large enough so that $x2^{-n}$ is not a multiple of π). Since, by l'Hopital's rule, $\lim_{y\to\infty}(\sin y)/y = 1$, we have $\lim_{n\to\infty} 2^n \sin(x2^{-n}) = x$. Hence the value of the given product is $\lim_{n\to\infty} P_n = (\sin x)/x$ if $x \neq 0$.

Problem 4. Let $S = \{0000000, 0000001, \ldots, 1111111\}$ be the set of all binary sequences of length 7. The **distance** of two elements $s_1, s_2 \in S$ is the number of places in which s_1 and s_2 differ. For example, 0001011 and 1001010 have distance 2, since they differ in positions 1 and 7. Show that if T is a subset of S having more than 16 elements then T contains two elements whose distance is at most 2.

Solution. With each element $t \in T$ we can associate a set $S_t \subset S$ consisting of the element t itself and the 7 elements of S that are obtained by switching exactly one of the digits of t. If the distance between any two elements of T were at least three, then the sets S_t would be disjoint, and we would have ($|A|$ denoting the cardinality of a set A) $|S| \geq \sum_{t\in T}|S_t| = 8|T|$, and therefore $|T| \leq |S|/8 = 2^7/8 = 16$. Thus, if $|T| > 16$, there must be two elements in T whose distance is at most 2.

Problem 5. Let a, b, c be real numbers > 1, and let

$$S = \log_a bc + \log_b ca + \log_c ab,$$

where $\log_b x$ denotes the base b logarithm of x. Find, with proof, the smallest possible value of S.

Solution. Setting $A = \log a$, $B = \log b$, $C = \log c$, we have $A, B, C > 0$ (since $a, b, c > 1$) and the sum S becomes $S = \frac{B}{A} + \frac{C}{A} + \frac{C}{B} + \frac{A}{B} + \frac{A}{C} + \frac{B}{C}$. By the arithmetic-geometric mean inequality, this sum is

$$\geq 6\left(\sqrt{\frac{B\cdot C\cdot C\cdot A\cdot A\cdot B}{A\cdot A\cdot B\cdot B\cdot C\cdot C}}\right)^{1/6} = 6.$$

Hence $S \geq 6$. The example $A = B = C = 1$ shows that this bound is attained, so 6 is the smallest possible value of S.

Problem 6. Suppose $0 \leq s < 1$, $\alpha, \beta > 0$, and $\lfloor\alpha\rfloor > \lfloor\beta\rfloor$. Let $\psi(\alpha, \beta; s)$ be the least positive integer n such that $\lfloor n\alpha + s\rfloor \neq \lfloor n\beta + s\rfloor$. Find an explicit formula for $\psi(\alpha, \beta; s)$ using the floor and ceiling functions. (The floor function $\lfloor x\rfloor$ denotes greatest integer $\leq x$ and the ceiling function $\lceil x\rceil$ denotes the least integer $\geq x$.)

Solution. Let $a = \lfloor\alpha\rfloor, b = \lfloor\beta\rfloor$, $\delta = \alpha - a$, $\gamma = \beta - b$. Then $a > b$ and $\lfloor n\alpha + s\rfloor = na + \lfloor n\delta + s\rfloor$, $\lfloor n\beta + s\rfloor = nb + \lfloor n\gamma + s\rfloor$, and thus

$$\lfloor n\alpha + s\rfloor - \lfloor n\beta + s\rfloor = n(a-b) + \lfloor n\delta + s\rfloor - \lfloor n\gamma + s\rfloor.$$

Since $a > b$ and $\lfloor n\gamma + s\rfloor \leq \lfloor n + s\rfloor \leq n$, the right-hand side is ≥ 0 for every positive integer n, and > 0 if and only if either (i) $a \geq b + 2$ or (ii) $a = b + 1$ and (at least) one of the conditions $n\delta + s \geq 1$ and $n\gamma + s < n$ holds. In the first case we have obviously $\Psi(\alpha, \beta; s) = 1$. In the second case,

$$\begin{aligned}\psi(\alpha, \beta; s) &= \min\{n \geq 1 : n\delta + s \geq 1 \text{ or } n\gamma + s < n\}\\ &= \min\left(\left\lceil\frac{1-s}{\delta}\right\rceil, \left\lfloor\frac{s}{1-\gamma}\right\rfloor + 1\right)\\ &= \min\left(\left\lceil\frac{1-s}{\alpha - \lfloor\alpha\rfloor}\right\rceil, \left\lfloor\frac{s}{1-\beta+\lfloor\beta\rfloor}\right\rfloor + 1\right).\end{aligned}$$

1997 UIUC UNDERGRAD MATH CONTEST

SOLUTIONS

Problem 1. Let abc represent a three digit number in base 10, with $a \geq c+2$. Let $abc - cba = efg$. Evaluate $efg + gfe$, for all a, b, c, as above.

Solution. Let n_1 denote the number abc, n_2 the number $efg = abc - cba$, and n_3 the number $efg + gfe$. Then $n_1 = 100a + 10b + c$ and $n_2 = 100(a-c) - (a-c) = 100(a-c-1) + 10 \cdot 9 + (10 - a + c)$, so that $e = a - c - 1$, $f = 9$, and $g = 10 - a + c$. (The given conditions on a, b, c ensure that e, f, g fall in the interval $[0, 9]$.) It follows that $n_3 = 101e + 20f + 101g = 101 \cdot 9 + 20 \cdot 9 = 1089$.

Problem 2. Each point in the plane is colored either orange or blue. Prove that one of these colors contains, for each positive value of d, a pair of points at distance d.

Solution. Suppose not. Then there exist positive numbers a and b such that no pair of orange points has distance a and no pair of blue points has distance b. Without loss of generality we may assume $a \leq b$. Now consider a blue point P; such a point has to exist, by our assumption that there are no two orange points at distance a from each other. Since no two blue points have distance b from each other, every point on the circle of radius b around P must be colored orange. Since $a \leq b$, there exist two points on this circle having distance a from each other. These two points are orange points at distance a, which contradicts our assumption. Thus, one of the colors must contain, for every positive distance d, a pair of points at distance d.

Problem 3. Mr. Wisenheimer evaluates on his calculator the expression $\frac{a}{b} - \frac{c}{d}$, where a, b, c, d are positive integers, each less than 1000. The calculator which is known to be accurate to within 10^{-11} for each arithmetic operation, gives the result 0.42857142857. Is Mr. Wisenheimer justified in reporting the answer as **exactly** 3/7? Explain.

Solution. Let $x = 0.42857142857$ be the decimal number displayed by the calculator, and let $\frac{r}{s} = \frac{a}{b} - \frac{c}{d}$ denote the number to be calculated, written as a reduced fraction. It is easy to check that the 11 digits of x after the decimal period represent the first 11 digits in the decimal representation of 3/7, so that x differs from 3/7 by at most 10^{-11}. On the other hand, since the calculator has an error of at most 10^{-11} for each arithmetic operation and the computation of $(r/s) = (a/b) - (c/d)$ involves three arithmetic operations, we know that x also differs from 3/7 by at most $3 \cdot 10^{-11}$. Hence (*) $|(3/7) - (r/s)| \leq 4 \cdot 10^{-11}$. Since b and d are positive integers, each less than 10^3, the denominator s in r/s is at most 10^6. Hence $(3/7) - (r/s) = (3s - 7r)/(7s)$ has denominator at most $7 \cdot 10^6$ and therefore is either 0 or at least $1/(7 \cdot 10^6)$. However, by (*) the second case is impossible, so r/s must be exactly equal to 3/7. Thus, Mr. Wisenheimer is correct in claiming that the result of his calculation is exactly 3/7.

Problem 4. Let $x_1 = x_2 = 1$, and $x_{n+1} = 1996x_n + 1997x_{n-1}$ for $n \geq 2$. Find (with proof) the remainder of x_{1997} upon division by 3.

Solution. First, an easy calculation (which is most conveniently done using congruences modulo 3, though one can do the problem without the use of congruences) shows that $x_1, \ldots, x_6$ have remainders $1, 1, 0, 2, 2, 0$, respectively, when divided by 3. Next, iterating the recurrence relation $x_n = 1996x_{n-1} + 1997x_{n-2}$ four times, one sees that for any $n \geq 6$, the remainder of x_n upon division by 3 is the same as that of x_{n-6}. (With congruences, this calculation amounts to $x_n \equiv x_{n-1} + 2x_{n-2} \equiv 3x_{n-2} + 2x_{n-3} \equiv 2x_{n-3} \equiv 4x_{n-6} \equiv x_{n-6}$.) By induction, it follows that the remainder of x_n upon division by 3 is equal to that of x_r, where r is the remainder of n upon division by 6. Since 1997 has remainder 5 when divided by 6, and x_5 has remainder 2, x_{1997} has remainder 2 when divided by 3.

Problem 5. Let f be a convex function with two continuous derivatives on $[0, 2\pi]$. Show that the integral $\int_0^{2\pi} f(x) \cos x dx$ is positive.

Solution. Since f is convex, $f''(x)$ is positive on $[0, 2\pi]$. Integrating by parts twice, we obtain

$$\begin{aligned}\int_0^{2\pi} f(x)\cos x &= (\sin x)f(x)\Big|_0^{2\pi} - \int_0^{2\pi} f'(x)\sin x dx \\ &= (\cos x)f'(x)\Big|_0^{2\pi} - \int_0^{2\pi} f''(x)\cos x dx \\ &= f'(2\pi) - f'(0) - \int_0^{2\pi} f''(x)\cos x dx = \int_0^{2\pi} f''(x)(1-\cos x)dx.\end{aligned}$$

The last integral is positive since the integrand $f''(x)(1 - \cos x)$ is strictly positive for $0 < x < 2\pi$.

Problem 6. Let $x_0 = 0$, $x_1 = 1$, and $x_{n+1} = \frac{x_n + nx_{n-1}}{n+1}$ for $n \geq 1$. Show that the sequence $\{x_n\}$ converges and find its limit.

Solution. Setting $d_n = x_{n+1} - x_n$, the recurrence relation for x_n translates into $d_n = -\frac{n}{n+1}d_{n-1}$ for $n \geq 1$. Iterating this identity gives

$$d_n = (-1)^n \frac{n}{n+1} \cdot \frac{n-1}{n} \cdots \frac{2}{3} \cdot \frac{1}{2} d_0 = \frac{(-1)^n}{n+1} d_0.$$

Hence

$$x_n = x_0 + \sum_{k=0}^{n-1} d_k = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1},$$

since $x_0 = 0$ and $d_0 = x_1 - x_0 = 1$. As an alternating series with decreasing terms, the series $\sum_{k=0}^{\infty} (-1)^k/(k+1)$ is convergent with sum $\ln(1+1) = \ln 2$, by the Taylor series expansion for $\ln(1+x)$. Hence the sequence $\{x_n\}$ converges with limit $\ln 2$.

1998 UIUC Undergraduate Math Contest

Solutions

Problem 1.

A sequence $a_0, a_1, a_2 \ldots$ of real numbers is defined recursively by

$$a_0 = 1, \quad a_{n+1} = \frac{a_n}{1 + na_n} \quad (n = 0, 1, 2, \ldots).$$

Find a general formula for a_n.

Solution.

Set $b_n = 1/a_n$. The given recurrence then takes the form

$$b_0 = 1, \quad b_{n+1} = b_n + n \quad (n = 0, 1, 2, \ldots).$$

Iterating this recurrence we obtain, for $n = 1, 2, \ldots,$

$$b_n = b_{n-1} + (n-1) = \cdots = b_0 + \sum_{k=0}^{n-1} k = 1 + \frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n + 1.$$

Hence

$$a_n = \frac{1}{b_n} = \left(\frac{1}{2}n^2 - \frac{1}{2}n + 1\right)^{-1} \quad (n = 1, 2, \ldots).$$

Problem 2.

Evaluate $\sum_{k=1}^n k2^{k-1}$ for $n = 1, 2, \ldots$

Solution.

Set $f(x) = \sum_{k=0}^n x^k$. Then $f'(x) = \sum_{k=1}^n kx^{k-1}$, so the given sum is equal to $f'(2)$. On the other hand, summing the geometric series in the definition of $f(x)$ gives $f(x) = (x^{n+1} - 1)/(x - 1)$, and differentiating this function we obtain

$$f'(x) = \frac{(x-1)(n+1)x^n - (x^{n+1} - 1)}{(x-1)^2}.$$

Hence

$$\sum_{k=1}^n k2^{k-1} = f'(2) = (n+1)2^n - 2^{n+1} + 1 = (n-1)2^n + 1.$$

Problem 3.

Given a nonempty finite set A of real numbers, let $m(A)$ denote the maximal element of A. For $n = 1, 2, \ldots$, let $f(n)$ be the sum of $m(A)$, where A runs over the $2^n - 1$ non-empty subsets of the set $\{1, 2, \ldots, n\}$. Give an explicit formula for $f(n)$.

Solution.

We have $f(n) = \sum_{k=1}^{n} kN(k)$, where $N(k)$ denotes the number of non-empty subsets of $\{1, 2, \ldots, n\}$ whose maximal element is k. Clearly, $N(1) = 1$, since the set $\{1\}$ is the only set of positive integers with maximal element 1. For $k \geq 2$, the sets counted by $N(k)$ are exactly those of the form $A = A_1 \cup \{k\}$, where A_1 is a (possibly empty) subset of $\{1, 2, \ldots, k-1\}$. Since there are exactly 2^{k-1} such sets A_1, it follows that $N(k) = 2^{k-1}$ for each k. Hence

$$f(n) = \sum_{k=1}^{n} k2^{k-1} = (n-1)2^n + 1,$$

by the result of the previous problem.

Problem 4.

Let $f(x)$ be a polynomial of degree n such that $f(k) = k/(k+1)$ for $k = 0, 1, \ldots, n$. Find $f(n+1)$.

Solution.

Set $g(x) = (x+1)f(x) - x$. Then $g(x)$ is a polynomial of degree $n+1$ which has roots at each of the $n+1$ numbers $0, 1, \ldots, n$. Hence $g(x)$ must be of the form

$$(*) \qquad g(x) = c \prod_{k=0}^{n} (x - k)$$

for some constant c. Setting $x = -1$ in the definition of $g(x)$ we obtain $g(-1) = 1$. By $(*)$ it follows that

$$1 = g(-1) = c \prod_{k=0}^{n} (-1-k) = c(-1)^{n+1}(n+1)!.$$

Hence $c = (-1)^{n+1}/(n+1)!$. Substituting this value into $(*)$ and setting $x = n+1$, we obtain

$$g(n+1) = c \prod_{k=0}^{n} (n+1-k) = \frac{(-1)^{n+1}}{(n+1)!}(n+1)! = (-1)^{n+1},$$

which implies

$$f(n+1) = \frac{g(n+1) + n + 1}{n+2} = \frac{n+1+(-1)^{n+1}}{n+2}.$$

Problem 5.

Let $x_1, x_2, \ldots, x_n$ be n real numbers satisfying

$$\sum_{k=1}^{n} x_k = 0, \quad \sum_{k=1}^{n} |x_k| = 1.$$

Prove that

$$\left| \sum_{k=1}^{n} \frac{x_k}{k} \right| \leq \frac{1}{2} - \frac{1}{2n}.$$

Solution.

The given conditions on x_k imply that, for any real number λ,

$$\left|\sum_{k=1}^{n} \frac{x_k}{k}\right| = \left|\sum_{k=1}^{n} x_k \left(\frac{1}{k} - \lambda\right)\right|$$
$$\leq \sum_{k=1}^{n} |x_k| \max_{k=1}^{n} \left|\frac{1}{k} - \lambda\right| = \max_{k=1}^{n} \left|\frac{1}{k} - \lambda\right|.$$

Taking $\lambda = 1/2 + 1/2n$, the maximum value of $|1/k - (1/2 + 1/2n)|$ is attained at $k = 1$ and $k = n$ and equal to $1/2 - 1/2n$. The desired inequality then follows.

Problem 6.

Suppose $n = a_1a_2 \ldots a_{1998}$ is the decimal representation of an integer n consisting of exactly 1998 non-zero digits $a_i \in \{1, 2, \ldots, 9\}$. Show that n is either divisible by 1998, or can be changed to an integer that is divisible by 1998 by replacing some, but not all, of the digits a_i by 0.

Solution.

Let $n_0 = 0$ and for $k = 1, 2, \ldots, 1998$ let n_k denote the number obtained by replacing all but the first k digits of n by the digit 0, i.e., $n_k = a_1a_2 \ldots a_k00 \ldots 0$. By the pigeon hole principle, two of the 1999 numbers $n_0, n_1, \ldots, n_{1998}$, say n_{k_1} and n_{k_2} with $k_2 > k_1$ must be congruent modulo 1998. It follows that the difference $n_{k_2} - n_{k_1}$ is divisible by 1998. Now $n_{k_2} - n_{k_1}$ is the number obtained from n by replacing the first k_1 and the last $(1998 - k_2)$ digits by 0. Since $k_2 > k_1$, we have $k_1 + (1998 - k_2) < 1998$, so not all of the 1998 digits are replaced by zero. Thus, the number $n_{k_2} - n_{k_1}$ has the required properties.

1999 UIUC Undergraduate Math Contest

Solutions

Problem 1.

Let a_n denote the integer closest to $\sqrt{n}$. (For example, $a_1 = a_2 = 1$ and $a_3 = a_4 = 2$ since $\sqrt{1} = 1$, $\sqrt{2} = 1.41\ldots$, $\sqrt{3} = 1.73\ldots$, and $\sqrt{4} = 2$.) Evaluate the sum

$$S = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{1980}}.$$

Solution. For any positive integer k, a_n is equal to k, if and only if $\sqrt{n}$ lies between $k - 1/2$ and $k + 1/2$, i.e., if and and only if n lies between $k^2 - k + 1/4$ and $k^2 + k + 1/4$. Since there are exactly $2k$ integer values in this range, and since $1980 = 44 \cdot 45 = 44^2 + 44$, it follows that $S = \sum_{k=1}^{44} (1/k) \cdot 2k = 88$.

Problem 2.

Let ABC be a triangle, and let BD and CE denote the angle-bisectors at B and C. Show that if BD and CE have the same length, then the triangle is isosceles (that is, the sides AB and AC have the same length).

Solution. Let $a = BC$, $b = AC$, $c = AB$ denote the three sides of the triangle, β and γ the angles of the triangle at B and C, and $d = BD = CE$ the (common) length of the angle-bisectors at these points. The area $\mathbf{A}$ of the triangle ABC is, on the one hand, $\mathbf{A} = (1/2)ac\sin\beta$. On the other hand, splitting ABC into the triangles BCD and BDA, which have areas $(1/2)ad\sin(\beta/2)$ and $(1/2)dc\sin(\beta/2)$, respectively, we obtain $\mathbf{A} = (1/2)d(a + c)\sin(\beta/2)$. Setting the two expressions for A equal and using the double angle formula $\sin\beta = 2\sin(\beta/2)\cos(\beta/2)$, it follows that (1) $2\cos(\beta/2) = d(1/a + 1/c)$. Similarly, interchanging the roles of B and C, we obtain (2) $2\cos(\gamma/2) = d(1/a + 1/b)$. If we now assume that b and c are not equal, say (without loss of generality) $b < c$, then $\beta < \gamma$ and so $\cos(\beta/2) > \cos(\gamma/2)$. However, by (1) and (2) this would imply $1/c > 1/b$, contradicting the assumption $b < c$. Hence b and c must be equal as claimed.

Problem 3.

Let a sequence $\{x_n\}$ be given by $x_1 = 1$ and $x_{n+1} = x_n^2 + x_n$ for $n = 1, 2, 3, \ldots$. Let $y_n = 1/(1 + x_n)$ and let $S_n = \sum_{k=1}^{n} y_k$ and $P_n = \prod_{k=1}^{n} y_k$ denote, respectively, the sum and the product of the first n terms of the sequence $\{y_k\}$. Evaluate $P_n + S_n$ for $n = 1, 2, 3, \ldots$.

Solution. From the given recurrence we obtain $x_{n+1} = x_n/y_n$, so that $y_n = x_n/x_{n+1}$ for all n. Hence $P_n = \prod_{k=1}^{n}(x_k/x_{k+1}) = x_1/x_{n+1} = 1/x_{n+1}$ for all n. Moreover, from the identity

$$y_n = \frac{1}{1 + x_n} = \frac{1}{x_n} - \frac{1}{(1 + x_n)x_n} = \frac{1}{x_n} - \frac{1}{x_{n+1}},$$

we see that $S_n = \sum_{k=1}^{n}(1/x_k - 1/x_{k+1}) = 1/x_1 - 1/x_{n+1} = 1 - 1/x_{n+1}$. Hence $P_n + S_n = 1$ for all n.

Problem 4.

Define a sequence $\{x_n\}$ by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2}^{\,x_n}$ for $n \geq 1$. Prove that the sequence $\{x_n\}$ converges and find its limit.

Solution. Since $x_1 = \sqrt{2} < 2$ and if $x_n < 2$ then $x_{n+1} = \sqrt{2}^{\,x_n} < \sqrt{2}^{\,2} = 2$, it follows by induction that (1) $x_n < 2$ for all n. Thus, the sequence $\{x_n\}$ is bounded from above. Next let $f(x) = \sqrt{2}^{\,x} - x$. Then $f'(x) = \sqrt{2}^x \log\sqrt{2} - 1 < 2\log\sqrt{2} - 1 < 0$ for $x < 2$, so $f(x)$ is decreasing for $x < 2$, and since $f(2) = 0$, this implies $f(x) > 0$, or equivalently $\sqrt{2}^x > x$, for $x < 2$. In view of (1), it follows that $x_{n+1} = \sqrt{2}^{\,x_n} > x_n$ for all n. Hence the sequence $\{x_n\}$ is monotone increasing and bounded from above and therefore must be convergent. Let L denote the limit of this sequence. By (1) we have (2) $L = \lim_{n\to\infty} x_n \leq 2$, and letting $n \to \infty$ on both sides of the recurrence $x_{n+1} = \sqrt{2}^{\,x_n}$, we obtain $L = \sqrt{2}^L$ or (3) $f(L) = 0$. Since $f(2) = 0$, $L = 2$ is a solution to (3). Moreover, $L = 2$ is the only solution satisfying (2), since $f(x)$ is decreasing for $x < 2$. Hence the limit of the sequence $\{x_n\}$ is 2.

Problem 5.

Prove that the series

$$\frac{1}{1} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \dots$$

converges and evaluate its sum.

Solution. Let S_n denote the sum of the first n terms of this sequence. Then

$$S_n = \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{[n/3]} \frac{3}{3k} = \sum_{k=[n/3]+1}^{n} \frac{1}{k}.$$

Let T_n denote the latter sum. Comparing this sum with an integral we see that

$$\log 3 - \log(1 + \frac{3}{n}) = \int_{n/3+1}^{n} \frac{1}{x} dx \leq T_n \leq \int_{n/3-1}^{n} \frac{1}{x} dx = \log 3 - \log(1 - \frac{3}{n})$$

Since $\log(1 \pm 3/n) \to 0$ as $n \to \infty$, we conclude that T_n, and therefore S_n, converges, and has limit $\log 3$. Hence the given infinite series converges with sum $\log 3$.

Problem 6.

Given positive integers n and m with $n \geq 2m$, let $f(n,m)$ be the number of binary sequences of length n (i.e., strings $a_1 a_2 \dots a_n$ with each a_i either 0 or 1) that contain the block 01 exactly m times. Find a simple formula for $f(n,m)$.

Solution. Every sequence of the required form can be written as $B_1C_101B_2C_201\dots01B_{m+1}C_{m+1}$, where each B_i is a block of 1's and each C_i a block of 0's, with empty blocks being allowed, and the sum of the lengths of the blocks B_i and C_i is $n-2m$. Moreover, the sequence is uniquely determined by the $(2m+2)$-tuple (1) $(b_1, c_1, b_2, c_2, \dots, b_{m+1}, c_{m+1})$ where b_i and c_i denote the number of elements in the blocks B_i and C_i, respectively. Conversely, any tuple of the form (1) with nonnegative integers b_i and c_i satisfying $\sum_{i=1}^{m+1}(b_i+c_i) = (n-2m)$ determines a sequence of the required type. Hence the number of such sequences is equal to the number of ways one can write $2n-m$ as a sum of $2m+2$) nonnegative integers, with order taken into account. The latter problem is equivalent to counting the number of ways of choosing $2n-m$ donuts from $2m+2$ varieties, a well-known combinatorial problem whose answer is given by the binomial coefficient $\binom{a}{b}$ with $a = (n-2m)+(2m+2)-1 = n+1$ and $b = (2m+2)-1 = 2m+1$. Hence $f(n,m) = \binom{n+1}{2m+1}$.

UIUC UNDERGRADUATE MATH CONTEST

APRIL 15, 2000, 10 am – 1 pm

SOLUTIONS

Problem 1

Suppose that a_1, a_2, ..., a_n are n given integers. Prove that there exist integers r and s with $0 \leq r < s \leq n$ such that $a_{r+1} + a_{r+2} + \ldots + a_s$ is divisible by n.

Solution. Let $S_0 = 0$ and for $m = 1, 2, \ldots, n$, let $S_n = a_1 + a_2 + \cdots a_n$. By the pigeonhole principle, two of these $n+1$ integers, say S_r and S_s (with $0 \leq r < s \leq n$), must leave the same remainder upon division by n. Hence $S_s - S_r = a_{r+1} + a_{r+2} + \cdots a_s$ is a multiple of n.

Problem 2

Let p be a point inside a triangle having sides of lengths a, b, c. Let h_a be the distance from p to the side of length a, and let h_b and h_c be defined analogously. Let $h := \min(h_a, h_b, h_c)$ and $s := (a + b + c)/2$. Prove that

$$h \leq \sqrt{(s-a)(s-b)(s-c)/s}.$$

Solution. We study the area of the triangle. Draw lines from p to the three vertices. The areas of the resulting three new triangles are $(1/2)ah_a$, $(1/2)bh_b$, $(1/2)ch_c$. Thus the area of the original triangle is *at least*

$$\frac{1}{2}(a+b+c)\min(h_a, h_b, h_c) = sh.$$

On the other hand, by Heron's formula, the area of the original triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

Thus, $sh \leq \sqrt{s(s-a)(s-b)(s-c)}$, which is equivalent to the asserted inequality.

Problem 3

Let f and g be twice continuously differentiable functions on $[0, 1]$ with $f(0) = g(0) = 0 = f(1) = g(1)$. Suppose that $0 < f(x) < g(x)$ for $0 < x < 1$ and that $f''(x) < 0$ for $0 < x < 1$. Show that

$$\int_0^1 f'(x)^2\,dx \leq \int_0^1 g'(x)^2\,dx.$$

Solution. Integrating by parts and using the assumptions $f(1) = g(1) = f(0) = g(0) = 0$, $f'' < 0$ and $f < g$, we have

$$\begin{aligned}\int_0^1 f'(x)^2 dx &= f(x)f'(x)\Big|_0^1 - \int_0^1 f(x)f''(x)dx < -\int_0^1 g(x)f''(x)dx \\ &= -g(x)f'(x)\Big|_0^1 + \int_0^1 g'(x)f'(x)dx = \int_0^1 g'(x)f'(x)dx\end{aligned}$$

Also, by Cauchy's inequality,

$$\left(\int_0^1 g'(x)f'(x)dx\right)^2 \leq \left(\int_0^1 g'(x)^2 dx\right)\left(\int_0^1 f'(x)^2 dx\right),$$

Thus,

$$\left(\int_0^1 f'(x)^2 dx\right)^2 \leq \left(\int_0^1 g'(x)^2 dx\right)\left(\int_0^1 f'(x)^2 dx\right),$$

and dividing by $\int_0^1 f'(x)^2 dx$ gives the result.

Problem 4

Prove that if a, b, and c are odd positive integers, then the polynomial $ax^2 + bx + c$ has no rational roots.

Solution. Suppose that $x = r/s$ is a rational solution in lowest terms of the polynomial equation $ax^2 + bx + c = 0$. Then $(*)$ $ar^2 + brs + cs^2 = 0$. Since the fraction r/s is in lowest terms, at most one of r and s can be even. If exactly one of r and s is even and the other is odd, then two of the three terms on the left of $(*)$ are even and the third term is odd, so their sum cannot equal 0. If both r and s are odd, then all three terms on the left of $(*)$ are odd, and their sum again cannot be equal to 0. Thus, the equation $ax^2 + bx + c = 0$ does not have a rational root.

Problem 5

Evaluate the infinite series

$$\frac{1}{2^1 - 2^{-1}} + \frac{1}{2^2 - 2^{-2}} + \frac{1}{2^4 - 2^{-4}} + \frac{1}{2^8 - 2^{-8}} + \cdots$$

Solution. Let $f(x) = \sum_{n=0}^{\infty}(x^{-2^n} - x^{2^n})^{-1}$, so that the given series is $f(1/2)$. Writing each term in this series as a product $x^{2^n}(1 - x^{2^{n+1}})^{-1}$ and expanding the second factor into a geometric series, we get

$$f(x) = \sum_{n=0}^{\infty} x^{2^n} \sum_{k=0}^{\infty} x^{2^{n+1}k} = \sum_{n,k=0}^{\infty} x^{2^n + 2^{n+1}k} = \sum_{n,k=0}^{\infty} x^{2^n(2k+1)}.$$

In the last series, the exponents $2^n(2k+1)$ are positive integers, and each positive integer occurs exactly once as such an exponent. Hence the last series is equal to $\sum_{m=1}^{\infty} x^m = x/(1-x)$, and so $f(x) = x/(1-x)$. The value of the given sum is therefore $f(1/2) = 1$.

[An alternative solution (found by David Dueber, Kaushik Roy, and Ken Scheiwe) consists in showing, by induction, that the sum of the first n terms of the given series is $1 - (2^{2^n} - 1)^{-1}$. Since this expression tends to 1 as $n \to \infty$, the sum of the series is equal to 1.]

Problem 6

Let $f(n)$ denote the number of 1's in the binary expansion of n. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)}.$$

Solution. Let S denote the sum of the given series (which converges since $f(n) \le 1 + \log_2 n$). Using the relations $f(2n) = f(n)$ and $f(2n+1) = f(2n) + 1 = f(n) + 1$ and splitting the series into odd and even parts, we obtain

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{f(2n)}{(2n)(2n+1)} + \sum_{n=0}^{\infty} \frac{f(2n+1)}{(2n+1)(2n+2)} \\ &= \sum_{n=1}^{\infty} \frac{f(n)}{(2n)(2n+1)} + \sum_{n=0}^{\infty} \frac{f(n)+1}{(2n+1)(2n+2)} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{(2n+1)(2n+2)} \right) + \sum_{n=1}^{\infty} \left(\frac{f(n)}{(2n)(2n+1)} + \frac{f(n)}{(2n+1)(2n+2)} \right) \\ &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \sum_{n=1}^{\infty} \frac{f(n)}{(2n)(n+1)} = \log 2 + \frac{1}{2} S. \end{aligned}$$

Hence, $S = \log 2 + S/2$, and so $S = 2 \log 2 = \log 4$.

UIUC UNDERGRADUATE MATH CONTEST
APRIL 21, 2001, 10 am – 1 pm
SOLUTIONS

Problem 1

Given a positive integer n, let n_1 be the sum of digits (in decimal) of n, n_2 the sum of digits of n_1, n_3 the sum of digits of n_2, etc. The sequence $\{n_i\}$ eventually becomes constant, and equal to a single digit number. Call this number $f(n)$. For example, $f(1999) = 1$ since for $n = 1999$, $n_1 = 28$, $n_2 = 10$, $n_3 = n_4 = \cdots = 1$. How many positive integers $n \leq 2001$ are there for which $f(n) = 9$?

Solution. Since an integer is divisible by 9 if and only if its sum of digits is divisible by 9, the numbers n with $f(n) = 9$ are exactly the multiples of 9. Since $2001 = 9 \cdot 222 + 3$, there are 222 such numbers below 2001.

Problem 2

Let x, y, and z be nonzero real numbers satisfying

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z}.$$

Show that $x^n + y^n + z^n = (x+y+z)^n$ for any odd integer n.

Solution. From the given relation one obtains, after clearing denominators and simplifying, $(x+y)(x+z)(y+z) = 0$. Hence $x = -y$, $x = -z$, or $y = -z$. In the first case, $x^n + y^n = 0$ for odd n, and so $x^n + y^n + z^n = (x+y+z)^n$. The other cases are analogous.

Problem 3

Suppose that an equilateral triangle is given in the plane, with none of its sides vertical. Let m_1, m_2, m_3 denote the slopes of the three sides. Show that

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = -3.$$

Solution. Let A, B, and C be the vertices of the triangle, labelled so that the path $ABCA$ is counter-clockwise, and let m_1, m_2, m_3 denote the slopes of the three sides AB, BC, and AC, respectively. Without loss of generality, we may assume that the vertex A is located at the origin. Then $m_1 = \tan\theta$, $m_2 = \tan(\theta + 2\pi/3)$, and $m_3 = \tan(\theta + \pi/3)$, where θ is the angle between the positive x axis and AB. (Note that $\theta \geq 0$ if the triangle lies entirely in the first quadrant; $\theta < 0$ if the triangle extends into the fourth quadrant.) Using the identity $\tan(x+y) = (\tan x + \tan y)/(1 - \tan x \tan y)$, we get, with $T = \tan\theta$,

$$m_1 m_2 = \frac{T(T-\sqrt{3})}{1+T\sqrt{3}}, \quad m_2 m_3 = \frac{(T-\sqrt{3})}{1+T\sqrt{3}} \cdot \frac{(T+\sqrt{3})}{1-T\sqrt{3}}, \quad m_3 m_1 = \frac{T(T+\sqrt{3})}{1-T\sqrt{3}}.$$

Adding these three terms and simplifying gives $m_1m_2+m_2m_3+m_3m_1 = -3$, independently of the value of T (and θ).

Problem 4

Let $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ be real numbers. Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \leq \frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}} + \frac{x_1}{x_n}.$$

Solution. Set $q_i = x_i/x_{i+1}$. Then $x_1/x_n = \prod_{i=1}^{n-1} q_i$, so the inequality to be proved can be written as

$$\sum_{i=1}^{n-1} q_i + \prod_{i=1}^{n-1} \frac{1}{q_i} \leq \sum_{i=1}^{n-1} \frac{1}{q_i} + \prod_{i=1}^{n-1} q_i,$$

or equivalently

$$(*) \qquad \sum_{i=1}^{n-1} \left(q_i - \frac{1}{q_i} \right) - \prod_{i=1}^{n-1} q_i + \prod_{i=1}^{n-1} \frac{1}{q_i} \leq 0.$$

Let $f(q_1, \ldots, q_{n-1})$ denote the function on the left of $(*)$. The hypothesis that the x_i are non-increasing implies that $q_i \geq 1$ for all i. Since $f(1, \ldots, 1) = 0$, to prove $(*)$ it therefore suffices to show that the partial derivatives of f are ≤ 0 when $q_i \geq 1$ for all i. This is indeed the case: we have

$$\begin{aligned} \frac{\partial f}{\partial q_i} &= 1 + \frac{1}{q_i^2} - \prod_{j \neq i} q_j - \frac{1}{q_i^2} \prod_{j \neq i} \frac{1}{q_j} \\ &= \left(1 + \frac{1}{q_i^2} \right) \left(1 - \prod_{j \neq i} q_j \right) \leq 0, \end{aligned}$$

since $\prod_{j \neq i} q_j \geq 1$. (When $n = 2$, this last product is empty, but in that case the sums and products on the left of $(*)$ reduce to a single term (corresponding to $i = 1$), and a direct computation shows that the derivative of f with respect to q_1 is equal to zero, so the last inequality remains valid for this case.)

Problem 5

Suppose that $q(x)$ is a polynomial satisfying the differential equation

$$7 \frac{d}{dx} \{xq(x)\} = 3q(x) + 4q(x+1), \quad -\infty < x < \infty.$$

Show that $q(x)$ is necessarily a constant.

Solution. The left-hand side of the given equation is $7xq'(x) + 7q(x)$, so the equation simplifies to

$$(*) \qquad 7xq'(x) = -4q(x) + 4q(x+1) = 4\int_x^{x+1} q'(t)dt.$$

The left-hand side of $(*)$ is zero at $x = 0$, so by the mean value theorem for integrals (which can be applied here since $q(x)$ is a polynomial and hence has continuous derivatives of all orders) there exists a number $x_1 \in (0,1)$ with $q'(x_1) = 0$. Setting $x = x_1$ in $(*)$, we obtain a number $x_2 \in (x_1, x_1+1)$ with $q'(x_2) = 0$. Repeating this process, we obtain an infinite sequence $x_1 < x_2 < \cdots$ of values x at which $q'(x) = 0$. Since q' is a polynomial, q' must be identically zero. Hence q is constant.

Alternative solution: The above solution was the one we had in mind when posing the problem. However, all students who correctly solved the problem, did so via the following approach (or a variant of it): Write $q(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_n \neq 0$. Then the left side of the differential equation is a polynomial of degree n with leading term $7a_n(n+1)x^n$, while the right-hand side has leading term $7a_n x^n$. Equating the coefficients of those terms, we obtain $7a_n(n+1) = 7a_n$; since $a_n \neq 0$, this can only hold when $n = 0$, i.e., when $q(x)$ is constant.

Problem 6

Evaluate the sum $\sum_{k=n}^{2n} \binom{k}{n} 2^{-k}$.

Solution. Let $S(n)$ denote the given sum. We claim that $S(n) = 1$ for all n. Since $S(1) = 1$, it suffices to show that $S(n+1) = S(n)$ for all n. Writing $k = n+1+h$ and using the identity $\binom{n+1+h}{n+1} = \binom{n+1+h}{h} = \binom{n+h}{h} + \binom{n+h}{h-1}$ for $h \geq 1$, we have

$$\begin{aligned}
2^{n+1}S(n+1) &= \sum_{h=0}^{n+1} \binom{n+1+h}{h} 2^{-h} = \sum_{h=0}^{n+1} \binom{n+h}{h} 2^{-h} + \sum_{h=1}^{n+1} \binom{n+h}{h-1} 2^{-h} \\
&= \sum_{h=0}^{n} \binom{n+h}{h} 2^{-h} + \binom{2n+1}{n+1} 2^{-n-1} + \sum_{h=0}^{n+1} \binom{n+1+h}{h} 2^{-h-1} - \binom{2n+2}{n+1} 2^{-n-2} \\
&= 2^n S(n) + 2^n S(n+1) + \left(\binom{2n+1}{n+1} - \frac{1}{2}\binom{2n+2}{n+1} \right) 2^{-n-1}.
\end{aligned}$$

Since $\binom{2n+2}{n+1} = \binom{2n+1}{n+1} + \binom{2n+1}{n} = 2\binom{2n+1}{n+1}$, the last term is zero, so we have $2^{n+1}S(n+1) = 2^n S(n) + 2^n S(n+1)$, and hence $S(n+1) = S(n)$, as claimed.

UIUC UNDERGRADUATE MATH CONTEST
APRIL 13, 2002, 10 am – 1 pm
SOLUTIONS

Problem 1

Without any numerical calculations, determine which of the two numbers e^π and π^e is larger.

Solution. Let $a = e^\pi$ and $b = \pi^e$. We will show that $a > b$. Since $\ln a = \pi \ln e = \pi$ and $\ln b = e \ln \pi$, and since taking logarithms preserves inequalities, we see that $a > b$ holds if and only if $(*)$ $(\ln e)/e > (\ln \pi)/\pi$. Now consider the function $f(x) = (\ln x)/x$. We have $f'(x) = (1 - \ln x)/x^2$, so f is decreasing for $x > e$, and since $\pi > e$, it follows that $f(\pi) < f(e)$ which is equivalent to $(*)$.

Problem 2

Let $0ABC$ be a tetrahedron with three right angles at the point O. Let S_A be the area of the face opposite to the point A, i.e., the area of the triangle OBC, and define S_B, S_C, and S_O analogously. Prove that $S_O^2 = S_A^2 + S_B^2 + S_C^2$.

Solution. Place the tetrahedron so that its vertices are located at $O = (0, 0, 0)$, $A = (a, 0, 0)$, $B = (0, b, 0)$, and $C = (0, 0, c)$. Clearly $S_A = bc/2$, $S_B = ac/2$, and $S_C = ab/2$. Moreover, the area S_O of the triangle ABC is $1/2$ times the area of the parallelogram determined by the vectors $AB = (-a, b, 0)$ and $AC = (-a, 0, c)$, which in turn is given by the magnitude of the cross product of these two vectors. Computing this cross product gives (bc, ac, ab), so

$$S_0^2 = (1/2)^2 \|(bc, ac, ab)\|^2 = \frac{1}{4}\left((bc)^2 + (ac)^2 + (ab)^2\right) = S_A^2 + S_B^2 + S_C^2.$$

Problem 3

Let $\theta_n = \arctan n$. Prove that, for $n = 1, 2, \ldots$,

$$\theta_{n+1} - \theta_n < \frac{1}{n^2 + n}.$$

Solution. Using the fact that $\arctan x$ has derivative $1/(1 + x^2)$, we obtain

$$\theta_{n+1} - \theta_n = \arctan(n+1) - \arctan n = \int_n^{n+1} \frac{dx}{1 + x^2}$$
$$< \int_n^{n+1} \frac{dx}{x^2} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

Problem 4

Determine, with proof, whether the (double) series

$$\sum_{(*)} \left(\frac{m}{n}\right)^{mn},$$

taken over all pairs (m, n) of positive integers satisfying

$$(*) \qquad n = 2, 3, \ldots, \quad m = 1, 2, \ldots, n-1$$

converges.

Solution. Split the range $(*)$ into the subranges (I) $n = 2, 3, \ldots, m \leq n/2$ and (II) $n = 2, 3, \ldots, n/2 < m \leq n-1$. It suffices to show that the series taken over each of these two ranges converge.

In the range (I), we have $(m/n)^{mn} \leq (1/2)^{mn}$. Summing this upper bound first over n (from $n = 2m$ to infinity) gives a geometric series with sum $(1/2)^{m(2m)}/(1-(1/2)^m)$ which is at most $(1/2)^m$ (with a lot to spare). Since $\sum_{n=1}^{\infty}(1/2)^m$ converges, so does the series $(*)$ over the subrange (I). To deal with the second subrange, we set $h = n - m$, so that $1 \leq h < n/2$ in the range (II). Using the bounds $(m/n) = (1 - h/n) \leq e^{-h/n}$ and $mn \geq n^2/2$, we obtain $(m/n)^{mn} \leq \exp\{-\frac{h}{n} \cdot \frac{1}{2}n^2\} = \exp\{-\frac{hn}{2}\}$. Summing the last term over n, from $n = 2h$ to infinity, gives again a geometric series, with sum $e^{-h^2}/(1-e^{-h/2})$ which is at most $e^{-h}(1-e^{-1/2})^{-1}$. Summing the latter bound, from $h = 1$ to infinity, we again obtain a convergent geometric series. Hence the series over the subrange (II) converges as well.

Problem 5

Let $a_1 = 2$, $a_2 = 4$, $a_3 = 8$, and for $n \geq 4$ define a_n to be last digit of the sum of the preceding **three** terms in the sequence. Thus the first few terms of this sequence of digits are (in concatenated form) 248468828.... Determine, with proof, whether or not the string 2002 occurs somewhere in this sequence.

Solution. First note that the sequence can be continued backwards in a unique manner by setting $a_{n-1} = a_{n+2} - a_{n+1} - a_n \mod 10$. Doing so, one finds that the first four terms prior to the given terms are 2, 0, 0, and 2. Thus, the string 2002 occurs in the extended sequence. To show that it also occurs in the original sequence (i.e., to the right of 2484...), note that the sequence is uniquely determined, backwards and forwards, by any three consecutive digits in the sequence. Since there are finitely many possibilities for such triples of consecutive digits, one such triple must occur again in the sequence, and the sequence is therefore is periodic (in both directions). In particular, any string that occurs somewhere in the extended sequence, occurs infinitely often and arbitrarily far out along the given sequence. Hence 2002 does occur in this sequence.

Problem 6

Call a set of integers A double-free if it does not contain two elements a and a' with $a' = 2a$. Determine, with proof, the size of the largest double-free subset of the set $\{1, 2, \ldots, 256\}$.

Solution. We will show that the maximal cardinality sought is 171. To prove that the cardinality cannot exceed 171, suppose $A \subset \{1, 2, \ldots, 256\}$ is double-free. Given any element $a \in A$, let a_0 denote the odd part of a, so that $a = a_0 2^i$ with a_0 odd and i a nonnegative integer. For each odd integer m, let A_m denote the set of $a \in A$ with $a_0 = m$. The sets A_m, $m = 1, 3, \ldots, 255$ partition the set A, so $|A| = |A_1| + |A_3| + \cdots + |A_{255}|$. To obtain an upper bound for $|A|$ we consider $|A_m|$ for different ranges of m.

If (1) $128 < m \leq 256$, then there can be at most one $a \in A$ with $a_0 = m$, namely $a = m$. Thus, the sum over $|A_m|$ for m in the range (1) is at most equal to the number of odd m in this range, i.e., 64.

If (2) $64 < m \leq 128$, then any $a \leq 256$ with $a_0 = m$ must be of the form $a = m$ or $a = 2m$, but because of the double-free condition at most one of these can belong to a. Hence $|A_m| \leq 1$ for m in the range (2), and the sum of $|A_m|$ over such m is at most 32.

If (3) $32 < m \leq 64$, then $a_0 = m$ implies that $a = m2^i$ with $i = 0$, 1, or 2, but the double-free condition again implies that at most two of these can belong to A. Hence $|A_m| \leq 2$ in the range (3), and the sum of $|A_m|$ over m in this range is at most $16 \cdot 2 = 32$.

Similarly, considering the ranges (4) $16 < m \leq 32$, (5) $8 < m \leq 16$, (6) $4 < m \leq 8$, (7) $2 < m \leq 4$ (i.e., $m = 3$), and (8) $m = 1$, we see that $|A_m|$ is at most 2 in the range (4), 3 in the ranges (5) and (6), 4 in the range (7), and 5 in the range 8, and the corresponding sums over $|A_m|$ are bounded by $8 \cdot 2 = 16$, $4 \cdot 3 = 12$, $2 \cdot 3 = 6$, $1 \cdot 4 = 4$, and $1 \cdot 5 = 5$, respectively. Adding up these bounds, we obtain

$$|A| \leq 64 + 32 + 32 + 16 + 12 + 6 + 4 + 5 = 171.$$

To show that this bound can be achieved, take A to be the set of integers $n \leq 256$ that are of the form $a_0 2^i$ with a_0 odd and $i = 0, 2, \ldots$. In this case, it is easy to check that the inequalities for $|A_m|$ in the above argument become equalities, and so we have $|A| = 171$.

UIUC UNDERGRADUATE MATH CONTEST
April 12, 2003
Solutions

1. Let

$$N = 9 + 99 + 999 + \cdots + \overbrace{99\ldots9}^{99}.$$

Determine the sum of digits of N.

Solution. The answer is 99. To see this, evaluate N explicitly as follows:

$$\begin{aligned} N &= (10-1) + (100-1) + \cdots + (\overbrace{10\ldots0}^{100} - 1) \\ &= \overbrace{11\ldots1}^{99}0 - 99 = \overbrace{11\ldots1}^{97}011. \end{aligned}$$

2. Evaluate

$$\frac{1}{1\cdot2\cdot3} + \frac{1}{3\cdot4\cdot5} + \frac{1}{5\cdot6\cdot7} + \cdots$$

Solution. Let S be the sum of the given series (which is easily seen to be convergent, e.g., by comparing it with the series $\sum_{n=1}^{\infty} n^{-2}$). We will show that $S = \ln 2 - 1/2$. Denoting by S_N the N-th partial sum of this series, we have

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{1}{2n(2n-1)(2n+1)} = \frac{1}{2}\sum_{n=1}^{N} \frac{1}{2n}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{2n-1} - 2\frac{1}{2n} + \frac{1}{2n+1}\right) \end{aligned}$$

Letting $U_N = \sum_{n=1}^{N} 1/(2n)$ and $V_N = \sum_{n=1}^{N} 1/(2n-1)$ denote the partial sums over the even resp. odd terms in the harmonic series, we can write the last expression as

$$S_N = \frac{1}{2}\left(V_N - 2U_N + V_N - 1 + \frac{1}{2N+1}\right) = V_N - U_N - \frac{1}{2} + \frac{1}{2(2N+1)}.$$

But

$$V_N - U_N = \sum_{n=1}^{2N} \frac{(-1)^{n+1}}{n},$$

which converges to $\ln 2$ as $N \to \infty$. Hence $S = \lim_{N\to\infty} S_N = \ln 2 - 1/2$, as claimed.

3. Prove that the limit

$$\lim_{n\to\infty} \frac{1}{n^2} \prod_{i=1}^{n} \left(n^2 + i^2\right)^{1/n}$$

exists and find its value.

Solution. We will show that the limit is equal to $2e^{-2+\pi/2}$. Let $P_n = n^{-2} \prod_{i=1}^{n} (n^2 + i^2)^{1/n}$. Factoring out $(n^2)^{1/n}$ from each term in the product, we see that $P_n = \prod_{i=1}^{n} (1 + (i/n)^2)^{1/n}$, and hence

$$\log P_n = \frac{1}{n} \sum_{i=1}^{n} \log \left(1 + \left(\frac{i}{n}\right)^2\right).$$

The term on the right is a Riemann sum for the integral $I = \int_0^1 \log(1+x^2)dx$, and therefore converges to this integral as $n \to \infty$, i.e., we have $\lim_{n\to\infty} \log P_n = I$. Hence the limit $\lim_{n\to\infty} P_n$ exists, and is equal to e^I. It remains to evaluate the integral I. This is a routine exercise in integration by parts:

$$\begin{aligned} I &= x \log(1 + x^2)\Big|_0^1 - \int_0^1 \frac{x(2x)}{1+x^2} dx \\ &= \log 2 - \int_0^1 \left(2 - \frac{2}{1+x^2}\right) dx \\ &= \log 2 - 2 + 2 \arctan x \Big|_0^1 = \log 2 - 2 + \frac{\pi}{2}. \end{aligned}$$

4. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers, and let b_n be the arithmetic mean of $a_1, a_2, \ldots, a_n$. Prove that if $\sum_{n=1}^{\infty} 1/a_n$ converges, then so does $\sum_{n=1}^{\infty} 1/b_n$.

Solution. Let $S_i = \sum_{2^i \le n < 2^{i+1}} a_n$. Then, for each $i \ge 0$ and $2^{i+1} \le n < 2^{i+2}$,

$$b_n = \frac{1}{n} \sum_{k=1}^{n} a_k \ge \frac{S_i}{2^{i+2}},$$

and so

$$\sum_{n=1}^{\infty} \frac{1}{b_n} \le \frac{1}{b_1} + \sum_{i=0}^{\infty} \sum_{2^{i+1} \le n < 2^{i+1}} \frac{2^{i+2}}{S_i} = \frac{1}{b_1} + 4 \sum_{i=0}^{\infty} \frac{2^{2i}}{S_i}.$$

Now, $S_i/2^i$ is the arithmetic mean of the 2^i numbers a_n, $2^i \le n < 2^{i+1}$. By the arithmetic-harmonic mean inequality, this is at least equal to the harmonic mean of these numbers, namely $2^i \left(\sum_{2^i \le n < 2^{i+1}} 1/a_n\right)^{-1}$. Hence,

$$\frac{2^{2i}}{S_i} \le \sum_{2^i \le n < 2^{i+1}} \frac{1}{a_n},$$

and it follows that

$$\sum_{n=1}^{\infty} \frac{1}{b_n} \le \frac{1}{b_1} + 4 \sum_{i=0}^{\infty} \frac{1}{a_n}.$$

Thus, the convergence of $\sum_{n=1}^{\infty} 1/a_n$ implies that of $\sum_{n=1}^{\infty} 1/b_n$.

5. Is it possible to arrange the numbers $1, 2, \ldots, 2003$ in a row so that each number, with the exception of the two numbers at the left and right end, is either the sum or the absolute value of the difference of the two numbers surrounding it?

   Solution. The answer is no, as can be seen by considering the parity of the numbers. The given condition implies that, except for the two integers at the left and right ends of the row, an integer n in the row must be surrounded by two odd integers or two even integers in case n is even, and by an odd and an even integer, in case n is odd. Thus, after reducing the numbers in the row modulo 2, the only blocks of three that can occur are 000, 011, 101, and 110. It follows that the entire row is determined (modulo 2) by its first two elements: If the row starts with 00, all elements in the row must be 0; if it starts with 01, the row is of the form 011011011...; if it starts with 10, it is of the form 101101101...; and if it starts with 11, it must be of the form 110110110.... In the first case, all numbers would have to be even, and in the other cases at least $2[2003/3] = 1334$ of the numbers would have to be odd. Since exactly 1002 of the given numbers are odd, we obtain a contradiction in all of these cases.

6. Find the smallest integer $n > 11$ for which there is a polynomial of degree n with the following properties:

   (a) $P(k) = k^{11}$ for $k = 1, 2, \ldots, n$;

   (b) $P(0)$ is an integer;

   (c) $P(-1) = 2003$.

   Solution. The answer is $n = 166$. To see this, suppose first that $P(x)$ is a polynomial of degree n satisfying the three conditions (a), (b), and (c). Consider the polynomial $Q(x) = P(x) - x^{11}$. Then $Q(x)$ has degree at most n, and condition (a) implies that $Q(x)$ has a root at each of the numbers $k = 1, 2, \ldots, n$. It follows that $Q(x)$ is of the form $Q(x) = C(x-1)(x-2)\ldots(x-n)$ for some constant C. To determine C, we use condition (c), which implies

$$2003 = P(-1) = Q(-1) + (-1)^{11} = C(-1)^n(n+1)! - 1.$$

Hence $C = 2004(-1)^n/(n+1)!$. Now,

$$P(0) = Q(0) = C(-1)^n n! = \frac{2004}{n+1},$$

so condition (b) holds if and only if $n+1$ is a divisor of 2004. Thus, any polynomial $P(x)$ of degree n satisfying all three conditions (a)–(c) is of the form $P(x) = C(x-1)(x-2)\ldots(x-n) + x^{11}$ with C as above and $n+1$ a divisor of 2004. Conversely, it is easy to see that any polynomial of this form satisfies (a)–(c). Therefore the number n sought in the problm is the smallest number $n > 11$ such that $n+1$ divides 2004, i.e., n is 1 less than the smallest divisor of 2004 that exceeds 13. Factoring 2004, we get $2004 = 2^2 \cdot 3 \cdot 167$. Hence, the smallest divisor of 2004 exceeding 13 is 167, and so $n = 166$.

UIUC UNDERGRADUATE MATH CONTEST
April 17, 2004
Solutions

1. Suppose a, b and c are integers such that the equation $ax^2+bx+c=0$ has a rational solution. Prove that at least one of the integers a, b and c must be even.

   Solution. We argue by contradiction. Suppose a, b, and c are all odd and that $x=p/q$, with $(p,q)=1$, is a rational solution of $ax^2+bx+c=0$. Clearing denominators, we obtain $(*)$ $ap^2+bpq+cq^2=0$. Since we assumed p and q are relatively prime, they cannot be both even. If p and q are both odd, then, in view of our initial assumption that a, b and c are odd, each term on the left of $(*)$ is odd, so the left-hand side is odd and we have a contradiction. If exactly one of p and q is odd, then exactly two of the three terms on the left of $(*)$ are even, and so the left-hand side is odd and we again arrive at a contradiction. Thus, a contradiction arises in either case, and a, b, and c therefore cannot all be odd.

2. Let F_n denote the Fibonacci sequence, defined by $F_1=1$, $F_2=1$, and $F_{n+1}=F_n+F_{n-1}$ for $n\geq 2$. Evaluate

$$\sum_{k=1}^{\infty}\frac{F_k}{3^k}.$$

   Solution. First note that, by induction, we have $F_n\leq 2^n$ for all n, so the given series is majorized by the geometric series $\sum_{k=1}^{\infty}(2/3)^k$, and hence is (absolutely) convergent. Let S denote the sum of this series. Using the Fibonacci recurrence for terms with $k\geq 3$, we obtain

$$\begin{aligned} S &= \frac{F_1}{3}+\frac{F_2}{3^2}+\sum_{k=3}^{\infty}\frac{F_{k-1}}{3^k}+\sum_{k=3}^{\infty}\frac{F_{k-2}}{3^k} \\ &= \frac{1}{3}+\frac{1}{9}+\frac{1}{3}\sum_{k=2}^{\infty}\frac{F_k}{3^k}+\frac{1}{9}\sum_{k=1}^{\infty}\frac{F_k}{3^k} \\ &= \frac{1}{3}+\frac{1}{9}+\frac{1}{3}\left(S-\frac{1}{3}\right)+\frac{1}{9}S. \end{aligned}$$

Solving for S we get

$$S = \frac{\frac{1}{3}}{1 - \frac{1}{3} - \frac{1}{9}} = \frac{3}{5}.$$

3. Define a sequence $\{a_n\}$ by $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, and

$$a_n = a_{n-1} + a_{n-2} - a_{n-3} + 1$$

for $n \geq 3$. Find, with proof, a_{2004}.

Solution. Let $b_n = a_n - a_{n-1}$. From the given recurrence for a_n we obtain

$$b_n = b_{n-2} + 1 \quad (n \geq 3)$$

with initial conditions $b_1 = 1$, $b_2 = 1$. This implies, by induction, $b_{2n} = b_{2n-1} = n$ for all $n \geq 1$. Hence

$$\begin{aligned} a_{2n} &= a_0 + \sum_{k=1}^{n}(a_{2k} - a_{2k-2}) = \sum_{k=1}^{n}(b_{2k} + b_{2k-1}) \\ &= \sum_{k=1}^{n}(2k) = 2\frac{n(n+1)}{2} = n(n+1). \end{aligned}$$

Hence $a_{2004} = 1002 \cdot 1003 = 1005006$.

4. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, where the a_k are real numbers. Suppose that $f(x)$ satisfies $|f(x)| \leq |\sin x|$ for all real x. Show that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

Solution. We have $f'(x) = \sum_{k=1}^{n} ka_k \cos kx$, and so $f'(0) = \sum_{k=1}^{n} ka_k$. Thus, the claim is equivalent to $|f'(0)| \leq 1$. Now, by the definition of the derivative, we have $f'(0) = \lim_{x\to 0} f(x)/x$, and since $|f(x)| \leq |\sin x| \leq |x|$ for all x, the inequality $|f'(0)| \leq 1$ follows.

5. Let

$$f(n) = \sum_{k=1}^{n} \left[\frac{n}{k}\right],$$

where $[x]$ denotes the greatest integer $\leq x$, and let $g(n) = (-1)^{f(n)}$. Find, with proof, $g(2004)$.

Solution. Note that $g(n) = 1$ or $g(n) = -1$ depending on whether $f(n)$ is even or odd. Since, for each k, $[n/k]$ counts the number of positive integers m for which $km \leq n$, the function $f(n)$ is equal to the number of pairs (k, m) of positive integers that satisfy $km \leq n$. Among these pairs, the number of those with $k \neq m$ is even since we can pair up (k, m) with (m, k). Hence, modulo 2, $f(n)$ is congruent to the number of remaining pairs in the above count, i.e., those of the form (k, k) with $k \le n$. Clearly, there are $[\sqrt{n}]$ such k, so we have $f(n) \equiv [\sqrt{n}]$ modulo 2, and therefore $g(n) = (-1)^{f(n)} = (-1)^{[\sqrt{n}]}$. Since $44^2 = 1936$ and $45^2 = 2025$, we have $[\sqrt{2004}] = 44$, so $g(2004) = (-1)^{44} = 1$.

6. Find, with proof, **all** functions $f(x)$ that are defined for real numbers x with $|x| < 1$, continuous at $x = 0$, and which satisfy

$$f(0) = 1, \quad f(x^2) = \frac{f(x)}{1+x} \quad (|x| < 1).$$

Solution. First note that the function $f(x) = 1/(1-x)$ satisfies the given conditions. We will show that this is the only solution. Suppose $f(x)$ is a solution, and let $g(x) = (1-x)f(x)$. Note first that since, by assumption, $f(x)$ is continuous at $x = 0$, $g(x)$ is also continuous at $x = 0$. From the given relation for $f(x)$ we obtain $g(0) = f(0) = 1$, and for $|x| < 1$,

$$g(x^2) = (1-x^2)f(x^2) = \frac{(1-x^2)f(x)}{1+x} = g(x).$$

Iterating this identity, we get

$$g(x) = g(x^{2^n})$$

for any positive integer n. Since, for $|x| < 1$, x^{2^n} tends to 0 as $n \to \infty$ and g is continuous at 0, it follows that

$$g(x) = \lim_{n \to \infty} g(x^{2^n}) = g(0) = 1$$

for all x with $|x| < 1$. Hence $f(x) = g(x)/(1-x) = 1/(1-x)$, so the function $1/(1-x)$ is indeed the only solution.

UIUC UNDERGRADUATE MATH CONTEST
April 16, 2005
Solutions

1. **For which positive integers n does the equation**

$$a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 0$$

have a solution in integers $a_i = \pm 1$? Explain!

Solution. If n is divisible by 4, then letting $a_1, a_2, \ldots, a_n$ be the pattern $(1, 1, -1, -1)$ repeated $n/4$ times, the terms a_ia_{i+1} in the above sum are alternately 1 and -1, and their sum is equal to 0. Thus, for all n divisible by 4, the given equation has a solution.

We now show that if n is not divisible by 4, there is no solution in integers ± 1. This is obvious in the case n is odd, since then the left-hand side of the equation consists of a sum of an odd number of terms, each ± 1, and thus cannot be equal to 0.

It remains to consider the case when $n = 2m$, where m is odd. Suppose there exist integers $a_i = \pm 1$, $i = 1, 2, \ldots, n$, for which the above equation holds. Set $a_{n+1} = a_1$, so that the equation can be written as $\sum_{i=1}^n a_ia_{i+1} = 0$. Since $n = 2m$ and each of the terms a_ia_{i+1} is ± 1, exactly m of these terms be equal to 1, and m must be equal to -1. Hence the product of all $2m$ terms must be equal to $(1)^m(-1)^m = -1$, since m was assumed to be odd. On the other hand, a direct calculation shows that the product is equal to

$$\prod_{i=1}^{n} a_ia_{i+1} = \prod_{i=1}^{n} a_i^2 = 1,$$

so we have reached a contradiction. Thus, no solution exists when $n = 2m$ with m odd.

2. **Evaluate the integral $I = \int_0^\pi \ln(\sin x)dx$.**

Solution. We will show that $I = -2\pi \ln 2$.

Using the identity $\sin x = 2\sin(x/2)\cos(x/2)$ we can write I as

$$\begin{aligned} I &= \int_0^\pi \ln 2 dx + \int_0^\pi \ln\sin(x/2)dx + \int_0^\pi \ln\cos(x/2)dx \\ &= \pi\ln 2 + 2\int_0^{\pi/2} \ln\sin y dy + 2\int_0^{\pi/2} \ln\cos y dy \\ &= \pi\ln 2 + 2I_1 + 2I_2, \end{aligned}$$

say. Setting $y = \pi/2 - u$ and using the relation $\cos(\pi/2 - u) = \sin u$, we see that $I_1 = I_2$, and since $\sin x = \sin(\pi - x)$, we have also

$$2I_1 = \int_0^{\pi/2} (\ln\sin x + \ln\sin(\pi - x))dx = \int_0^\pi \ln\sin x dx = I.$$

Hence the above relation implies $I = \pi\ln 2 + 4I_1 = \pi\ln 2 + 2I$, and solving for I gives $I = -\pi\ln 2$, as claimed.

3. **Suppose 3 players, P_1, P_2, P_3, seated at a round table, take turns rolling a die. Player P_1 rolls first, followed by P_2, etc. Once a player has rolled a 6, the game is stopped and that player is declared the winner. If no 6 has been obtained after each of P_1, P_2, P_3 have rolled the die once, player P_1 gets to roll again, followed by P_2, etc. Find the probability that the first player, P_1, wins this game.**

   Solution. Let N denote the number of "rounds" in the game, i.e., the number of rolls needed until a six shows up (including the roll at which the six shows up). Then the first player wins if and only if N is equal to one of the values $1, 4, 7, \ldots$, i.e., if N is of the form $N = 3k+1$, $k = 0, 1, \ldots$. Now, $N = n$ for a given value n if and only no 6 is rolled in the first $n-1$ rolls and a 6 appears in the n-th roll; the probability for this event is $p_n = (5/6)^{n-1}(1/6)$. Hence the probability that player P_1 wins is

$$\begin{aligned} \sum_{k=0}^\infty p_{3k+1} &= \sum_{k=0}^\infty (5/6)^{3k+1-1}(1/6) \\ &= \frac{1}{6}\sum_{k=0}^\infty ((5/6)^3)^k = \frac{1}{6(1-(5/6)^3)} = \frac{36}{91}. \end{aligned}$$

4. **Prove that, for any real numbers x and y in the interval $(0, 1)$,**

$$\left(\frac{x+y}{2}\right)^{x+y} \le x^x y^y.$$

Solution. Let $f(x) = \log x^x = x \log x$. Taking logarithms and dividing both sides by 2, the given inequality is equivalent to

$$f\left(\frac{1}{2}(x+y)\right) \leq \frac{1}{2}(f(x) + f(y)). \tag{1}$$

Now note that $f'(x) = \log x + 1$, and $f''(x) = 1/x > 0$ for positive x, so the function $f(x)$ is convex (i.e., concave up) for $x > 0$. Since any convex function satisfies (1), the result follows. (The fact that (1) holds for any convex function f is easily seen by sketching the graph of a convex function and comparing the value of the function at the midpoint of an interval $[x, y]$, with the average of the values of the function at the two end points x and y.)

5. **Determine, with proof, whether the series**

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.8+\sin n}}$$

converges or diverges.

Solution. We show that the series diverges. Note that $\sin x \leq -\sqrt{3}/2$ whenever x falls into one of the intervals

$$I_k = [(2k+4/3)\pi, (2k+5/3)\pi], \quad k = 0, \pm 1, \pm 2, \ldots$$

Each of these intervals has length $\pi/3 > 1$ and the gap between two successive intervals has length $< (5/3)\pi < 6$. Hence, among any 7 consecutive integers n at least one must fall into one of the intervals I_k; for this value of n we have $1.8 + \sin n < 1.8 - \sqrt{3}/2 < 1$, so the corresponding term in the above series is greater than $1/n$. Therefore the above series is bounded from below by

$$\sum_{m=0}^{\infty} \sum_{n=7m+1}^{7m+7} \frac{1}{n^{1.8+\sin n}} \geq \sum_{m=0}^{\infty} \frac{1}{7m+7} = \frac{1}{7} \sum_{m=1}^{\infty} \frac{1}{m} = \infty,$$

and hence diverges.

6. **Let $a_1, \ldots, a_n$ be a set of positive integers such that the product $\prod_{i=1}^{n} a_i$ has fewer than n distinct prime divisors. Prove that there exists a nonempty subset $I \subset \{1, \ldots, n\}$ such that $\prod_{i \in I} a_i$ is a perfect square.**

Solution. Let $p_1, \ldots, p_m$ denote the prime divisors of $\prod_{i=1}^{n} a_i$. Then each a_i can be written as $a_i = \prod_{j=1}^{m} p_j^{\alpha_{ij}}$, with nonnegative integers α_{ij}, and for any subset $I \subset \{1, 2, \ldots, n\}$, the product $\prod_{i \in I} a_j$ has prime factorization $\prod_{j=1}^{m} p^{\alpha_{Ij}}$, with $\alpha_{Ij} = \sum_{i \in I} \alpha_{ij}$. Such a product is a perfect square if and only if the exponents α_{Ij}, $j = 1, \ldots, m$, are all even, i.e., if and only if the system of congruences

$$\sum_{i \in I} \alpha_{ij} \equiv 0 \bmod 2, \quad j = 1, \ldots, m \tag{1}$$

holds. Setting $\epsilon_i = 1$ if $i \in I$, and $\epsilon_i = 0$ otherwise, (1) can be written as

$$\sum_{i=1}^{n} \alpha_{ij} \epsilon_i \equiv 0 \bmod 2, \quad j = 1, \ldots, m \tag{2}$$

Thus, we need to show that, if $m < n$, then the latter system has a solution $\epsilon_i \in \{0, 1\}$, with $\epsilon_i \neq 0$.

To this end we consider first the system

$$\sum_{i=1}^{n} \alpha_{ij} x_i = 0, \quad j = 1, \ldots, m, \tag{3}$$

in the variables $x_1, \ldots, x_n$. This is a system of m linear equations with integer coefficients in n variables. Since, by assumption, $m < n$, by elementary linear algebra this system has a solution in integers $x_1, \ldots, x_n$ that are not all zero. Dividing through by the greatest common divisor, we may further assume that the x_i's do not have a common prime divisor and, in particular, are not all even. Hence, defining $\epsilon_i = 1$ if x_i is even, and $\epsilon_i = 0$ otherwise, and reducing both sides of (3) modulo 2, we obtain a nontrivial solution $(\epsilon_1, \ldots, \epsilon_n)$ to (2).

UIUC UNDERGRADUATE MATH CONTEST
April 8, 2006
Solutions

1. *Determine,* without numerical calculations, *which of the two numbers* $\sqrt{2005}^{\sqrt{2006}}$ *and* $\sqrt{2006}^{\sqrt{2005}}$ *is larger.*

   Solution. We will show that the first of the two numbers, $\sqrt{2005}^{\sqrt{2006}}$, is the larger one. Taking the $(\sqrt{2005}\cdot\sqrt{2006})$-th root of the two given numbers, this amounts to showing that $\sqrt{2005}^{1/\sqrt{2005}}$ is larger than $\sqrt{2006}^{1/\sqrt{2006}}$. This will follow provided we can show that the function $f(x) = x^{1/x}$ is decreasing in an interval that includes $\sqrt{2005}$ and $\sqrt{2006}$. To do this, we compute the derivative of $f(x)$: Writing $f(x) = \exp\{(\ln x)/x\}$, we have, by the chain rule,

   $$f'(x) = \exp\left\{\frac{\ln x}{x}\right\}\left(\frac{x(1/x) - (\ln x)\cdot 1}{x^2}\right) = x^{1/x}\frac{1-\ln x}{x^2}.$$

   Thus we see that $f'(x)$ is negative, and hence $f(x)$ is decreasing, for $x > e$. Since $\sqrt{2005} > e$, this is what we need, with plenty of room to spare.

2. *Let* $f(x) = e^{x^2}\sin x$. *Find, with proof,* $f^{(2006)}(0)$, *the* 2006*th derivative of* f *at* 0.

   Solution. We use the connection between derivatives at 0 and coefficients of Taylor series: if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the Taylor expansion of f at 0, then $a_n = f^{(n)}(0)/n!$. Now, the Taylor series of $f(x) = e^{x^2}\sin x$ is the product of the Taylor series for e^{x^2} and $\sin x$, which are $\sum_{n=0}^{\infty} x^{2n}/n!$ and $\sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$, respectively. Since the first of these series involves only even powers of x, and the second involves only odd powers of x, their product contains only odd powers of x. Hence, all even-indexed coefficients in the Taylor series for $f(x)$ are 0 and so, in particular, $f^{(2006)}(0) = a_{2006}(2006)! = 0$.

3. *Evaluate the series*

   $$\sum_{n=0}^{\infty}\frac{1}{2006^{2^n} - 2006^{-2^n}} = \frac{1}{2006^1 - 2006^{-1}} + \frac{1}{2006^2 - 2006^{-2}} + \frac{1}{2006^4 - 2006^{-4}} + \cdots$$

   and express it as a rational number.

Solution. We claim that the given series is equal to 1/2005.

More generally, let

$$S(x) = \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}, \quad S_N(x) = \sum_{n=0}^{N} \frac{x^{2^n}}{1 - x^{2^{n+1}}}.$$

Note that setting $x = 1/2006$ in $S(x)$ gives the series of the problem. We will show that, for any x with $0 < x < 1$,

$$S(x) = \frac{x}{1 - x}, \tag{$*$}$$

and so, in particular, $S(1/2006) = (1/2006)(1 - 1/2006) = 1/2005$, as claimed.

We give two proofs for $(*)$.

First proof of $(*)$. We use a "telescoping" argument, based on the elementary identity

$$\frac{x^{2^n}}{1 - x^{2^{n+1}}} = \frac{1}{1 - x^{2^n}} - \frac{1}{1 - x^{2^{n+1}}}.$$

Substituting this into the partial sums $S_N(x)$ (we work with partial sums rather than the infinite series $S(x)$ in order to avoid possible convergence problems), we get

$$\begin{aligned} S_N(x) &= \left(\frac{1}{1-x} - \frac{1}{1-x^2} \right) + \left(\frac{1}{1-x^2} - \frac{1}{1-x^4} \right) \\ &\qquad + \cdots + \left(\frac{1}{1-x^{2^N}} - \frac{1}{1-x^{2^{N+1}}} \right) \\ &= \frac{1}{1-x} - \frac{1}{1-x^{2^{N+1}}}. \end{aligned}$$

As $N \to \infty$ here, the last term tends to 1, and we obtain

$$S(x) = \lim_{N \to \infty} S_N(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x},$$

proving $(*)$.

Second proof of $(*)$. A completely different proof of $(*)$ goes as follows: First expand each term in $S(x)$ into a geometric series:

$$S(x) = \sum_{n=0}^{\infty} x^{2^n} \sum_{k=0}^{\infty} x^{2^{n+1}k} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} x^{2^n(1+2k)}.$$

Note that all terms in this double series are positive, so we can rearrange the terms in this series. Since each positive integer has a unique decomposition as a power of 2 times an odd positive integer, the exponents $2^n(1 + 2k)$ in the latter double series correspond in a one-to-one manner to the positive integers, and the latter double sum, and hence $S(x)$, is therefore exactly $\sum_{m=1}^{\infty} x^m = x/(1 - x)$. This again gives $(*)$.

4. *For any positive integer n, define a sequence $\{n_k\}_{k=0}^{\infty}$ as follows: Set $n_0 = n$, and for each $k \geq 1$, let n_k be the sum of the (decimal) digits of n_{k-1}. For example, for $n = 1729$ we get the sequence $1729, 19, 10, 1, 1, 1, \ldots$. In general, for any given starting value n, the resulting sequence $\{n_k\}$ eventually stabilizes at a single digit value. Let $f(n)$ denote this value; for example, $f(1729) = 1$. Determine $f(2^{2006})$.*

   Solution. The key to this problem is the fact (which underlies the well-known disibility test by 9) that the sum of the decimal digits of a number has the same remainder modulo 9 as the number itself. Thus, each of the numbers n_k in the given sequence has the same remainder modulo 9. In particular, $f(n)$ has the same remainder modulo 9 as n, and since $f(n)$ must be among $\{1, 2, \ldots, 9\}$, the remainder of n modulo 9 determines $f(n)$ uniquely. Thus, it remains to determine the remainder of 2^{2006} upon division by 9. This is easy using congruence calculus: We have $2^6 = 64 \equiv 1 \bmod 9$, and so

$$2^{2006} = 4 \cdot 2^{2004} = 4 \cdot (2^6)^{334} \equiv 4 \cdot 1^{334} = 4 \bmod 9.$$

   Hence $f(2^{2006}) = 4$.

5. *Let $D = \{d_1, d_2, \ldots, d_{10}\}$ be a set of 10 distinct positive integers. Show that any sequence of 2006 integers from D contains a block of one or more consecutive terms whose product is the square of a positive integer.*

   Solution. Let a set D and a sequence $\{a_i\}_{i=1}^{2006}$ with $a_i \in D$ be given as in the problem. For $n = 1, 2, \ldots, 2006$ let $P_n = \prod_{i=1}^{n} a_i$ denote the product of the first n terms and set $P_0 = 1$,

   Note that any block of consecutive terms from the sequence $\{a_i\}$ is of the form $\prod_{i=m+1}^{n} a_i = P_n/P_m$ for some integers m and n with $0 \leq m < n \leq 2006$. Thus, the problem amounts to showing that, for a suitable choice of (m, n) with $0 \leq m < n \leq 2006$, P_n/P_m is a perfect square. Since each a_i is among the numbers $d_1, d_2, \ldots, d_{10}$, each P_n is of the form $P_n = \prod_{i=1}^{10} d_i^{\alpha_{in}}$, where the exponents α_{in} are nonnegative integers. (With $\alpha_{i0} = 0$ for $i = 1, 2, \ldots, 10$ this also holds for P_0.)

   Note that, by the definition of P_n as the product of the first n terms of the sequence $\{a_i\}$, the exponents α_{in} are non-decreasing in n, for each i. Thus, for $0 \leq m < n \leq 2006$, $P_n/P_m = \prod_{i=m+1}^{n} a_i = \prod_{i=1}^{n} d_i^{\alpha_{in} - \alpha_{im}}$, where the exponents $\alpha_{in} - \alpha_{im}$ are nonnegative integers. Hence P_n/P_m will certainly be a perfect square if the numbers $\alpha_{kn} - \alpha_{km}$, $k = 1, \ldots, 10$, are all even.

   The latter condition holds if and only if the vectors $\boldsymbol{\alpha}_n = (\alpha_{1n}, \ldots, \alpha_{10n})$ and $\boldsymbol{\alpha}_m = (\alpha_{1m}, \ldots, \alpha_{10m})$ have the same parity in each component. Now, since each $\boldsymbol{\alpha}_n$, $n = 0, 1, 2, \ldots, 2006$, is a vector with 10 components, there are $2^{10} = 1024$ possible parity combinations for these components. Since we have $2007 > 1024$ such vectors, by the pigeonhole principle two of these must have the same parity combination. Denoting the indices of these two vectors by m and n (ordered so that $0 \leq m < n \leq 2006$), we then have that P_n/P_m is a perfect square, as claimed.

6. *Given a real number α with $0 \leq \alpha < 1$, define an α-step a move of unit length in the xy-plane in the direction $2\pi\alpha$ (measured counterclockwise with respect to the horizontal). For example, if you are located at the origin, a $(1/2)$-step (corresponding to an angle of π, or 180 degrees) will put you at position $(-1, 0)$, a $(1/3)$-step (120 degrees) will place you at the point $(-1/2, \sqrt{3}/2)$, a $(1/4)$-step you will place you at $(0, 1)$, and after performing all three of these steps, you will be located at $(-1 + (-1/2) + 0, 0 + \sqrt{3}/2 + 1) = (-3/2, 1 + \sqrt{3}/2)$.*

   Suppose you start at the origin and perform a sequence of (p/q)-steps, where p and q range over all pairs of integers (p, q) with $1 \leq p < q \leq 2006$, giving a total of $2005 \cdot 2006/2 = 2011015$ steps of unit length. Where will you be located at the end of these 2011015 steps?

   Solution. At first glance, this problem looks impossibly difficult, but it becomes doable when approached the right way.

   If we think of the xy-plane as the complex plane with a point (x, y) corresponding to the complex number $z = x+iy$, then moving by an α-step corresponds to adding $e^{2\pi i\alpha}$ to the complex number corresponding to your current location. Thus, the location after performing the given sequence of moves is the determined by the sum

$$S = \sum_{q=2}^{2006} \sum_{p=1}^{q-1} e^{2\pi i \frac{p}{q}},$$

   so it remains to evaluate this double sum. To this end note that each of the inner sums here is a finite geometric series which is easily evaluated:

$$\sum_{p=1}^{q-1} \left(e^{2\pi i/q}\right)^p = \frac{e^{2\pi i(q/q)} - e^{2\pi i/q}}{e^{2\pi i/q} - 1} = -1.$$

   Hence, for each value of q, the corresponding inner sum contributes an amount -1 to S, and since there are 2005 values of q, we get $S = -2005$. This corresponds to the location $(-2005, 0)$ in the xy-plane.

U OF I UNDERGRADUATE MATH CONTEST
April 14, 2007
Solutions

1. *Let*
$$f(n) = (1^2 + 1)1! + (2^2 + 1)2! + \cdots + (n^2 + 1)n!.$$
Find a simple general formula for $f(n)$.

Solution. We will show by induction that

$$(*) \qquad f(n) = n(n+1)!$$

for $n = 1, 2, \ldots$. For $n = 1$, $(*)$ holds trivially. Let now $n \geq 1$, and assume that $(*)$ holds for this value of n. Then

$$\begin{aligned} f(n+1) &= f(n) + ((n+1)^2 + 1)(n+1)! \\ &= n(n+1)! + (n^2 + 2n + 2)(n+1)! \\ &= (n^2 + 3n + 2)(n+1)! \\ &= (n+1)(n+2)(n+1)! = (n+1)(n+2)!, \end{aligned}$$

which proves $(*)$ with $n + 1$ in place of n, completing the induction.

2. *Prove that for every odd integer n the sum $1^n + 2^n + \cdots + n^n$ is divisible by n^2.*

Solution. For $n = 1$, the assertion is trivially true. Suppose therefore that n is greater than 1 and odd. Then

$$1^n + 2^n + \cdots + n^n = \sum_{k=1}^{(n-1)/2} (k^n + (n-k)^n) + n^n$$

$$= \sum_{k=1}^{(n-1)/2} \left(k^n + n^n + \binom{n}{1} n^{n-1}(-k)^1 + \cdots + \binom{n}{n-1} n^1 (-k)^{n-1} + (-k)^n \right) + n^n.$$

Since n is odd, the terms k^n and $(-k)^n$ cancel each other out. Of the remaining terms, all divisible by n^2, so the assertion follows.

3. *For any positive integer k let $f_1(k)$ denote the sum of the squares of the digits of k (when written in decimal), and for $n \geq 2$ define $f_n(k)$ iteratively by $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{2007}(2006)$.*

Solution. Starting from $k = 2006$ and iterating the map "sum of squares of the digits" we obtain the chain $2006 \to 40 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4 \to 16$, after which the sequence repeats itself, with period 8. Thus, $f_1(2006) = 40$, $f_2(2006) = 16$, etc., and $f_{n+8}(2006) = f_n(2006)$ for all integers $n \geq 1$. Since $2007 = 8 \cdot 250 + 7$, it follows that follows that $f_{2007}(2006) = f_7(2006) = 42$.

4. *Determine, with proof, whether the series*

$$\sum_{n=1}^{\infty}\left(e-\left(1+\frac{1}{n}\right)^{n}\right)$$

converges.

Solution. We claim that the series diverges. To show this, we first derive a bound for $\ln(1+x)$, using the Taylor expansion $\ln(1+x)=\sum_{n=1}^{\infty}(-x)^{n+1}/n$. For $0<x<1$ the latter series is an alternating series with decreasing terms, so the successive partial sums of this series alternately undershoot and overshoot the limit, $\ln(1+x)$. In particular, we have, for $0<x<1$,

$$\ln(1+x)\leq x-\frac{x^2}{2}+\frac{x^3}{3}\leq x-\frac{x^2}{2}+\frac{x^2}{3}=x-\frac{x^2}{6}.$$

It follows that, for $n\geq 2$,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n}&=\exp\left\{n\ln\left(1+\frac{1}{n}\right)\right)\}\\ &\leq\exp\left\{n\left(\frac{1}{n}-\frac{1}{6n^2}\right)\right\}=e^{1-1/(6n)}.\end{aligned}$$

Hence, if a_n denotes the n-th term of the given series, we have the lower bound

$$a_n=e-\left(1+\frac{1}{n}\right)^{n}\geq e-e^{1-1/(6n)}=e\left(1-e^{-1/(6n)}\right)$$

By another application of the alternating series properties, we see that, for $0\leq x\leq 1$,

$$e^{-x}=\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\leq 1-x+\frac{x^2}{2}\leq 1-\frac{x}{2}.$$

Combining this with the above inequality for a_n gives

$$a_n\geq e\left(1-e^{-1/(6n)}\right)\geq e\left(1-\left(1-\frac{3}{n}\right)\right)=\frac{e}{12n},$$

for $n\geq 2$. Since the harmonic series $\sum_{n=1}^{\infty}1/n$ diverges, it follows that the series over a_n diverges as well.

Alternate solution (due to Ben Kaduk). By the binomial theorem, we have for $n\geq 2$

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n}&=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}\\ &=1+\sum_{k=1}^{n}\frac{1}{k!}\frac{n(n-1)\dots(n-k+1)}{n^k}\\ &\leq 1+\frac{1}{1!}+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\sum_{k=3}^{n}\frac{1}{k!}\\ &\leq\sum_{k=0}^{\infty}\frac{1}{k!}-\frac{1}{2n}=e-\frac{1}{2n}.\end{aligned}$$

Hence $a_n \geq 1/(2n)$ for $n \geq 2$, and comparison with the harmonic series yields the divergence of $\sum_{n=1}^{\infty} a_n$.

5. *Suppose $P_1, \ldots, P_{12}$ are points on the unit circle $x^2 + y^2 = 1$, and let*

$$S = S(P_1, \ldots, P_{12}) = \sum_{1 \leq i < j \leq 12} |P_i P_j|^2,$$

where $|P_i P_j|$ denotes the distance between P_i and P_j. In other words, S is the sum of the squares of the pairwise distances between the points $P_1, \ldots, P_{12}$. Determine, with proof, the largest possible value of S among all choices of the points $P_1, \ldots, P_{12}$ on the unit circle.

Solution. We represent the points P_i by complex numbers z_i. Points on the unit circle correspond to complex numbers of absolute value 1, and the distance between two such points is the absolute value of the difference between the corresponding complex numbers. We thus can write

$$S = S(z_1, \ldots, z_{12}) = \sum_{1 \leq i < j \leq 12} |z_i - z_j|^2$$

and the problem reduces to that of maximizing this function subject to the condition that $|z_i| = 1$ for all i.

Since $|z_i - z_i| = 0$ and $|z_i - z_j| = |z_j - z_i|$, if we extend the above sum over $1 \leq i < j \leq 12$ to *all* pairs of indices (i, j), with $1 \leq i, j \leq 12$, we count each summand twice. Thus, S is exactly equal to half this extended sum, i.e.,

$$S = \frac{1}{2} \sum_{i=1}^{12} \sum_{j=1}^{12} |z_i - z_j|^2$$

Using the assumption that $|z_i| = 1$, we can expand the summands into

$$\begin{aligned} |z_i - z_j|^2 &= (z_i - z_j)(\overline{z_i} - \overline{z_j}) \\ &= |z_i|^2 + |z_j|^2 - z_i \overline{z_j} - \overline{z_i} z_j = 2 - z_i \overline{z_j} - \overline{z_i} z_j. \end{aligned}$$

Substituting this into the above identity for S, we get

$$\begin{aligned} S &= \frac{1}{2} \sum_{i=1}^{12} \sum_{j=1}^{12} (2 - z_i \overline{z_j} - \overline{z_i} z_j) \\ &= \frac{1}{2} \left(2 \cdot 12^2 - 2 \left| \sum_{i=1}^{12} z_i \right|^2 \right) \end{aligned}$$

Thus, $S \leq 12^2 = 144$ for any choice of the numbers z_i (subject to $|z_i| = 1$), and $S = 144$ whenever

$$\sum_{i=1}^{12} z_i = 0.$$

The latter condition can be achieved, for example, by choosing the first 6 points, $z_1, \ldots, z_6$, arbitrarily on the unit circle, and letting the remaining 6 points be the points located diametrically opposite the 6 chosen points, i.e., $z_7 = -z_1, \ldots, z_{12} = -z_6$. Hence 144 is the maximal value of S.

6. *Let a_n ($n = 0, 1, \ldots$) be a bounded sequence of positive integers that satisfies*

$$a_n \left(a_{n-1}^2 + a_{n-2}^2 + \cdots + a_{n-2007}^2\right) = a_{n-1}^3 a_1 + a_{n-2}^3 a_2 + \cdots + a_{n-2007}^3 a_{2007} \quad (n \geq 2007).$$

Show that the sequence eventually becomes periodic.

Solution. Let $\mathbf{a}_n$ denote the 2007-tuple $(a_n, \ldots, a_{n-2006})$. The given recurrence can be rewritten as

$$a_n = \frac{a_{n-1}^3 a_1 + a_{n-2}^3 a_2 + \cdots + a_{n-2007}^3 a_{2007}}{a_{n-1}^2 + a_{n-2}^2 + \cdots + a_{n-2007}^2}. \tag{1}$$

Thus the value of $\mathbf{a}_{n-1}$, along with that of the (constant) tuple $\mathbf{a}_{2007} = (a_{2007}, \ldots, a_1)$, completely determines that of a_n, hence, by induction, the values of a_{n+k} and $\mathbf{a}_{n+k}$ for all $k \geq 0$.

Since the numbers a_n are bounded positive integers, there are only finitely many possible values for the tuple $\mathbf{a}_n$. By the pigeonhole principle, it therefore follows that there exist positive integers $n < m$ with $\mathbf{a}_n = \mathbf{a}_m$. In view of the above remark, this implies $a_{n+k} = a_{m+k}$ for all integers $k \geq 0$. Thus, a_n is eventually periodic with period $m - n$.

Note. In the original version of this problem, the assumption that the a_n be *positive* was missing. This assumption ensures that the denominator in (1) is positive, so the given recurrence can be written in the above form. Ben Kaduk constructed a example showing that, i if the a_n are allowed to be 0, the conclusion need not not hold.

U OF I UNDERGRADUATE MATH CONTEST
March 8, 2008
Solutions

1. *Does there exist a multiple of* 2008 *whose decimal representation involves only a single digit (such as* 11111 *or* 22222222*)?*

   Solution. The answer is yes; specifically, we will show that there exists a multiple of 2008 of the form $888\dots 8$.

   Given a digit $d \in \{1, 2, \dots, 9\}$, let $N_{d,k}$ be the number whose decimal representation consists of k digits d. Note that

   $$N_{d,k} = d\sum_{i=0}^{k-1} 10^i = \frac{d(10^k - 1)}{9}.$$

   Thus, a given positive integer m has a multiple of this form if and only if the congruence (1) $d(10^k - 1) \equiv 0 \bmod 9m$ has a solution k. We apply this with $d = 8$ and $m = 2008$. Then (1) is equivalent to (2) $10^k - 1 \equiv 0 \bmod 9(2008/8) = 9 \cdot 251$. Since $10^k \equiv 1^k = 1 \bmod 9$ for any positive integer k, (2) is equivalent to (3) $10^k \equiv 1 \bmod 251$. Now, 251 is prime, so by Fermat's Theorem, we have $10^{251-1} \equiv 1 \bmod 251$. Thus, (3) holds for $k = 250$, and so the number $N_{8,250} = \underbrace{88\dots 8}_{250}$ is divisible by 2008.

2. *What is the maximal value of the integral* $\int_0^1 f(x)x^{2008}dx$ *among all nonnegative continuous functions* f *on the interval* $[0, 1]$ *satisfying* $\int_0^1 f(x)^2 dx = 1$*?*

   Solution. Let $I = \int_0^1 f(x)x^{2008}dx$ be the integral whose maximum we seek. Applying the integral version of Cauchy-Schwarz with the functions $f(x)$ and x^{2008}, we get, for any f with $\int_0^1 f(x)^2 dx = 1$,

   $$I^2 \le \int_0^1 f(x)^2 dx \int_0^1 x^{4016} dx = \frac{1}{4017}.$$

   Thus, $I \le 1/\sqrt{4017}$. Moreover, this upper bound is achieved by taking $f(x) = cx^{2008}$, with $c = \sqrt{4017}$ (so that f satisfies $\int_0^1 f(x)^2 dx = 1$). Thus, $1/\sqrt{4017}$ is the maximal value for I.

3. *Find, with proof, all functions* $f : \mathbf{R} \to \mathbf{R}$ *satisfying*

   $$|f(x + y) - f(x - y) - y| \le y^2$$

   for all $x, y \in \mathbf{R}$.

Solution. Any function of the form $f_c(x) = x/2 + c$, where c is a constant, satisfies $f_c(x+y) - f_c(x-y) - y \equiv 0$ for all x and y, and hence trivially satisfies the above inequality. We will show that these are the only such functions.

Suppose $f(x)$ is solution to the given inequality. Set $g(x) = f(x) - x/2$. Then

$$\begin{aligned}|g(x+y) - g(x-y)| &= |f(x+y) - (x+y)/2 - f(x-y) + (x-y)/2 - y| \\ &= |f(x+y) - f(x-y) - y| \le y^2\end{aligned}$$

for all $x, y \in \mathbf{R}$. Dividing by y and letting $y \to 0$, we conclude

$$\lim_{y\to 0} \frac{g(x+y) - g(x-y)}{y} = \lim_{y\to 0} y = 0,$$

for all $x \in \mathbf{R}$. Thus g is a differentiable function, with derivative equal to 0 everywhere. It follows that g must be a constant function, say, $g(x) = c$ for all x. Therefore $f(x) = x/2 + g(x) = x/2 + c$, as claimed.

4. *Let $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, and for $n \ge 4$ define a_n to be the last digit of the sum of the preceding* **three** *terms in the sequence. Thus the first few terms of this sequence of digits are (in concatenated form) 124734419447.... Determine, with proof, whether or not the string 1001 occurs in this sequence. (Hint: Do* **not** *attempt this by brute force!)*

   Solution. First note that the sequence can be continued backwards in a unique manner by setting $a_{n-1} = a_{n+2} - a_{n+1} - a_n \mod 10$. Doing so, one finds that the first four terms prior to the given terms are 1, 0, 0, and 1. Thus, the string 1001 occurs in the extended sequence. To show that it also occurs in the original sequence (i.e., to the right of 1247...), note that the sequence is uniquely determined, backwards and forwards, by any three consecutive digits in the sequence. Since there are finitely many possibilities for such triples of consecutive digits, one such triple must occur again in the sequence, and the sequence is therefore is periodic (in both directions). In particular, any string that occurs somewhere in the extended sequence, occurs infinitely often and arbitrarily far out along the given sequence. Hence 1001 does occur infinitely often in this sequence. (While this term occurs immediately to the left of the given initial string 1247..., its first occurrence to the right is at the 120-th term. This would be hard to discover by a hand calculation!)

5. *Let n be a positive integer, and denote by S_n the set of all permutations of $\{1, 2, \dots, n\}$. Given a permutation $\sigma \in S_n$, define its* **perturbation index** *$P(\sigma)$ as*

   $$P(\sigma) = \#\{k \in \{1, \dots, n\} : \sigma(k) \ne k\};$$

   i.e., $P(\sigma)$ denotes the number of elements in $\{1, \dots, n\}$ that are "perturbed" by σ, in the sense of being mapped to a different element. Find the average perturbation index of a permutation in S_n, i.e.,

   $$\frac{1}{\#S_n} \sum_{\sigma \in S_n} P(\sigma).$$

Solution. We have

$$\begin{aligned}
\sum_{\sigma\in S_n} P(\sigma) &= \sum_{\sigma\in S_n} \sum_{\substack{k=1\\ \sigma(k)\neq k}}^{n} 1\\
&= \sum_{k=1}^{n} \sum_{\substack{\sigma\in S_n\\ \sigma(k)\neq k}} 1\\
&= \sum_{k=1}^{n} \#\{\sigma\in S_n : \sigma(k)\neq k\}\\
&= \sum_{k=1}^{n} (n! - \#\{\sigma\in S_n : \sigma(k)= k\})\\
&= \sum_{k=1}^{n} (n! - (n-1)!)\\
&= (n-1)n!,
\end{aligned}$$

since, for any k, there are exactly $(n-1)!$ permutations in S_n that fix k. Since $\#S_n = n!$, it follows that the average perturbation index is $n-1$.

6. *Let $\mathcal{A}$ be a collection of 100 distinct, nonempty subsets of the set $\{0,1,\dots,9\}$. Show that there exist two (distinct) sets $A, A' \in \mathcal{A}$ whose symmetric difference has at most two elements. (The symmetric difference of two sets A and A' is defined as the set of elements that are in one of the two sets, but not in both, i.e., the set $(A\cup A')\setminus(A\cap A')$.)*

   Solution. Let $A\Delta B$ denote the symmetric difference of two sets A and B, and let $d(A,B) = |A\Delta B|$ denote the number of elements in $A\Delta B$. It is easy to see that the function d satisfies the triangle inequality:

   $$d(A,C) \le d(A,B) + d(B,C). \tag{1}$$

   Now, let $\mathcal{A}$ be a collection of subsets of $\{0,1,\dots,9\}$ with $|\mathcal{A}| = 100$. In the above terminology, we need to show that if $\mathcal{A}$ has at least 100 elements then it contains two elements, A and A', such that $d(A,A') \le 2$.

   Given $A\in\mathcal{A}$, define a "neighborhood" of A by

   $$\mathcal{U}(\mathcal{A}) = \{B \subset \{0,1,\dots,9\} : B\neq\emptyset, d(B,A)\le 1\};$$

   i.e., $\mathcal{U}(\mathcal{A})$ consists all nonempty subsets of $\{0,1,\dots,9\}$ whose symmetric difference with A has at most 1 element. We are going to estimate the sum of the cardinalities of these "neighborhoods",

   $$S = \sum_{A\in\mathcal{A}} |\mathcal{U}(\mathcal{A})|.$$

To this end, note that $\mathcal{U}(\mathcal{A})$ consists of the following sets: (i) the set A itself, (ii) any proper nonempty subset of A obtained by removing exactly one element from A, and (iii) any set obtained by adding to A exactly one element from $\{0, 1, \ldots, 9\}$.

If A has k elements with $k \geq 2$, then there are exactly k sets of type type (ii), and $(10 - k)$ sets of type (iii), so $\mathcal{U}(\mathcal{A})$ has exactly $1 + k + (10 - k) = 11$ elements. If A has 1 element, then there is no set of type (ii) (since removing the single element of A would leave an empty set), so in that case $\mathcal{U}(\mathcal{A})$ has 10 elements. Setting

$$\mathcal{A}_1 := \{A \in \mathcal{A} : |A| = 1\}, \quad \mathcal{A}_2 := \mathcal{A} \setminus \mathcal{A}_1$$

we therefore have

$$\begin{aligned} S &= 10|\mathcal{A}_1| + 11|\mathcal{A}_2| \\ &= 11|\mathcal{A}| - |\mathcal{A}_1| \\ &\geq 11 \cdot 100 - 10 = 1090 \end{aligned} \tag{2}$$

by our assumption $|\mathcal{A}| = 100$ and the trivial bound $|\mathcal{A}_1| \leq 10$, since there are 10 one element subsets of $\{0, 1, \ldots, 9\}$.

On the other hand, if the sets $\mathcal{U}(\mathcal{A})$, $A \in \mathcal{A}$, were pairwise disjoint, we would have

$$\begin{aligned} S &= \left| \bigcup_{A \in \mathcal{A}} \mathcal{U}(\mathcal{A}) \right| \\ &\leq |\{B \subset \{0, 1, \ldots, 9\} : B \neq \emptyset\}| \\ &= 2^{10} - 1 = 1023, \end{aligned} \tag{3}$$

contradicting the lower bound (2). Thus, two of these sets, say $\mathcal{U}(\mathcal{A})$ and $\mathcal{U}(\mathcal{A}')$, must have a nonempty intersection. By the definition of the neighborhoods $\mathcal{U}$ this means that there exists a nonempty subset $B \subset \{0, 1, \ldots, 9\}$ such that $d(B, A) \leq 1$ and $d(B, A') \leq 1$. By the triangle inequality (1), this implies

$$d(A, A') \leq d(A, B) + d(B, A') \leq 1 + 1 \leq 2.$$

which is what we wanted to show.