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ROMANIAN MATHEMATICAL COMPETITIONS

SOCIETATEA DE ȘTIINȚE MATEMATICE DIN ROMÂNIA

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MIHAI BĂLUNĂ
National College "*Mihai Viteazul*"
Bucharest, Romania

MIRCEA BECHEANU
University of Bucharest,
Dept. of Mathematics
Bucharest, Romania

BOGDAN ENESCU
National College "*B. P. Hasdeu*"
Buzău, România

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SECTION 1

49^{TH} NATIONAL MATHEMATICAL OLYMPIAD

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A. PROPOSED PROBLEMS

7^{th} Form

7.1. Let n be a positive integer and $x_1, x_2, ..., x_n$ be integer numbers such that

$$x_1^2 + x_2^2 + \ldots + x_n^2 + n^3 \leq (2n-1)(x_1 + x_2 + \ldots + x_n) + n^2 .$$

Show that :

a) $x_1, x_2, ..., x_n$ are non-negative integers

b) the number $x_1 + x_2 + ... + x_n + n + 1$ is not a perfect square.

S. Smarandache

7.2. Show that there is no positive integer n such that $n + k^2$ is a perfect square for at least n positive integer values of k.

V. Zidaru

7.3. In the exterior of the triangle ABC with $m(\angle B) > 45°$, $m(\angle C) > 45°$ one constructs the right isosceles triangles ACM and ABN such that $m(\angle CAM) = m(\angle BAN) = 90°$ and, in the interior of ABC, the right isosceles triangle BCP, with $m(\angle P) = 90°$. Show that the triangle MNP is a right isosceles triangle.

B. Enescu

7.4. Let $ABCD$ be a rectangle and let $E \in (BD)$ such that $m(\angle DAE) = 15°$. Let $F \in AB$ such that $EF \perp AB$. It is known that $EF = \frac{1}{2} AB$ and $AD = a$. Find the measure of the angle $\angle EAC$ and the length of the segment (EC).

S. Peligrad

8^{th} Form

8.1. Let a be a real number and $A = \{(x, y) \in \mathbf{R} \times \mathbf{R} | x + y = a\}$, $B = \{(x, y) \in \mathbf{R} \times \mathbf{R} | x^3 + y^3 < a\}$. Find all values of a such that $A \cap B = \varnothing$.

R. Ilie

8.2. Let $P(X)=a_{1998}X^{1998}+a_{1997}X^{1997}+\ldots+a_1X+a_0$ be a polynomial with real coefficients such that $P(0)\neq P(-1)$, and let a, b be real numbers.

Let $Q(X)=b_{1998}X^{1998}+b_{1997}X^{1997}+\ldots+b_1X+b_0$ be the polynomial with real coefficients obtained by taking $b_k=aa_k+b$, $\forall\, k=0, 1, 2,\ldots, 1998$. Show that if $Q(0)=Q(-1)\neq 0$, then the polynomial Q has no real roots.

M. Fianu and *Şt. Alexe*

8.3. In the right-angled trapezoid $ABCD$, $AB \parallel CD$, $m(\angle A)=90°$, $AD = DC = a$ and $AB = 2a$. On the perpendiculars raised in C and D on the plane containing the trapezoid one considers points E and F (on the same side of the plane) such that $CE = 2a$ and $DF = a$.

Find the distance from the point B to the plane (AEF) and the measure of the angle between the lines AF and BE.

R. Popovici and *N. Solomon*

8.4. Let $ABCD$ be an arbitrary tetrahedron. The bisectors of the angles $\angle BDC$, $\angle CDA$ and $\angle ADB$ intersect BC, CA and AB, in the points M, N, P, respectively.

a) Show that the planes (ADM), (BDN) and (CDP) have a common line d.

b) Let the points $A'\in(AD)$, $B'\in(BD)$ and $C'\in(CD)$ be such that $(AA')\equiv(BB')\equiv(CC')$; show that if G and G' are the centroids of ABC and $A'B'C'$ then the lines GG' and d are either parallel or identical.

M. Miculiţa

9th Form

9.1. Find the integer numbers a, b, c such that the function $f:\mathbf{R}\to\mathbf{R}$, $f(x)=ax^2+bx+c$ satisfies the equalities :

$$f(f(1))=f(f(2))=f(f(3)).$$

C. Mortici and *M. Chiriţă*

9.2. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AC-BD|\leq|AB-CD|.$$

When does the equality hold ?

D. Miheţ

9.3. Find the rational roots (if any) of the equation

$$abx^2+(a^2+b^2)x+1=0,\ (a,\ b\in\mathbf{Z}).$$

D. Popescu

9.4. Let $A_1A_2...A_n$ be a regular polygon ($n > 4$), T be the common point of A_1A_2 and $A_{n-1}A_n$ and M be a point in the interior of the triangle A_1A_nT.

Show that the equality

$$\sum_{i=1}^{n-1} \frac{\sin^2(\angle A_iMA_{i+1})}{d(M, A_iA_{i+1})} = \frac{\sin^2(\angle A_1MA_n)}{d(M, A_1A_n)}$$

holds if and only if M belongs to the circumcircle of the polygon.

D. Brânzei

10th Form

10.1. Let $n \geq 2$ be an integer and $M = \{1, 2, ..., n\}$. For every $k \in \{1, 2, ..., n-1\}$ we denote by $x_k = \frac{1}{n+1} \sum_{\substack{A \subset M \\ |A|=k}} (\min A + \max A)$.

Show that $x_1, x_2, \ldots, x_{n-1}$ are integer numbers, not all divisible by 4.

V. Zidaru

10.2. Let $a \geq 1$ be a real number and z be a complex number such that $|z + a| \leq a$ and $|z^2 + a| \leq a$. Show that $|z| \leq a$.

D. Şerbănescu

10.3. Let $ABCD$ be a tetrahedron and A', B', C' be arbitrary points on the edges $[DA]$, $[DB]$, $[DC]$, respectively. One considers the points $P_c \in [AB]$, $P_a \in [BC]$, $P_b \in [AC]$ and $P'_c \in [A'B']$, $P'_a \in [B'C']$, $P'_b \in [A'C']$ such that

$$\frac{P_cA}{P_cB} = \frac{P'_cA'}{P'_cB'} = \frac{AA'}{BB'}; \frac{P_aB}{P_aC} = \frac{P'_aB'}{P'_cC'} = \frac{BB'}{CC'}; \frac{P_bC}{P_bA} = \frac{P'_bC'}{P'_aA'} = \frac{CC'}{AA'}.$$

Prove that :

a) the lines AP_a, BP_b, CP_c have a common point P and the lines $A'P'_a$, $B'P'_b$, $C'P'_c$ have a common point P' ;

b) $\frac{PC}{PP_c} = \frac{P'C'}{P'P'_c}$;

c) if A', B', C' are variable points on the edges $[DA]$, $[DB]$, $[DC]$, then the line PP' is always parallel to a fixed line.

Mihai Miculiţa

10.4. Let $n \geq 2$ and $0 < x_1 < x_2 < \ldots < x_n$ be integer numbers. Let

$$s_k = \sum_{\substack{A \subset \{x_1, x_2, \ldots, x_n\} \\ A \neq \emptyset}} \frac{1}{\prod_{a \in A} a}, \quad k = 1, 2, \ldots, n.$$

Show that if s_n and s_{n-1} are positive integers, then s_k is a positive integer for every k.

D. Miheț

11th Form

11.1. The non-zero matrices $A_0, A_1, \ldots, A_n \in \mathcal{M}_2(\mathbf{R})$, $n \geq 2$ have the following properties : $A_0 \neq aI_2$, $\forall\ a \in \mathbf{R}$ and $A_0A_k = A_kA_0$, $\forall\ k = 1, 2, \ldots, n$. Show that :

a) $det\left(\sum_{k=1}^{n} A_k^2\right) \geq 0$;

b) if $det\left(\sum_{k=1}^{n} A_k^2\right) = 0$ and $A_2 \neq aA_1$, $\forall\ a \in \mathbf{R}$, then $\sum_{k=1}^{n} A_k^2 = O_2$.

V. Pop

11.2. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that the sequence $x_n = \sum_{k=1}^{n} a_k^2$ is convergent and the sequence $y_n = \sum_{k=1}^{n} a_k$ is unbounded. Prove that the sequence $(b_n)_{n\geq 1}$, $b_n = y_n - [y_n]$ ($[y_n]$ is the integer part of y_n) is divergent.

B. Enescu

11.3. For the differentiable function $f : \mathbf{R} \to \mathbf{R}$ the inequality

$$f'(x) \leq f'\left(x + \frac{1}{n}\right)$$

holds for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$. Prove that f' is a continuous function.

M. Piticari

11.4. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that for every $a, b \in \mathbf{R}$, $a < b$ there exists $c_1, c_2 \in [a, b]$, $c_1 \leq c_2$ with $f(c_1) = \min_{x\in[a,b]} f(x)$, $f(c_2) = \max_{x\in[a,b]} f(x)$.

Show that the function f is an increasing function.

C. Mortici

12th Form

12.1. Let a, b be positive real numbers such that $a + b < 1$ and $f : [0, +\infty] \to [0, +\infty]$ be an increasing function, such that for every $x \geq 0$,

$$\int_0^x f(t)dt = \int_0^{ax} f(t)dt + \int_0^{bx} f(t)dt .$$

Prove that $f(x) = 0, \forall x \geq 0$.

M. Piticari

12.2. a) Let $p \geq 2$ be a prime integer number and

$$G_p = \bigcup_{n \in \mathbf{N}} \left\{ z \in \mathbf{C} \,\middle|\, z^{p^n} = 1 \right\}.$$

Show that G_p is a subgroup of the multiplicative group $\mathbf{C}^*$.

b) Let H be an infinite subgroup of ($\mathbf{C}^*$, $\cdot$). Prove that every subgroup of H (different from H) is finite if and only if there exists a prime number p such that $H = G_p$.

***

12.3. A ring A is a called a boolean ring if $x^2 = x$, for every $x \in A$. Prove that :

a) One can define a structure of boolean ring on a finite set with $n \geq 2$ elements if and only if $n = 2^k$, with $k \in \mathbf{N}^*$.

b) It is possible to define a structure of boolean ring on the set of the natural numbers.

M. Andronache, S. Dăscălescu, I. Savu

12.4. Let $k \subseteq \mathbf{C}$ be a field (with the standard operations) such that :

a) k has exactly two endomorphisms f and g.

b) $f(x) = g(x) \Rightarrow x \in \mathbf{Q}$.

Prove that there exists a square-free positive integer $d \neq 1$ such that $k = \mathbf{Q}\left[\sqrt{d}\right]$.

M. Ţena

B. SOLUTIONS

7th Form

7.1. Let n be a positive integer and $x_1, x_2, ..., x_n$ be integer numbers such that

$$x_1^2 + x_2^2 + \ldots + x_n^2 + n^3 \leq (2n-1)(x_1 + x_2 + \ldots + x_n) + n^2 .$$

Show that : a) $x_1, x_2, ..., x_n$ are non-negative integers

b) the number $x_1 + x_2 + \ldots + x_n + n + 1$ is not a perfect square.

S. Smarandache

Solution. It is easy to see that the given inequality is equivalent to the following one :

$$(x_1-n)(x_1-n+1)+(x_2-n)(x_2-n+1)+\ldots+(x_n-n)(x_n-n+1)\le 0 .$$

Since the product of two consecutive integer numbers is nonnegative, we deduce that

$$(x_1-n)(x_1-n+1)=(x_2-n)(x_2-n+1)=\ldots=(x_n-n)(x_n-n+1)=0,$$

so $x_k\in\{n-1,n\}$, for every k. Thus a) is proven and for b), let us notice that

$$n(n-1)\le x_1+x_2+\ldots+x_n\le n^2, \text{ therefore}$$

$$n^2<1+n^2\le 1+n+x_1+x_2+\ldots+x_n\le 1+n+n^2<(n+1)^2 .$$

It follows that $1+n+x_1+x_2+\ldots+x_n$ cannot be a perfect square.

7.2. Show that there is no positive integer n such that $n+k^2$ is a perfect square for at least n positive integer values of k.

V. Zidaru

Solution. Suppose there exists $n\in\mathbf{N}^*$ and positive integers $k_1<k_2<\ldots<k_n$, such that $n+k_i^2$ is a perfect square, for every $i=1, \ldots, n$.

Let $n+k_i^2=m_i^2$, $i = 1, \ldots, n$. Then $m_1<m_2<\ldots<m_n$ and $m_1+k_1<m_2+k_2<\ldots<m_n+k_n$. But $n=(m_i+k_i)(m_i-k_i)$, $i = 1, \ldots, n$ so n has at least n different divisors greater that 1, which is a contradiction (the set $\{2, 3, \ldots, n\}$ has $n-1$ elements).

7.3. In the exterior of the triangle ABC with $m(\angle B)>45°$, $m(\angle C)>45°$ one constructs the right isosceles triangles ACM and ABN such that $m(\angle CAM)=m(\angle BAN)=90°$ and, in the interior of ABC, the right isosceles triangle BCP, with $m(\angle P)=90°$. Show that the triangle MNP is a right isosceles triangle.

B. Enescu

Solution. Let Q be the midpoint of BC. Then $\Delta BNP\sim\Delta BAQ$. Indeed, $\frac{BN}{BA}=\frac{BP}{BQ}=\sqrt{2}$ and $m(\angle NBP)=m(\angle ABQ)=45°+m(\angle ABP)$ (since from the conditions $m(\angle B)>45°$, $m(\angle C)>45°$ we know that $P\in \mathrm{Int}ABC$).

In a similar way we prove that $\Delta CMP\sim\Delta CAQ$.

It follows that $\frac{NP}{AQ} = \frac{MP}{AQ} = \sqrt{2}$, so

$NP = MP$, and

$$m(\angle NPB) = m(\angle AQB),$$

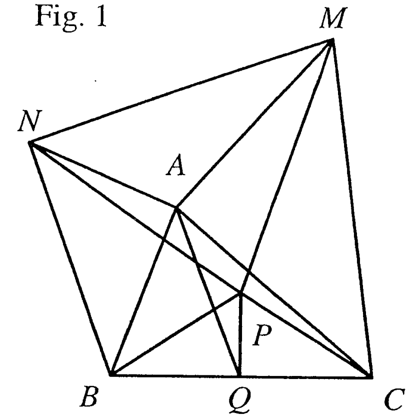
$$m(\angle MPC) = m(\angle AQC).$$

But $m(\angle AQB) + m(\angle AQC) = 180°$ and $m(\angle BPC) = 90°$, therefore

$$m(\angle MPN) = 90°,$$

hence ΔMPN is a right angled isosceles triangle.

Fig. 1

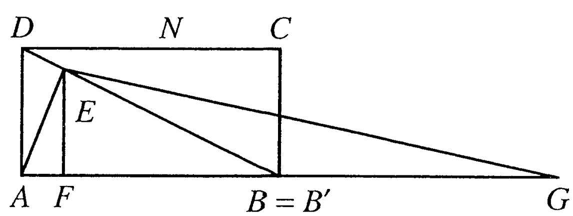


7.4. Let $ABCD$ be a rectangle and let $E \in (BD)$ such that $m(\angle DAE) = 15°$. Let $F \in AB$ such that $EF \perp AB$. It is known that $EF = \frac{1}{2} AB$ and $AD = a$. Find the measure of the angle $\angle EAC$ and the length of the segment (EC).

S. Peligrad

Solution. a) Let $G \in AB$ such that $AE \perp EG$. Clearly, $m(\angle AGE) = 15°$. Let B' be the midpoint of (AG). We shall prove that $B = B'$. In the right triangle EAG we have $EB' = AB' = B'G$, so $\angle B'GE \equiv \angle B'EG$ and $m(\angle EB'A) = 30°$. In the right triangle EFB' we obtain $EF = \frac{1}{2} EB'$ (since $m(\angle AB'E) = 30°$) hence $EF = \frac{1}{4} AG$.

It follows that $AB' = \frac{1}{2} AG$, that is $B = B'$.



We obtain that $AB = BG = EB$, hence ΔAEB is isosceles and

$$m(\angle EAB)=\frac{1}{2}(180°-m(\angle EBA))=75°.$$

Since $m(\angle CAB)=m(\angle DBA)=30°$ it follows that

$$m(\angle EAC)=75°-30°=45°.$$

b) From $AB=ADctg30°=a\sqrt{3}$, $BE=AB$, $BF=BE\cdot cos30°=\frac{3a}{2}$, $EF=\frac{AB}{2}=\frac{a\sqrt{3}}{2}$ and the trapezoid $EFBC$ we get

$$EC^2=BF^2+(BC-EF)^2=(4-\sqrt{3})a^2,$$

therefore $EC=a\sqrt{4-\sqrt{3}}$.

8th Form

8.1. Let a be a real number and $A=\{(x,y)\in\mathbf{R}\times\mathbf{R}\,|\,x+y=a\}$, $B=\{(x,y)\in\mathbf{R}\times\mathbf{R}\,|\,x^3+y^3<a\}$. Find all values of a such that $A\cap B=\varnothing$.

R. Ilie

Solution. Note that $A\cap B=\varnothing$ if and only if for every x, $y\in\mathbf{R}$, the equality $x+y=a$ implies $x^3+y^3\geq a$, which is equivalent to :

$$\forall x\in\mathbf{R},\ x^3+(a-x)^3\geq a$$

and, furthermore, to

$$\forall x\in\mathbf{R},\ 3a\left[\left(x-\frac{a}{2}\right)+\frac{a^2-4}{12}\right]\geq 0.$$

It follows that $a=0$ verifies the condition and $a<0$ obviously doesn't (take $x=\frac{a}{2}+1$).

For $a>0$ we obtain $a^2-4\geq 0$, hence $a\geq 2$.

In conclusion, $a\in\{0\}\cup[2,+\infty)$.

8.2. Let $P(X)=a_{1998}X^{1998}+a_{1997}X^{1997}+\ldots+a_1X+a_0$ be a polynomial with real coefficients such that $P(0)\neq P(-1)$, and let a, b be real numbers.

Let $Q(X)=b_{1998}X^{1998}+b_{1997}X^{1997}+\ldots+b_1X+b_0$ be the polynomial with real coefficients obtained by taking $b_k=aa_k+b$, $\forall\, k=0,\ 1,\ 2,\ldots,\ 1998$. Show that if $Q(0)=Q(-1)\neq 0$, then the polynomial Q has no real roots.

M. Fianu and *Şt. Alexe*

Solution. A short computation leads to

$$Q(X)=aP(X)+b\left(X^{1998}+X^{1997}+\ldots+1\right).$$

From $Q(0)=Q(-1)$, we get $a\left(P(0)-P(-1)\right)=0$, hence $a=0$, so $Q(X)=b\left(X^{1998}+X^{1997}+\ldots+1\right)$, with $b\neq 0$ (since $Q(0)\neq 0$).

Now, clearly Q has no positive roots. For $x\leq 1$, we have

$$x^{1998}+x^{1997}+\ldots+x+1=x^{1997}(x+1)+x^{1995}(x+1)+\ldots+x(x+1)+1\geq 1$$

and for $x\in(-1,\,0)$,

$$x^{1998}+x^{1997}+\ldots+x+1=x^{1998}+x^{1996}(x+1)+\ldots+x^2(x+1)+x+1>0.$$

We conclude that Q has no real roots.

8.3. In the right-angled trapezoid *ABCD*, *AB* $\parallel$*CD*, $m(\angle A)=90°$, $AD=DC=a$ and $AB=2a$. On the perpendiculars raised in C and D on the plane containing the trapezoid one considers points E and F (on the same side of the plane) such that $CE=2a$ and $DF=a$.

Find the distance from the point B to the plane (*AEF*) and the measure of the angle between the lines *AF* and *BE*.

R. Popovici and *N. Solomon*

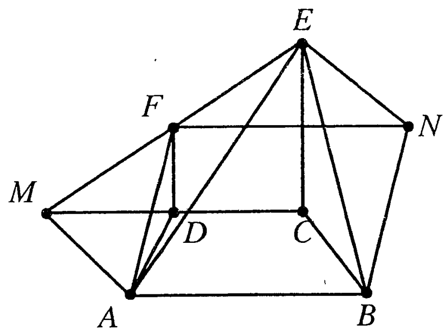
Solution. Let M be the intersection point of *EF* and *CD*. In the triangle *MEC*, *FD* $\parallel$ *EC* and $FD=\frac{1}{2}EC$ hence $MD=\frac{1}{2}MC=DC$. Clearly *MA* $\parallel$ *BC*, so the distance from B to (*AEF*) is equal to the distance from C to (*AEF*).

The last distance can be computed expressing in two ways the volume of the pyramid *AMCE* :

$$\frac{S[AMC]\cdot EC}{3}=\frac{S[AME]\cdot x}{3},$$

where x is the distance from C to (AEF). After a short computation, we obtain

$$x = \frac{2a\sqrt{3}}{3}.$$



Let N be the reflection of F in CE. Then $FN = 2a$, $BN = AF = a\sqrt{2}, EN = a\sqrt{2}$ and $BE = a\sqrt{6}$. It follows

$$m\angle(AF, BE) = m\angle(BN, BE) = 30^\circ.$$

8.4. Let $ABCD$ be an arbitrary tetrahedron. The bisectors of the angles $\angle BDC$, $\angle CDA$ and $\angle ADB$ intersect BC, CA and AB, in the points M, N, P, respectively.

a) Show that the planes (ADM), (BDN) and (CDP) have a common line d.

b) Let the points $A' \in (AD)$, $B' \in (BD)$ and $C' \in (CD)$ be such that $(AA') \equiv (BB') \equiv (CC')$; show that if G and G' are the centroids of ABC and $A'B'C'$ then the lines GG' and d are either parallel or identical.

M. Miculița

Solution. a) Since DM is the bisector of $\angle BDC$ we have $\frac{BM}{CM} = \frac{BD}{CD}$. Similarly : $\frac{CN}{AN} = \frac{CD}{AD}, \frac{AP}{PB} = \frac{DA}{DB}$, whence $\frac{BM}{CM} \cdot \frac{CN}{AN} \cdot \frac{AP}{BP} = 1$ and, from *Ceva*'s theorem, it follows that AM, BN and CP have a common point Q, therefore

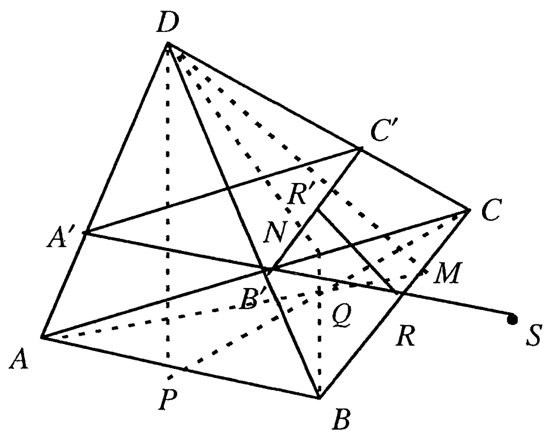
$$(ADM) \cap (BDN) \cap (CDP) = DQ.$$

b) Let R, R' be the midpoints of BC and $B'C'$. Consider the reflection of B' in R and denote it by S. Then $BSCB'$ is a parallelogram, the triangle $CC'S$ is isosceles ($CS = CC'$), $C'S \parallel DM$

($m(\angle CC'S) = \frac{1}{2}(180° - m(\angle C'CS)) = \frac{1}{2}m(\angle BDC) = m(\angle CDM)$)

and $C'S \parallel RR'$, therefore $RR' \parallel DM$ or RR' coincides with DM.

Since $G \in AR$ and $GA = 2GR$, $G' \in A'R'$, $G'A' = 2G'R'$ it follows $GG' \parallel (ADM)$ (or $GG' \subset (ADM)$).



Analogously, $GG' \parallel (BDN)$ or $GG' \subset (BDN)$.
The conclusion is now obvious.

9th Form

9.1. Find the integer numbers a, b, c such that the function $f: \mathbf{R} \to \mathbf{R}$, $f(x) = ax^2 + bx + c$ satisfies the equalities :

$$f(f(1)) = f(f(2)) = f(f(3)).$$

C. Mortici and *M. Chiriță*

Solution. Note that

$$f(f(x)) - f(f(y)) = af^2(x) + bf(x) + c - af^2(y) - bf(y) - c =$$
$$= (f(x) - f(y))(af(x) + af(y) + b).$$

Since $f(f(2)) - f(f(1)) = 0$ and $f(f(3)) - f(f(2)) = 0$, we obtain

$$(3a + b)(5a^2 + 3ab + 2ac + b) = 0$$
$$(5a + b)(13a^2 + 5ab + 2ac + b) = 0.$$

Now, if $3a+b=5a+b=0$, it follows $a=b=0$ and $c\in\mathbf{Z}$.

Suppose $3a+b=0$ and $13a^2+5ab+2ac+b=0$. Since $b=-3a$, we get $2(c-a)=3$ which is impossible. Analogously, the equalities $5a+b=0$ and $5a^2+3ab+2ac+b=0$ lead to a contradiction.

From $5a^2+3ab+2ac+b=13a^2+5ab+2ac+b=0$, it follows $8a^2+2ab=0$, that is $b=-4a$ (clearly $a=0$ implies $b=0$ and $c\in\mathbf{Z}$) and $2c=7a+4$. Obviously a must be even, say $a=2k$ ($k\in\mathbf{Z}$) and we obtain

$$b=-8k,\ c=7k+2.$$

9.2. Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AC-BD|\le|AB-CD|.$$

When does the equality hold ?

D. Miheţ

Solution. Let E and F be the midpoints of the diagonals AC and BD. In every quadrilateral the following relation holds :

$$AC^2+BD^2+4EF^2=AB^2+BC^2+CD^2+DA^2, \qquad (Euler).$$

Since $ABCD$ is a cyclic quadrilateral, we have *Ptolemeus* identity

$$AB\cdot CD+AD\cdot BC=AC\cdot BD$$

hence

$$(AC-BD)^2+4EF^2=(AB-CD)^2+(AD-BC)^2.$$

Let us prove now that $4EF^2\ge(AD-BC)^2$, obtaining thus the stated inequality. Let M be the midpoint of AB. In the triangle MEF we have $MF=\frac{1}{2}AD$, $ME=\frac{1}{2}BC$ and, from the triangle's inequality, $EF\ge|ME-MF|$, hence $2EF\ge|BC-AD|$ and $4EF^2\ge(AD-BC)^2$.

The equality holds if and only if the points M, E, F are collinear, which happens if and only if $AB \parallel CD$ that is $ABCD$ is either an isosceles trapezoid or a rectangle.

9.3. Find the rational roots (if any) of the equation

$$abx^2+(a^2+b^2)x+1=0,\ (a,\ b\in\mathbf{Z}).$$

D. Popescu

Solution. Suppose $ab\neq 0$. Then the equation has the discriminant $\Delta=(a^2+b^2)^2-4ab$. Obviously, the equation has rational roots if and only if Δ is a perfect square.

If $a = b$, $\Delta = 4a^2(a^2 - 1)$, which is a square only for $a^2 = 1$. We obtain thus $a = b = 1$ or $a = b = -1$; in both cases the equation has the two roots equal to -1.

If $a \neq b$, observe that for $ab < 0$,

$$(a^2 + b^2)^2 < \Delta < (a^2 + b^2 + 1)^2$$

and for $ab > 0$,

$$(a^2 + b^2 - 1)^2 < \Delta < (a^2 + b^2)^2,$$

so Δ cannot be a square of an integer number.

Finally, if $a = b = 0$, the equation has no roots, and if $a = 0$ and $b \neq 0$ (respectivelly $b = 0$ and $a \neq 0$) the equation has the rational root $x = -\frac{1}{b^2}$ (respectivelly $x = -\frac{1}{a^2}$).

9.4. Let $A_1A_2\dots A_n$ be a regular polygon ($n > 4$), T be the common point of A_1A_2 and $A_{n-1}A_n$ and M be a point in the interior of the triangle A_1A_nT.

Show that the equality

$$\sum_{i=1}^{n-1} \frac{sin^2(\angle A_iMA_{i+1})}{d(M, A_iA_{i+1})} = \frac{sin^2(\angle A_1MA_n)}{d(M, A_1A_n)}$$

holds if and only if M belongs to the circumcircle of the polygon.

D. Brânzei

Solution. Let us consider the inversion of center M and radius $k > 0$. The points $A_1, A_2, \dots, A_n$ are transformed into the points $B_1, B_2, \dots, B_n$, which are collinear if and only if M belongs to the circumcircle of the polygon (and in that case, $B_2, B_3, \dots, B_{n-1}$ are situated in this order on the segment $[B_1B_n]$). If we denote by a the length of the polygon's side, we have

$$\frac{sin^2(\angle A_iMA_{i+1})}{d(M, A_iA_{i+1})} = \frac{A_iA_{i+1}\, sin^2(\angle A_iMA_{i+1})}{2S[A_iMA_{i+1}]} = \frac{a\,sin(\angle A_iMA_{i+1})}{MA_i \cdot MA_{i+1}} =$$

$$= \frac{aMB_i \cdot MB_{i+1}\, sin(\angle A_iMA_{i+1})}{(MA_i \cdot MB_i)(MA_{i+1} \cdot MB_{i+1})} = \frac{2aS(B_iMB_{i+1})}{k^2},$$

where S[XYZ] denotes the area of the triangle XYZ. Analogously,

$$\frac{\sin^2(\angle A_1MA_n)}{d(M, A_1A_n)} = \frac{2aS[MB_1B_n]}{k^2}.$$

Thus, the equality enounced in the problem is equivalent to :

$$\sum_{i=1}^{n-1} S[MB_iB_{i+1}] = S[MB_1B_n],$$

which holds if and only if the points B_1, B_2,…, B_n are collinear (in this order).

The conclusion is now obvious.

10th Form

10.1. Let $n \geq 2$ be an integer and $M = \{1, 2, \ldots, n\}$. For every $k \in \{1, 2, \ldots, n-1\}$ we denote by $x_k = \frac{1}{n+1}\sum_{\substack{A\subset M\\|A|=k}}(\min A + \max A)$.

Show that $x_1, x_2, \ldots, x_{n-1}$ are integer numbers, not all divisible by 4.

V. Zidaru

Solution. Let us notice that there are $\binom{n-j}{k-1}$ subsets of M having k elements and the smallest element being j. It follows

$$\sum_{\substack{A\subset M\\|A|=k}} \min A = 1\cdot\binom{n-1}{k-1} + 2\cdot\binom{n-2}{k-1} + \ldots + (n-k+1)\cdot\binom{k-1}{k-1}.$$

Analogously, there are $\binom{n-j}{k-1}$ subsets of M having k elements, the greatest one being j, so

$$\sum_{\substack{A\subset M\\|A|=k}} \max A = n\binom{n-1}{k-1} + (n-1)\binom{n-2}{k-1} + \ldots + k\cdot\binom{k-1}{k-1}.$$

Thus, $\sum_{\substack{A\subset M\\|A|=k}}(\min A + \max A) = (n+1)\left[\binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{k-1}{k-1}\right].$

Writing now $\binom{n-1}{k-1}=\binom{n}{k}-\binom{n-1}{k-1}$; $\binom{n-2}{k-1}=\binom{n-1}{k}-\binom{n-2}{k-1}$ and so on, we obtain $x_k=\binom{n}{k}\in\mathbf{N}$.

If all x_k would be divisible by 4, so would be their sum. But $x_1+x_2+\ldots+x_{n-1}=2^n-2$ which is not divisible by 4 if $n\ge 2$.

10.2. Let $a \ge 1$ be a real number and z be a complex number such that $|z+a|\le a$ and $|z^2+a|\le a$. Show that $|z|\le a$.

D. Şerbănescu

Solution. Since $|z+a|\le a$, we obtain $|z+a|^2\le a^2$, or $(z+a)(\bar{z}+a)\le a^2$ that is $|z|^2+a(z+\bar{z})\le 0$. Analogously,

$$|z|^4+a(z^2+\bar{z}^2)\le 0.$$

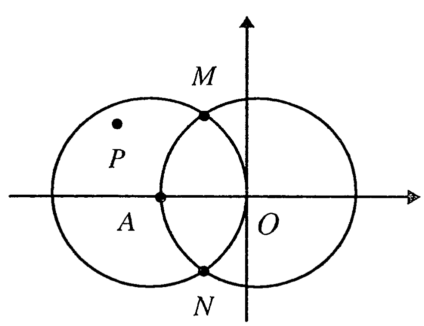
Let $|z|=r$ and $\bar{z}+z=t$. Then $t^2=z^2+\bar{z}^2+2r^2$ and we obtain :

$$r^2+at\le 0 \text{ and } r^4+a(t^2-2r^2)\le 0.$$

From $t\le -\frac{r^2}{a}$ it follows $t^2\ge -\frac{r^4}{a^2}$. Introducing in the second inequality we have.

$$0\ge r^4+at^2-2ar^2\ge r^4+\frac{r^4}{a}-2ar^2=r^4\left(1+\frac{1}{a}\right)-2ar^2.$$

This implies $0\ge r^2\left(1+\frac{1}{a}\right)-2a$ and $r^2\le\frac{2a^2}{a+1}$. Finally, we observe that $\frac{2a^2}{a+1}\le a^2$ (this is equivalent to $2a^2\le a^3+a^2$, that is $a^2\le a^3$, obvious since $a\ge 1$). It follows $r\le a$, as requested.



Second solution. In the complex plane let $A(-a)$ and the discs $D_0=\mathscr{D}(0,a)$, $D=\mathscr{D}(A,a)$.

If the point $P(z)$ lies in $D - D_0$ then $P \in \text{int}\angle MON$, whence $\arg z \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$ and $\arg z^2 \in \left(\frac{4\pi}{3}, 2\pi\right) \cup \left[0, \frac{2\pi}{3}\right)$.

It follows $z^2 \in D \cap D_0$, so $\left|z^2\right| = |z|^2 \le a$. On the other hand $z \in D - D_0 \Rightarrow |z| > a$ and $|z|^2 > a^2$, which is impossible because

$$a \ge 1 \Rightarrow a \le a^2.$$

10.3. Let $ABCD$ be a tetrahedron and A', B', C' be arbitrary points on the edges $[DA]$, $[DB]$, $[DC]$, respectively.

One considers the points $P_c \in [AB]$, $P_a \in [BC]$, $P_b \in [AC]$ and $P_c' \in [A'B']$, $P_a' \in [B'C']$, $P_b' \in [A'C']$ such that

$$\frac{P_cA}{P_cB} = \frac{P_c'A'}{P_c'B'} = \frac{AA'}{BB'}; \frac{P_aB}{P_aC} = \frac{P_a'B'}{P_c'C'} = \frac{BB'}{CC'}; \frac{P_bC}{P_bA} = \frac{P_b'C'}{P_a'A'} = \frac{CC'}{AA'}.$$

Prove that :

a) the lines AP_a, BP_b, CP_c have a common point P and the lines $A'P_a'$, $B'P_b'$, $C'P_c'$ have a common point P';

b) $\frac{PC}{PP_c} = \frac{P'C'}{P'P_c'}$;

c) if A', B', C' are variable points on the edges $[DA]$, $[DB]$, $[DC]$, then the line PP' is always parallel to a fixed line.

Mihai Miculița

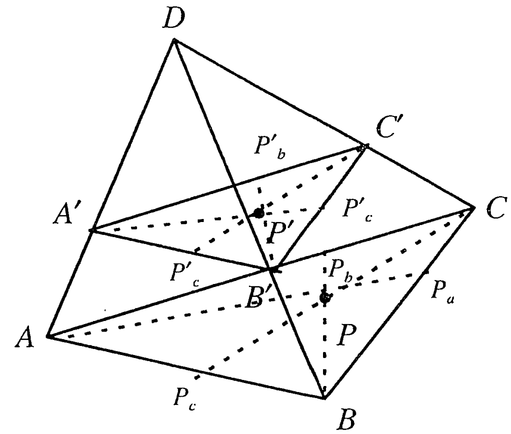
Solution. i) The conclusion easily follows from

$$\frac{P_cA}{P_cB} \cdot \frac{P_aB}{P_aC} \cdot \frac{P_bC}{P_bA} = \frac{AA'}{BB'} \cdot \frac{BB'}{CC'} \cdot \frac{CC'}{AA'} = 1 \text{ and}$$

$$\frac{P_c'A'}{P_c'B'} \cdot \frac{P_a'B'}{P_a'C'} \cdot \frac{P_b'C'}{P_b'A'} = \frac{AA'}{BB'} \cdot \frac{BB'}{CC'} \cdot \frac{CC'}{AA'} = 1.$$

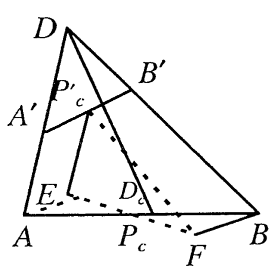
ii) Using *van Aubel*'s relation one gets

$$\frac{PC}{PP_c}=\frac{P_bC}{P_bA}+\frac{P_aC}{P_aB}=\frac{CC'}{BB'}+\frac{CC'}{AA'}$$



and $\dfrac{P'C'}{P'P_c'}=\dfrac{CC'}{AA'}+\dfrac{CC'}{BB'}$.

iii) Using the parallelograms $AA'P_c'E$ and $BB'P_c'F$ we get the collinear points E, P_c and F and, in the triangle $P_c'EF$, $\dfrac{P_cE}{P_cF}=\dfrac{P_c'E}{P_c'F}$. This shows that the straight line $P_c'P_c$ bisects the angle $EP_c'F$ and therefore is parallel to the bisector DD_c of angle $\angle ADB$.



On the other hand it is a well known fact that (ii) implies the existence of the parallel planes α, β, γ such that $P_cP_c'\subset\alpha, PP'\subset\beta$ and $CC'\subset\gamma$; moreover this planes are uniquely determined because $P_c'P_c$ and CC' are not parallel.

Since $P_cP_c'\|DD_c$ it follows that $\gamma=(CDD_c)$ and therefore

$$PP'\parallel(CDD_c).$$

In the same way $PP' \parallel (ADD_a)$, so PP' is parallel to the common straight line of the planes (ADD_a) and (CDD_c).

10.4. Let $n \geq 2$ and $0 < x_1 < x_2 < \ldots < x_n$ be integer numbers. Let

$$s_k = \sum_{\substack{A \subset \{x_1, x_2, \ldots, x_n\} \\ A \neq \emptyset}} \frac{1}{\prod_{a \in A} a}, \quad k = 1, 2, \ldots, n.$$

Show that if s_n and s_{n-1} are positive integers, then s_k is a positive integer for every k.

D. Miheţ

Solution. It is easy to check that

$$s_n = \left(1 + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_2}\right) \ldots \left(1 + \frac{1}{x_n}\right) - 1$$

and $s_{n-1} = \left(1 + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_2}\right) \ldots \left(1 + \frac{1}{x_{n-1}}\right) - 1 = \frac{x_n}{x_n + 1} s_n - 1 = s_n - \frac{s_n}{x_n + 1} - 1$.

Since $s_n \leq \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right) \ldots \left(1 + \frac{1}{n}\right) = n + 1$ and $x_n + 1 \geq n + 1$ it follows that $x_k = k$ for every $k = \overline{1, n}$, whence

$$s_k = \left(1 + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_2}\right) \ldots \left(1 + \frac{1}{x_k}\right) - 1 = k \text{ for every } k = \overline{1, n}.$$

11th Form

11.1. The non-zero matrices $A_0, A_1, \ldots, A_n \in \mathcal{M}_2(\mathbf{R})$, $n \geq 2$ have the following properties : $A_0 \neq aI_2$, $\forall\, a \in \mathbf{R}$ and $A_0 A_k = A_k A_0$, $\forall\, k = 1, 2, \ldots, n$. Show that :

a) $det\left(\sum_{k=1}^{n} A_k^2\right) \geq 0$;

b) if $det\left(\sum_{k=1}^{n} A_k^2\right) = 0$ and $A_2 \neq aA_1$, $\forall\, a \in \mathbf{R}$, then $\sum_{k=1}^{n} A_k^2 = O_2$.

V. Pop

Solution. From the condition $A_0 \cdot A_k = A_k \cdot A_0$, $k = 1, 2, \ldots, n$, we deduce (via direct computation) the existence of constants α_k, β_k such that

$$A_k = \alpha_k \cdot A_0 + \beta_k \cdot I_2 \,,\; k = 1,2,\ldots,n \,.$$

If all $\alpha_k = 0$, then $\sum_{k=1}^{n} A_k^2 = \left(\sum_{k=1}^{n} \beta_k^2\right) \cdot I_2$, whence

$$det\left(\sum_{k=1}^{n} A_k^2\right) = \left(\sum_{k=1}^{n} \beta_k^2\right)^2 \geq 0 \,.$$

If $\alpha = \sum_{k=1}^{n} \alpha_k^2 \neq 0$, then

$$\sum_{k=1}^{n} A_k^2 = \left(\sum_{k=1}^{n} \alpha_k^2\right) A_0^2 + 2\left(\sum_{k=1}^{n} \alpha_k \beta_k\right) A_0 + \left(\sum_{k=1}^{n} \beta_k^2\right) I_2 = P(A_0),$$

where $P(X) = \left(\sum_{k=1}^{n} \alpha_k^2\right) X^2 + 2\left(\sum_{k=1}^{n} \alpha_k \beta_k\right) X + \left(\sum_{k=1}^{n} \beta_k^2\right)$ is a polynomial whose discriminant

$$\Delta = 4\left(\sum_{k=1}^{n} \alpha_k \beta_k\right)^2 - 4\left(\sum_{k=1}^{n} \alpha_k^2\right) \cdot \left(\sum_{k=1}^{n} \beta_k^2\right)$$

is less or equal to zero (from *Cauchy-Schwarz* inequality).

So $P(z) = \alpha \cdot (z - z_0)(z - \bar{z}_0)$ with $z_0 \in \mathbf{C}$ and $\alpha \in \mathbf{R}^*$, therefore

$$P(A_0) = \alpha \cdot (A_0 - z_0 I_2)(A_0 - \bar{z}_0 I_2) \,.$$

It follows that $det\, P(A_0) = \alpha^2 \left|det(A_0 - z_0 I_2)\right|^2 \geq 0$, thus proving a).

For b) let us notice that from $A_2 \notin \{a \cdot A_1 \mid a \in \mathbf{R}\}$ it follows that $\alpha \neq 0$ and $\Delta < 0$, so $z_0 \in \mathbf{C} \setminus \mathbf{R}$. Since at least one of $det(A_0 - z_0 I_2)$ and $det(A_0 - \bar{z}_0 I_2)$ must be zero, so both are (since they are conjugate complex numbers). Taking $A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$, the condition $det(A_0 - zI_2) = 0$ is equivalent to

$$z^2 - z(a_0 + d_0) + a_0 d_0 - b_0 c_0 = 0 \,, \quad (*)$$

whence

$$P(A_0) = \alpha(A_0 - z_0 I_2)(A_0 - \bar{z}_0 I_2) = \alpha\left(A_0^2 - (z + \bar{z}_0)A_0 + z\bar{z}_0 I_2\right) =$$
$$= \alpha\left((a_0 + d_0)A_0 - (a_0 d_0 - b_0 c_0)I_2 - (z + \bar{z}_0)A_0 + z\bar{z}_0 I_2\right) = 0,$$

(because z_0 and $\bar{z}_0$ are the roots of equation (*)).

11.2. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that the sequence $x_n = \sum_{k=1}^{n} a_k^2$ is convergent and the sequence $y_n = \sum_{k=1}^{n} a_k$ is unbounded. Prove that the sequence $(b_n)_{n\geq 1}$, $b_n = y_n - [y_n]$ ($[y_n]$ is the integer part of y_n) is divergent.

B. Enescu

Solution. Let $x = \lim_{n\to\infty} x_n$. We have

$\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty}(x_n - x_{n-1}) = x - x = 0$ whence $\lim_{n\to\infty} a_n = 0$.

Let us assume $(b_n)_{n\geq 1}$ is convergent and let $b = \lim_{n\to\infty} b_n$. Since $\lim_{n\to\infty}(b_{n+1} - b_n) = b - b = 0$ and $b_n = y_n - [y_n]$ we deduce

$$\lim_{n\to\infty}(y_{n+1} - y_n - [y_{n+1}] + [y_n]) = 0.$$

But $y_{n+1} - y_n = a_{n+1}$ and $\lim_{n\to\infty} a_{n+1} = 0$. It follows that

$$\lim_{n\to\infty}([y_{n+1}] - [y_n]) = 0.$$

A sequence of integer numbers that is convergent is (excluding some initial terms) constant, therefore there exists $n_0 \in \mathbf{N}$ such that for every $n \geq n_0$, $[y_{n+1}] = [y_n]$. This clearly contradicts the fact that (y_n) is unbounded.

In conclusion, $(b_n)_{n\geq 1}$ must be divergent.

11.3. For the differentiable function $f: \mathbf{R} \to \mathbf{R}$ the inequality

$$f'(x) \leq f'\left(x + \frac{1}{n}\right)$$

holds for every $x \in \mathbf{R}$ and every $n \in \mathbf{N}^*$. Prove that f' is a continuous function.

M. Piticari

Solution. For every $n \in \mathbf{N}^*$, let us consider the function $f_n : \mathbf{R} \to \mathbf{R}$,

$$f_n(x) = n \cdot \left[f\left(x + \frac{1}{n}\right) - f(x) \right].$$

Obviously f_n is a differentiable function for every $n \in \mathbf{N}^*$ and

$$f_n'(x)=n\cdot\left[f'\left(x+\frac{1}{n}\right)-f'(x)\right]\geq 0,$$

for every $x\in\mathbf{R}$. It follows that f_n is an increasing function for every $n\in\mathbf{N}^*$. Now let $x_1, x_2\in\mathbf{R}$, $x_1<x_2$. We have $f_n(x_1)\leq f_n(x_2)$, $\forall n\in\mathbf{N}^*$, that is :

$$\frac{f\left(x_1+\frac{1}{n}\right)-f(x_1)}{\frac{1}{n}}\leq\frac{f\left(x_2+\frac{1}{n}\right)-f(x_2)}{\frac{1}{n}}.$$

Letting $n\to\infty$, we obtain $f'(x_1)\leq f'(x_2)$ that is f' is an increasing function, and since f' has the intermediate values property (theorem of *Darboux*) we deduce that f' is continuous.

11.4. Let $f:\mathbf{R}\to\mathbf{R}$ be a continuous function such that for every a, $b\in\mathbf{R}$, $a<b$ there exists c_1, $c_2\in[a, b]$, $c_1\leq c_2$ with $f(c_1)=\min_{x\in[a,b]}f(x)$, $f(c_2)=\max_{x\in[a,b]}f(x)$.

Show that the function f is an increasing function.

C. Mortici

Solution. Let us assume that f is not increasing. Thus we can find $a<b$ such that $f(a)>f(b)$. Then the set $A=\{x\in[a, b]\mid f(x)=f(a)\}$ is non-empty and bounded. Let $\alpha=\sup A$; we deduce that $\alpha\leq b$ and that for every $n\in\mathbf{N}^*$, we can find $u_n\in A$ such that $\alpha-\frac{1}{n}\leq u_n\leq\alpha$, hence $\lim_{n\to\infty}u_n=\alpha$. Since f is continuous, $\lim_{n\to\infty}f(u_n)=f(\alpha)$. But $u_n\in A$ implies $f(u_n)=f(a)$, hence $f(\alpha)=f(a)$, and clearly $\alpha<b$.

We state that $f(x)<f(a), \forall x\in(\alpha, b]$. Indeed, if there exists $z\in(\alpha, b]$ such that $f(z)\geq f(a)$, from the continuity of f we deduce the existence of $z'\in[z, b)$ such that $f(z')=f(a)$, leading thus to $z'\in A$ and $z'>\alpha=$ $=\sup A$, which is a contradiction.

Therefore $f(x)<f(a), \forall x\in(\alpha, b]$, and this contradicts the property of the function f for the interval $[\alpha, b]$.

This shows that the initial supposition is false.

12^{th} Form

12.1. Let a, b be positive real numbers such that $a + b < 1$ and f: [0, +∞] → [0, +∞] be an increasing function, such that for every $x \geq 0$,

$$\int_0^x f(t)dt = \int_0^{ax} f(t)dt + \int_0^{bx} f(t)dt.$$

Prove that $f(x) = 0, \forall x \geq 0$.

M. Piticari

Solution. We have

$$\int_0^{ax} f(t)dt = a\int_0^x f(ay)dy \text{ and } \int_0^{bx} f(t)dt = b\int_0^x f(by)dy,$$

whence

$$\int_0^x [f(t) - af(at) - bf(bt)]dt = 0, \text{ for every } x.$$

Let $h : [0,\infty) \to \mathbf{R}$, $h(t) = f(t) - af(at) - bf(bt)$. Since f is an increasing function and $a, b \in (0,1)$, we obtain $f(at) \leq f(t)$, $f(bt) \leq f(t)$ so

$$h(t) \geq f(t)(1 - a - b), \text{ for every } t \geq 0.$$

Finally, let $t_0 \in [0,\infty)$ and $x > t_0$. We have :

$$0 = \int_0^x h(t)dt \geq \int_0^x f(t)(1 - a - b)dt = (1 - a - b)\int_0^{t_0} f(t)dt +$$

$$+(1 - a - b)\int_{t_0}^x f(t)dt \geq (1 - a - b)(x - t_0)f(t_0), \text{ whence } f(t_0) \leq 0.$$

Since $f(t) \geq 0$, $\forall t \geq 0$ we deduce $f = 0$.

12.2. a) Let $p \geq 2$ be a prime integer number and

$$G_p = \bigcup_{n\in\mathbf{N}} \left\{z \in \mathbf{C} \,\middle|\, z^{p^n} = 1\right\}.$$

Show that G_p is a subgroup of the multiplicative group $\mathbf{C}^*$.

b) Let H be an infinite subgroup of $(\mathbf{C}^*, \cdot)$. Prove that every subgroup of H (different from H) is finite if and only if there exists a prime number p such that $H = G_p$.

***

Solution. For every $m\in \mathbf{N}^*$, let us denote by $U_m = \{z\in \mathbf{C} \mid z^m = 1\}$. It is known that U_m is the unique subgroup of $(\mathbf{C}^*, \cdot)$ having m elements.

a) Let $x, y \in G_p = \bigcup_{m\in\mathbf{N}^*} U_{p^m}$. Then one can find $n, r\in\mathbf{N}^*$ such that $x\in U_{p^n}$, $y\in U_{p^r}$ and, putting $m = max(n,r)$, it follows that $x, y\in U_{p^m}$, hence $xy^{-1}\in U_{p^m}\subset G_p$, that is G_p a subgroup of $(\mathbf{C}^*, \cdot)$.

b) Let p be a prime number and $H = G_p$. We shall prove that every subgroup X of H, $(X\neq H)$ is finite. It is clear that for a subgroup X of H there exists for every $n\in\mathbf{N}$ a number $\alpha(n)\in\mathbf{N}$ such that $X\cap U_{p^n} = U_{p^{\alpha(n)}}$, with $\alpha(n)\leq n$ ($X\cap U_{p^n}$ is a finite subgroup of $\mathbf{C}^*$, therefore $X\cap U_{p^n} = U_m$, where m is a divisor of p^n).

If the set $A = \{\alpha(n) \mid n\in\mathbf{N}\}$ is unbounded then for every $n\in\mathbf{N}$ there exists $m\in\mathbf{N}$ with $n\leq\alpha(m)$, whence $U_{p^n}\subseteq U_{p^{\alpha(m)}} = X\cap U_{p^m} = X$. It follows that $H = \bigcup_{n\in\mathbf{N}} U_{p^n}\subseteq X$, which is a contradiction. Therefore A is bounded above and, putting $m = max(A)$, we deduce $X = U_{p^m}$.

Conversely, suppose that H is an infinite subgroup of $(\mathbf{C}^*, \cdot)$ having only finite subgroups. Firstly we remark that every $g\in H$ must have a finite order (otherwise $X = \{g^{2n} \mid n\in\mathbf{Z}\}$ would be an infinite subgroup of H and $g\notin X$). For every $g\in H$ let $\mathcal{P}(g)$ be the set of the prime divisors of ord(g) and let $\mathcal{P} = \bigcup_{g\in H}\mathcal{P}(g)$.

We will prove that $\mathcal{P} = \{p\}$, whence $H = G_p$ (because H is infinite, H must contain all U_{p^α}). Indeed, if $|\mathcal{P}|\geq 2$ then there are possible two cases.

Case 1: $\mathcal{P}$ is infinite. In this case let $p\in\mathcal{P}$ and

$$X = \{g\in H \mid ordg \text{ is not divisible by } p\}.$$

Then $X\neq H$ and X is an infinite subgroup of H, which is a contradiction.

Case 2 : P is finite. In this case there exists $p \in P$ such that the set $\{\alpha \in \mathbf{N} \mid p^{\alpha} \textit{ divides } ord(g) \textit{ for some } g \in H\}$ is infinite (otherwise H would be finite). But this would imply $G_p \subset H$ and $G_p \neq H$ which is also a contradiction.

12.3. A ring A is a called a boolean ring if $x^2 = x$, for every $x \in A$. Prove that :

a) One can define a structure of boolean ring on a finite set with $n \geq 2$ elements if and only if $n = 2^k$, with $k \in \mathbf{N}^*$.

b) It is possible to define a structure of boolean ring on the set of the natural numbers.

M. Andronache, S. Dăscălescu, I. Savu

Solution. a) "$\Rightarrow$" In a boolean ring A, $x + x = 0$, for every $x \in A$. We will prove using induction that the number n of the elements of the finite group $(A, +)$ must be of the form 2^k, $k \in \mathbf{N}$. If $n = 1$ then $n = 2^0$. Let now $A = \{x_1, x_2, \ldots, x_n\}$ and H be a subgroup of A such that $H \neq A$ and $|H| = \max\{|G| \mid G = \text{subgroup of } A,\ G \neq A\}$. Let $X \in A \setminus H$. Then $H_1 = H \cup (H + X)$ is a subgroup of A which includes (strictly) H, therefore $H_1 = A$. Therefore $|A| = |H_1| = 2|H|$ and, using the induction hypothesis ($|H| = 2^k$) we get $|A| = 2^{k+1}$.

"$\Leftarrow$" For $n = 2^k$ we can take $A = \mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2$ (k times) with the ring structure of direct product of rings.

b) Let $B = \{X \subset \mathbf{N} \mid X \text{ finite}\}$, $C = \{X \subset \mathbf{N} \mid \mathbf{N} \setminus X \text{ finite}\}$ and $A = B \cup C$. It is easy to check that $(A, \Delta, \cap)$ is a boolean ring (here $X \Delta Y = (X \setminus Y) \cup (X \setminus Y)$). Let $p_1 < p_2 < \ldots < p_n < \ldots$ be the positive prime numbers and $f : A \to \mathbf{N}$, $f(\varnothing) = 0$, $f(\mathbf{N}) = 1$, $f(X) = 2\prod_{a \in X} p_a$ if $X \in B, X \neq \varnothing$ and $f(Y) = 2\prod_{a \in \mathbf{N} \setminus Y} p_a - 1$ if $Y \in \mathbf{C}$, $Y \neq \mathbf{N}$. Clearly f is injective and if the range of f is $\{n_0, n_1, \ldots, n_k, \ldots\}$ then the function $g : A \to \mathbf{N}$, $g(X) = k$ if $f(X) = n_k$ is bijective. The required structure of boolean ring is now given by the operations

$$m \oplus n = g\left(g^{-1}(m) \Delta g^{-1}(n)\right), \quad m \otimes n = g\left(g^{-1}(m) \cap g^{-1}(n)\right).$$

12.4. Let $k \subseteq \mathbf{C}$ be a field (with the standard operations) such that :

a) k has exactly two endomorphisms f and g.

b) $f(x) = g(x) \Rightarrow x \in \mathbf{Q}$.

Prove that there exists a square-free positive integer $d \neq 1$ such that $k = \mathbf{Q}\left[\sqrt{d}\right]$.

M. Ţena

Solution. It is well known that $\mathbf{Q} \subset K$ and $\mathbf{Q}$ has only one endomorphism, therefore $\mathbf{Q} \neq K$. Let $g = 1_K$ and f be the two endomorphisms of K. Since f is injective and $f \circ f$ is an endomorphism it follows that $f \circ f = 1_K$ ($f \circ f = f$ would lead to $f = 1_K$).

Let $x \in K \setminus \mathbf{Q}$ and $a = x + f(x)$, $b = xf(x)$. Then

$$f(a) = f(x) + x = a - g(a) \text{ and } f(b) = xf(x) = b = g(b),$$

so $a, b \in \mathbf{Q}$. Thus x and $f(x)$ are the roots of the equation $x^2 - ax + b = 0$, therefore $x = m + n\sqrt{d}$ for $m, n \in \mathbf{Q}$ and some squarefree integer $d_x \neq 1$, whence $\sqrt{d_x} \in K$.

Taking into account that $f^2\left(\sqrt{d}\right) = f\left(\left(\sqrt{d}\right)^2\right) = f(d_x) = d_x$ and $f\left(\sqrt{d_x}\right) \neq \sqrt{d_x}$ we see that $f\left(\sqrt{d_x}\right) = -\sqrt{d_x}$.

In the same way, for $y \in K \setminus \mathbf{Q}$ we can find a squarefree integer d_y such that $y \in \mathbf{Q}\left(\sqrt{d_y}\right)$ and $f\left(\sqrt{d_y}\right) = -\sqrt{d_y}$. This leads to

$$f\left(\sqrt{d_x}\sqrt{d_y}\right) = \left(-\sqrt{d_x}\right)\left(-\sqrt{d_y}\right) = \sqrt{d_x}\sqrt{d_y},$$

which implies $\sqrt{d_x}\sqrt{d_y} \in \mathbf{Q}$, therefore $d_x = d_y$.

Thus the squarefree integer d_x associated to every $X \in K \setminus \mathbf{Q}$ is the same integer d, whence $K \subset \mathbf{Q}\left(\sqrt{d}\right)$.

The inclusion $\mathbf{Q}\left(\sqrt{d}\right) \subset K$ is obvious.

SECTION 2

SELECTION EXAMINATIONS FOR THE 39TH I.M.O.

A. PROPOSED PROBLEMS

A1. First Round, Vaslui, 1998, March 27th

Problem 1. A word of length n is an ordered sequence $x_1\ x_1\ \dots\ x_n$, where x_i is a letter of the alphabet $\{a, b, c\}$. Denote by A_n the set of words of length n which do not contain any block $x_i x_{i+1}$, $i = 1, 2, \dots, n-1$, of the form aa or bb and by B_n the set of words of length n in which none of the subsequences $x_i x_{i+1}\ x_{i+2}$, $i = 1, 2, \dots, n-2$, contains all the letters a, b, c.

Prove that $|B_{n+1}| = 3|A_n|$.

Vasile Pop

Problem 2. The volume of a parallelepiped is 216 cm^3 and its total area is 216 cm^3. Prove that the parallelepiped is a cube.

Bogdan Enescu

Problem 3. Let m, $m \geq 2$, be an integer number. Find the smallest positive integer n, $n > m$, such that for any partition with two classes of the set $\{m, m+1, \dots, n\}$ at least one of these classes contains three numbers a, b, c (not necessarly different) such that $a^b = c$.

Ciprian Manolescu

Problem 4. Consider in the plane a finite set of segments such that the sum of their lengths is less than $\sqrt{2}$. Prove that there exists an infinite unit square grid covering the plane such that the lines defining the grid do not intersect any of the segments.

Vasile Pop

A2. Second Round, Bucharest, 1998, April 25th

Problem 5. We are given an isosceles triangle ABC such that $BC = a$ and $AB = AC = b$. The variable points M and N are given by the conditions: $M \in (AC)$, $N \in (AB)$ and

$$a^2 \cdot AM \cdot AN = b^2 \cdot BN \cdot CM.$$

The straight lines BM and CN intersect in P. Find the locus of the variable point P.

Dan Brânzei

Problem 6. All the vertices of a convex pentagon have both coordinates integer numbers.Prove that the area of the pentagon is at least $\frac{5}{2}$.

Bogdan Enescu

Problem 7. Find all positive integers *x*, *n* such that $x^n + 2^n + 1$ is a divisor of $x^{n+1} + 2^{n+1} + 1$.

Laurențiu Panaitopol

A3. Third Round, Bucharest, 1998, May 1th

Problem 8. Let $n \geq 2$ be an integer. Show that there exists a subset $A \subset \{1, 2, \ldots, n\}$ such that :

(i) The number of elements of *A* is at most $2\left[\sqrt{n}\right]+1$;

(ii) $\{|x-y| \mid x, y \in A \text{ and } x \neq y\} = \{1,2,\ldots,n-1\}$.

Radu Todor

Problem 9. An infinite arithmetic progression whose terms are positive integers contains the square of an integer and the cube of an integer. Show that it contains the sixth power of an integer.

(Proposed Problem for the 38th I.M.O., 1997)

Problem 10. Show that for any positive integer *n* the polynomial $f(X) = \left(X^2 + X\right)^{2^n} + 1$ cannot be decomposed as a product of two integer non-constant polynomials.

Marius Cavachi

A4. Fourth Round, Bucharest, 1998, May 22th

Problem 11. Let *ABC* be an equilateral triangle and $n \geq 2$ be an integer. Denote by $\mathcal{A}$ the set of $n-1$ straight lines which are parallel to *BC* and divide the surface [*ABC*] into *n* polygons having the same area and denote by $\mathcal{P}$ the set of $n-1$ straight lines parallel to *BC* which divide the surface [*ABC*] into *n* polygons having the same perimeter.

Prove that the intersection $\mathcal{A} \cap \mathcal{P}$ is empty.

Laurențiu Panaitopol

Problem 12. Let $n \geq 3$ be a prime number and $a_1 < a_2 < \ldots < a_n$ be integers. Prove that $a_1, a_2, \ldots, a_n$ is an arithmetic progression if and only if there exists a partition of the set $\mathbf{N} = \{0, 1, 2, \ldots\}$ with classes $A_1, A_2, \ldots, A_n$ such that $a_1 + A_1 = a_2 + A_2 = \ldots = a_n + A_n$ (where $a_i + A_i = \{a_i + x | x \in A_i\}$).

Vasile Pop

Problem 13. Let n be a positive integer and P_n be the set of the integer polynomials of the form $a_0 + a_1X + \ldots + a_nX^n$, where $|a_i| \leq 2$ for $i = 0, 1, \ldots, n$. Find, for each positive integer k, the number of elements of the set $A_n(k) = \{f(k) | f \in P_n\}$.

Marian Andronache

A5. Fifth Round, Bucharest, 1998, May 23th

Problem 14. Find all the functions $u : \mathbf{R} \to \mathbf{R}$ which have the property : there exists a strictly monotonic function $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x+y) = f(x)u(y) + f(y) \text{ for every } x, y \in \mathbf{R}.$$

Vasile Pop

Problem 15. Find all the positive integers k which fulfil the following condition : if f is an integer polynomial such that $0 \leq f(a) \leq k$ for every $a \in \{0, 1, 2, \ldots, k+1\}$ then $f(0) = f(1) = f(2) = \ldots = f(k+1)$.

(Proposed problem for the 38th I.M.O.)

Problem 16. The lateral surface of a cylinder of revolution is divided by $n-1$ planes parallel to the base and m parallel generators into mn cases ($n \geq 1$, $m \geq 3$). Two cases will be called neighbouring cases if they have a common side. Prove that there is possible to write a real number in each case such that each number is equal to the sum of the numbers of the neighbouring cases and not all the numbers are zero if and only if there exists integers k, l such that $(n+1)$ does not divide k and

$$\cos\frac{2l\pi}{m} + \cos\frac{k\pi}{n+1} = \frac{1}{2}.$$

Ciprian Manolescu

B. SOLUTIONS

Problem 1. A word of length n is an ordered sequence $x_1\, x_1\, \ldots\, x_n$, where x_i is a letter of the alphabet $\{a, b, c\}$. Denote by A_n the set of words of length n which do not contain any block $x_i x_{i+1}$, $i = 1, 2, \ldots, n-1$, of the form aa or bb and by B_n the set of words of length n in which none of the subsequences $x_i\, x_{i+1}\, x_{i+2}$, $i = 1, 2, \ldots, n-2$, contains all the letters a, b, c.

Prove that $|B_{n+1}| = 3|A_n|$.

Vasile Pop

Solution. We will prove that the numbers $a_n = |A_n|$ and $b_n = |B_n|$ satisfy the same linear recurrence relation and $b_2 = 3a_1, b_3 = 3a_2$ (this easly leads to the conclusion).

In order to do this denote by A_n' the set of the words from A_n which end by a or b and denote by $A_n'' = A_n - A_n'$.

Clearly $|A_{n+1}'| = |A_n'| + 2|A_n''|$ *and* $|A_{n+1}''| = |A_n'| + |A_n''| = |A_n|$ therefore

$$|A_{n+1}| = 2(|A_n'| + |A_n''|) + |A_n''| = 2|A_n| + |A_{n-1}|, \text{ whence}$$

$$a_{n+1} = 2a_n + a_{n-1} \text{ for every } n \geq 2.$$

Denote now (for $n \geq 2$) by B_n' the set of the words from B_n which end by two different letters and denote by $B_n'' = B_n - B_n'$.

We get $|B_{n+1}''| = |B_n'| + 2|B_n''|$ *and* $|B_{n+1}''| = |B_n'| + |B_n''| = |B_n|$, so

$$|B_{n+1}| = 2(|B_n'| + |B_n''|) + |B_n''| = 2|B_n| + |B_{n-1}| \text{ and}$$

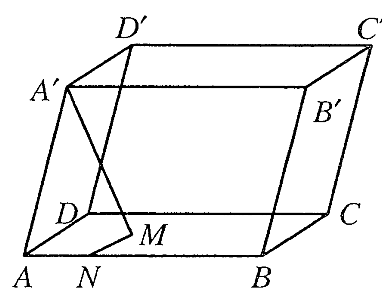
$$b_{n+1} = 2b_n + b_{n-1} \text{ for every } n \geq 3.$$

The proof is ended by noticing that $a_1 = 3$, $a_2 = 7$, $b_2 = 9$ and $b_3 = 21$.

Problem 2. The volume of a parallelepiped is 216 cm^3 and its total area is 216 cm^3. Prove that the parallelepiped is a cube.

Bogdan Enescu

Solution. Let $a = AB$, $b = AA'$, $c = AD$, $S_1 = S[ABB'A']$, $S_2 = S[ABCD]$, $S_3 = S[ADD'A']$ and V be the volume of the parallelepiped. We will prove that $V^2 \leq S_1 S_2 S_3$ and the equality holds if and only if the parallelepiped is rectangular.



Let $A'M \perp (ABCD)$ and $A'N \perp AB$.

We get $V = A'M \cdot S_2 \le A'N \cdot S_2 = \frac{S_1}{a} \cdot S_2$. In the same way $V \le \frac{S_1 S_3}{b}$, $V \le \frac{S_2 S_3}{c}$ therefore $V^3 \le \frac{S_1^2 S_2^2 S_3^2}{abc}$. Since

$$V = S_2 \cdot A'M \le ac \cdot AA' = abc,$$

it follows $V^4 \le abcV^3 \le S_1^2 S_2^2 S_3^2$, so $V^2 \le S_1 S_2 S_3$ and the equality is equivalent to $AB \perp AD \perp AA' \perp AB$.

On the other hand $S_1 S_2 S_3 \le \left(\frac{S_1 + S_2 + S_3}{3}\right)^3 = 36^3 = 216^2 = V^2$ whence $S_1 = S_2 = S_3$ and the rectangular parallelepiped is a cube.

Problem 3. Let m, $m \ge 2$, be an integer number. Find the smallest positive integer n, $n > m$, such that for any partition with two classes of the set $\{m, m+1, \dots, n\}$ at least one of these classes contains three numbers a, b, c (not necessarly different) such that $a^b = c$.

Ciprian Manolescu

Solution. We will firstly try to establish an upper bound for the searchen n. Suppose that (A, B) is a partition with two classes of the set $\{m, m+1, \dots, n\}$ such that the equation $x^y = z$ has no solution neither in A no in B and $m \in A$. Then $m_1 = m^m \in B$ (otherwise we have the solution $(m, m, m^m) \in A^3$), $m_2 = (m^m)^{m^m} = m^{m^{m+1}} \in A$ (otherwise we have the solution $(m^m, m^m, m^{m^{m+1}}) \in B^3$) and $m_3 = m^{m+1} \in B$ (because m and $m^{m^{m+1}}$ belong to A). Consider now the number

$$M = m^{m^{m+2}} = (m^m)^{m^{m+1}} = \left(m^{m^{m+1}}\right)^m.$$

If $M \in B$ then we have the solution $(m^m, m^{m+1}, M) \in B^3$ and if $M \in A$ then we have the solution $\left(m^{m^{m+1}}, m, M\right) \in A^3$. This proves that if $n \ge M$ then, for every partition of the set $\{m, m+1, \dots, n\}$ with two classes, the equation $x^y = z$ has at least a solution in one off the classes, so the searchen n must be at most M.

We will prove now that if $n \le M - 1$ then it is possible to find a partition with two classes (A, B) of the set $\{m, m+1, \dots, n\}$ such that the equation $x^y = z$ has no solution neither in A no in B. In order to do this we remark that we can study only the case $n = M-1$.

Consider $A = \{m, m+1, \ldots, m_1 - 1\} \cup \{m_2, m_2+1, \ldots, M-1\}$ and

$$B = \{m_1, m_1+1, \ldots, m_2 - 1\}.$$

If $x, y \in B$ then $x^y \geq m_1^{m_1} = m_2$, whence $x^y \notin B$.

If $x \geq m_2$ and $x, y \in A$ then $x^y \geq m_2^{m} = M$ therefore $x^y \notin A$.

If x, $y < m_1$ and x, $y \in A$ then $x^y \geq m^m = m_1$ and $x^y < m_1^{m_1} = m_2$, so $x^y \notin A$.

If $x < m_1, y \geq m_2$ and x, $y \in A$ then $x^y \geq m^{m_2} \geq M$ implies $x^y \notin A$.

Problem 4. Consider in the plane a finite set of segments such that the sum of their lengths is less than $\sqrt{2}$. Prove that there exists an infinite unit square grid covering the plane such that the lines defining the grid do not intersect any of the segments.

Vasile Pop

Solution. Take an arbitrary rectangular system of coordinates xOy and denote by α_i the angle between the segment S_i and Ox.

The orthogonal projections of the segments onto Ox and Oy have total length $p_x = \sum l(S_i)|\cos\alpha_i|$ and $p_y = \sum l(S_i)|\sin\alpha_i|$ respectively.

If we rotate xOy through an angle t these length become $p_x(t) = \sum l(S_i)|\cos(\alpha_i + t)|$ and $p_y(t) = \sum l(S_i)|\sin(\alpha_i + t)|$. Since $p_x(0) = p_y\left(\frac{\pi}{2}\right)$ and $p_x\left(\frac{\pi}{2}\right) = p_y(0)$, the equation $p_x(t) = p_y(t)$ has at least a solution $t_0 \in \left[0; \frac{\pi}{2}\right]$. Taking into account that

$$p_x(t_0) + p_y(t_0) = \sum l(S_i)\left(|\cos(\alpha_i + t_0)| + |\sin(\alpha_i + t_0)|\right) \leq \sqrt{2}\sum l(S_i) < 2$$

we get $p_x(t_0) = p_y(t_0) < 1$.

Denote now by $x'Oy'$ the rectangular system of coordinates obtained from xOy by a rotation through angle t_0. The conditions $p_x(t_0) < 1$ and $p_y(t_0) < 1$ lead to the existence of two points $x_0' \in [0;1)$, $y_0' \in [0;1)$ such that

$\{x'\} \neq x_0$ and $\{y'\} \neq y_0$ for every point $M(x', y')$ belonging to any of the segments S_i.

The infinite unit square grid with origin (x_0', y_0') and lines parallel to Ox' and Oy' clearly satisfies the required conditions.

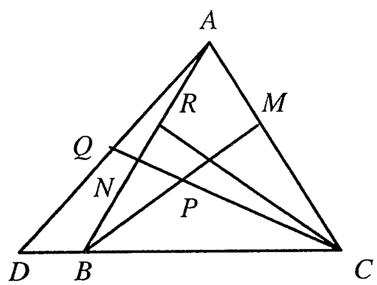
Problem 5. We are given an isosceles triangle ABC such that $BC = a$ and $AB = AC = b$. The variable points M and N are given by the conditions : $M \in (AC)$, $N \in (AB)$ and

$$a^2 \cdot AM \cdot AN = b^2 \cdot BN \cdot CM .$$

The straight lines BM and CN intersect in P. Find the locus of the variable point P.

Dan Brânzei

First solution. Consider D on the half-line $(CB$ such that $AD = CD$. From the similar triangles ABC and DAC we get $CD = \frac{b^2}{a}$. The common point Q of the lines AD and CN satisfies



$$\frac{QA}{QD} \cdot \frac{CD}{CB} \cdot \frac{NB}{NA} = 1,$$

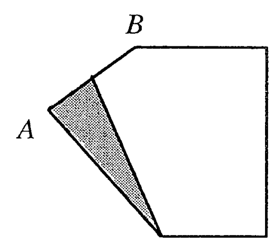
therefore $\frac{QA}{QD} = \frac{a^2}{b^2} \cdot \frac{NA}{NB} = \frac{MC}{MA}$ and further $\Delta BMC \sim \Delta CQA$. This proves that $\angle MBC \equiv \angle NCA$, whence $m\angle(BPC) = 180° - m\angle(B)$. This shows that P describes the arcBC of the circle which is tangent in B and C to AB and AC respectively.

Second solution. Let R be the point of (AB) such that $AR = AM$. It follows $\frac{RA}{RB} \cdot \frac{AN}{NB} = \frac{AM}{MC} \cdot \frac{AN}{NB} = \frac{AC^2}{BC^2}$, therefore $\angle ACR \equiv \angle BCN$ (*Steiner*) whence $\angle ABM \equiv \angle BCN$ and the solution finishes as above.

Problem 6. All the vertices of a convex pentagon have both coordinates integer numbers.Prove that the area of the pentagon is at least $\frac{5}{2}$.

Bogdan Enescu

Solution. We will prove that there exists a point with integer coordinates which lies in the interior of the pentagon. Because there are only for types of parity of a pair of integers, ar least two of the vertices of the pentagon must have coordinates of the same parity, therefore the midpoint of the segment determined by them has also integer coordinates. If this segment is a side AB of the pentagon then we can replace the pentagon by a smaller one by deleting the triangle shown in the figure. Since the pentagonal surface contains only a finite number of points with integer coordinates it follows that the previous "move" can be repeated only a finite number of times and at some moment we will get a midpoint with integer coordinates in the interior of the pentagon.



The conclusion follows now using the fact that the area of a triangle whose vertices have integer coordinates is at least $\frac{1}{2}$ (this area is half the value of a determinant with integer elements) and the decomposition of the pentagon into five such triangles.

Problem 7. Find all positive integers x, n such that $x^n + 2^n + 1$ is a divisor of $x^{n+1} + 2^{n+1} + 1$.

Laurenţiu Panaitopol

Solution. It is obviuos that $x = 1$ or $x = 2$ cannot give solutions of the problem, hence $x \geq 3$. From the equality :

$$x^{n+1} + 2^{n+1} + 1 = x\left(x^n + 2^n + 1\right) + 2^{n+1} - x \cdot 2^n - x + 1$$

we obtain that $x^n + 2^n + 1 \mid x^{n+1} + 2^{n+1} + 1$ if and only if

$$x^n + 2^n + 1 \mid (x-2)2^n + x - 1 .$$

Therefore, one necessarilly has

$$(1) \qquad x^n + 2^n + 1 \leq (x-2)2^n + x - 1 .$$

The equality (1) can be expressed under the form :

$$(2) \qquad \left(\frac{x}{2}\right)^n + 1 \leq (x-2) + \frac{x}{2} \cdot \frac{1}{2^{n-1}} - \frac{1}{2^{n-1}} ,$$

where $\frac{x}{2} \geq \frac{3}{2}$. Note $\frac{x}{2} = 1 + y, y \geq \frac{1}{2}$. Then $(1+y)^n \geq 1 + ny$, by *Bernoulli*'s inequality. Therefore, we obtain from (2) :

$$(3) \qquad 2 + ny \leq 2y + \frac{1}{2^{n-1}} y .$$

It follows that :

$$n \leq 2 + \frac{1}{2^{n-1}} - \frac{2}{y} < 2 + \frac{1}{2^{n-1}} .$$

Hence, $n \leq 2$. In the case $n = 2$, it follows $x^2 + 5 \leq 5x - 9$ and this is impossible. In the case $n = 1$, the condition of the problem is equivalent with:

$$\frac{x^2+5}{x+3} = \frac{x^2-9+14}{x+3} = x + 3 + \frac{14}{x+3} \in \mathbf{N} .$$

We obtain $x+3=7$ or $x+3=14$ and then $x = 4$ or $x = 11$. Therefore the solutions are the pairs (x, n) of the form (4, 1) or (11, 1).

Problem 8. Let $n \geq 2$ be an integer. Show that there exists a subset $A \subset \{1, 2, \ldots, n\}$ such that :

(i) The number of elements of A is at most $2[\sqrt{n}]+1$;

(ii) $\{|x-y| \mid x, y \in A \textit{ and } x \neq y\} = \{1, 2, \ldots, n-1\}$.

Radu Todor

Solution. The required subset can be taken for instance

$$\{1,2,\ldots,k-1\} \cup \left\{pk \,\middle|\, 1 \leq p \leq \left[\frac{n}{k}\right]\right\} \cup \{n\}, \text{ where } k = [\sqrt{n}].$$

This set has at most $2k+1$ elements, because $n < (k+1)^2$ implies $\frac{n}{k} \leq \frac{k^2+2k}{k} = k+2$, and $\left[\frac{n}{k}\right] = k+2$ holds only if $n = k(k+2)$.

On the other hand, each number of the form $qk + r$, $r \in \overline{1, k-1}$, $q \in \overline{0, \left[\frac{n}{k}\right]-1}$ can be written in the form $(k+1)q - (k-r)$, each number of the form qk, $q \in \overline{1, \left[\frac{n}{k}\right]-1}$ can be written in the form $(q+1)k - qk$ and each number of the set $\left\{k\left[\frac{n}{k}\right], k\left[\frac{n}{k}\right]+1, k\left[\frac{n}{k}\right]+2, \ldots, n-1\right\}$ can be written in the form $n-r$, $r \in \overline{1, k-1}$ because if the last set is not empty then n is not divisible by k and $0 \leq n-k\left[\frac{n}{k}\right] \leq n-k \cdot \frac{n-(k-1)}{k}=k-1$.

Problem 9. An infinite arithmetic progression whose terms are positive integers contains the square of an integer and the cube of an integer. Show that it contains the sixth power of an integer.

(Proposed Problem for the 38th I.M.O., 1997)

First solution. Let r be the step of the progression. We shall prove the assertion by induction on r. If $r = 1$, there is nothing to prove.

Let a be the first term of the progression and d be the greatest common divisor of a and r, i.e. $d = \gcd(a, r)$. Write $r = dq$.

Case 1. $\gcd(d, q) = 1$.

We have $a+ir=x^2$, $a+jr=y^3$. Then $x^2 \equiv y^3 \equiv a(mod\, r)$. Hence, we also have $x^2 \equiv y^3 \equiv a(mod\, q)$. Because q is coprime with a, it ia also coprime with x and y. Then, there exists a positive integer n such that $ny \equiv x(mod\, q)$. Consequently $n^6 y^6 \equiv x^6(mod\, q)$, which can be rewritten

$$n^6 a^2 \equiv a^3(mod\, q);$$

moreover, because $\gcd(a, q) = 1$ we have

$$(1) \qquad n^6 \equiv a(mod\, q).$$

From the condition $\gcd(d, q) = 1$, we can find an integer number k such that $n+kq \equiv 0(mod\, d)$. Then

$$(2) \qquad (n+kq)^6 \equiv 0 \equiv a(mod\, d).$$

Using the binomial formula and the congruence (1), we also have

$$(3) \qquad (n+kq)^6 \equiv a(mod\, q).$$

Because d and q are coprime and $r = dq$, from (2) and (3) we get

$$(n+kq)^6 \equiv a(mod\, r).$$

Clearly, k could have been chosen arbitrarly large. Thus the last condition shows that the sequence $a+hr, h=0,1,2\ldots$ contains the sixth power of the integer $n + kq$.

Case 2. gcd(d, q) > 1.

Let p be a common prime divisor of d and q. Let α, β be the exponents of p in the decomposition of a and r respectively. Because $d \mid q$ and $q \mid r$, it follows $\beta > \alpha \geq 1$. Therefore each term of the progression is divisible by p^{α}. Since x^2 and y^3 are in the progression, α is divisible by 2 and 3 ; so $\alpha = 6\delta, \delta \geq 1$. The progression $\frac{a+hr}{p^6}$, h = 0, 1, 2, ... with step $\frac{r}{p^{\alpha}} < r$ has integer terms and contains the numbers $\left(\frac{x}{p^3}\right)^2, \left(\frac{y}{p^2}\right)^3$. By the induction hypothesis it contains a term z^6. Therefore $(pz)^6$ is a term in the original progression.

Second solution. We start with the same notations like in the first solution and we also note $a = db$.

It is obvious the following result : every positive integer l, $l \geq 7$ can be represented under the form $l = 3u + 2v$ with u, $v > 0$.

From the equalities $a + ir = x^2$ and $a + jr = y^3$ we obtain

$$(4) \qquad d(b+iq) = x^2 \text{ and } d(b+jq) = y^3 .$$

Then, for every $u > 0$, $v > 0$ we obtain

$$d^{3u}(b+iq)^{3u} = x^{6u} \textit{ and } d^{2v}(b+jq)^{2v} = y^{6v} .$$

Multiply the last equalities and obtain :

$$d^{3u+2v}(b+iq)^{3u}(b+jq)^{2v} = x^{6u} \cdot y^{6v} = (x^u \cdot y^v)^6 = n^6 ,$$

where $x^u y^v = n$. Taking into account that

$$(b+iq)^{3u}(b+jq)^{2v} = b^{3u+2v} + cq ,$$

where c is an integer number we obtain the equality :

$$(5) \qquad d^{3u+2v}\left(b^{3u+2v} + cq\right) = n^6 .$$

Using *Fermat*'s theorem and the property of b, q to be relatively prime we have the general forms :

$$b^{6k\varphi(q)+1} = b^{6k\varphi(q)} \cdot b = (1+lq)b = b + blq$$

where l is convenable chosen number.

Because $6k\varphi(q)+1 \geq 7$ we can find positive integers u, v such that $6k\varphi(q)+1 = 3u+2v$.

Then, from (5) we obtain :

$$d^{3u+2v}\left(b^{3u+2v}+cq\right)=d^{6k\varphi(q)+1}\left(b^{6k\varphi(q)+1}+cq\right)=$$

$$=d^{6k\varphi(q)}\cdot d\left(b+blq+c\right)=m^{6}\cdot d\left(b+Bq\right).$$

Using (5) we have :

$$d\left(b+Bq\right)\cdot m^{6}=n^{6}\Rightarrow\left(a+Br\right)m^{6}=n^{6}.$$

Because $m^6 \mid n^6$, it follows $m \mid n$. Let p be the quotient $\frac{n}{m}$. It follows

$$a+Br=p^{6}$$

and this proves the required property.

Third solution. We observe that the statement of the problem is in fact equivalent to the following : there exists a positive integer z such that $z^2 \geq y$ and $y^3 - x^2$ divides $z^6 - y^3$. Let

$y=p_1^{a_1}\ldots p_k^{a_k}q_1^{b_1}\ldots q_l^{b_l}r_1^{2c_1}\ldots r_m^{2c_m}Y$ and $x=p_1^{\alpha_1}\ldots p_k^{\alpha_k}q_1^{\beta_1}\ldots q_k^{\beta_k}r_1^{3c_1}\ldots r_m^{3c_m}X$,

where $p_1,\ldots,p_k$ are the primes which have a greater exponent in y^3 than in x^2, $q_1,\ldots,q_l$ are the primes which have a greater exponent in x^2 than in y^3, $r_1,\ldots,r_m$ are the primes which have the same exponent in y^3 as in x^2 and $(X,Y)=1$.

It follows that $y^3-x^2 = p_1^{2\alpha_1}\ldots p_k^{2\alpha_k}q_1^{3b_1}\ldots q_l^{3b_l}r_1^{6c_1}\ldots r_m^{6c_m}\cdot$

$$\cdot\left(p_1^{3a_1-2\alpha_1}\ldots p_k^{3a_k-2\alpha_k}Y^3-q_1^{2\beta_1-3b_1}\ldots q_l^{2\beta_l-3b_l}X^2\right).$$

Let $D=p_1^{3a_1-2\alpha_1}\ldots p_k^{3a_k-2\alpha_k}Y^3-q_1^{2\beta_1-3b_1}\ldots q_l^{2\beta_l-3b_l}X^2$.

Since $p_1,\ldots,p_k$ and Y are relatively prime to $q_1,\ldots,q_l$ and X it follows that $p_1,\ldots,p_k$ and Y are invertible (mod$D$), therefore

$$Y=q_1^{2\beta_1-3b_1}\ldots q_l^{2\beta_l-3b_l}p_1^{2\alpha_1-3a_1}\ldots p_k^{2\alpha_k-3a_k}\left(XY^{-1}\right)^2(mod\,D).$$

Let Z be such that $Z\equiv XY^{-1}(mod\,D)$. This gives

$$Y^3=q_1^{6\beta_1-9b_1}\ldots q_l^{6\beta_l-9b_l}p_1^{6\alpha_1-9a_1}\ldots p_k^{6\alpha_k-9a_k}Z^6(mod\,D)$$

whence

$$y^3=p_1^{3a_1}\ldots p_k^{3a_k}q_1^{3b_1}\ldots q_l^{3b_l}r_1^{6c_1}\ldots r_m^{6c_m}\cdot$$

$$\cdot q_1^{6\beta_1-9b_1}\ldots q_l^{6\beta_l-9b_l}p_1^{6\alpha_1-3a_1}\ldots p_k^{6\alpha_k-3a_k}Z^6\left(mod\left(y^3-x^2\right)\right)$$ or

$$y^3 = \left(p_1^{\alpha_1}\dots p_k^{\alpha_k} q_1^{\beta_1-b_1}\dots q_l^{\beta_l-b_l} Z\right)^6 \left(mod\left(y^3-x^3\right)\right)$$

Since Z can be chosen large enough, this ends the proof.

Problem 10. Show that for any positive integer n the polynomial $f(X)=\left(X^2+X\right)^{2^n}+1$ cannot be decomposed as a product of two integer non-constant polynomials.

Marius Cavachi

Solution. In the case $n = 0$ the conclusion is obvious, so assume $n \geq 1$. We will associate to each polynomial

$$g = a_0 + a_1X + \dots + a_nX^n \in \mathbf{Z}[X],$$

the polynomial

$$\overline{g} = \hat{a}_0 + \hat{a}_1X + \dots + \hat{a}_nX^n \in \mathbf{Z}_2[X]$$

with coefficients in the field of the congruence classes (mod2).

Since

$$\left(X^2+X+\hat{1}\right)^{2^n} = \left[\left(X^2+X\right)^2+\hat{1}\right]^{2^{n-1}} = \left[\left(X^2+X\right)^{2^2}+\hat{1}\right]^{2^{n-2}} = \dots = \left(X^2+X\right)^{2^n}+\hat{1}$$

it follows that $\hat{f} = \left(X^2+X+\hat{1}\right)^{2^n}$.

Suppose now that f is factorable in the form $f = gh$, with $g, h \in \mathbf{Z}[X]$. It follows that $\hat{f} = \hat{g}\cdot\hat{h}$ and, since $X^2+X+\hat{1}$ is irreducible over $\mathbf{Z}_2$, $\hat{g} = \left(X^2+X+\hat{1}\right)^p$, $\hat{h} = \left(X^2+X+\hat{1}\right)^{2^n-p}$, $1 \leq p \leq 2^n-1$. This leads to $g = \left(X^2+X+1\right)^p + 2u(X)$, $h = \left(X^2+X+1\right)^{2^n-p} + 2v(X)$ where u and v are integer polinomials.

Let ε be one of the roots of X^2+X+1.

Replacing X with ε in the equality

$$f(X) = \left(X^2+X\right)^{2^n}+1 = \left[\left(X^2+X+1\right)^{2^n}+2u(X)\right]\left[\left(X^2+X+1\right)^{2^n-p}+2v(X)\right]$$

we get $2 = 2u(\varepsilon)\cdot 2v(\varepsilon)$, therefore $u(\varepsilon)\cdot v(\varepsilon) = \frac{1}{2}$.

Taking now into account that uv is an integer polynomial and $\varepsilon^2 = -\varepsilon - 1$ we see that $u(\varepsilon)\cdot v(\varepsilon)$ is a complex number of the form $a + b\varepsilon$, with $a, b \in \mathbf{Z}$, so the equality $u(\varepsilon)\cdot v(\varepsilon) = \frac{1}{2}$ is impossible.

Problem 11. Let ABC be an equilateral triangle and $n \geq 2$ be an integer. Denote by $\mathcal{A}$ the set of $n-1$ straight lines which are parallel to BC and divide the surface $[ABC]$ into n polygons having the same area and denote by $\mathcal{P}$ the set of $n-1$ straight lines parallel to BC which divide the surface $[ABC]$ into n polygons having the same perimeter.

Prove that the intersection $\mathcal{A} \cap \mathcal{P}$ is empty.

Laurenţiu Panaitopol

Solution. Let $\mathcal{A} = \{B_1C_1, B_2C_2, \ldots, B_{n-1}C_{n-1}\}$ where $B_1, \ldots, B_{n-1} \in (AB),\ C_1, \ldots, C_{n-1} \in (AC), AB_1 < AB_2 < \ldots < AB_{n-1}$ and $\mathcal{P} = \{D_1E_1, D_2E_2, \ldots, D_{n-1}E_{n-1}\}$ where $D_1, \ldots, D_{n-1} \in (AB),\ E_1, \ldots, E_{n-1} \in (AC), AD_1 < AD_2 < \ldots < AD_{n-1}$.

Let $x_i = \dfrac{AB_i}{AB}$ and $y_i = \dfrac{AD_i}{AB}$ for $i \in \overline{1, n-1}$.

The definition of $\mathcal{A}$ leads to $x_i^2 = \dfrac{area[AB_iC_i]}{area[ABC]} = \dfrac{i}{n}$ therefore

$$x_i = \sqrt{\frac{i}{n}}.$$

The definition of $\mathcal{P}$ gives $B_iC_i + 2B_iB_{i+1} = B_{i+2}C_{i+2} + 2B_{i+1}B_{i+2}$ for $i \in \overline{0, n-2}$ (where $B_0 = C_0 = A, B_n = B, C_n = C$), therefore, because $B_iC_i = AB_i$, $y_i + 2(y_{i+1} - y_i) = y_{i+2} + 2(y_{i+2} - y_{i+1})$, for $i \in \overline{0, n-2}$ with $y_0 = 0$ and $y_n = 1$.

The linear recurrence $3y_{i+2} - 4y_{i+1} + y_i = 0$ for $i \in \overline{0, n-2}$ implies the existence of coefficients α, β such that $y_i = \alpha + \dfrac{\beta}{3^i}$ for $i \in \overline{0, n}$ and the conditions $y_0 = 0$, $y_n = 1$ lead to $\alpha = -\beta = \dfrac{3^n}{3^n - 1}$ and $y_i = \dfrac{3^{n-i}(3^i - 1)}{3^n - 1}$ for $i \in \overline{0, n}$.

Let us suppose $x_l = y_k$ for some k, $l \in \overline{1, n-1}$, that is $l(3^n - 1)^2 = n \cdot 3^{2n-2k}(3^k - 1)^2$.

Let $d = (n, k)$ and $n = dm$, $m > 1$ (because $d \leq k < n$).

It is known that $\left(3^n-1,3^k-1\right)=3^d-1$ (indeed 3^d-1 divides both 3^n-1 and 3^k-1 ; if some integer p divides 3^n-1 and 3^k-1 then $(p, 3) = 1$ and p divides $3^{an}-1-\left(3^{bk}-1\right)=3^{bk}\left(3^{an-bk}-1\right)$ so, choosing the positive integers a, b such that $an-bk=d$, p divides 3^d-1). This implies that n is divisible by $\left(\frac{3^n-1}{3^d-1}\right)^2$, therefore $\sqrt{n}\geq\frac{3^n-1}{3^d-1}>3^{n-d}\geq 3^{\frac{n}{2}}$, whence $n > 3^n$. But this is a contradiction, because an easy induction shows that $3^n > n$ for every positive integer n, so the supposition $\mathcal{A}\cap\mathcal{P}\neq\emptyset$ is false.

Problem 12. Let $n \geq 3$ be a prime number and $a_1 < a_2 <\ldots< a_n$ be integers. Prove that $a_1, a_2,\ldots, a_n$ is an arithmetic progression if and only if there exists a partition of the set $\mathbf{N} = \{0, 1, 2, \ldots\}$ with classes $A_1, A_2,\ldots, A_n$ such that $a_1+A_1=a_2+A_2=\ldots=a_n+A_n$ (where $a_i+A_i=\left\{a_i+x \mid x\in A_i\right\}$).

Vasile Pop

Solution. If $a_1, a_2,\ldots, a_n$ is an arithmetic progression with step r then the partition $\left(A, A-r, A-2r,\ldots, A-(n-1)r\right)$, with

$$A=\bigcup_{k\geq 0}\left\{knr+(n-1)r+i \mid i\in\overline{0,r-1}\right\}$$

fulfils the required condition.

For the converse, denote $r_i=a_n-a_{n-i}$ and $B_i=A_{n-i}$ for $i\in\overline{0,n-1}$, so $B_i=B_0+r_i$ for $i \geq 1$. We will call segment of length k of the set B_i every set $\{a, a+1,\ldots, a+k-1\}\subset B_i$ such that $a-1\notin B_i$ and $a+k\notin B_i$.

We will firstly prove that each B_i is an union of segments of the same length $r = r_1$.

From $x\in B_0 \Rightarrow x + r\in B_1$ we get $x\in B_0 \Rightarrow x + r \notin B_0$, therefore all the segments of B_0 must have lengths less that $r + 1$. If B_0 contains segments of lengths less than r, let S be the first of these. Then $S + r\subset B_1$ and "between" S and $S + r$ there exists a segment S' of some B_i, $i \neq 0$. But, in this case the segment $S'-r_i$ would be in B_0, would have length less that r and would be "before" S, which contradicts the way S was chosen.

This proves that all the segments of B_0 have length r and, from $B_i=B_0+r_i$, this is also true for every B_i.

We will now prove that the first segment of B_i is $S_i = \{ir, ir+1, \ldots, ir+r-1\}$ for every $i \in \overline{0, n-1}$. Since $x \in B_i$ for $i \geq 1 \Rightarrow x \geq r_i \geq r$ it follows that B_0 must contain the segment $S_0 = \{0, 1, \ldots, r-1\}$ and therefore B_1 contains the segment $S_1 = \{r, r+1, \ldots, 2r-1\}$.

Suppose now that there exists $k < n$ ($k \geq 2$) such that $S_0 \in B_0$, $S_1 \in B_1, \ldots, S_{k-1}$ and $S_k \notin B_k$. Then S_k must be a segment of some B_i (since all the segments have the same length), i must be less than k and B_i must be B_0 (because the second segment of every B_i, $i \geq 1$ must come "after" the second segment of B_0).

This leads to $S_{k+1} \in B_0$, $S_{k+2} \in B_2, \ldots, S_{2k-1} \in B_{k-1}$ and, repeating the above judgement if necessary, the first segment of B_k must be of the form S_{lk}, $l \geq 2$. This leads to $r_k = lk$, therefore $S_{(l+1)k} = S_k + r_k \in B_k$. The segment S_{lk+1} cannot be in B_0 (it would lead to $S_{(l+1)k} \in B_{k-1}$) or in any of the $B_i's$, $i \in \overline{0, k-1}$ (the set $\{S_0, S_1, \ldots, S_{lk+1}\}$ would contain more segments from B_i than segments from B_0) therefore $S_{lk+1} \in B_{k+1}$.

In the same way $S_{lk+2} \in B_{k+2}, \ldots, S_{lk+k-1} \in B_{2k-1}$ and the sequence of segments from $(B_k, B_{k+1}, \ldots, B_{2k-1})$ will repeat itself a number of times before the appearence of a segment from a new set (which might be B_0 or B_{2k}).

We notice that a judgement as above shows that each time when a segment from a new set B_{sk} appears, then he must be followed immediately by segments from the sets $B_{sk+1}, B_{sk+2}, \ldots, B_{sk+k-1}$, so the number n of $B_i's$ must be a multiple of k, $1 < k < n$, a contradiction with the premises.

Thus $S_i \in B_i$ for every $i \in \overline{0, n-1}$, therefore $r_i = ir$ for every $i \in \overline{1, n-1}$ and $a_1, a_2, \ldots, a_n$ is an arithmetic progression.

Remark. If $n = pq$, $p, q \geq 2$ then the converse is false. This can be seen from the example

$$B_0 = \{2nk \mid k \in \mathbf{N}\} \cup \{2nk + p \mid k \in \mathbf{N}\} \text{ and } r_i = i + p\left[\frac{i}{p}\right] \text{ for } i \in \overline{1, n-1}$$

which corresponds to the periodic sequence of segments of length 1 obtained by repeating the block

$$[B_0 B_1 \ldots B_{p-1} B_0 \ldots B_{p-1}][B_p B_{p+1} \ldots B_{2p-1} B_p \ldots B_{2p-1}] \ldots$$
$$\ldots [B_{n-p} B_{n-p+1} \ldots B_{n-1} B_{n-p} \ldots B_{n-1}]$$

Problem 13. Let n be a positive integer and P_n be the set of the integer polynomials of the form $a_0 + a_1X + \ldots + a_nX^n$, where $|a_i| \le 2$ for $i = 0, 1, \ldots, n$. Find, for each positive integer k, the number of elements of the set $A_n(k) = \{f(k) | f \in P_n\}$.

Marian Andronache

Solution. For each fixed k consider the fixed number

$$n(k) = 2 + 2k + 2k^2 + \ldots + 2k^n$$

and the set $B_n(k) = A_n(k) + n(k) = \{x + n(k) | x \in A_n(k)\}$; clearly $|B_n(k)| = |A_n(k)|$. On the other hand $B_n(k) = \{g(k) | g \in \mathbf{R}_n\}$ where $\mathbf{R}_n$ is the set of the integer polynomials $b_0 + b_1X + \ldots + b_nX^n$ with $0 \le b_0, \ldots, b_n \le 4$.

If $k \ge 5$ then $B_n(k)$ is the set of the integers which can be written in the base k with the digits 0, 1, 2, 3, 4, so $|B_n(k)| = 5^{n+1}$.

If $k = 1$ then $\min B_n(1) = 0$, $\max B_n(1) = 4n + 4$ and $B_n(1) = \overline{0, 4n+4}$ because each number x from $\overline{1, 4n+3}$, $x = 4p + r$, $0 \le p \le n$, $0 \le r \le 3$ can be written $4 + 4\cdot 1 + 4\cdot 1^2 + \ldots + 4\cdot 1^{p-1} + r\cdot 1^p$, therefore $|B_n(1)| = 4n + 5$.

If $2 \le k \le 4$ then :

$$\min B_n(k) = 0,\ \max B_n(k) = 4(1 + k + k^2 + \ldots + k^n) = 4\cdot\frac{k^{n+1}-1}{k-1}$$

and we will prove that $B_n(k) = \overline{0, 4\cdot\frac{k^{n+1}-1}{k-1}}$ using induction on n.

For $n = 0$, $B_0(k) = \{0,1,2,3,4\} = \overline{0, 4\cdot\frac{k-1}{k-1}}$. Suppose now that $B_n(k) = \overline{0, 4\cdot\frac{k^{n+1}-1}{k-1}}$ for every $n < m$ and take $x \in \overline{1, 4\cdot\frac{k^{m+1}-1}{k-1} - 1}$.

If $4(k^m + k^{m-1} + \ldots + k^{m-q+1}) \le x < 4(k^m + k^{m-1} + \ldots + k^{m-q} + k^{m-q})$ for some $q \in \overline{1, m}$, then $x - 4(k^m + k^{m-1} + \ldots + k^{m-q+1}) < 4k^{m-q} \le 4\frac{k^{m-q+1}-1}{k-1}$ and, using the induction statement, $x = 4(k^m + k^{m-1} + \ldots + k^{m-q+1}) + g(k)$ with $g \in \mathbf{R}_{m-q}$.

If $x < 4k^m$ then $qk^m \le x < (q+1)k^m$ for some $q \in \{0,1, 2, 3\}$ and $x - qk^m < k^m \le 4\dfrac{k^m - 1}{k-1}$ so, as above $x = qk^m + g(k)$ with $g \in \mathbf{R}_{m-1}$, which ends the induction.

In conclusion, $|B_n(k)| = 4 \cdot \dfrac{k^{n+1}-1}{k-1} + 1$ for $2 \le k \le 4$.

Problem 14. Find all the functions $u : \mathbf{R} \to \mathbf{R}$ which have the property : there exists a strictly monotonic function $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x+y) = f(x)u(y) + f(y) \text{ for every } x, y \in \mathbf{R}.$$

Vasile Pop

Solution. The given property says that

$$f(x+y) = f(x)u(y) + f(y) \text{ and } f(y+x) = f(y)u(x) + f(x),$$

therefore

$$f(x)(u(y)-1) = f(y)(u(x)-1), \qquad (1).$$

From $f(0+y) = f(0)u(y) + f(y)$ we get $f(0)u(y) = 0$, whence $f(0) = 0$ or $u \equiv 0$. But $u \equiv 0$ would lead to $f(x+y) = f(x)$ which means that f would be constant, impossible. Therefore $f(0) = 0$ and $f(x) \ne 0$, so (1) leads to $\dfrac{u(x)-1}{f(x)} = \dfrac{u(y)-1}{f(y)}$ for every x, $y \ne 0$. This shows that there exists a real number c such that $u(x) - 1 = cf(x)$ for every $x \ne 0$. Relation (1) also shows that $0 = f(1)(u(0)-1)$ whence $u(0) = 1$ so $u(x) = 1 + cf(x)$ for every $x \in \mathbf{R}$.

If $c = 0$, then $u \equiv 1$, which fulfils the condition for $f = 1_{\mathbf{R}}$.

If $c \ne 0$, the property becomes $f(x+y) = f(x)(1 + cf(y)) + f(y)$ or

$$1 + cf(x+y) = (cf(x)+1)(cf(y)+1).$$

Let $g : \mathbf{R} \to \mathbf{R}$, $g(x) = cf(x) + 1$; g is strictly monotonic and $g(x+y) = g(x)g(y)$ for every $x, y \in \mathbf{R}$. Since $g(x) = g\left(\dfrac{x}{2} + \dfrac{x}{2}\right) = g^2\left(\dfrac{x}{2}\right)$ and $g(x) \ne 0$ for every x (otherwise $g \equiv 0$) it is legitimate to use the function $h : \mathbf{R} \to \mathbf{R}$ $h(x) = \ln g(x)$ and the monotonic function h fulfils the *Cauchy* condition $h(x+y)=h(x)+h(y)$. This shows that there exists a constant $k \neq 0$ such that $h(x) = kx$, therefore $g(x)=e^{kx}, f(x)=\frac{e^{kx}-1}{c}$ and $u(x)=e^{kx}$, which fulfil the required property.

In conclusion the searchen functions are those of the form $u(x)=a^x$ for some $a > 0$.

Problem 15. Find all the positive integers k which fulfil the following condition : if f is an integer polynomial such that $0 \leq f(a) \leq k$ for every $a \in \{0, 1, 2, \ldots, k+1\}$ then $f(0)=f(1)=f(2)=\ldots=f(k+1)$.

(Proposed problem for the 38[th] I.M.O.)

Solution. If f is an integer polynomial which satisfies $0 \leq f(a) \leq k$ for every $a \in \overline{0, k+1}$ then $|f(k+1) - f(0)|$ is divisible by $k+1$ and less than $k+1$, therefore $f(k+1) = f(0)$. This shows that

$$f(X)-f(0)=X(X-k-1)g(X)$$

for some integer polynomial g.

We notice that $|a(a-k-1)| > k$ if a is an integer different from 0, 1, k and $k+1$.

Therefore if $k \geq 3$ then $|f(a) - f(0)| \leq k$ and $|a(a-k-1)| > k$ for every $a \in \overline{2, k-1}$, so the integer $|g(a)|$ must be 0, that is

$$g(X)=(X-2)(X-3)\ldots(X-k+1)h(X)$$

for some integer polynomial h. In this case

$$|f(1)-f(0)|=k(k-2)!h(1) \text{ and } |f(k)-f(0)|=k(k-2)!h(k).$$

This proves that if $k \geq 4$ then, as above, $h(1)=h(k)=0$, so

$$f(X)-f(0)=X(X-1)(X-2)\ldots(X-k)(X-k-1)p(X)$$

for some integer polynomimal P, which means that $f(a)=f(0)$ for every $a \in \overline{0, k+1}$, so every $k \geq 4$ fulfils the required condition.

If $k \geq 3$ we have the counterexamples

$$f(X)=X(2-X) \; for \; k=1$$
$$f(X)=X(3-X) \; for \; k=2$$
$$f(X)=X(4-X)(X-2)^2 \; for \; k=3.$$

Problem 16. The lateral surface of a cylinder of revolution is divided by $n-1$ planes parallel to the base and m parallel generators into mn cases ($n \geq 1$, $m \geq 3$). Two cases will be called neighbouring cases if they have a common side. Prove that there is possible to write a real number in each case such that each number is equal to the sum of the numbers of the neighbouring cases and not all the numbers are zero if and only if there exists integers k, l such that $(n+1)$ does not divide k and

$$\cos\frac{2l\pi}{m}+\cos\frac{k\pi}{n+1}=\frac{1}{2}.$$

Ciprian Manolescu

Solution. We will number the rings 1, 2,..., n (going downwards) and the columns 1, 2,..., m (anticlockwise). We will associate to each ring the polynomial

$$P_i(X)=a_{i1}+a_{i2}X+\ldots+a_{im}X^{m-1},$$

where a_{ij} is the number written in the ring i and column j. We will also consider

$$P_0(X)=P_{n+1}(X)=0.$$

The condition that every number equals the sum of the numbers placed in the neighbouring cases translates into

$$P_i(X)=P_{i-1}(X)+P_{i+1}(X)+\left(X^{m-1}+X\right)P_i(X)\left(mod\,X^m-1\right) \text{ for every } i\in\overline{1,n}$$

$$\Leftrightarrow P_{i+1}(X)=\left(1-X-X^{m-1}\right)P_i(X)-P_{i-1}(X)\left(mod\,X^m-1\right) \text{ for every } i\in\overline{1,n}.$$

Consider the sequence of polynomials $\left(Q_i(X)\right)_{i\in\overline{0,n+1}}$ given by $Q_0(X)=0, Q_1(X)=1$ and

$$Q_{i+1}(X)=\left(1-X-X^{m-1}\right)Q_i(X)-Q_{i-1}(X)\left(mod\,X^m-1\right).$$

It is easy to see that $P_i(X)=Q_i(X)\cdot P_1(X)$ for every $i\in\overline{1,n+1}$ and that all the mn numbers are nil if and only if $P_1=0$. Therefore the existence of the numbers from the problem is equivalent to the existence of a polynomial $P_1\in\mathbf{R}[X]$ such that

$$P_1\neq 0\left(mod\,X^m-1\right) \text{ and } P_1Q_{n+1}=0\ \left(mod\,X^m-1\right).$$

This in turn is equivalent to the fact that the polynomials Q_{n+1} and X^m-1 are not relatively prime in $\mathbf{Q}[X]$, that is there exists a complex number ε such that $\varepsilon^m=1$ and $Q_{n+1}(\varepsilon)=0$.

Consider the sequence $(x_k)_{k\in\overline{0,n+1}}$, $x_k = Q_k(\varepsilon)$. This sequence fulfils the condition $x_0 = 0$, $x_1 = 1$ and $x_{k+1} = x_k(1-\varepsilon-\varepsilon^{m-1}) - x_{k-1}$, therefore, denoting by $\alpha = 1-\varepsilon-\varepsilon^{m-1}$ and by r_1, r_2 the roots of the equation $x^2 - \alpha x + 1 = 0$ we get $x_k = \dfrac{r_1^k - r_2^k}{r_1 - r_2}$ if $r_1 \neq r_2$ and $x_k = kr_1^{k-1}$ if $r_1 = r_2$.

In the case $r_1 = r_2$ the condition $Q_{n+1}(\varepsilon) = 0$ becomes $x_{n+1} = (n+1)r_1^n = 0$ which is impossible because $r_1 r_2 = 1$.

In the case $r_1 \neq r_2$, $Q_{n+1}(\varepsilon) = 0 \Leftrightarrow x_{n+1} = 0 \Leftrightarrow r_1^{n+1} = r_2^{n+1} \Leftrightarrow \exists\ \omega \neq 1$ such that $\omega^{n+1} = 1$ and $r_2 = \omega r_1 \Leftrightarrow \exists \omega \neq 1, \exists r \in \mathbf{C}$ such that $r(1+\omega) = \alpha$ *and* $r^2\omega = 1 \Leftrightarrow \exists \omega \neq 1$ such that $\dfrac{(1+\omega)^2}{\omega} = \alpha^2 \Leftrightarrow \exists \omega \neq 1$ such that

$$\overline{\omega} + 2 + \omega = (1-\varepsilon-\overline{\varepsilon})^2.$$

This way we came to the conclusion that the cases can be completed as asked $\Leftrightarrow \exists k, l \in \mathbf{Z}$ such that $(n+1) \nmid k$ and

$$2 + 2\cos\frac{2k\pi}{n+1} = \left(1 - 2\cos\frac{2l\pi}{m}\right)^2.$$

The last equality is the same as

$$4\cos^2\frac{2k\pi}{n+1} = \left(1 - 2\cos\frac{2l\pi}{m}\right)^2$$

which reads (replacing k by $n+1-k$ if necessary)

$$\cos\frac{2k\pi}{n+1} + \cos\frac{2l\pi}{m} = \frac{1}{2}.$$

SECTION 3

THE SELECTION EXAMINATION FOR THE JUNIOR BALKAN MATHEMATICAL OLYMPIAD

IAŞI, 27 - 28 May, 1998

A. PROPOSED PROBLEMS

Problem 1. Let

$$A=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{1997\cdot 1998} \text{ and}$$

$$B=\frac{1}{1000\cdot 1998}+\frac{1}{1001\cdot 1997}+\ldots+\frac{1}{1998\cdot 1000}.$$

Prove that $\frac{A}{B}$ is an integer number.

Bogdan Enescu

Problem 2. In the rectangle $ABCD$ one considers the variable points $M\in(AB)$, $N\in(BC)$, $P\in(CD)$ and $Q\in(AD)$. Let p and σ be the perimeter and respectively the area of the quadrilateral $MNPQ$. Prove that :

i) $p\geq AC+AB$;

ii) if $p=AC+BD$ then $\sigma\leq\frac{1}{2}S_{ABCD}$;

iii) if $p=AC+BD$ then $MP^2+NQ^2\geq AC^2$.

Problem 3. Let n be a positive integer. Find all the integers that can be written in the form

$$\frac{1}{a_1}+\frac{2}{a_2}+\ldots+\frac{n}{a_n}$$

with $a_1, a_2,\ldots, a_n\in\mathbf{N}^*$.

Gh. Iurea

Problem 4. Determine all the pairs of integers (x, y) such that

$$(x+1)(x+2)(x+3)+x(x+2)(x+3)+x(x+1)(x+3)+x(x+1)(x+2)=y^{2^x}.$$

A. Zanoschi

Problem 5. Let ABC be fixed triangle. One of the vertices of the variable quadrilateral $DEFG$ coincides with one of the vertices of the triangle and the other three vertices are on the segments $[AB]$, $[BC]$ and $[CA]$ respectively. It is known that $DF\perp EG$ and that $DEFG$ is circumscribed about a circle.

Find the locus of the point $M = DF \cap EG$.

Dan Brânzei

Problem 6. Find the smallest value of n for which one can find $x_1, x_2, \ldots, x_n \in \mathbf{N}$ such that

$$x_1^4 + x_2^4 + \ldots + x_n^4 = 1998 .$$

Gh. Iurea

B. SOLUTIONS

Problem 1. Observe that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$ and, therefore

$$A = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{1997} - \frac{1}{1998} =$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{1997} + \frac{1}{1998} - 2\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{1998}\right) =$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{1997} + \frac{1}{1998} - 1 - \frac{1}{2} - \ldots - \frac{1}{999} =$$

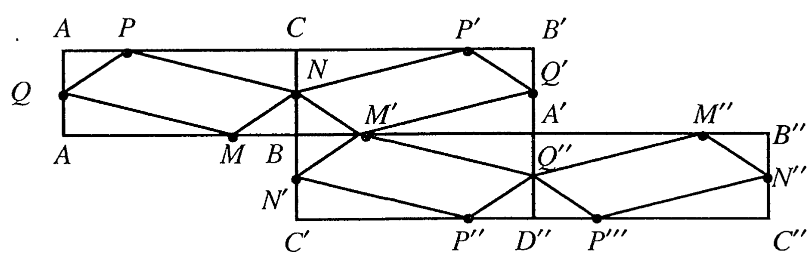
$$= \frac{1}{1000} + \frac{1}{1001} + \ldots + \frac{1}{1998}.$$

Then $2A = \left(\dfrac{1}{1000} + \dfrac{1}{1998}\right) + \left(\dfrac{1}{1001} + \dfrac{1}{1997}\right) + \ldots + \left(\dfrac{1}{1998} + \dfrac{1}{1000}\right) =$

$$= \frac{2998}{1000 \cdot 1998} + \frac{2998}{1001 \cdot 1997} + \ldots + \frac{2998}{1998 \cdot 1000} = 2998B .$$

Hence $\dfrac{A}{B} = 1499 \in \mathbf{Z}$.

Problem 2. Reflect $ABCD$ in the side BC. We obtain the rectangle $BA'D'C$. Reflect $BA'D'C$ in the side BA' obtaining $BA'D''C'$ and, finally,



$BA'D''C'$ in the side $A'D''$ obtaining $A'D''C''B$ (see figure).

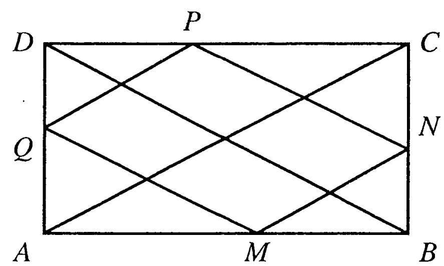
The points M, N, P, Q go succesively, into M', M'' etc.

Now, clearly

$$MN + NP + PQ + QM = PN + NM' + M'Q'' + Q''P' \geq PP''' = DD'' = AC + BD.$$

Furthermore, the equality holds iff the points P, N, M', Q'', P''' are collinear, which is equivalent to the fact that $MNPQ$ is a parallelogram, having the sides parallel to the diagonals of the rectangle $ABCD$.

Thus, if $p = AC + BD$, we have (see figure) $PN \parallel BD \parallel MQ$, $MN \parallel \parallel AC \parallel PQ$. Let $k = \frac{AM}{AB}$. It follows $S[AMQ] = k^2 S[ABD] = \frac{1}{2}k^2 S[ABCD]$.



We have $\frac{BM}{AB} = 1 - k$ and $S[MNB] = \frac{1}{2}(1-k)^2 S[ABCD]$. Since

$$\sigma = S[ABCD] - 2S[MAQ] - 2S[MNB]$$

the inequality to be proven becomes $1 - k^2 - (1-k)^2 \leq \frac{1}{2}$ or $\left(k - \frac{1}{2}\right)^2 \geq 0$, obvious. The equality $\sigma = \frac{1}{2}S[ABCD]$ holds if and only if M, N, P, Q are the midpoints of the rectangle's sides.

The second inequality follows immediately, being equivalent to

$$2\left(MN^2 + MQ^2\right) \geq AC^2$$

that is $2\left(k^2 + (1-k)^2\right) \geq 1$.

Problem 3. Clearly, if $k = \frac{1}{a_1} + \frac{2}{a_2} + \ldots + \frac{n}{a_n}$ then

$$k \leq 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

We will prove that every integer $k \in \left\{1,2,\ldots,\frac{n(n+1)}{2}\right\}$ can be written in the given form.

For $k = 1$ take $a_1 = a_2 = \ldots = a_n = \frac{n(n+1)}{2}$. For $k = n$ take $a_1 = a_2 = \ldots = a_n = n$.

For $1 < k < n$, take $a_{k-1} = 1$ and $a_i = \frac{n(n+1)}{2} - k + 1$ for $i \neq k - 1$.

Then $\sum_{i=1}^{n} \frac{i}{a_i} = \frac{k-1}{1} + \sum_{\substack{i=1 \\ i \neq k-1}}^{n} \frac{i}{a_i} = k - 1 + \frac{\frac{n(n+1)}{2} - k + 1}{\frac{n(n+1)}{2} - k + 1} = k$.

. For $n < k \leq \frac{n(n+1)}{2}$, k belongs to one of the intervals

$$\left(n + (n-1) + (n-2) + \ldots + (n-i+1), n + (n-1) + \ldots + (n-i+1) + (n-i)\right], i \in \overline{1, n-1}$$

therefore k can be written as a sum

$k = n + p_1 + p_2 + \ldots + p_i$, where $n - 1 \geq p_1 > p_2 > \ldots > p_i \geq 1$.

We take then

$$a_{p_1+1} = a_{p_2+1} = \ldots = a_{p_i+1} = 1$$

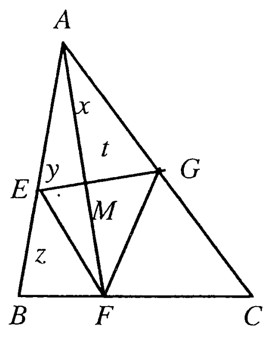
and $a_j = j$ for the rest of the indices.

Problem 4. If $x \geq 1$ then y^{2^x} is the square of an integer. On the other hand, the numbers x, $x+1$, $x+2$, $x+3$ are of the form $4k$, $4k+1$, $4k+2$, $4k+3$ (not necessarily in this order) so three of the four terms of the left side are divisible by 4 and the fourth is of the form $4k+2$. Thus, the left part of the equality cannot be a perfect square, so the equation has no solution in this case .

If $x \leq -4$, then left side is negative, while the right side is positive (we must impose the condition $y \geq 0$ if x is negative).

It follows that we must check only $x \in \{-3, -2, -1, 0\}$, and we obtain two solutions: $x = -2$, $y = 16$ and $x = 0$, $y = 6$.

Problem 5. The quadrilateral $DEFG$ is circumscribed if and only if $DE + FG = EF + DG$. Let $MD = x$, $ME = y$, $MF = z$ and $MG = t$. Then :



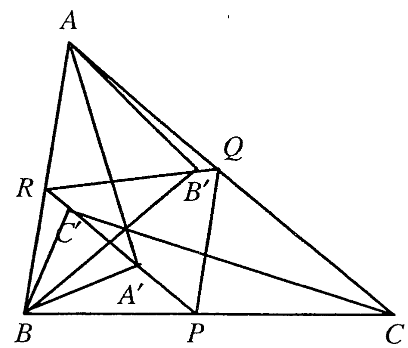
$$\sqrt{x^2+y^2}+\sqrt{z^2+t^2}=\sqrt{y^2+z^2}+\sqrt{x^2+t^2}$$

which leads to $\left(x^2+y^2\right)\left(z^2+t^2\right)=$

$$=\left(y^2+z^2\right)\left(x^2+t^2\right) \text{ so } \left(x^2-z^2\right)\left(t^2-y^2\right)=0,$$

that is one of the diagonals of $DEFG$ passes through the midpoint of the other.

Suppose $D = A$. One must analyze the two cases : $MG = ME$ (hence M lies on the bissector of $\angle A$) and $DM = MF$ (hence M lies on the segment joing the midpoints of the sides (AB) and (AC)).



We also notice that M cannot reach every position on that segments.

If, for instance, $AB \le BC \le CA$ then the required locus is the union of the segments $(AA']$, $(BB']$, $(CC']$, $(RB']$, $[A'C']$ and (PQ), where P, Q, R are the triangle's sides midpoints and AA', BB', CC' are the triangle's side angles bissectors (see figure).

Problem 6. Observe that if $x \in \mathbf{N}$, then $x^4 = 16k$ or $x^4 = 16k+1$ (depending on the parity of x).

Since $1998 = 16 \cdot 24 + 14$, it follows that $n \ge 14$.

If $n = 14$, x_1, x_2, ..., x_{14}, must be odd numbers. Let $a_k = \dfrac{x_k^4 - 1}{16}$, $k = 1, \ldots, 14$. Then $a_k \in \{0, 5, 39, 150, 410, \ldots\}$ and $a_1 + a_2 + \ldots + a_{14} = 124$.

It follows $a_k \in \{0, 5, 39\}$, $k = 1, 2, \ldots, 14$ and because $124 = 5 \cdot 24 + 4$, be the number of terms a_k equal to 39 is 1 or at least 6. In the first case we must have at least 17 terms equal to 5 and the second case is impossible because $6 \cdot 39 = 234 > 124$.

In conclusion, $n \ge 15$. This is indeed the minimal value of n because

$$1998 = 5^4+5^4+3^4+3^4+3^4+3^4+3^4+3^4+3^4+3^4+3^4+2^4+1^4+1^4+1^4.$$

THE INTERNATIONAL CONTEST

BAIA-MARE, september 1997

A. PROPOSED PROBLEMS

Problem 1. For every positive integer x denote by $S(x)$ the set

$$S(x)=\left\{a_n \middle| \ a_0=x \text{ and } a_{n+1}=a_n!+(n+2)! \text{ for every } n\geq 0\right\}.$$

Prove that there exists a sequence $(x_k)_{k\geq 0}$ such that

$$\bigcup_{k\geq 0} S(x_k)=\mathbf{N}^* \text{ and } S(x_k)\cap S(x_j)=\varnothing$$

for every $k\neq j$.

N. Vornicescu

Problem 2. Let ABC be a triangle and D, E and F be the contacts of the incircle to the sides of the triangle. Prove that

$$\frac{2pr}{R}\leq DE+EF+DF\leq p$$

and that the equalities hold in the same time.

D. Brânzei

Problem 3. Let n be a positive integer. Find all the monotonic functions $f:\mathbf{R}\to\mathbf{R}$ which fulfil the condition

$$f(x+f(y))=f(x)+y^n \text{ for every } x, y\in\mathbf{R}.$$

V. Pop

Problem 4. Let a, b, c be positive numbers such that $abc=1$ and let

$$S(x,y)=\frac{a^xb^x}{a^y+b^y+a^xb^x}+\frac{b^xc^x}{b^y+c^y+b^xc^x}+\frac{c^xa^x}{c^y+a^y+c^xa^x},$$

for every $x, y>0$.

Prove that if $x\leq 2y$ then $S(x,y)\leq 1$ and if $x>2y$ then $S(x,y)\geq 1$.

V. Pop and *D. Popa*

B. SOLUTIONS

Problem 1. Clearly the sequence must be chosen such that, for instance $x_0 = 1$, x_1 is the first positive integer which does not show up in $S(x_0)$, that is $x_1 = 2$, x_2 is the first positive integer which does not appear in $S(x_0)\bigcup S(x_1)$ and, in general, x_{k+1} is the smallest positive integer which does not belong to $S(x_0)\bigcup S(x_1)\bigcup\ldots\bigcup S(x_k)$. This easily leads to $\bigcup_{i\geq 0} S(x_i) = \mathbf{N}^*$.

Suppose now that $S(x_i)\bigcap S(x_j)\neq\varnothing$ for some $i \neq j$. Then there exists $a_n \in S(x_i)$ and $b_m \in S(x_j)$ such that $a_n = b_m$. Since $x_i\notin S(x_j)$ and $x_j\notin S(x_i)$ it follows that $m, n \geq 1$ and $a_{n-1}!+(n+1)! = b_{m-1}!+(m+1)!$, (1).

If $n = 1$ then $x_i!+2 = b_{m-1}!+(m+1)!$ which is impossible ($m \geq 2 \Rightarrow 3 \mid b_{m-1}!+(m+1)! \Rightarrow x_i = 1 \Rightarrow x_i!+2 < b_{m-1}!+(m+1)!$ and $m = 1 \Rightarrow x_i!+2 = x_j!+2 \Rightarrow x_i = x_j$). In the same way $m \geq 2$.

This shows that $a_{n-1} = a_{n-2}!+n! > n+1$ (with the exception of the case $n = 2$ and $a_0 = 1$, but then $a_{n-1}!+(n+1)! < b_{m-1}!+(m+1)!$).

In the case $a_{n-1} > b_{m-1}$ we get $a_{n-1}! > b_{m-1}!$ and therefore $(n+1)! < (m+1)!$, that is $n < m$, so $b_{m-1} > m+1 > n+1$. Dividing by $(n+2)!$ relation (1) we get that the integer

$$\frac{b_{m-1}!}{(n+2)!}+\frac{(m+1)!}{(n+2)!}-\frac{a_{n-1}!}{(n+2)!}$$

is equal to $\frac{1}{n+2}$, false.

In the same way $a_{n-1} < b_{m-1}$ leads to a contradiction.

Finally, in the case $a_{n-1} = b_{m-1}$ we get $(n+1)! = (m+1)!$, hence $m = n$, so $x_i = x_j$, impossible.

Problem 2. Let $D\in(BC)$, $E\in(CA)$, $F\in(AB)$. It is easly seen that $AE = AF = p-a, EF = 2(p-a)sin\frac{A}{2}$ and the similar.

Since $2sin\frac{A}{2} = 2\sqrt{\frac{(p-b)(p-c)}{bc}} \leq \frac{p-b}{b}+\frac{p-c}{c}$ it follows

$$DE+EF+DF \le \left(\frac{p-a}{a}+\frac{p-b}{b}\right)(p-c)+$$

$$+\left(\frac{p-b}{b}+\frac{p-c}{c}\right)(p-a)+\left(\frac{p-a}{a}+\frac{p-c}{c}\right)(p-b)=p.$$

For the other inequality we notice that using the incenter I we get

$$EF^2=2r^2-2r^2\cos(\angle FIE)=2r^2(1+\cos A)=4r^2\cos^2\frac{A}{2},$$

so the inequality is equivalent to

$$p \le R\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right), \text{ that is}$$

$$\sin A+\sin B+\sin C \le \sin\frac{B+C}{2}+\sin\frac{A+C}{2}+\sin\frac{A+B}{2}.$$

This follows immediately from

$$\frac{\sin x+\sin y}{2} \le \sin\frac{x+y}{2} \text{ for every } x, y \in (0, \pi).$$

The equalities take place if and only if $p-a=p-b=p-c$ and, respectively, $A = B = C$, that is if and only if the triangle is equilateral.

Problem 3. If $y_1, y_2 \ge 0$ and $f(y_1)=f(y_2)$ then $y_1^n=y_2^n$, so $y_1=y_2$. Hence $f\big|_{[0,\infty)}$ is injective. Take now $x_0 > 0$ such that $x_0+f(0)>0$. Since $f(x_0+f(0))=f(x_0)$ it follows that $f(0)=0$ and the given condition leads to $f(f(y))=y^n$, $\forall y\in \mathbf{R}$, (1).

The condition shows also that $f(f(x+f(y)))=f(f(x)+y^n)$ and this, combined with (1), leads to

$$(x+f(y))^n=f(y^n+f(x))=f(y^n)+x^n=f(f(f(y)))+x^n=f^n(y)+x^n.$$

Relation (1) shows also that $f(f(1))=1$ and the previous relation gives for $x = 1$ and $y=f(1)$ that $n = 1$. This leads to $f(f(y))=y, \forall y\in \mathbf{R}$, therefore f is bijective and $f=f^{-1}$.

The condition from the premise becomes $f(x+f(y))=f(x)+y$, $\forall$ $x, y\in \mathbf{R}$, that is $f(x+z)=f(x)+f^{-1}(z)$ $\forall$ $x, z\in \mathbf{R}$ hence

$$f(x+z)=f(x)+f(z)\ \forall\ x,\ z\in \mathbf{R}.$$

This proves that f is of the form $f(x)=cx$, and $f=f^{-1}$ implies that

$$c=\pm 1.$$

Thus the solutions are :

- $f_1(x)=x,\ \forall x\in \mathbf{R}\ and f_2(x)=-x,\ \forall x\in \mathbf{R}$, in the case $n=1$;
- no solution, in the case $n\geq 2$.

Problem 4. Denote that

$$a^y+b^y=a^{\frac{x+y}{3}}b^{\frac{x+y}{3}}\left(a^{\frac{y-2x}{3}}+b^{\frac{y-2x}{3}}\right)+\left(a^{\frac{2y-x}{3}}-b^{\frac{2y-x}{3}}\right)\left(a^{\frac{x+y}{3}}-b^{\frac{x+y}{3}}\right).$$

In the case $x\leq 2y$ this leads to $a^y+b^y\geq a^{\frac{x+y}{3}}b^{\frac{x+y}{3}}\left(a^{\frac{y-2x}{3}}+b^{\frac{y-2x}{3}}\right)$

and therefore

$$S(x,y)\leq \sum \frac{a^xb^x}{a^{\frac{x+y}{3}}b^{\frac{x+y}{3}}\left(a^{\frac{y-2x}{3}}+b^{\frac{y-2x}{3}}\right)+a^xb^x}=$$

$$=\sum \frac{1}{a^{\frac{y-2x}{3}}b^{\frac{y-2x}{3}}\left(a^{\frac{y-2x}{3}}+b^{\frac{y-2x}{3}}\right)+1}=\sum \frac{c^{\frac{y-2x}{3}}}{a^{\frac{y-2x}{3}}+b^{\frac{y-2x}{3}}+c^{\frac{y-2x}{3}}}=1.$$

In the case $x>2y$ we get

$$a^y+b^y\leq a^{\frac{x+y}{3}}b^{\frac{x+y}{3}}\left(a^{\frac{y-2x}{3}}+b^{\frac{y-2x}{3}}\right)$$

and the previous inequality is reversed.

THE 49[TH] NATIONAL MATHEMATICAL OLYMPIAD, 1998

The First Round in the City of Bucharest

PROPOSED PROBLEMS[1]

9[th] Form

Problem 1. For any ordered pair (a, b) of real numbers consider the set

$$A(a,b) = \{a + bm \mid m \in \mathbf{Q}\}.$$

Prove that for any ordered pair (a, b), one has

$$A(a,b) \cap \{\sqrt{5}, \sqrt{6}, \sqrt{7}\} \neq \varnothing .$$

Valentin Matrosenco

Problem 2. a) Prove that for any real number x, $x \neq k\pi$ where $k \in \mathbf{Z}$, the following equality holds :

$$\frac{\sin 5x}{\sin x} = 4\cos^2 2x + 2\cos 2x - 1 .$$

b) Prove that there exist real numbers a, b such that

$$\frac{\sin 5x}{\sin x} = 16\sin(x+a)\sin(x-a)\sin(x+b)\sin(x-b)$$

for any real number x, $x \neq k\pi$.

Iaroslav Chebici

Problem 3. Let $\angle AOB$ be a right angle and C, D be points on the half-lines $(OA$, $(OB$ respectively.

a) Prove that : $OA \cdot OC + OB \cdot OD \leq AB \cdot CD$.

b) Let us suppose that C, D belong to the segments (OA), (OB) and let M, N be the midpoints of the segments (AB), (CD) respectively. Prove that the above inequality is an equality if and only if O, M, N are collinear points.

Greta Marinescu and *Marcel Chiriţă*

Problem 4. Solve in integer numbers the equation

$$\left[\frac{x}{y} - \frac{y}{x}\right] = \frac{x^2}{y} + \frac{y}{x^2} .$$

Valentin Matrosenco

[1] The solutions of the problems from this section are available in Gazeta Matematică, vol. 103, no. 4, pp. 153-162.

10^{th} Form

Problem 1. Let x, y be complex numbers, $n \geq 2$ be a positive integer such that $\left|x^{n-2}y\right| = 2$ *and* $x^n = y^n = x + y$. Prove that $x = y$.

Maria Elena Panaitopol

Problem 2. Consider in space the points A, B, the plane π and the circle $\mathcal{C}$ in the plane π.

a) Find the points M, $M \in \pi$ such that $MA + MB$ is minimal.

b) Find the points N, $N \in \pi$ such that $NA^2 + NB^2$ is minimal.

c) Find the points P, $P \in \mathcal{C}$ such that $PA^2 + PB^2$ is minimal.

***

Problem 3. a) Let a be a real number, $a > 0$, $a \neq 1$. Prove that the real function $f(x) = a^{(1-a)x+1}$ is increasing.

b) Let a, b be positive real numbers. Solve the inequation :

$$a^x \cdot a^{1-ax} + b^x \cdot b^{1-bx} > a^a + b^b.$$

Valentin Matrosenco

Problem 4. Find all functions $f : \mathbf{N} \to \mathbf{N}$ which satisfy the conditions :

a) $f(x+y) = f(x) + f(y) + 2xy$, $\forall x, y \in \mathbf{N}$;

b) for all x, $x \in \mathbf{N}$, $f(x)$ is a perfect square.

Marcel Chiriţă and *Marian Andronache*

11^{th} Form

Problem 1. Let $(b_n)_{n\geq 0}$ be a sequence of positive real numbers such that $\lim\limits_{n\to\infty} \frac{b_n}{n} = \infty$. Prove that :

i) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \sum\limits_{k=1}^{n} \frac{1}{\sqrt{b_k}} = 0$; ii) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n + b_k} = 0$.

Virgil Nicula

Problem 2. Find all 2×2 matrices X with real entries such that :

$$X^3 - 4X^2 + 5X = \begin{pmatrix} 10 & 20 \\ 5 & 10 \end{pmatrix}.$$

Iaroslav Chebici and *Marcel Chiriţă*

Problem 3. Let A, B be complex matrices of dimensions 2×2 such that there exists a positive integer with the property :

$$A^n B^n = B^n A^n .$$

Prove that for all positive integers p, q one has :

$$\left(A^n B^p - B^p A^n\right)\left(A^q B^n - B^n A^q\right) = O .$$

Marian Andronache

Problem 4. Let f: $(0,\infty)\to\mathbf{R}$ be a monotonic function such that

$$\lim_{x\to\infty}\left(f(2x) - f(x)\right) = 0 .$$

Prove that for every positive real number a, $\lim_{x\to\infty}\left(f(ax) - f(x)\right) = 0$.

Marcel Chiriță and *Marian Andronache*

12th Form

Problem 1. Find all primitives of the function f: $[-1,1]\to\mathbf{R}$,

$$f(x) = arcsin\sqrt{1-x^2} .$$

***

Problem 2. Let G be a group. For each positive integer we consider the set

$$H_n = \left\{x \in G \middle| \ x^n = 1\right\}.$$

i) Show that H_2 is a subgroup of G if and only if

$xy = yx$, for every $x, y \in H_2$.

ii) If p is a prime number such that H_p has at most p elements then H_p is a subgroup of G.

Marcel Țena

Problem 3. Let f: $\mathbf{R}\to\mathbf{R}$ be a differentiable function such that $f'(x)$ is a bounded function. Prove that for any positive integer k, there exists a real number c such that

$$f(c)f'(c) = c^{2k-1} .$$

Marcel Chiriță and *Valentin Matrosenco*

Problem 4. Let G be a finite group of order n. Prove that the following conditions are equivalent :

i) every subset $A \subset G$ which has the property $x, y \in A \Rightarrow x^{-1}y^{-1} \in A$ is a subgroup of G.

ii) n is not divisible by 3.

Marian Andronache and *Ion Savu*

THE 49TH NATIONAL MATHEMATICAL OLYMPIAD, 1998

The Second Round in the City of Bucharest

PROPOSED PROBLEMS[2]

9th Form

Problem 1. Let $a, b \in \left(0, \frac{\pi}{2}\right)$. Prove that :

$$\sqrt{x\sin^2 a + y\sin^2 b} + \sqrt{x\cos^2 a + y\cos^2 b} \geq \sqrt{x} + \sqrt{y}$$

for every $x, y \geq 0$ if and only if $a + b = \frac{\pi}{2}$.

L. Panaitopol

Problem 2. Let ABC be an isosceles triangle ($AB = AC$), D be the midpoint of (BC) and E be the midpoint of (AB).

a) Find the locus of the points M such that

$$MC^2 + MB^2 = 2MA^2 + BC^2.$$

b) If P the common point of the locus and AD, then $EP \perp AC$.

C. Ţuţu

Problem 3. Let A be a finite set which has at least two elements and $f: A \to A$ be a function with the property : for every set $B \subset A$ which has at least two elements, $f(B) \neq B$.

Prove that there exists an unique $a \in A$ such that $f(a) = a$.

M. Andronache and *I. Savu*

Problem 4. Let O be a point on the median $[AA_1]$ of the triangle ABC. Prove that :

a) there exists an unique point $M \in (BC)$ which fulfils the condition : if $N \in (AC)$ such that $MN \parallel BO$ and $P \in (AB)$ such that $NP \parallel CO$ then $PM \parallel AO$;

b) if M is the point from above then

$$\frac{BM}{BC} + \frac{CN}{CA} + \frac{AP}{AB} = 1.$$

V. Matrosenco, M. Chiriţă and *M. Andronache*

[2] The solutions of the problems from this section are available in Gazeta Matematica, vol. 103, no. 5-6, pp. 203.

10^{th} Form

Problem 1. Prove that

$$\frac{n}{3!}+\frac{n(n-1)}{4!}+\frac{n(n-1)(n-2)}{5!}+\ldots+\frac{n(n-1)(n-2)\ldots 3}{n!}=$$

$$=\frac{2^{n+3}-(n^2+7n+14)}{2(n+1)(n+2)},\ \textit{for every } n\geq 3..$$

***

Problem 2. Let $a,\ b,\ c\in(0,\infty)\setminus\{1\}$ be mutually distinct numbers. Find the positive integer solutions of the system :

$$a^x=bc,\ b^y=ac,\ c^z=ab$$

supposing that they exist.

M. Chiriţă and *V. Matrosenco*

Problem 3. Let $x,\ y,\ z$ be complex numbers such that $|x|=|y|=|z|=$ $=1$. Prove that $3\leq|-x+y+z|+|x-y+z|+|x+y-z|\leq 6$.

M. Chiriţă and *M. Andronache*

Problem 4. Let $ABCDA'B'C'D'$ be a cube.

a) Find the locus of the points M which lie inside the square $A'B'C'D'$ and fulfil the condition $MA+MC=MB+MD$.

b) Let $N\in(AB)$, $P\in(C'D')$, $Q\in(A'D')$ and $R\in(BC)$ be variable points. Denote by X and Y the midpoints of the segments (NP) and (QR). If the segment (XY) has a constant length $k>0$, find the locus of its midpoint.

L. Panaitopol

11^{th} Form

Problem 1. Let $a,\ b,\ c$ be complex numbers and $A=\begin{pmatrix} c & 1 & -1\\ 0 & b & 0\\ 1 & 1 & a\end{pmatrix}$.

Prove that $A^2=O_3$ if and only if there exists two sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ of complex numbers such that $A^n=x_nI_3+y_nA$ for every $n\geq 1$.

I. Chiţescu

Problem 2. Find all the functions $f:\mathbf{R}\to\mathbf{R}$, which are continuous in $x_0=0$ and have the properties: $f(2x)-f(x)\leq 3x^2+x$ and $f(3x)-f(x)\geq 8x^2+2x$ for every $x\in\mathbf{R}$.

M. Chiriţă and *V. Matrosenco*

Problem 3. Let $A\in\mathcal{M}_n(\mathbf{C})$ be such that $det(A+XY)=det(A+YX)$ for every $X, Y\in\mathcal{M}_n(\mathbf{C})$. Prove that there exists $a\in\mathbf{C}$ such that $A=aI_n$.

M. Andronache and *I. Savu*

Problem 4. Let f: $[0,\infty)\to[0,\infty)$, be a continous, non-identical nil function such that $f(f(x))=(x^2+x+1)f(x), for\ every\ x\in[0,\infty)$.

a) Prove that f is bijective.

b) Study the limits $\lim\limits_{x\to0}\frac{f(x)}{x}$ *and* $\lim\limits_{x\to\infty}\frac{f(x)}{x}$.

M. Chiriţă and *M. Andronache*

12th Form

Problem 1. a) Compute $\int_0^{\frac{\pi}{4}} tg^{2n}x dx$ and prove that

$$\lim_{n\to\infty}\left(1-\frac{1}{3}+\frac{1}{5}-\ldots+\frac{(-1)^{n-1}}{2n-1}\right)=\frac{\pi}{4}.$$

***

Problem 2. Let A be a ring such that for every $x\in A$, $x^2=1\ or\ x^2=x$. Prove that if A has at least two invertible elements then A is isomorphic to $\mathbf{Z}_3$.

M. Ţena

Problem 3. Let A be a ring such that $x^2=0\Rightarrow x=0$. Denote by M the set $\{a\in A\mid a^2=a\}$. Prove that :

a) if $a, b\in M$ then $a+b-2ab\in M$;

b) if M is finite then $|M|=2^k$ for some positive integer k.

M. Andronache and *I. Savu*

Problem 4. a) Let f: $[0,1]\to\mathbf{R}$ be a differentiable function having a continuous derivative. Prove that if not all the values of f are positive then

$$\int_0^1 f(t)dt\le\int_0^1|f'(t)|dt.$$

b) Let g : $[0,1]\to\mathbf{R}$ be a differentiable function having a continuous derivative. Prove that :

$$\int_0^1\left(g(t)-|g'(t)|\right)dt\le g(x)\le\int_0^1\left(g(t)+|g'(t)|\right)dt,\ \text{for every } x\in[0,1].$$

S. Rădulescu and *P. Alexandrescu*

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Societatea de Științe Matematice din România

Str. Academiei nr. 14, 70109 București, România
Email : socmat@math.math.unibuc.ro

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