MIHAI BĂLUNĂ MIRCEA BECHEANU
BOGDAN ENESCU

R.M.C.

2001

ROMANIAN MATHEMATICAL COMPETITIONS

SOCIETATEA DE ŞTIINŢE MATEMATICE DIN ROMÂNIA

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The 52nd National Mathematical Olympiad
2nd Round (county level), March, 2001

7th Grade

Problem 1. A natural number is called "good" if can be written both as a sum of two consecutive natural numbers and as a sum of three consecutive natural numbers. Show that:
a) 2001 is "good", but 3001 is not "good";
b) the product of two "good" numbers is "good";
c) if the product of two numbers is "good", then at least one of them is "good".

Bogdan Enescu

Problem 2. Let $n = 1234567891011\ldots99100101$.
a) Find the first three digits of the number $\sqrt{n}$.
b) Compute the sum of the digits of the number n.
c) Show that $\sqrt{n}$ is irational.

Valer Pop

Problem 3. Let ABC be a triangle and the points D,E and F such that :
i) B and E are separated by the line AC,
ii) D and C are separated by the line AB,
iii) A and F are not separated by the line BC,
iv) $\Delta ADB \sim \Delta CEA \sim \Delta CFB$.
Prove that :
a) $\Delta BDF \sim \Delta FEC$;
b) the segments AF and DE have the same midpoint.

Dan Brânzei

Problem 4. We consider the convex quadrilateral ABCD and the points M∈(AB), N∈(CD) such that $\frac{AM}{BM} = \frac{DN}{CN} = k$. Show that BC is parallel to AD if and only if $MN = \frac{1}{k+1}AD + \frac{k}{k+1}BC$.

***

8th Grade

Problem 1. a) Find integer numbers m and n such that

$$9m^2+3n=n^2+8.$$

b) Let a and b be positive integers. Compare the numbers

$$x=a^{a+b}+(a+b)^a \text{ and } y=a^a+(a+b)^{a+b}.$$

Florin Nicoară and Valer Pop

Problem 2. Let x, y, z non–zero real numbers such that xy, yz, zx are rational.

a) Show that the number $x^2+y^2+z^2$ is rational.

b) If the number $x^3+y^3+z^3$ is also rational, show that x, y, z are rational.

Marius Ghergu

Problem 3. The points A, B, C, D are taken such that they are not in the same plane and:

$$AB=BD=CD=AC=\sqrt{2}\ AD=\frac{\sqrt{2}}{2}\ BC=a.$$

Show that:

a) There exists a point on the segment [BC] equally distanced from the points A,B,C,D.

b) $2m(\angle(AD,BC))=3m(\angle((ABC),(BCD)))$.

c) $6[d(A,DC)]^2=7[d(A,(BCD))]^2$.

Ion Trandafir

Problem 4. In the right parallelepiped ABCDA'B'C'D' we have: $AB=a$, $BC=b$, $AA'=c$. We denote by E and F the projections of the point D on AC and A'C, respectively, and by P and Q the projections of the point C' on B'D' and BD', respectively.

Show that the planes (DEF) and (C'PQ) are perpendicular if and only if $b^2=a^2+c^2$.

Sorin Peligrad

9th Grade

Problem 1. We consider the equation :

$$x^2+(a+b+c)x+\lambda(ab+bc+ca)=0,$$

where a, b, c are positive real numbers and $\lambda\in\mathbf{R}$ is a parameter. Show that:

a) if $\lambda\leq\frac{3}{4}$, the equation has real roots;

b) if a, b, c are the lengths of a triangle's sides and $\lambda\geq 1$, then the equation does not have real roots.

***

Problem 2. In the system of coordinates xOy we consider the lines having the equations: d_1: $2x-y-2=0$, d_2: $x+y-4=0$, d_3: $y=2$ and d_4: $x-4y+3=0$.

Find the vertices of the triangles which have the medians d_1, d_2, d_3 and in which d_4 is one of the altitudes.

Lucian Dragomir

Problem 3. Let k and $n_1<n_2<\ldots<n_k$ be odd positive integers. Show that:

$$n_1^2-n_2^2+n_3^2-n_4^2+\ldots+n_k^2\geq 2k^2-1.$$

Titu Andreescu

Problem 4. We consider a function $f:\mathbf{Z}\to\mathbf{Z}$ with the property:

$$f(m^2+f(n))=f^2(m)+n,$$

for every $m, n\in\mathbf{Z}$. Show that:

a) $f(0)=0$;

b) $f(1)=1$;

c) $f(n)=n$, for every $n\in\mathbf{Z}$.

Lucian Dragomir

10^{th} Grade

Problem 1. Show that if $(a_n)_{n\geq 1}$ is a sequence of non–zero real numbers such that :

$$a_1C_n^1 + a_2C_n^2 + ... + a_nC_n^n = a_n 2^{n-1}\text{, for every } n\in\mathbf{N}^*,$$

then $(a_n)_{n\geq 1}$ is an arithmetical progression.

Lucian Dragomir

Problem 2. We say that the pair of complex numbers $(z_1, z_2)\in\mathbf{C}^*\times\mathbf{C}^*$ has the property (P) if there exists a real number $a\in[-2,2]$ such that $z_1^2 - az_1z_2 + z_2^2 = 0$. Show that if (z_1, z_2) has the property (P), then for every natural number n, the pair (z_1^n, z_2^n) has this property.

Dorin Andrica

Problem 3. We consider the cyclic pentagon ABCDE. Denote by H_1, H_2, H_3, H_4, H_5 the orthocenters of the triangles ABC, BCD, CDE, DEA, EAB and with M_1, M_2, M_3, M_4, M_5 the midpoints of the segments DE, EA, AB, BC and CD, respectively. Show that the lines H_1M_1, H_2M_2, H_3M_3, H_4M_4 and H_5M_5 are concurrent.

Dinu Şerbănescu

Problem 4. Solve the equation:

$$2^{lg\,x} + 8 = (x-8)^{\frac{1}{lg\,2}}$$

Daniel Jinga

11^{th} grade

Problem 1. Let $A\in M_2(\mathbf{R})$ with $\det A = d \neq 0$, such that $\det(A + dA^*) = 0$.

Prove that $\det(A - dA^*) = 4$.

Daniel Jinga

Problem 2. Let $n\in\mathbf{N}$, $n\geq 2$. For every matrix $A\in M_n(\mathbf{C})$, we denote by $m(A)$ the number of all its non–zero minors. Show that:

a) $m(I_n) = 2^n - 1$.

b) if $A\in M_n(\mathbf{C})$ is nonsingular, then $m(A)\geq 2^n - 1$.

Marius Ghergu

Problem 3. Let $f:\mathbf{R}\to\mathbf{R}$ a function which transforms every closed and bounded interval in a closed and bounded interval and every open and bounded interval in an open and bounded interval.

Show that f is continuous.

Mihai Piticari

Problem 4. Show that:

a) the sequence $x_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+n}$, $n\geq 1$, is monotonic;

b) there exists a sequence $(a_n)_{n\geq 1}$ with values 0 or 1 such that

$$\lim_{n\to\infty}\left(\frac{a_1}{n+1} + \frac{a_2}{n+2} + ... + \frac{a_n}{n+n}\right) = \frac{1}{2}.$$

Radu Gologan

12th grade

Problem 1. For every $n \in \mathbf{N}^*$ we consider $H_n = \left\{ \frac{k}{n!} \mid k \in \mathbf{Z} \right\}$.

a) Prove that H_n is a subgroup of the group $(\mathbf{Q},+)$ and that

$$\mathbf{Q} = \bigcup_{n \in \mathbf{N}^*} H_n.$$

b) Prove that if $G_1, G_2, \ldots, G_m$ are subgroups of the group $(\mathbf{Q},+)$ and $G_i \neq \mathbf{Q},\ \forall\ 1 \le i \le m$, then

$$G_1 \cup G_2 \cup \ldots \cup G_m \neq \mathbf{Q}.$$

Marian Andronache and Ion Savu

Problem 2. Let K be a commutative field with 8 elements. Prove that there exists $a \in K$ such that

$$a^3 = a + 1.$$

Mircea Becheanu

Problem 3. Let $f : [0,1] \to \mathbf{R}$, be a continuous function, with the property that for every third degree polynomial function $P: [0,1] \to [0,1]$ we have:

$$\int_0^1 f(P(x))\,dx = 0.$$

Show that $f(x) = 0$, for every $x \in [0,1]$.

Mihai Piticari

Problem 4. a) Show that: $\ln(1+x) \le x$, for every $x \ge 0$.

b) Let $a > 0$. Prove that: $\lim_{n \to \infty} n \int_0^1 \frac{x^n}{a + x^n} dx = \ln \frac{a+1}{a}$.

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The 52nd National Mathematical Olympiad Final Round, April 7–13, 2001, Târgu Mureş

7th Grade

Problem 1. Show that there exist no integer numbers a and b such that $a^3 + a^2b + ab^2 + b^3 = 2001$.

***

Problem 2. Let a and b be real, positive and distinct numbers. We consider the set:

$$M = \{ax + by \mid x, y \in \mathbf{R}, x > 0, y > 0, x + y = 1\}.$$

Prove that :

i) $\frac{2ab}{a+b} \in M$;

ii) $\sqrt{ab} \in M$.

Romeo Ilie

Problem 3. We consider a right trapezoid $ABCD$, in which $AB \parallel CD$, $AB > CD$, $AD \perp AB$ and $AD > CD$. The diagonals AC and BD intersect in O. The parallel through O to AB intersects AD in E and BE intersects CD in F. Prove that $CE \perp AF$ if and only if $AB \cdot CD = AD^2 - CD^2$.

***

Problem 4. We consider the acute angle ABC. On the half–line $(BC$ we consider the distinct points P and Q whose projections on the line AB are the points M and N. Knowing that $AP = AQ$ and $AM^2 - AN^2 = BN^2 - BM^2$, find the angle ABC.

Mircea Fianu

8^{th} Grade

Problem 1. Determine the real numbers a and b such that $a+b\in\mathbf{Z}$ and $a^2+b^2=2$.

Romeo Ilie

Problem 2. For every rational number $m > 0$ we consider the function $f_m:\mathbf{R}\to\mathbf{R}$, $f_m(x)=\frac{1}{m}x+m$. Denote by G_m the graph of the function f_m. Let p, q and r be rational positive numbers.

a) Show that if p and q are distinct, then $G_p \cap G_q$ is nonempty.

b) Show that if $G_p \cap G_q$ is a point with integer coordinates, then p and q are integer numbers.

c) Show that if p, q, r are consecutive natural numbers, then the area of the triangle determined by the intersections of G_p, G_q and G_r is equal to 1.

Mircea Fianu

Problem 3. We consider the points A, B, C, D, not in the same plane, such that $AB\perp CD$ and $AB^2+CD^2=AD^2+BC^2$.

a) Prove that $AC \perp BD$.

b) Prove that if $CD < BC < BD$, then the angle between the planes (ABC) and (ADC) is greater than 60°.

Sorin Peligrad

Problem 4. In the cube $ABCDA'B'C'D'$, with side a, the plane $(AB'D')$ intersects the planes $(A'BC)$, $(A'CD)$, $(A'DB)$ after the lines d_1, d_2, and d_3, respectively.

a) Show that the lines d_1, d_2, d_3 pairwisely intersect.

b) Determine the area of the triangle formed by the three lines.

***

9^{th} Grade

Problem 1. Let A be a set of real numbers which verifies:

a) $1\in A$;

b) $x\in A \Rightarrow x^2\in A$;

c) $x^2-4x+4 \in A \Rightarrow x\in A$.

Show that $2000+\sqrt{2001}\in A$.

Lucian Dragomir

Problem 2. Let ABC a right triangle (A=90°) and $D\in(AC)$ such that BD is the bisector of B. Prove that $BC-BD=2AB$ if and only if

$$\frac{1}{BD}-\frac{1}{BC}=\frac{1}{2AB}.$$

Dan Brânzei

Problem 3. Let $n\in\mathbf{N}^*$ and $v_1, v_2,\ldots, v_n$ vectors in the plane with lengths less or equal to 1. Prove that there exist $\varepsilon_1, \varepsilon_2,\ldots, \varepsilon_n\in\{-1,1\}$ such that

$$|\varepsilon_1 v_1+\varepsilon_2 v_2+\ldots+\varepsilon_n v_n| \le \sqrt{2}.$$

Mihai Băluną

Problem 4. Determine the ordered systems (x, y, z) of positive rational numbers for which $x+\frac{1}{y}$, $y+\frac{1}{z}$ and $z+\frac{1}{x}$ are integers.

Mircea Becheanu

10^{th} Grade

Problem 1. Let a and b be complex non–zero numbers and z_1, z_2 the roots of the polynomial $X^2 + aX + b$. Show that $|z_1 + z_2| = |z_1| + |z_2|$ if and only if there exists a real number $\lambda \geq 4$ such that $a^2 = \lambda b$.

Valentin Matrosenco

Problem 2. In the tetrahedron $OABC$ we denote by α, β, γ the measures of the angles $\angle BOC$, $\angle COA$ and $\angle AOB$, respectively. Prove the inequality:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma < 1 + 2 \cos \alpha \cos \beta \cos \gamma.$$

Dinu Şerbănescu

Problem 3. Let m, k be positive integers, $k < m$ and M a set with m elements. Prove that maximal number of subsets $A_1, A_2, \ldots, A_p$ of M for which $A_i \cap A_j$ has at most k elements, for every $1 \leq i < j \leq p$, equals

$$p_{max} = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{k+1}.$$

Mihai Manea

Problem 4. Let $n \geq 2$ an even integer and a, b real numbers such that $b^n = 3a + 1$. Show that the polynomial $P(X) = (X^2 + X + 1)^n - X^n - a$ is divisible by $Q(X) = X^3 + X^2 + X + b$ if and only if $b = 1$.

Cristinel Mortici

11^{th} Grade

Problem 1. Let f: $\mathbf{R} \to \mathbf{R}$ a continuous function, derivable on $\mathbf{R} \setminus \{x_0\}$, having finite side derivatives in x_0. Show that there exists a derivable function g: $\mathbf{R} \to \mathbf{R}$, a linear function h: $\mathbf{R} \to \mathbf{R}$ and $\alpha \in \{-1, 0, 1\}$ such that:

$$f(x) = g(x) + \alpha |h(x)|, \ \forall x \in \mathbf{R}.$$

Aurelian Gheondea

Problem 2. We consider a matrix $A \in M_n(\mathbf{C})$, with rank r, where $n \geq 2$ and $1 \leq r \leq n-1$.

a) Show that there exist $B \in M_{n,r}(\mathbf{C})$, $C \in M_{r,n}(\mathbf{C})$, with rank$B$ = rank$C = r$, such that $A = BC$;

b) Show that the matrix A verifies a polynomial equation of degree $r+1$, with complex coefficients.

Mircea Becheanu and Ion Savu

Problem 3. Let f: $\mathbf{R} \to [0, \infty)$ a function with the property:

$$|f(x) - f(y)| \leq |x - y|, \text{ for every } x, y \in \mathbf{R}.$$

Show that:

a) if $\lim_{n\to\infty} f(x+n) = \infty$, for every $x \in \mathbf{R}$, then $\lim_{x\to\infty} f(x) = \infty$;

b) if $\lim_{n\to\infty} f(x+n) = a$, $a \in [0, \infty)$, for every $x \in \mathbf{R}$, then $\lim_{x\to\infty} f(x) = a$.

Mihai Piticari and Sorin Rădulescu

Problem 4. The continuous function f: $[0,1] \to \mathbf{R}$ has the property:

$$\lim_{n\to\infty} n\left(f\left(x + \frac{1}{n}\right) - f(x)\right) = 0, \text{ for every } x \in [0,1).$$

Show that:

a) for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have:

$$\sup\{ x \in [0, \lambda) \mid |f(x) - f(0)| \leq \varepsilon x \} = \lambda;$$

b) f is a constant function.

Cristinel Mortici

12^{th} Grade

Problem 1. a) We consider the polynomial $P(X)=X^5\in\mathbf{R}[X]$. Show that for every $\alpha\in\mathbf{R}^*$, the polynomial $P(X+\alpha)-P(X)$ has no real roots.
b) Let $P\in\mathbf{R}[X]$ be a polynomial of degree $n\geq 2$, with real and distinct roots. Show that there exists $\alpha\in\mathbf{Q}^*$ such that the polynomial $P(X+\alpha)-P(X)$ has only real roots.

Radu Gologan

Problem 2. Let A be a finite ring. Show that there exist two natural numbers m, p, $m>p\geq 1$, such that

$$a^m=a^p, \forall a\in A.$$

Ion Savu

Problem 3. Let $f:[-1,1]\to\mathbf{R}$ be a continuous function. Show that:
a) if $\int_0^1 f(\sin(x+\alpha))dx=0$, for every $\alpha\in\mathbf{R}$, then $f(x)=0, \forall x\in[-1,1]$;
b) if $\int_0^1 f(\sin nx)dx=0$, for every $n\in\mathbf{Z}$, then $f(x)=0, \forall x\in[-1,1]$.

Dorin Andrica and Mihai Piticari

Problem 4. Let $f:[0,\infty)\to\mathbf{R}$ a periodical function, with period 1, integrable on $[0,1]$. For a strictly increasing and unbounded sequence $(x_n)_{n\geq 0}$, $x_0=0$, with $\lim\limits_{n\to\infty}(x_{n+1}-x_n)=0$, we denote $r(n)=\max\{k \mid x_k\leq n\}$.

a) Show that:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{r(n)}(x_k-x_{k-1})f(x_k)=\int_0^1 f(x)dx.$$

b) Show that:

$$\lim_{n\to\infty}\frac{1}{\ln n}\sum_{k=1}^{n}\frac{f(\ln k)}{k}=\int_0^1 f(x)dx.$$

Cristinel Mortici

Selection examinations for the 42^{nd} IMO, 2001
First round, April 12^{th}, Târgu Mureş

Problem 1. Show that if a, b, c are complex numbers such that

$$(a+b)(a+c)=b$$
$$(b+c)(b+a)=c$$
$$(c+a)(c+b)=a$$

then a, b, c are real numbers.

Mihai Cipu

Problem 2. a) Let $f, g:\mathbf{Z}\to\mathbf{Z}$ be one to one maps. Show that the function $h:\mathbf{Z}\to\mathbf{Z}$, defined by $h(x)=f(x)g(x)$, for all $x\in\mathbf{Z}$, cannot be a surjective function.
b) Let $f:\mathbf{Z}\to\mathbf{Z}$ be a surjective function. Show that there exist surjective functions $g, h:\mathbf{Z}\to\mathbf{Z}$ such that $f(x)=g(x)h(x)$, for all $x\in\mathbf{Z}$.

Ion Savu

Problem 3. The sides of a triangle have lengths a, b, c. Show that:

$$(-a+b+c)(a-b+c)+(a-b+c)(a+b-c)+(a+b-c)(-a+b+c)\leq$$
$$\leq\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).$$

Mircea Becheanu

Problem 4. Three schools have 200 students each. Every student has at least one friend in each school (if the student a is a friend of the student b then b is a friend of a).

It is known that there exists a set E of 300 students (among the 600) such that for any school S and any two students $x, y\in E$ which are not în the school S, the numbers of friends in S of x and y are different.

Show that one can find a student in each school such that they are friend to each other.

Antal Bege

Second round, May 19th, Bucharest

Problem 5. Find all polynomials with real coefficients $P(X)$ such that
$$P(x)\cdot P(2x^2-1)=P(x^2)\cdot P(2x-1),$$
for every $x\in\mathbf{R}$.

Nicolai Nikolov (Bulgaria)

Problem 6. The vertices A, B, C, and D of a square lie outside a circle centered in M. Let AA', BB', CC', DD' be tangents to the circle. We assume that the segments AA', BB', CC', DD' are the consecutive sides of a quadrilateral p in which a circle is inscribed. Prove that p has an axis of symmetry.

Dan Brânzei

Problem 7. Find the least number n with the property : from any n half lines in the space sharing a common origin, one can pick two such that the angle between them is acute.

Mihai Bălună

Problem 8. Prove that there are finitely many positive integers that cannot be written as a sum of distinct squares.

Radu Todor (IMO 200 Shortlist)

Third round, May 25th, Bucharest

Problem 9. Let n be a positive integer and $f(X)=a_0+a_1X+\ldots+a_mX^m$, with $m\geq 2$, a polynomial with integer coefficients, such that :
(1) $a_2, a_3, \ldots, a_m$ are divisible by all prime factors of n,
(2) a_1 and n are relatively prime.
Prove that for any positive integer k, there exists a positive integer c, such that $f(c)$ is divisible by n^k.

Ion Savu

Problem 10. Let p and q be relatively prime positive integers. A subset S of $\{0, 1, 2,\ldots\}$ is called *ideal* if $0\in S$ and, for each element $n\in S$, the integers $n+p$ and $n+q$ belong to S. Determine the number of ideal subsets of $\{0, 1, 2,\ldots\}$.

41st IMO, Jury

Fourth round, May 26th, Bucharest

Problem 11. Find all pairs (m, n) of positive integers, with $m, n\geq 2$, such that a^n-1 is divisible by m for each $a\in\{1, 2, \ldots, n\}$.

L.Panaitopol

Problem 12. Prove that there is no function $f:(0, \infty)\to(0, \infty)$ such that
$$f(x+y)\geq f(x)+yf(f(x)),$$
for every $x, y\in(0, \infty)$.

Bulgaria

Problem 13. The tangents at A and B to the circumcircle of the acute triangle ABC intersect the tangent at C in the points D and E, respectively. The line AE intersects BC in P and the line BD intersects AC in R. Let S be the midpoint of the segment AP. Show that the angles $\angle ABQ$ and $\angle BAS$ are equal.

United Kingdom (IMO 2000 Shortlist)

Problem 14. Let P be a convex polyhedron, with vertices $V_1, V_2,\ldots, V_p$. The distinct vertices V_i and V_j are called neighbours if they belong to the same face of the polyhedron. In each vertex V_i an integer number $v_i(0)$ is written and next, the sequences $(v_i(n))_{n\geq 0}$ are defined as follows: $v_i(n+1)$ is the arithmetic mean of the numbers $v_j(n)$, for all vertices V_j which are neighbours with V_i.
Prove that if all $v_i(n)$, $1\leq i\leq p$, $n\in\mathbf{N}$, are integer numbers, then there exists $M\in\mathbf{N}$ and $k\in\mathbf{Z}$ such that $v_i(n)=k$, for every $n\geq M$ and every $i=1, 2,\ldots,p$.

Barbu Berceanu

Selection Examinations for the 5th JBMO
1st Selection Examination, April 12th, 2001, Târgu Mureş

Problem 1. Let ABC be an arbitrary triangle. A circle passes through B and C and intersects the lines AB and AC in D and E, respectively. The projections of the points B and E on CD are denoted by B' and E', respectively. The projections of the points D and C on BE are denoted by D' and C', respectively. Prove that the points B', D', E' and C' lie on the same circle.

Dan Brânzei

Problem 2. Find $n\in\mathbf{Z}$ such that the number $\sqrt{\frac{4n-2}{n+5}}$ is rational.

Dan Popescu

Problem 3. In the interior of a circle centered in O a number of 1200 points $A_1, A_2, \ldots, A_{1200}$ are considered, such that for every i, j with $1\le i\le j\le 1200$, the points O, A_i and A_j are not collinear. Prove that there exist the points M and N on the circle, with $m(\angle MON)=30°$, such that in the interior of the angle $\angle MON$ lie exactly 100 points.

Bogdan Enescu

Problem 4. Three students write on the blackboard next to each other three two–digit squares. In the end, they observe that the 6–digit number thus obtained is also a square. Find this number!

Mircea Becheanu

2nd Selection Examination, May 19th, 2001, Buzău

Problem 5. Let $ABCD$ be a rectangle. We consider the points $E\in CA$, $F\in AB$, $G\in BC$ such that $DE\perp CA$, $EF\perp AB$ and $EG\perp BC$. Solve in the set of rational numbers the equation $AC^x = EF^x + EG^x$.

Dan Brânzei

Problem 6. Let A be a non–empty subset of $\mathbf{R}$ with the property that for every real numbers x, y, if $x+y\in A$, then $xy\in A$. Prove that $A=\mathbf{R}$.

Eugen Păltănea

Problem 7. Let $ABCD$ be a quadrilateral inscribed in the circle O. For a point $E\in O$, its projections K, L, M, N on the lines DA, AB, BC, CD, respectively, are considered. Prove that if N is the orthocenter of the triangle KLM for some point E, different from A, B, C, D, then this holds for every point E of the circle O.

Dan Brânzei

Problem 8. Determine the positive integers $a<b<c<d$ with the property that each of them divides the sum of the other three.

Dinu Şerbănescu

3rd Selection Examination, May 20th, 2001, Buzău

Problem 9. Let n be a non–negative integer. Find the non–negative integers a, b, c, d such that

$$a^2+b^2+c^2+d^2=7\cdot 4^n .$$

Laurenţiu Panaitopol

Problem 10. Let $ABCDEF$ be a hexagon with $AB\|DE$, $BC\|EF$, $CD\|FA$ and in which the diagonals AD, BE and CF are congruent. Prove that the hexagon can be inscribed in a circle.

Dan Brânzei

Problem 11. Let $n\ge 2$ be a positive integer. Find the positive integers x such that

$$\sqrt{x+\sqrt{x+\ldots+\sqrt{x}}}<n,$$

for any number of radicals.

Ion Dobrotă

Problem 12. Determine a right parallelepiped with minimal area, if its volume is strictly greater than 1000, and the lengths of its sides are integer numbers.

Dinu Şerbănescu

The 52nd National Mathematical Olympiad
2nd Round
Solutions

7th Grade

Problem 1. *A natural number is called "good" if can be written both as a sum of two consecutive natural numbers and as a sum of three consecutive natural numbers. Show that:*

a) 2001 is "good", but 3001 is not "good";

b) the product of two "good" numbers is "good";

c) if the product of two numbers is "good", then at least one of them is "good".

Solution. a) It is easy to see that a number is "good" if and only if is odd and divisible by 3. Thus, 2001 is "good", but 3001 is not, since it is not divisible by 3.

b) The product a two odd and divisible by 3 numbers is also odd and divisible by 3, hence the conclusion.

c) If the product of two numbers is odd and divisible by 3, then both factors are odd and at least one of them is divisible by 3.

Problem 2. *Let $n = 1234567891011...99100101$.*

a) Find the first three digits of the number $\sqrt{n}$.

b) Compute the sum of the digits of the number n.

c) Show that $\sqrt{n}$ is irational.

Solution. a) We observe that the number n has 195 digits. By applying the algorithm of extracting the square root, we find that the first three digits of $\sqrt{n}$ are equal to 1.

b) The requested sum is 903.

c) Since the digits' sum is 903, 3 divides n but 9 does not divide n, hence $\sqrt{n}$ cannot be a rational number.

Problem 3. *Let ABC be a triangle and the points D,E and F such that:*

i) B and E are separated by the line AC,

ii) D and C are separated by the line AB,

iii) A and F are not separated by the line BC,

iv) $\Delta ADB \sim \Delta CEA \sim \Delta CFB$.

Prove that :

a) $\Delta BDF \sim \Delta FEC$;

b) the segments AF and DE have the same midpoint.

Solution .a) From the similarity of the triangles ADB and CFB we obtain $\frac{BD}{AB}=\frac{FB}{BC}$, hence $\frac{BD}{FB}=\frac{AB}{BC}$, and it results that $\Delta ABC \sim \Delta DBF$. Analogously, $\Delta ABC \sim \Delta EFC$ and conclusion is obvious.

b) From the above similarities we obtain $\frac{DF}{AC}=\frac{DB}{AB}=\frac{AE}{AC}$, hence $AE=DF$. Analogously, $AD=EF$, so the quadrilateral $AEFD$ is a parallelogram and its diagonals have the same midpoint.

Problem 4. *We consider the convex quadrilateral ABCD and the points $M\in(AB)$, $N\in(CD)$ such that* $\frac{\mathrm{AM}}{\mathrm{BM}}=\frac{\mathrm{DN}}{\mathrm{CN}}=k$. *Show that* BC is parallel to AD *if and only if* $\mathrm{MN}=\frac{1}{k+1}\mathrm{AD}+\frac{k}{k+1}\mathrm{BC}$.

Solution. If $BC \parallel AD$ the conclusion follows immediately. Conversely, let $P\in(AC)$ such that $MP \parallel BC$. It results $PN \parallel AD$ and from the hypothesis, $MP+PN=MN$. It follows that $BC \parallel AD$.

8th Grade

Problem 1. ***a)*** *Find the integer numbers m and n such that*

$$9m^2+3n=n^2+8.$$

b) *Let a and b be positive integers. Compare the numbers*

$$x=a^{a+b}+(a+b)^a \text{ and } y=a^a+(a+b)^{a+b}.$$

Solution. a) We observe that $9m^2+3n=n^2+8$ if and only if $(6m-2n+3)(6m+2n-3)=23$. Checking all ways in which 23 can be written as a product of integer numbers, we obtain the solutions (2;7),(2;-4),(-2;-4) and (-2;7).

b) We observe that $x-y=a^a(a^b-1)-(a+b)^a((a+b)^b-1)$.

From $a^a<(a+b)^a$ and $0\le a^b-1<(a+b)^b-1$, it results $x<y$.

Problem 2. *Let x, y, z non–zero real numbers such that xy, yz, zx are rational.*

a) Show that the number $x^2+y^2+z^2$ is rational.

b) If the number $x^3+y^3+z^3$ is also rational, show that x, y, z are rational.

Solution. a) If $xy\in\mathbf{Q}$ and $yz\in\mathbf{Q}$, it results $\frac{x}{z}\in\mathbf{Q}$, but $xz\in\mathbf{Q}$, hence $x^2\in\mathbf{Q}$. Similarly, y^2 and z^2 are rational, hence their sum is also rational.
b) We observe that $x(x^3+y^3+z^3)=(x^2)^2+(xy)y^2+(xz)z^2\in\mathbf{Q}$, hence $x\in\mathbf{Q}$. Analogously for y and z.

Problem 3. *The points A, B, C, D are taken such that they are not in the same plane and:*

$$AB=BD=CD=AC=\sqrt{2}\,AD=\frac{\sqrt{2}}{2}\,BC=a.$$

Show that:

a) There exists a point on the segment [BC] equally distanced from the points A, B, C, D.

b) $2m(\angle(AD,BC))=3m(\angle((ABC),(BCD)))$.

c) $6[d(A,DC)]^2=7[d(A,(BCD))]^2$.

Solution . a) It results that ΔBDC is right in $\angle$ D, and ΔBAC is right in $\angle$ A. Therefore, if M is the midpoint of [BC], then MA=MB=MC=MD.
b) We can see that $m(\angle(AD,BC))=90^\circ$ and $m(\angle(ABC),(BCD))=60^\circ$.
c) Construct $AQ\perp MD$, $Q\in MD$, and it results that $AQ\perp(ABC)$, hence $d(A,(BCD))=AQ=\frac{a\sqrt{6}}{4}$. Construct $QP\perp DC$, $P\in DC$. It results that $AP\perp DC$ hence $d(A,DC)=AP=\frac{a\sqrt{7}}{4}$.

Problem 4. *In the right parallelepiped ABCDA'B'C'D' we have: AB=a, BC=b, AA'=c. We denote by E and F the projections of the point D on AC and A'C, respectively, and by P and Q the projections of the point C' on B'D' and BD', respectively.*

Show that the planes (DEF) and (C'PQ) are perpendicular if and only if $b^2=a^2+c^2$.

Solution. We observe that $A'C\perp(DEF)$ and $BD'\perp(C'PQ)$. It results that $(DEF)\perp(C'PQ)$ if and only if $A'C\perp BD'$ and then $A'C\perp BD'$ if and only if $b^2=a^2+c^2$.

9th Grade

Problem1. *We consider the equation :$x^2+(a+b+c)x+\lambda(ab+bc+ca)=0$, where a, b, c are positive real numbers and $\lambda\in\mathbf{R}$ is a parameter. Show that:*

a) if $\lambda\le\frac{3}{4}$, the equation has real roots;

b) if a, b, c are the lengths of a triangle's sides and $\lambda\ge 1$, then the equation does not have real roots.

Solution.a) We observe that $\Delta=(a+b+c)^2-4\lambda(ab+bc+ca)$.
Using the inequality $(a+b+c)^2\ge 3(ab+bc+ca)$, we obtain

$$\Delta\ge(3-4\lambda)(ab+bc+ca)\ge 0.$$

b) Since a, b, c are the lengths of a triangle's sides, we have $|a-b|<c$ hence $(a-b)^2<c^2$. Analogously we obtain $(b-c)^2<a^2$ and $(c-a)^2<b^2$ and, by adding, we find $(a+b+c)^2<4(ab+bc+ca)$, hence

$$\Delta<4(1-\lambda)(ab+bc+ca)\le 0.$$

This means the equation has no real roots.

Problem 2. *In the system of coordinates xOy we consider the lines having the equations: d_1: $2x-y-2=0$, d_2: $x+y-4=0$, d_3: $y=2$ and d_4: $x-4y+3=0$.*

Find the vertices of the triangles which have the medians d_1, d_2, d_3 and in which d_4 is one of the altitudes.

Solution. Let $A\in d_1$, $B\in d_2$, $C\in d_3$. It results that there exists $a, b, c\in\mathbf{R}$ such that the points have coordinates $A(a, 2a-2)$, $B(b, 4-b)$, $C(c, 2)$. By computing the coordinates of the midpoints of the sides of the triangle ABC, we obtain $b=2a-2$, $c=8-3a$, hence we have $A(a,2a-2)$, B(2a–2, 6–2a), C(8–3a, 2), with $a\in\mathbf{R}$. From the condition $d_4 \perp BC$ and $A\in d_4$, we obtain a=1, hence A(1, 0), B(0, 4), C(5, 2).

Problem 3. *Let k and $n_1 < n_2 < ... < n_k$ be odd positive integers. Show that:*

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + ... + n_k^2 \geq 2k^2 - 1.$$

Solution. We prove the statement by induction on $k \geq 1$. The case $k=1$ is obvious. We suppose that

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + ... + n_k^2 \geq 2k^2 - 1.$$

and we prove

$$n_1^2 - n_2^2 + n_3^2 - n_4^2 + ... + n_k^2 - n_{k+1}^2 + n_{k+2}^2 \geq 2(k+2)^2 - 1 = 2k^2 - 1 + 8k + 8.$$

Thus, is sufficient to prove that

$$n_{k+2}^2 - n_{k+1}^2 \geq 8k + 8,$$

or that $(n_{k+2} - n_{k+1})(n_{k+2} + n_{k+1}) \geq 2(4k+4)$. This is obvious since $n_{k+2} - n_{k+1} \geq 2$, and from the hypothesis it results that $n_k \geq 2k - 1$, hence $n_{k+2} + n_{k+1} \geq 4k + 4$.

Problem 4. *We consider a function $f: \mathbf{Z} \to \mathbf{Z}$ with the property:*

$$f(m^2 + f(n)) = f^2(m) + n,$$

for every m, $n\in\mathbf{Z}$. Show that:

a) $f(0)=0$;

b) $f(1)=1$;

c) $f(n)=n$, for every $n\in\mathbf{Z}$.

Solution. For m=0 we obtain $f(f(n))=f^2(0)+n$ (1). From this, it results that if $f(x)=f(y)$ then $x=y$. Also, we deduce that there exists k such that $f(k)=0$ (namely $k=f(-f^2(0))$). Let $f(0)=l$. For $m=k$ and $n=0$, it results $f(k^2+l)=0=f(k)$, hence $k^2+l=k$. For $n=k$ in the relation (1) we obtain $f(f(k))=f(0)=l=l^2+k$. It results $k=l=0$, hence $f(0)=0$. For m = 1 and n = 0 in the initial relation one gets: $f(1)=f^2(1)$, hence $f(1)=1$. Also, we obtain $f(2)=f(1+f(1))=1+f(1)=2$. By induction it results $f(n)=n$, for every $n\in\mathbf{N}$, and then $f(n)=n$, for every $n\in\mathbf{Z}$ (one obtains $f(-m)=-f(m)$).

10th Grade

Problem 1. *Show that if $(a_n)_{n\geq 1}$ is a sequence of non–zero real numbers such that :*

$$a_1C_n^1 + a_2C_n^2 + ... + a_nC_n^n = a_n 2^{n-1}, \text{ for every } n\in N^*,$$

then $(a_n)_{n\geq 1}$ is an arithmetical progression.

Solution. It is easy to see that $a_2=2a_1$ and $a_3=3a_1$. We prove by induction that $a_n=na_1$, for every $n\in\mathbf{N}$ Suppose $a_k=k\,a_1$, for $k\leq n-1$,we have

$$a_n(2^{n-1}-1) = a_1(1\cdot C_n^1 + 2\cdot C_n^2 + ... + (n-1)\cdot C_n^{n-1}) = a_1\left(\sum_{k=1}^{n} kC_n^k - nC_n^n\right) =$$

$$= a_1\left(\sum_{k=1}^{n} nC_{n-1}^{k-1} - n\right) = a_1(n2^{n-1} - n)$$

hence $a_n=na_1$. Obviously, the sequence $(a_n)_{n\geq 1}$ is an arithmetical progression.

Problem 2. *We say that the pair of complex numbers $(z_1, z_2)\in C^*\times C^*$ has the property (P) if there exists a real number $a\in[-2,2]$ such that $z_1^2 - az_1z_2 + z_2^2 = 0$. Show that if (z_1, z_2) has the property (P), then for every natural number n, the pair (z_1^n, z_2^n) has this property.*

Solution. Let $\frac{z_1}{z_2} = t$. We obtain $t^2-at+1=0$. Since $a\in[-2,2]$, there exists $\alpha\in[0,\pi]$ such that $a=2\cos\alpha$ and it results that $t=\cos\alpha \pm i\sin\alpha$. It follows that

$$\frac{z_1^n}{z_2^n} = t^n = \cos n\alpha \pm i\sin n\alpha, \text{ for every } n\in\mathbf{N}.$$

Hence $z_1^{2n} - 2\cos na\, z_1^n z_2^n + z_2^{2n} = 0$, i.e.$(z_1^n, z_2^n)$ has the property (*P*).

Problem 3. *We consider the cyclic pentagon ABCDE. Denote by H_1, H_2, H_3, H_4, H_5 the orthocenters of the triangles ABC, BCD, CDE, DEA, EAB and with M_1, M_2, M_3, M_4, M_5 the midpoints of the segments DE, EA, AB, BC and CD, respectively. Show that the lines H_1M_1, H_2M_2, H_3M_3, H_4M_4 and H_5M_5 are concurrent.*

Solution. We use complex numbers. If the points A, B, C, D and E corespond to the complex numbers a, b, c, d, e, respectively, it is known that H_1 coresponds to h_1=a+b+c, etc. , and M_1 coresponds to m_1=(d+e)/2. A point P coresponding to p is on the line H_1M_1 if and only if there exists t such that p=(1–t)(a+b+c)+t(d+e)/2. It is easy to check that the point P coresponding to (a+b+c+d+e)/3 belongs to all lines H_iM_i.

Problem 4. *Solve the equation:*

$$2^{lgx} + 8 = (x-8)^{\frac{1}{lg2}}$$

Solution. Clearly, x>8. Let $2^{\lg x} + 8 = y$; it results $2^{\lg x} = y - 8$ and $(x-8)^{\frac{1}{\lg 2}} = y$, so we get:

$$\lg x + \log_2(x-8) = \lg y + \log_2(y-8)\ .$$

We deduce x=y, hence $2^{\lg x} + 8 = x$. If we denote $2^{\lg x} = t$, it results $2^t+8=10^t$, with the unique solution t=1. It follows that x=10.

11th grade

Problem 1. *Let* $A \in M_2(\mathbf{R})$ *with* $\det A = d \neq 0$, *such that* $\det(A + dA^*) = 0$. *Prove that* $\det(A - dA^*) = 4$.

Solution.

Let $A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$, $A^* = \begin{pmatrix} q & -n \\ -p & m \end{pmatrix}$, $d = mq - np, m,n,p,q \in \mathbf{R}$. By direct computation, we have:

$$det(A + dA^*) = \begin{vmatrix} m+qd & n(1-d) \\ p(1-d) & q+md \end{vmatrix} = d\left[(d-1)^2 + (m+q)^2\right]$$

and the condition from the enounce becomes d = 1 and $m + q$ = 0. Then

$$det(A - dA^*) = det(A - A^*) = \begin{vmatrix} m-q & 2n \\ 2p & q-m \end{vmatrix} =$$
$$= -(m+q)^2 + 4(mq-np) = 4d = 4.$$

Problem 2. *Let* $n \in N$, $n \geq 2$. *For every matrix* $A \in M_n(C)$, *we denote by m(A) the number of all its non–zero minors. Show that:*

a) $m(I_n) = 2^n - 1$.

b) if $A \in M_n(C)$ *is nonsingular, then* $m(A) \geq 2^n - 1$.

Solution.a) We consider the columns $i_1 < \ldots < i_k \in \{1,2,\ldots,n\}$ of the matrix I_n. We observe that only non–zero minor of order k is formed by the rows $i_1 <\ldots< i_k$, hence for any k columns we have only one non–zero minor. It results that the number of the non–zero minors of order k is C_n^k and that their sum is $2^n - 1$.

b) Let's fix $k \in \{1,\ldots,n\}$ and consider the columns $i_1 < \ldots < i_k \in \{1,\ldots,n\}$. If all minors of order k which contain these columns are zero, then every minor of order k + 1 which contains these columns is equal to zero. Analogously, every minor of order k + 2 which contains these columns is zero and finally, detA = 0, which is a contradiction. Hence there exists one minor of order k with elements from these columns which is non–zero. It results that we have at least C_n^k minors of order k which are non–zero (we can choose C_n^k systems of k columns from n) and then the total number is at least $2^n - 1$.

Problem 3. *Let* $f: \mathbf{R} \to \mathbf{R}$ *a function which transforms every closed and bounded interval in a closed and bounded interval and every open and bounded interval in an open and bounded interval.*
Show that f is continuous.

Solution. Let $a < b$, and suppose that $f((a,b)) = (c,d)$. Since $f([a,b])$ is a closed and bounded interval and $f([a,b])=(c,d)\cup\{f(a),f(b)\}$, it follows that $f(a)=c$ and $f(b)=d$ or $f(a)=d$ and $f(b)=c$. Anyway, $f(a)\neq f(b)$ thus f is an injective function .

Next, if we take w between $f(a)$ and $f(b)$, since $f((a,b))=(f(a),f(b))$ or $f((a,b))=(f(b),f(a))$, it results that there exists x_w in (a,b) with $f(x_w)=w$. Hence f has the intermediate values property.

In conclusion, f is continuous, being injective and with the intermediate values property.

Problem 4. *Show that:*

a) the sequence $x_n=\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{n+n}$, *n≥1, is monotonic;*

b) there exists a sequence $(a_n)_{n\geq 1}$ *with values 0 or 1 such that*

$$\lim_{n\to\infty}\left(\frac{a_1}{n+1}+\frac{a_2}{n+2}+...+\frac{a_n}{n+n}\right)=\frac{1}{2}.$$

Solution.a) $x_{n+1}-x_n=\frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1}=\frac{1}{2n+1}-\frac{1}{2n+2}>0.$

b) Obviously , x_n>1/2 and we inductively choose $a_i\in\{0,1\}$ such that

$$\frac{1}{2}\leq\frac{a_1}{n+1}+\frac{a_2}{n+2}+...+\frac{a_n}{n+n}\leq\frac{1}{2}+\frac{1}{2n},$$

for every $n\in\mathbf{N}^*$.

For $n = 2$ we choose $a_1=a_2=1$. If $a_1,\dots,a_n$ are already chosen, we proceed in the following way to choose a_{n+1}:

–if $\frac{a_1}{n+2}+...+\frac{a_n}{2n+1}\geq\frac{1}{2}$ we choose $a_{n+1}=0$ and the relation (1) remains true for n+1.

–if $\frac{a_1}{n+2}+...+\frac{a_n}{2n+1}<\frac{1}{2}$, we choose $a_{n+1}=1$ and it remains to show that $b_n=\frac{a_1}{n+2}+...+\frac{a_n}{2n+1}+\frac{1}{2n+2}\geq\frac{1}{2}$.

Indeed, using the induction hypothesis, we have:

$$b_n=-a_1\left(\frac{1}{n+1}-\frac{1}{n+2}\right)-...-a_n\left(\frac{1}{2n}-\frac{1}{2n+1}\right)+$$

$$+\frac{a_1}{n+1}+...+\frac{a_n}{2n}+\frac{1}{2n+2}\geq-\left(\frac{1}{n+1}-\frac{1}{n+2}\right)-...-\left(\frac{1}{2n}-\frac{1}{2n+1}\right)+\frac{1}{2}+\frac{1}{2n+2}=$$

$$=\frac{1}{2n+1}-\frac{1}{n+1}+\frac{1}{2n+2}+\frac{1}{2}=\frac{1}{2n+1}-\frac{1}{2n+2}+\frac{1}{2}>0.$$

12th grade

Problem 1. *For every* $n\in N^*$ *we consider* $H_n=\left\{\frac{k}{n!}\mid k\in\mathbf{Z}\right\}$.

a) Prove that H_n *is a subgroup of the group* $(\mathbf{Q},+)$ *and that*

$$\mathbf{Q}=\bigcup_{n\in\mathbf{N}^*}H_n.$$

b) Prove that if $G_1, G_2, ..., G_m$ *are subgroups of the group* $(\mathbf{Q},+)$ *and* $G_i\neq\mathbf{Q},\ \forall\ 1\leq i\leq m$, *then*

$$G_1\cup G_2\cup...\cup G_m\neq\mathbf{Q}.$$

Solution. a) Let $x,y\in H_n$, $x=\frac{k}{n!}$, $y=\frac{p}{n!}$, hence $x+y=\frac{k+p}{n!}\in H_n$, so it follows that H_n is closed under addition.

Let $x\in H_n$, $x=\frac{k}{n!}\Rightarrow -x=\frac{-k}{n!}\in H_n$, hence $H_n\leq\mathbf{Q}$. If $x\in\mathbf{Q}$, $x=\frac{p}{q}$, then $x\in H_q$, hence $=\bigcup_{n\in N^*}H_n$.

b) We consider $A=\left\{\frac{1}{n!}\mid n\in\mathrm{N}^*\right\}$. If we suppose that $\mathbf{Q}=G_1\cup...\cup G_m$, since A is infinite, it results that there exists i such that $G_i\cap A$ is also infinite. We observe that $H_n\subset H_{n+1}$ and if

$\frac{1}{n!} \in G_i$, it results that $H_n \subset G_i$. Let $n < r$ such that $\frac{1}{r!} \in G_i \cap A$. Then $H_r \subset G_i \Rightarrow H_n \subset G_i$. Hence $\bigcup_{n\in N^*} H_n \subset G_i$ or $\mathbf{Q} \subset G_n$, contradiction.

Problem 2. *Let K be a commutative field with 8 elements. Prove that there exists $a \in K$ such that*

$$a^3 = a + 1.$$

Solution. The group $(K^*,\cdot)$ has 7 elements, hence for every $x \in K^*$, we have $x^7=1$. It follows that the polynomial $f = x^8 - x$ has as roots all the elements of K. Since K is a commutative field, f has 8 factors of degree one. We consider $g = x^3 - x - 1$; it is easy to see that $f = g\cdot(x^5 + x^3 + x^2 + x)$, since $-1=1$ in K. Then g contains 3 linear factors of f, hence has roots in K. If $a \in K$ is a root of g, then we have $a^3 = a + 1$.

Problem 3. *Let $f : [0,1] \to \mathbf{R}$, be a continuous function, with the property that for every third degree polynomial function $P: [0,1] \to [0,1]$ we have:*

$$\int_0^1 f(P(x))\mathrm{d}x = 0.$$

Show that $f(x) = 0$, for every $x \in [0,1]$.

Solution. We suppose that f is not the zero function. Let $x_0 \in (0,1)$ with $f(x_0) \neq 0$, and suppose $f(x_0) > 0$.

Since f is continuous in x_0, there exists an interval $[x_0 - \varepsilon, x_0 + \varepsilon] \subset [0,1]$ such that $f(x) > 0$, for every $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$.

We consider the polynomial function $P:[0,1] \to [0,1]$, $P(x) = ax^3 + b$, with $P(0) = x_0 - \varepsilon$, $P(1) = x_0 + \varepsilon$. We have:

$$0 = \int_0^1 f(P(x))dx = f(P(c)) > 0,$$

with $c \in [0,1]$ from the mean value theorem, contradiction.

Problem 4. *a) Show that: $\ln(1+x) \le x$, for every $x \ge 0$.*

b) Let $a > 0$. Prove that: $\lim_{n\to\infty} n \int_0^1 \frac{x^n}{a + x^n} dx = \ln \frac{a+1}{a}$.

Solution. a) Take $f(x) = \ln(1+x) - x$ with $f'(x) \le 0$, hence $f(x) \le f(0) = 0$.

b) We have:

$$n\int_0^1 \frac{x^n}{a+x^n}dx = \int_0^1 \frac{x\cdot nx^{n-1}}{a+x^n}dx = \int_0^1 x(\ln(a+x^n))'dx = \ln(a+1) - \int_0^1 \ln(a+x^n)dx,$$

and

$$\ln a \le \int_0^1 \ln(a+x^n)dx = \int_0^1 \left(\ln a + \ln\left(1 + \frac{x^n}{a}\right)\right)dx \le \ln a + \int_0^1 \frac{x^n}{a}dx \to \ln a.$$

The 52nd National Mathematical Olympiad
Final Round
Solutions

7th Grade

Problem 1. Show that there exist no integer numbers a and b such that $a^3 + a^2b + ab^2 + b^3 = 2001$.

Solution. We observe that the equality can be written as $(a^2 + b^2)(a+b) = 1\cdot 3\cdot 23\cdot 29$. It results that the numbers $a^2 + b^2$ and $a+b$ are odd, hence a, b have different parities and $a^2 + b^2$ is a number with the form $4k+1$, $k \in \mathbf{N}$. Thus, $a^2 + b^2$ can be only 1, 29, $3\cdot 23 = 69$ or 2001. We obtain

- $a^2 + b^2 = 1 \Rightarrow a, b \in \{0, -1, 1\} \Rightarrow a + b \neq 2001$;
- $a^2 + b^2 = 29 \Rightarrow a, b \in \{-5, 5, -2, 2\} \Rightarrow a + b \neq 69$;
- $a^2 + b^2 = 69$, impossible;
- $a^2 + b^2 = 2001$, impossible.

Problem 2. *Let a and b be real, positive and distinct numbers. We consider the set:*

$$M = \{ax + by \mid x, y \in \mathbf{R}, x > 0, y > 0, x + y = 1\}.$$

Prove that :

i) $\dfrac{2ab}{a+b} \in M$; *ii)* $\sqrt{ab} \in M$.

Solution.

i) $\dfrac{2ab}{a+b}=ax+by$ $\dfrac{2ab}{a+b}=ax+(1-x)b \Rightarrow x=\dfrac{b}{a+b}\in(0;1)$ and $y=\dfrac{a}{a+b}\in(0;1)$

ii) $\sqrt{ab}=ax+by \Rightarrow \sqrt{ab}=ax+(1-x)b \Rightarrow$

$x=\dfrac{\sqrt{ab}-b}{a-b} \Rightarrow x=\dfrac{\sqrt{b}}{\sqrt{a}+\sqrt{b}}\in(0;1)$ and $y=\dfrac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\in(0;1)$

Problem 3. *We consider a right trapezoid ABCD, in which AB || CD, AB > CD, AD⊥AB and AD >CD. The diagonals AC and BD intersect in O. The parallel through O to AB intersects AD in E and BE intersects CD in F. Prove that CE⊥AF if and only if* $AB\cdot CD = AD^2 - CD^2$.

Solution. Denote $CE \cap AB = \{G\}$. Let $AH \parallel CG, H \in DC$.

Since $OE \parallel DC \parallel AB \Rightarrow \dfrac{DF}{AB}=\dfrac{DE}{EA}=\dfrac{CO}{OA}=\dfrac{DC}{AB} \Rightarrow DF = DC$

Since $OE \parallel DC \parallel AB \Rightarrow \dfrac{DC}{AG}=\dfrac{DE}{EA}=\dfrac{CO}{OA}=\dfrac{DC}{AB} \Rightarrow AG = AB$

Because $AHCG$ is a parallelogram $\Rightarrow CH = AB$.

$CE \perp AF \Leftrightarrow AF \perp AH \Leftrightarrow AD^2 = DF\cdot DH \Leftrightarrow$

$AD^2 = DC^2 + AB\cdot DC \Leftrightarrow AB\cdot DC = AD^2 - DC^2$.

Problem 4. *We consider the acute angle ABC. On the half-line (BC we consider the distinct points P and Q whose projections on the line AB are the points M and N. Knowing that AP=AQ and* $AM^2 - AN^2 = BN^2 - BM^2$, *find the angle ABC.*

Solution. The points M and N are different, ordered A, N, M, B. The relation $AM^2 - AN^2 = BN^2 - BM^2$
is equivalent to

$$(AN+MN)^2 - AN^2 - (BM+MN)^2 + BM^2 = 0,$$

hence $2MN\,(AN - BM) = 0$, so, it follows that $AN=BM$ (analogously for the ordering A,M,N,B).

Let T be the midpoint of the segment PQ. Since $AQ = AP$ it results $AT\perp PQ$. If $TS\perp AB$ then TS is the midline in the trapezoid $QNMP$ hence $SN = SM$. Therefore $AS=SB$, hence the triangle ATB is a right isosceles triangle. It follows that $m(\angle ABC)= 45^\circ$.

8th Grade

Problem 1. *Determine the real numbers a and b such that* $a+b \in \mathbb{Z}$ *and* $a^2+b^2=2$.

Solution. $(a+b)^2 \le 2(a^2+b^2) \Rightarrow (a+b)^2 \le 4 \Rightarrow |a+b| \le 2$

$\Rightarrow a+b \in \{-2;-1;0;1;2\}$. Denote $a+b=k \Rightarrow 2a^2 - 2ak + k^2 - 2 = 0$.

For the determined values of k we obtain :

$$(a,b)\in\left\{(1,-1),(-1,1),\left(\frac{1+\sqrt3}{2},\frac{1-\sqrt3}{2}\right),\left(\frac{1-\sqrt3}{2},\frac{1+\sqrt3}{2}\right),\left(\frac{-1+\sqrt3}{2},\frac{-1-\sqrt3}{2}\right)\right\}$$

$$\cup\left\{\left(\frac{-1-\sqrt3}{2},\frac{-1+\sqrt3}{2}\right),(1,1),(-1,-1)\right\}$$

Problem 2. *For every rational number m > 0 we consider the function* $f_m: \mathbf{R}\to\mathbf{R}$, $f_m(x)=\dfrac{1}{m}x+m$. *Denote by* G_m *the graph of the function* f_m.
Let p, q and r be rational positive numbers.

a) Show that if p and q are distinct, then $G_p \cap G_q$ *is nonempty.*

b) Show that if $G_p \cap G_q$ *is a point with integer coordinates, then p and q are integer numbers.*

c) Show that if p, q, r are consecutive natural numbers, then the area of the triangle determined by the intersections of G_p, G_q *and* G_r *is equal to 1.*

Solution. If $(\alpha,\beta)\in G_p\cap G_q$, then $\frac{1}{p}\alpha+p=\beta$ and $\frac{1}{q}\alpha+q=\beta$.

Solving the system, we obtain $\alpha=pq$ and $\beta=p+q$, hence $G_p\cap G_q=\{(pq, p+q)\}$.

b)Let $p=\frac{a}{b}$, $q=\frac{c}{d}$, fractions in the lowest terms. We have $p+q=\frac{ad+bc}{bd}\in\mathbf{N}$ and $pq=\frac{ac}{bd}\in\mathbf{N}$, so it results $b \mid d$ and $d \mid b$. We deduce that $b=d=1$, hence $p,q\in\mathbf{Z}$.

c) Let $r=q+1=p+2$. We have $f_p(0)=p$, $f_q(0)=p+1$, $f_r(0)=p+2$, $G_p\cap G_q=\{(pq, p+q)\}=\{A\}$, $G_p\cap G_r=\{(pr, p+r)\}=\{B\}$, $G_r\cap G_q=\{(rq, r+q)\}=\{C\}$, and $\{P\}=G_p\cap Oy$, $\{Q\}=G_q\cap Oy$, $\{R\}=G_r\cap Oy$. We have:

$$\text{area}(ABC)=\text{area}(CQR)-\text{area}(ABRQ)=$$
$$=\text{area}(CQR)-(\text{area}(BPR)-\text{area}(AQP))=$$
$$=\frac{qr\cdot RQ}{2}-\left(\frac{pr\cdot PR}{2}-\frac{pq\cdot PQ}{2}\right)=\frac{(p+2)(p+1)}{2}-\frac{p(p+2)\cdot 2}{2}+\frac{p(p+1)}{2}=1.$$

Problem 3. *We consider the points A, B, C, D, not in the same plane, such that $AB\perp CD$ and $AB^2+CD^2=AD^2+BC^2$.*

a) Prove that $AC\perp BD$.

b) Prove that if $CD<BC<BD$, then the angle between the planes (ABC) and (ADC) is greater than 60°.

Solution. a) From $AB^2+CD^2=AD^2+BC^2 \Rightarrow AD\perp BC$. Let $AO\perp(DBC)$. It results that O is the ortocenter of the triangle BCD, hence $AC\perp BD$.

b) Let $DN\perp AC$, hence $BN\perp AC$. From $BN<BC$, $DN<DC$ it results that BD is the longest side of the triangle BDN, hence the conclusion.

Problem 4. *In the cube ABCDA'B'C'D', with side a, the plane (AB'D') intersects the planes (A'BC), (A'CD), (A'DB) after the lines d_1, d_2, and d_3, respectively.*

a) Show that the lines d_1, d_2, d_3 pairwisely intersect.

b) Determine the aria of the triangle formed by the three lines.

Solution.

a) Let E and G be the centers of the squares $ABB'A'$ and $ADD'A'$, respectively, and F the centroid of the triangle $AB'D'$. Then $d_1=EF$, $d_2=FG$, $d_3=EG$.

b) We have $EF=FG=\frac{a\sqrt{6}}{6}$, $GE=\frac{a\sqrt{2}}{2}$, hence the required area equals $\frac{a^2\sqrt{3}}{24}$.

9th Grade

Problem 1. *Let A be a set of real numbers which verifies:*

a) $1\in A$;

b) $x\in A\Rightarrow x^2\in A$;

c) $x^2-4x+4\in A\Rightarrow x\in A$.

Show that $2000+\sqrt{2001}\in A$.

Solution. Let $x\in A$; from b) we have $x^2\in A$, hence $[(x+2)-2]^2\in A$, and from c) it results that $x+2\in A$ (*). From $1\in A$ and (*) it follows inductively that A contains all odd positive integers. Thus, $2001=[(\sqrt{2001}+2)-2]^2\in A$, hence $\sqrt{2001}+2\in A$. Using again the remark (*), we obtain the conclusion.

Problem 2. *Let ABC a right triangle (A=90°) and $D\in(AC)$ such that BD is the bisector of B. Prove that $BC-BD=2AB$ if and only if $\frac{1}{BD}-\frac{1}{BC}=\frac{1}{2AB}$.*

Solution. Let $u=m(\angle ABD)$. Then

$$BD=\frac{c}{\cos u},\ BC=\frac{c}{\cos 2u}$$

hence the relations from the enounce become

$$\frac{1}{\cos 2u}=\frac{1}{\cos u}+2 \text{ and } \cos u=\cos 2u+\frac{1}{2}$$

If we denote $\cos u = x$ we obtain the equations $4x^3 + 2x^2 - 3x - 1 = 0$ and $4x^2 - 2x - 1 = 0$. Since $4x^3 + 2x^2 - 3x - 1 = (x+1)(4x^2 - 2x - 1)$ and $x \neq -1$, the two equations are equivalent.

Problem 3. *Let $n \in N^*$ and $v_1, v_2, \ldots, v_n$ vectors in the plane with lengths less or equal to 1. Prove that there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{-1, 1\}$ such that*

$$|\varepsilon_1 v_1 + \varepsilon_2 v_2 + \ldots + \varepsilon_n v_n| \leq \sqrt{2}.$$

Solution. If $n = 1$, the statement is obvious, and for $n = 2$ we have

$$|v_1 + v_2|^2 + |v_1 - v_2|^2 = 2(|v_1|^2 + |v_2|^2) \leq 4 \Rightarrow |v_1 + v_2| \leq \sqrt{2} \text{ or } |v_1 - v_2| \leq \sqrt{2}.$$

Now, let $n \geq 3$. We use induction on n. Suppose the property true for $n - 1$ and consider a system of n non–zero vectors $v_1, v_2, \ldots, v_n$. Two of the vectors $\pm v_1, \pm v_2, \pm v_3$, for instance v_1 and v_2, determine an angle of at most 60°. Then $|v_1 - v_2| \leq 1$; applying the induction hypothesis for $v_1 - v_2$ and $v_3, v_4, \ldots, v_n$ it results that there exists $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1} \in \{-1, 1\}$ such that

$$|\varepsilon_1 v_1 - \varepsilon_1 v_2 + \varepsilon_2 v_3 + \ldots + \varepsilon_{n-1} v_n| \leq \sqrt{2}.$$

Problem 4. *Determine the ordered systems (x, y, z) of positive rational numbers for which $x + \frac{1}{y}$, $y + \frac{1}{z}$ and $z + \frac{1}{x}$ are integers.*

Solution. Let $x + \frac{1}{y} = a$, $y + \frac{1}{z} = b$, $z + \frac{1}{x} = c$. Then

$$y = \frac{1}{a - x},\ z = \frac{1}{b - y} = \frac{a - x}{ab - 1 - bx},\text{ hence}$$

$$\frac{a - x}{ab - 1 - bx} + \frac{1}{x} = c \Leftrightarrow (bc - 1)x^2 + (a - b + c - abc)x + ab - 1 = 0.(*)$$

If $bc = 1$ then $b = 1$, $c = 1$, hence $a = 1$, and it follows that the numbers x, y, z cannot be all positive (likewise we exclude the cases $ac = 1$, $ab = 1$). If $bc > 1$, the discriminant of the equation is

$\Delta = (a - b + c - abc)^2 - 4(bc - 1)(ab - 1) = (abc - a - b - c)^2 - 4$.

Since x must be rational, it is necessary that Δ is a square, which is only possible if $abc - a - b - c = \pm 2$.

Since $abc - a - b - c = a(bc - 1) - b - c \geq bc - 1 - b - c = (b - 1)(c - 1) - 2 \geq -2$, the relation $abc - a - b - c = -2$ cannot hold unless all previous inequalities become equalities, which leads to $ab = 1$ or $ac = 1$ or $bc = 1$, impossible.

The same inequalities show that the equality $abc - a - b - c = 2$ cannot hold if $a, b, c \geq 3$, hence at least one of numbers is 1 or 2.

If $a = 1$ then $(b - 1)(c - 1) = 4$, hence $b = c = 3$ or $\{b, c\} = \{2, 5\}$. Therefore

- $a = 1, b = 2, c = 5 \Rightarrow (x, y, z) = (1/3, 3/2, 2)$;
- $a = 1, b = 3, c = 3 \Rightarrow (x, y, z) = (1/2, 2, 1)$;
- $a = 1, b = 5, c = 2 \Rightarrow (x, y, z) = (2/3, 3, 1/2)$.

If $a = 2$ then $(2b - 1)(2c - 1) = 9$, hence $b = c = 2$ or $\{b, c\} = \{1, 5\}$.
For $a = b = c = 2$ we obtain $(x, y, z) = (1, 1, 1)$ and the other cases are obtained by performing circular permutations.

10th Grade

Problem 1. *Let a and b be complex non–zero numbers and z_1, z_2 the roots of the polynomial $X^2 + aX + b$. Show that $|z_1 + z_2| = |z_1| + |z_2|$ if and only if there exists a real number $\lambda \geq 4$ such that $a^2 = \lambda b$.*

Solution. We observe that the roots of the equation are non–zero, and the condition $|z_1 + z_2| = |z_1| + |z_2|$ is equivalent to the following: there exists the real number $r > 0$ such that $z_1 = r z_2$.

If this is true, we have

$$\frac{a^2}{b} = \frac{(z_1 + z_2)^2}{z_1 z_2} = \frac{(r+1)^2}{r} \geq 4.$$

Conversely, if $a^2 = \lambda b$, $\lambda \geq 4$, then the discriminant of the equation is $\Delta = a^2 - 4b = = a^2(\lambda - 4)/\lambda$, and

$$z_{1,2} = \frac{a}{2}(-1 \pm \sqrt{\frac{\lambda - 4}{\lambda}}) \Rightarrow$$

$$\frac{z_1}{z_2} = \frac{\sqrt{\lambda} \pm \sqrt{\lambda - 4}}{\sqrt{\lambda} \mp \sqrt{\lambda - 4}} = \frac{(\sqrt{\lambda} \pm \sqrt{\lambda - 4})^2}{4} = r > 0.$$

Problem 2. *In the tetrahedron OABC we denote by α, β, γ the measures of the angles $\angle BOC$, $\angle COA$ and $\angle AOB$, respectively. Prove the inequality:*

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma < 1 + 2 \cos \alpha \cos \beta \cos \gamma.$$

Solution. The relation is equivalent to

$$\cos^2\alpha - 2\cos\alpha\cos\beta\cos\gamma + \cos^2\beta\cos^2\gamma <$$
$$< 1 - \cos^2\beta - \cos^2\gamma + \cos^2\beta\cos^2\gamma \Leftrightarrow$$
$$\Leftrightarrow (\cos\alpha - \cos\beta\cos\gamma)^2 < \sin^2\beta\sin^2\gamma \Leftrightarrow$$
$$\Leftrightarrow -\sin\beta\sin\gamma < \cos\alpha - \cos\beta\cos\gamma < \sin\beta\sin\gamma \Leftrightarrow$$
$$\Leftrightarrow \cos(\beta + \gamma) < \cos\alpha < \cos|\beta - \gamma|.$$

Since $\alpha, \beta, \gamma \in (0, \pi)$, the last inequality is equivalent to

$$|\beta - \gamma| < \alpha < \min\{\beta + \gamma, 2\pi - \beta - \gamma\},$$

which results from Euler's inequalities for the plane angles of a trihedral angle.

Problem 3. *Let m, k be positive integers, $k < m$ and M a set with m elements. Prove that maximal number of subsets $A_1, A_2, \ldots, A_p$ of M for which $A_i \cap A_j$ has at most k elements, for every $1 \leq i < j \leq p$, equals*

$$p_{max} = \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{k+1}.$$

Solution. If we consider $A_1, A_2, \ldots, A_p$ be the subsets of M which have at most $k+1$ elements, we obtain a family with $\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \ldots + \binom{m}{k+1}$ sets which verifies the relation from the enounce.

Let now $A_1, A_2, \ldots, A_p$ be a family of sets which verifies the relation from the enounce. For each A_i with $|A_i| \geq k + 1$ we consider a set $B_i \subset A_i$ such that $|B_i| = k + 1$. Then the function $A_i \mapsto B_i$, defined for the sets of the family which have at least $k + 1$ element and with values in set of subsets of M which have $k + 1$ elements is injective. It results that the family contains at most $\binom{m}{k+1}$ sets which have more than $k + 1$ elemente and, obviously, at most $\binom{m}{0} + \binom{m}{1} + \ldots + \binom{m}{k}$ sets which have no more than $k + 1$ elements.

Problem 4. *Let $n \geq 2$ an even integer and a, b real numbers such that $b^n = 3a + 1$. Show that the polynomial $P(X) = (X^2 + X + 1)^n - X^n - a$ is divisible by $Q(X) = X^3 + X^2 + X + b$ if and only if $b = 1$.*

Solution. If $b = 1$ then $a = 0$ and $Q(X) = (X + 1)(X^2 + 1) \mid P(X)$.

Conversely, observe that $b \neq 0$ (otherwise $P(\varepsilon) = 0$, where ε is a cubic root of the unity, hence $|a| = |\varepsilon^n| = 1$, contradiction with the hypothesis), hence the roots z_1, z_2, z_3 of Q are non–zero. From

$$z_1^2 + z_1 + 1 = -\frac{b}{z_1} = z_2 z_3$$

and similar equalities we obtain

$$0 = P(z_1) + P(z_2) + P(z_3) = \sum z_1^n z_2^n - \sum z_1^n - 3a =$$
$$= (1 - z_1^n)(1 - z_2^n)(1 - z_3^n) - 1 - 3a + z_1^n z_2^n z_3^n =$$
$$= (1 - z_1^n)(1 - z_2^n)(1 - z_3^n) - 1 - 3a + b^n = (1 - z_1^n)(1 - z_2^n)(1 - z_3^n).$$

It results that, for instance, $z_1^n = 1$.

Case 1. If $z_1 = 1$ then $b = -3$ and $P(1) = 3^n - a - 1 = 0$, which contradicts the hypothesis.

Case 2. If $z_1 = -1$ then $b = 1$ and $a = 0$.

Case 3. If $z_1 \in \mathbf{C} \setminus \mathbf{R}$ then $|z_1| = 1$, hence $z_2 = \bar{z}_1 = 1/z_1$ and $z_3 = -b$. It results that $Q(-b) = 0$, hence $b = 1$ or $b = 0$. As we saw, $b = 0$ leads to a contradiction hence we obtain $b = 1$ and $a = 0$.

Second solution. It is easy to see that if b=1 then $Q \mid P$. Conversely, suppose $Q \mid P$. In this case $b \neq 0$ hence $Q \mid P$ if and only if $Q \mid X^nP$. We have

$$X^nP = (X^3 + X^2 + X)^n - X^{2n} - aX^n = (Q-b)^n - X^{2n} - aX^n.$$

Hence, $Q \mid X^{2n}+aX^n-b^n$. Let x_1, x_2, x_3 be the roots of Q. Then x_1^n, x_2^n, x_3^n are the roots of the quadratic equation $t^2 + at - b^n = 0$. Since its discriminant equals $a^2+4b^n > 0$, the quadratic equation has distinct roots, say t_1, t_2. It follows that we have either $x_1^n = x_2^n = t_1, x_3^n = t_2$ or $x_1^n = x_2^n = x_3^n = t_1$. The conclusion is now easy to obtain using the equalities $t_1+t_2=-a$, $t_1t_2=-b^n$, $x_1x_2x_3 = -b$ and by analizing the cases: (i) $x_1^nx_2^nx_3^n = t_1^2t_2$ and (ii) $x_1^nx_2^nx_3^n = t_1^3$.

11th Grade

Problem 1. *Let f: **R**→**R** a continuous function, derivable on **R**\{x_0}, having finite side derivatives in x_0. Show that there exists a derivable function g: **R**→**R**, a linear function h: **R**→**R** and α ∈{–1,0,1} such that:*

$$f(x) = g(x)+\alpha|h(x)|,\ \forall x \in \mathbf{R}.$$

Solution. Let $f'_s(x_0), f'_d(x_0)$ the side derivatives of f in x_0. We determine m such that $g(x) = f(x) - m|x - x_0|$ is derivabile on **R**. It is sufficient to impose the derivability in x_0. Since f is continuous, we deduce: $g'_s(x_0) = f'_s(x_0) + m, g'_d(x_0) = f'_d(x_0) - m,$ hence

$$m = \frac{f'_d(x_0) - f'_s(x_0)}{2}.$$

Problem 2. *We consider a matrix $A \in M_n(C)$, with rank r, where $n \geq 2$ and $1 \leq r \leq n-1$.*

a) Show that there exist $B \in M_{n,r}(C)$, $C \in M_{r,n}(C)$, with rankB= =rankC = r, such that A =BC;

b) Show that the matrix A verifies a polynomial equation of degree r +1, with complex coefficients.

Solution. a) Let $A = (a_1,...,a_n)$, where $a_1,....,a_n \in M_{n,1}$(**C**) are the columns of A. If *rank A = r*, we choose r of these columns which contain a determinant of order r different from zero (linear independent columns). Let these columns be $a_{i1},...,a_{ir}$ and we define $B = (a_{i1},...,a_{ir})$, $B \in M_{n,r}$(**C**). We consider the systems $Bx = a_k$, $1 \leq k \leq n$ and $x \in M_{r,1}$(**C**). These systems are compatibile and we choose x_k a solution of the system $Bx = a_k$, $1 \leq k \leq n$.
We define $C = (x_1,...,x_n) \in M_{r,n}$(**C**). Then

$$BC = B(x_1,...,x_n) = (Bx_1,...,Bx_n) = (a_1,...,a_n) = A$$

and since rank $A = r$, it results $r = \text{rank}(BC) \leq \text{rank}C \leq r$, hence rank$C = r$ and similarly, we obtain rank $B = r$.
a) Let $M = CB \in M_r$(**C**) . Using Hamilton–Cayley's theorem, M verifies an equation of degree r, $\alpha_r M^r + ... + \alpha_1 M + \alpha_0 I_r = 0$. Then we have

$$\alpha_r BM^rC + ... + \alpha_1 BMC + \alpha_0 BC = 0.$$

But

$$BM^kC = BCB...CBC = (BC)^{k+1} = A^{k+1}, \text{ so that}$$

$$\alpha_r A^{r+1} + ... + \alpha_1 A^2 + \alpha_0 A = 0.$$

Problem 3. *Let f: **R**→[0, ∞) a function with the property:*

$$|f(x) - f(y)| \leq |x-y|, \text{ for every } x, y \in \mathbf{R}.$$

Show that:

*a) if $\lim_{n\to\infty} f(x+n) = \infty$, for every x∈**R**, then $\lim_{x\to\infty} f(x) = \infty$;*

*b) if $\lim_{n\to\infty} f(x+n) = a$, a∈[0,∞), for every x∈**R**, then $\lim_{x\to\infty} f(x) = a$.*

Solution. a) We suppose that f does not have the limit ∞. Then there exists $(x_n)_n, x_n \to \infty$ such that $f(x_n) \to l \in [0,\infty)$. We have: $|f(x_n) - f([x_n])| \leq |x_n - [x_n]| < 1$, hence $\lim_{n\to\infty}|f(x_n) - f([x_n])| \leq 1$, that is $\infty \leq 1$.
b) We suppose that f does not have the limit a at infinity. We consider tha case when there exists $(x_n)_n, x_n \to \infty$ with

$f(x_n) \to b \in [0,\infty), b \neq a$. Let $(x_{p_n}) \subseteq (x_n)$ such that $\lim_{n\to\infty}\{x_{p_n}\} = c$. We have

$$|f(x_{p_n}) - f([x_{p_n}] + c)| \leq |x_{p_n} - [x_{p_n}] - c| = |\{x_{p_n}\} - c|,$$

and then $\lim_{n\to\infty}|f(x_{p_n}) - f([x_{p_n}] + c)| \leq \lim_{n\to\infty}|\{x_{p_n}\} - c| \Leftrightarrow |b - a| \leq 0$, hence $b = a$, contradiction.

Problem 4. *The continuous function $f: [0,1] \to \mathbf{R}$ has the property:*

$$\lim_{n\to\infty} n\left(f\left(x + \frac{1}{n}\right) - f(x)\right) = 0, \text{ for every } x \in [0,1).$$

Show that:
a) for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have:

$$\sup\{x \in [0,\lambda) \mid |f(x) - f(0)| \leq \varepsilon x\} = \lambda;$$

b) f is a constant function.

Solution. Let $s = \sup A$, $0 \leq s \leq \lambda$. We suppose by way of contradiction that $s < \lambda$. It results that $|f(s) - f(0)| \leq \varepsilon s$. Let $n_0 \in \mathbf{N}$ such that $s + \frac{1}{n_0} < \lambda$ and $\left|f\left(s + \frac{1}{n_0}\right) - f(s)\right| \leq \frac{\varepsilon}{n_0}$. Then we have:

$$\left|f\left(s + \frac{1}{n_0}\right) - f(0)\right| \leq \left|f\left(s + \frac{1}{n_0}\right) - f(s)\right| + |f(s) - f(0)| \leq$$

$$\leq \frac{\varepsilon}{n_0} + \varepsilon s = \varepsilon\left(s + \frac{1}{n_0}\right),$$

hence $s + \frac{1}{n_0} \in A$, contradiction. It follows that $s = \lambda$.

b) We proved that $\lambda = \sup A$, hence $|f(\lambda) - f(0)| \leq \varepsilon\lambda \leq \varepsilon$. We obtained that $|f(\lambda) - f(0)| \leq \varepsilon, \forall \varepsilon > 0, \lambda \in (0,1)$, hence f is constant.

12th Grade

Problem 1. *a) We consider the polynomial $P(X) = X^5 \in \mathbf{R}[X]$. Show that for every $\alpha \in \mathbf{R}^*$, the polynomial $P(X+\alpha) - P(X)$ has no real roots.*

b) Let $P \in \mathbf{R}[X]$ be a polynomial of degree $n \geq 2$, with real and distinct roots. Show that there exists $\alpha \in \mathbf{Q}^*$ such that the polynomial $P(X+\alpha) - P(X)$ has only real roots.

Solution. a) $(x+\alpha)^5 = x^5, x \neq 0 \Leftrightarrow \left(\frac{x+\alpha}{x}\right)^5 = 1$. It results that $\frac{x+\alpha}{x} \in \mathbf{C} \setminus \mathbf{R}$ and then $x \in \mathbf{C} \setminus \mathbf{R}$.

b) Let $x_1 < \ldots < x_n$ be the roots of P and $y_k \in (x_k, x_{k+1})$ the roots of its derivative. Let

$$\beta = \min\{y_k - x_k, x_{k+1} - y_k \mid 0 \leq k \leq n-1\}.$$

On each interval $[x_k, x_{k+1}]$ and for every $0 < \alpha < \beta$, we define $g(x) = P(x+\alpha) - P(x)$. Since y_k is the only extremal point for P on the interval $[x_k, x_{k+1}]$, if we suppose, for instance, that it is point where P reaches its maximal value, we have: $g(x_k) = P(x_k + \alpha) - P(x_k) > 0$, $g(y_k) = P(y_k + \alpha) - P(y_k) < 0$, hence there exists $z_k \in (x_k, x_{k+1})$ with $g(z_k) = 0$. Any $\alpha \in (0,\beta) \cap \mathbf{Q}$ will do.

Problem 2. *Let A be a finite ring. Show that there exist two natural numbers m, p, $m > p \geq 1$, such that*

$$a^m = a^p, \forall a \in A.$$

Solution. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $a \in A$. Since $\{a, a^2, \ldots, a^{n+1}\} \subset A$, there exists $i < j$ such that $a^i = a^{i+j}$. It follows that $a^k = a^{k+jl}, \forall k \geq i$ and $l \in \mathbf{N}$. We apply this for every element of the ring A:

$\exists i_1, j_1$ such that $a_1^k = a_1^{k+j_1 l}, \forall k \geq i_1, l \in \mathbf{N}$

....

$\exists i_n, j_n$ such that $a_n^k = a_n^{k+j_n l}, \forall k \geq i_n, l \in \mathbf{N}$.

We now choose $p \geq max\{i_1,...,i_n\}$ and $q = cmmmc\{j_1,...,j_n\}$ and it results that $a^p = a^{p+q}, \forall a \in A$.

Problem 3. *Let $f:[-1,1]\to\mathbf{R}$ be a continuous function. Show that:*

a) if $\int_0^1 f(sin(x+\alpha))dx = 0$, for every $\alpha\in\mathbf{R}$, then $f(x)=0, \forall x\in[-1,1]$;

b) if $\int_0^1 f(sin\, nx)dx = 0$, for every $n\in\mathbf{Z}$, then $f(x)=0, \forall x\in[-1,1]$.

Solution. We have $\int_0^1 f(sin(x+\alpha))dx = \int_\alpha^{\alpha+1} f(sinx)dx = F(\alpha+1)-F(\alpha)$, where F is a primitive of the function $f\circ sin$. There exists $k\in\mathbf{R}$ such that $F(x)=l(x)+\frac{k}{2\pi}x$ and l is a function having the period 2π. Since F has the period 1, it is bounded and it results $k = 0$. Thus, F is continuous and has the periods 1 and 2π, hence F is constant. In conclusion, f is identically zero.

b) We have $F(n)=F(0)$ and F has period 2π. Let $x_0\in\mathbf{R}$ and $(x_0-\varepsilon, x_0+\varepsilon)$ a nighbourhood of x_0. Let $m,n\in\mathbf{Z}$ such that $m+2n\pi\in(x_0-\varepsilon,x_0+\varepsilon)$. We have $F(m+2n\pi)=F(m)=F(0)$, hence there exists $(x_n)_n\to x_0$ with $F(x_n)=F(0)$. Since F is continuous, it results that it is constant and thus f is identically zero.

Problem 4. *Let $f:[0,\infty)\to\mathbf{R}$ a periodical function, with period 1, integrable on [0,1]. For a strictly increasing and unbounded sequence $(x_n)_{n\geq0}$, $x_0=0$, with $\lim_{n\to\infty}(x_{n+1}-x_n)=0$, we denote $r(n)=max\{k \mid x_k\leq n\}$.*

a) Show that: $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{r(n)}(x_k-x_{k-1})f(x_k)=\int_0^1 f(x)dx$.

b) Show that: $\lim_{n\to\infty}\frac{1}{ln\,n}\sum_{k=1}^{n}\frac{f(ln\,k)}{k}=\int_0^1 f(x)dx$.

Solution. a) We have $a_n=\frac{1}{n}\sum_{p=1}^{n}\left(\sum_{p-1<x_k\leq p}(x_{k+1}-x_k)f(x_k)\right)=\frac{1}{n}\sum_{p=1}^{n}s_p$.

Using Cesaro-Stolz' theorem, we have: lim a_n = lim s_n. Now,

$$s_n=\sum_{n-1<x_k\leq n}(x_{k+1}-x_k)f(x_k)=\sum_{0<x_k-(n-1)\leq1}(y_{k+1}-y_k)f(y_k),$$

with $y_k=x_k-(n-1)$, represents the Riemann sum associated to the function f and division $(y_k)_{r(n-1)<k\leq r(n)}$ of the interval [0,1], whose norm goes to zero, for $n\to\infty$. Therefore, $lim\, s_n=\int_0^1 f(x)dx$.

b) For $x_n=ln\,n$, it results that

$$z_n=\frac{1}{n}\sum_{k=1}^{[e^n]}ln\frac{k+1}{k}f(ln\,k)\to I,\quad I=\int_0^1 f(x)dx.$$

We also have $\lim_{n\to\infty}s_{[ln\,n]}=I$, hence

$$\frac{1}{[ln\,n]}\sum_{k=1}^{[e^{[ln\,n]}]}ln\frac{k+1}{k}f(ln\,k)\to I\Rightarrow\frac{1}{ln\,n}\sum_{k=1}^{[e^{[ln\,n]}]}ln\frac{k+1}{k}f(ln\,k)\to I.$$

Then

$$\frac{1}{lnn}\sum_{k=1}^{n}ln\frac{k+1}{k}f(lnk)=\frac{1}{lnn}\sum_{k=1}^{[e^{[lnn]}]}ln\frac{k+1}{k}f(lnk)+\frac{1}{lnn}\sum_{k=[e^{[lnn]}]+1}^{n}ln\frac{k+1}{k}f(lnk)$$

and we show that $\lim_{n\to\infty}\sum_{k=[e^{[ln\,n]}]+1}^{n}ln\frac{k+1}{k}f(ln\,k)=0$.

If we denote $M=\sup|f(x)|$, we have:

$$\left|\sum_{k=[e^{[ln\,n]}]+1}^{n}ln\frac{k+1}{k}f(ln\,k)\right|\leq M\cdot\sum_{k=[e^{[ln\,n]}]+1}^{n}ln\frac{k+1}{k}=M\,ln\frac{n}{[e^{[ln\,n]}]+1}\to0,$$

for $n\to\infty$.

Finally, $\left|\frac{1}{lnn}\sum_{k=1}^{n}\left(\frac{1}{k}-ln\frac{k+1}{k}\right)f(lnk)\right|\leq M\cdot\frac{1}{lnn}\sum_{k=1}^{n}\left(\frac{1}{k}-ln\frac{k+1}{k}\right)=M\cdot\frac{1+...+\frac{1}{n}-ln(n+1)}{lnn}\to0$,

hence the conclusion.

Selection examinations for the 42nd IMO
Solutions

Problem 1. *Show that if a, b, c are complex numbers such that*

$$(a+b)(a+c)=b$$
$$(b+c)(b+a)=c$$
$$(c+a)(c+b)=a$$

then a, b, c are real numbers.

Solution. Let $P(x)=x^3-sx^2+qx-p$ the polynomial with the roots a, b, c. We have $s=a+b+c$, $q=ab+bc+ca$, $p=abc$. The given equalities are equivalent to:

$$\begin{cases} sa+bc=b \\ sb+ca=c \\ sc+ab=a \end{cases} \qquad (1)$$

so, by adding them, we obtain $q=s-s^2$. Multiplying the equalities in (1) with a, b, c, respectively, and by adding them we obtain $s(a^2+b^2+c^2)+3p=q$ or, after a short computation, $3p=-3s^3+s^2+s$ (2).
If we write the given equations under the form

$$(s-c)(s-b)=b,\quad (s-a)(s-c)=c,\quad (s-b)(s-a)=a,$$

we obtain by multiplying $((s-a)(s-b)(s-c))^2=abc$, and, by performing standard computations and using (2), we finally get

$$s(4s-3)(s+1)^2=0.$$

If $s=0$, then $P(x)=x^3$, so $a=b=c=0$. If $s=-1$, then $P(x)=x^3+x^2-2x-1$, which has the roots $2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{6\pi}{7}$ (this is not obvious, but we can see that P changes its sign on the intervals $(-2,-1)$, $(-1,0)$, $(1,2)$, hence its roots are real). Finally, if $s=3/4$, then $P(x)=x^3-\frac{3}{4}x^2+\frac{3}{16}x-\frac{1}{64}$, which have the roots $a=b=c=1/4$.

Second solution. Substract the second equation from the first. We obtain

$(a+b)(a-b)=b-c$. Analogously, $(b+c)(b-c)=c-a$ and $(c+a)(c-a)=a-b$. We can see that if two of the numbers are equal, then all three are equal and the conclusion is obvious. Suppose the numbers are distinct. Then, after multiplying the equalities above, we obtain $(a+b)(b+c)(c+a)=1$, and next:

$$b(b+c)=c(c+a)=a(a+b)=1.$$

Now, if one of the numbers is real, it follows immediately that all three are real. Suppose all numbers are not real. The $\arg a$, $\arg b$, $\arg c\in(0,2\pi)$. Two of the numbers $\arg a$, $\arg b$, $\arg c$ are contained in either $(0,\pi)$ or $\in[\pi,2\pi)$. Suppose these are $\arg a$, $\arg b$ and that $\arg a\le \arg b$. Then $\arg a\le\arg(a+b)\le\arg b$ and $\arg a\le\arg a(a+b)\le\arg(a+b)\le\arg b$. This is a contradiction, since $a(a+b)=1$.

Problem 2. *a) Let f, g : **Z** → **Z** be one to one maps. Show that the function h : **Z** → **Z**, defined by h(x) = f(x) g(x), for all x∈**Z**, cannot be a surjective function.*
*b) Let f : **Z** → **Z** be a surjective function. Show that there exist surjective functions g, h : **Z** → **Z** such that f(x) = g(x) h(x), for all x∈**Z**.*

Solution. a) Suppose h is a surjective function and let $a, b\in\mathbf{Z}$ such that $h(a)=1$ and $h(b)=-1$. Obviously, $a\ne b$. Since $f(a)g(a)=1$ and $f(b)g(b)=-1$ it follows that $f(a), g(a), f(b), g(b)\in\{-1,1\}$. If $f(a)\ne f(b)$ and $g(a)\ne g(b)$ then $f(a)f(b)=g(a)g(b)=-1$, but then $f(a)f(b)g(a)g(b)=1$, which is false. Therefore $f(a)=f(b)$ or $g(a)=g(b)$, contradicting the injectivity of f and g.
b) Let a_0 be an integer such that $f(a_0)=0$. We define $g(a_0)=h(a_0)=0$. For every positive integer n, let a_n, b_n be the integer numbers such that $f(a_n)=n^2$ and $f(b_n)=-n^2$. We define $g(a_n)=n$, $h(a_n)=n$, $g(b_n)=-n$, $h(b_n)=n$. Thus, all integers are in the range of g and all non-negative integers are in the range of h. For any positive integer n, let c_n be the integer such that $f(c_n)=n(n+1)$. Since $a_m\ne c_n$ and $b_m\ne c_n$ for every m, we can define $g(c_n)=-(n+1)$ and $h(c_n)=-n$. Thus, the range of h covers all **Z**. In the case that the set $A=\mathbf{Z}-\bigcup_{n\in N}\{a_n,b_n,c_n\}$ is nonempty, we define $g(k)=f(k)$ and $h(k)=1$ for all $k\in A$.

Problem 3. *The sides of a triangle have lengths a, b, c. Show that:*

$$(-a+b+c)(a-b+c)+(a-b+c)(a+b-c)+(a+b-c)(-a+b+c)\le$$
$$\le\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).$$

Solution. It is easy to see that there exists a triangle *MNP* whose sides are $m=\sqrt{a}, n=\sqrt{b}, p=\sqrt{c}$. For instance, the inequality $\sqrt{a}+\sqrt{b}>\sqrt{c}$ is equivalent to $a+b+2\sqrt{ab}>c$, which is obvious. Now, if we denote by s, r and R the semiperimeter, inradius and circumradius of the triangle *MNP*, we have the well-known formulae:

$$m+n+p=2s$$
$$mn+np+pm=s^2+r^2+4Rr$$
$$mnp=4Rrs$$

By standard transformations, the original inequality is equivalent to:

$$\left(-m^2+n^2+p^2\right)\left(m^2-n^2+p^2\right)+\left(m^2-n^2+p^2\right)\left(m^2+n^2-p^2\right)+$$
$$+\left(m^2+n^2-p^2\right)\left(-m^2+n^2+p^2\right)\le mnp(m+n+p)\Leftrightarrow$$
$$\Leftrightarrow 2\left(m^2n^2+n^2p^2+p^2m^2\right)-\left(m^4+n^4+p^4\right)\le mnp(m+n+p)\Leftrightarrow$$
$$\Leftrightarrow 4\left(m^2n^2+n^2p^2+p^2m^2\right)-\left(m^2+n^2+p^2\right)^2\le mnp(m+n+p)\Leftrightarrow$$
$$\Leftrightarrow 4(m+n+p)^2(mn+np+pm)-9(m+n+p)mnp-(m+n+p)^2\le 0\Leftrightarrow$$
$$\Leftrightarrow (m+n+p)^3+9mnp\ge 4(mn+np+pm)(m+n+p)\Leftrightarrow$$
$$\Leftrightarrow 8s^3+36Rrs\ge 8s\left(s^2+r^2+4Rr\right)\Leftrightarrow 4Rrs\ge 8sr^2\Leftrightarrow R\ge 2r.$$

Thus, the required inequality is equivalent to Euler's inequality in the triangle *MNP*.

Second solution. We can improve the required inequality: for any positive real numbers a, b, c,

$$\sum(-a+b+c)(a-b+c)\le\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right),$$

and the equality holds if $a=b=c$ or if two of the numbers are equal and the third is zero. The left hand side of the inequality can be written as:

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(-\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}-\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right).$$

If we denote $\sqrt{a}=x, \sqrt{b}=y, \sqrt{c}=z$, the inequality is equivalent to

$$(-x+y+z)(x-y+z)(x+y-z)\le xyz.$$

If x, y, z are the lengths of a triangle's sides, the inequality is equivalent to Euler's inequality. If not, it is easy to see that the left hand side is negative, and the result follows. For a complete discussion see problem 2, 41st IMO (M.Becheanu & B.Enescu: 41 st IMO in "Gazeta Matematica", vol. CV, No. 10, 2000, presupunem. 386–395).

Third solution. The numbers a, b, c, are supposed arbitrary positive numbers. After standard computations, the required inequality becomes:

$$2(ab+bc+ca)\le a^2+b^2+c^2+a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}.$$

Set $a=x^2, b=y^2, c=z^2$. We obtain:

$$x^4+y^4+z^4+x^2yz+xy^2z+xyz^2\ge 2\left(x^2y^2+y^2z^2+z^2x^2\right).$$

By the AM–GM inequality we have:

$$2x^2y^2\le x^3y+xy^3,$$

hence it suffices to prove that:

$$x^4+y^4+z^4+x^2yz+xy^2z+xyz^2\ge x^3y+y^3z+z^3x+xy^3+yz^3+zx^3.$$

This can be written under the form:

$$x^2(x-y)(x-z)+y^2(y-z)(y-x)+z^2(z-x)(z-y)\ge 0.$$

Assume WLOG, $x\ge y\ge z$. Then $x^2(x-y)(x-z)\ge y^2(y-z)(x-y)$ and $z^2(z-x)(z-y)\ge 0$, which proves the claim.

Problem 4. *Three schools have 200 students, each. Every student has at least one friend în each school (if the student a is a friend of the student b then b is a friend of a).*

It is known that there exists a set E of 300 students (among the 600) such that for any school S and any two students x, y∈E which are not în the school S, the numbers of friends in S of x and y are different.

Show that one can find a student in each school such that they are friend to each other.

Solution. Let A, B, C be the sets of students of the given schools and let $A_1 = A \cap E, B_1 = B \cap E, C_1 = C \cap E$. Since $|A_1| + |B_1| + |C_1| = 300$, we may assume that $|A_1| \le 100$, thus $|B_1| + |C_1| \ge 200$. For a student $x \in B_1 \cap C_1$, denote by $a(x)$ the number of friends of x in A. When x runs over $B_1 \cap C_1$, the numbers $a(x)$ are pairwisely distinct, hence there exists $x \in B_1 \cap C_1$ such that $a(x) \ge 200$. Let $b \in B_1$ such that $a(b)=200$. It suffices to take $c \in C$ which is a friend of b and $a \in A$ which is a friend of c. Since all $x \in A$ are friends of b it follows that $\{a, b, c\}$ is a required set of three students.

Problem 5. *Find all polynomials with real coefficients $P(X)$ such that*

$$P(x) \cdot P(2x^2 - 1) = P(x^2) \cdot P(2x - 1),$$

for every $x \in \mathbf{R}$.

Solution. Let $\deg P = n$. It is clear that $P(2x-1) = 2^n P(x) + R(x)$, where either $R(x)=0$, or $\deg R < n$. It follows that

$$P(x)(2^n P(x^2) + R(x^2)) = P(x^2)(2^n P(x) + R(x)).$$

Then $P(x)R(x^2) = P(x^2)R(x)$. If we assume that $R \ne 0$ and we denote $\deg R = m$, it follows that $n+2m=m+2n$, hence $m=n$, which is a contradiction. Therefore, $P(2x-1) = 2^n P(x)$ or, equivalently, $P(2x+1) = 2^n P(x+1)$. Thus, the polynomial $Q(x)=P(x+1)$ verifies the condition $Q(2x)=2^n Q(x)$. After writing $Q(x)$ in the form $Q(x) = \sum_{i=1}^{n} a_i x^i$, we obtain $2^i a_i = 2^n a_i$, hence $a_i=0$ for $0 \le i \le n-1$. The conclusion is: $P(x)=c$ or $P(x)=c(x-1)^n$, where $c \in \mathbf{R}$ and $n \ge 1$. Conversely, it is clear that these polynomials satisfy the condition of the problem.

Second solution. Let $D(x) = \gcd(P(x), P(2x-1))$. Then $P(x)=D(x)F(x)$ and $P(2x-1)=D(x)G(x)$, where F and G are relatively prime polynomials. One also has $P(x^2)=D(x^2)F(x^2)$ and $P(2x^2-1)=D(x^2)G(x^2)$.

From the given condition we obtain $D(x)\, F(x)\, D(x^2)\, G(x^2) = D(x)\, G(x)\, D(x^2)\, F(x^2)$, therefore it follows that $F(x)G(x^2)=G(x)\, F(x^2)$. Since $F(x)$ and $G(x)$ are relatively prime polynomials, the same are $F(x^2)$ and $G(x^2)$, thus $F(x^2)$ divides $F(x)$ and $G(x^2)$ divides $G(x)$. It follows that F and G are constant, whence $P(2x-1) = 2^n P(x)$. The solution ends as the previous one.

Problem 6. *The vertices A, B, C, and D of a square lie outside a circle centered in M. Let AA', BB', CC', DD' be tangents to the circle. We assume that the segments AA', BB', CC', DD' are the consecutive sides of a quadrilateral p in which a circle is inscribed. Prove that p has an axis of symmetry.*

Solution. Let O be the centre of the square $ABCD$, a the length of its side and r the radius of the circle centered in M. We have:

$$AA'^2 + CC'^2 = AM^2 + CM^2 - 2r^2.$$

Using the median theorem in the triangle MAC we obtain:

$$AA'^2 + CC'^2 = 2MO^2 + a^2 - 2r^2.$$

In the same way:

$$BB'^2 + DD'^2 = 2MO^2 + a^2 - 2r^2.$$

Since p is circumscribed to a circle, we have $AA' + CC' = BB' + DD'$ (Pithot's theorem). Since $AA'^2 + CC'^2 = BB'^2 + DD'^2$, we obtain $AA' \cdot CC' = BB' \cdot DD'$. By a quadratic equation argument we obtain either $AA'=BB'$ and $CC'=DD'$, or $AA'=DD'$ and $CC'=BB'$. In both cases, the quadrilateral has a line of reflection, which is the perpendicular bisector of one of the diagonal segments.

Problem 7. *Find the least number n with the property : from any n half lines in the space sharing a common origin, one can pick two such that the angle between them is acute.*

Solution. We will prove that the required number is $n = 7$. It is clear that the angles of the six rays of an orthogonal frame are 90° or 180°, hence $n \geq 7$. Assume by contradiction that we are given 7 rays emanating from a point and such that the angle between any two of them is at least 90°. We may replace the 7 rays by unit vectors v_1, $v_2, \ldots, v_7$ and let $w_i = v_i - a_i v_7$, $1 \leq i \leq 6$, where $a_i = v_i \cdot v_7$ is the scalar product. All vectors w_i are orthogonal to v_7, since

$$w_i \cdot v_7 = (v_i - a_i v_7) \cdot v_7 = v_i \cdot v_7 - a_i v_i \cdot v_7 = a_i - a_i = 0.$$

Therefore, w_i are all in the plane orthogonal to v_7. Moreover,

$$w_i \cdot w_j = (v_i - a_i v_7) \cdot (v_j - a_j v_7) = v_i \cdot v_j - a_i a_j - a_i a_j + a_i a_j \leq v_i \cdot v_j \leq 0,$$

for all $1 \leq i < j \leq 6$ and at most one w_i is zero. therefore, there exist five non–zero vectors in the plane such that their reciprocal angles are equal or greater than 90°. This is a contradiction.

Problem 8. *Prove that there are finitely many positive integers that cannot be written as a sum of distinct squares.*

Solution. (Author's solution)
Suppose we have a positive integer N with the following properties:

$$N = a_1^2 + a_2^2 + \ldots + a_m^2 \text{ and } 2N = b_1^2 + b_2^2 + \ldots + b_n^2,$$

where $a_1, a_2, \ldots, a_m, b_1, \ldots, b_n$ are positive integers such that none of the fractions $a_\alpha / a_\beta, a_\alpha / b_\delta, b_\gamma / a_\beta, b_\gamma / b_\delta$, is a power of 2 (including $2^0=1$) for all $\alpha \neq \beta$ and $\gamma \neq \delta$. We will prove that every positive integer $P > \sum_{k=0}^{4N-2} (2kN+1)^2$ can be represented as a sum of distinct perfect squares. Write P in the form

$$p = 4Nq + r, \quad 0 \leq r \leq 4N-1.$$

Since $r \equiv \sum_{k=0}^{r-1} (2kN+1)^2 \pmod{4N}$ and the latter sum is less than P, we may write $P = \sum_{k=0}^{r-1} (2kN+1)^2 + 4Nt$, for some positive integer t if $r \geq 1$. If $r=0$, we just take $P=4Nt$, where $t=q$. Let

$$t = \sum_i 2^{2u_i} + \sum_j 2^{2v_j+1}$$

be the binary expansion of t. Then

$$P = \begin{cases} \sum_{k=0}^{r-1} (2kN+1)^2 + \sum_{i,\alpha} \left(2^{u_i+1} a_\alpha\right)^2 + \sum_{j,\gamma} \left(2^{v_j+1} b_\gamma\right)^2, \text{if } r \geq 1, \\ \sum_{i,\alpha} \left(2^{u_i+1} a_\alpha\right)^2 + \sum_{j,\gamma} \left(2^{v_j+1} b_\gamma\right)^2, \text{if } r = 0. \end{cases}$$

This formula shows that P can be represented as a sum of distinct perfect squares. It remains only to show that a positive integer N as above exists, and 29 is such a number:

$$29=2^2+5^2, \quad 58=3^2+7^2.$$

Problem 9. *Let n be a positive integer and $f(X) = a_0 + a_1 X + \ldots + a_m X^m$, with $m \geq 2$, a polynomial with integer coefficients, such that:*
(1) $a_2, a_3, \ldots, a_m$ are divisible by all prime factors of n,
(2) a_1 and n are relatively prime.
Prove that for any positive integer k, there exists a positive integer c, such that $f(c)$ is divisible by n^k.

Solution. . We first prove the statement in the case that $n=p$ is a prime number . For $k=1$, we choose x_1 such that $f(x_1) \equiv 0 \pmod p$ i.e. $a_1 x_1 + a_0 \equiv 0 \pmod p$. This is possible since $a_1 \not\equiv 0 \pmod p$.

Now, suppose we found x_k such that p^k divides $f(x_k)$ and p^{k+1} does not. We have $f(x_k + tp^k) = f(x_k) + mp^{k+1} + ta_1 p^k$. Since $f(x_k) = lp^k$ for some l with $(l, p)=1$, it is sufficient to find t such that $l + ta_1 \equiv 0 \pmod p$ and this is possible since $a_1 \not\equiv 0 \pmod p$.

Finally, let $n = p_1^{r_1} \ldots p_s^{r_s}$ be the prime factorization of n. From the arguments above it follows that for every i, $1 \leq i \leq s$, there exists x_i such that $(p_i^{r_i})^k$ divides $f(x_i)$. It is known that if a divides $f(b)$ then a divides $f(b+\lambda a)$ for every integer λ. So, we search for λ such that

$$x_1 + \lambda (p_1^{r_1})^k \equiv \ldots \equiv x_s + \lambda (p_s^{r_s})^k.$$

The existence of such λ follows from the chinese remainder theorem.

Second solution. We can see that for $a, b \in Z$, $f(a) \equiv f(b) \pmod{n^k}$ if and only if $a \equiv b \pmod{n^k}$. Indeed, $f(a) \equiv f(b)$ is equivalent to

$$(a-b)[a_1 + a_2(a+b) + \ldots + a_m(a^{m-1} + \ldots + b^{m-1})] \equiv 0 \pmod{n^k}.$$

Since no prime factor of n divides the second paranthesis, it follows that $a \equiv b \pmod{n^k}$. Now, if $x, y \in \{1, 2, \ldots, n^k\}$ and $x \neq y$ it follows that $f(x) \not\equiv f(y) \pmod{n^k}$ therefore there exists $x_k \in \{1, 2, \ldots, n^k\}$ such that n^k divides $f(x_k)$.

Problem 10. *Let p and q be relatively prime positive integers. A subset S of {0, 1, 2,...} is called ideal if $0 \in S$ and, for each element $n \in S$, the integers n+p and n+q belong to S. Determine the number of ideal subsets of {0, 1, 2,...}.*

Solution. (The solution is from 41st IMO Shortlist)
Every integer z has a unique representation $z=px+qy$ with integer x, y such that $0 \le x \le q-1$. The last inequality defines a vertical strip in R^2; every lattice point (x, y) in this strip corresponds bijectively to an integer z via the equation $px+qy = z$. Fill the strip with grid–line pattern. In the unit square $[x, x+1]\times[y, y+1]$, write the corresponding integer $px+qy$.

Let S be an ideal set. All the elements of S have been written in some squares (in the strip in question). Put markers in those squares. Every integer in a square above the line $y=0$ corresponds to a nonnegative combination $px+qy$ and hence belongs to S. Thus there is no freedom in marking the squares except for those contained entirely in the right triangle Δ limited by the lines $y=0$, $x=q$ and $px+qy=0$.

If a number $z= px+qy$ appears in Δ, in a square Q not adjacent to the horizontal (resp. vertical) leg of Δ, then the number $z+q$ (resp. $z+p$) appears in the square immediately above Q (resp. right to Q). So the condition defining an ideal set translates into the following: together with any marked square Q in Δ, the whole portion of Δ upwards and rightwards of Q has to be marked. In other words, the marked the marked portion of Δ should be a union of grid rectangles, each of them having a vertex at $(q,0)$.

The polygonal line bordering the marked part of Δ from the unmarked part is then a grid–line path from (0,0) to $(q, -p)$ situated above the line $px+qy=0$. We call such a path an ideal path. All that remains is to count the ideal paths.

Let Γ denote the set of all paths of length $p+q$ from (0,0) to $(q, -p)$. Then the cardinal of Γ equals $\binom{p+q}{p}$. Let E and S denote the unit moves, east and south, respectively. Then each path $\gamma \in \Gamma$ gives rise to a sequence $D_1D_2\ldots D_{p+q}$ where $D_i \in \{E, S\}$, such that q of the D_i's are E and p are S. For a path $\gamma=D_1D_2\ldots D_{p+q}$, let P_i be the point, called a vertex of γ, reached after tracing $D_1D_2\ldots D_i$ from (0,0), and let l_i be the line parallel to $px+qy=0$, passing through P_i, $i=1,\ldots, p+q$. Since p and q are coprime, we see that the lines $l_1, l_2,.., l_{p+q}$ are all distinct (*).

Two paths are said to be equivalent if one is obtained from the other by a circular shift of the coding sequence $D_1D_2\ldots D_{p+q}$. For $\gamma \in \Gamma$, the equivalence class containing γ has $p+q$ elements. If $\gamma=D_1D_2\ldots D_{p+q}$, let m be such that l_m is the lowest among the l_i's . In view of (*), such a m is unique. Then the path $D_m\ldots D_{p+q}D_1\ldots D_{m-1}$ is above the line $px+qy=0$. Every other cyclic shift gives rise to a path with at least one vertex below the line $px+qy=0$. Thus each equivalence class contains exactly one ideal path, so the number of ideal paths equals $\frac{1}{p+q}\binom{p+q}{p}$.

(The reader should also consult problem 81 in " Challenging mathematical problems with elementary solutions", by Y.M. Iaglom and I.M. Yaglom, Dover, 1987)

Problem 11. *Find all pairs (m, n) of positive integers, with $m, n \ge 2$, such that a^n-1 is divisible by m for each $a \in \{1, 2, \ldots, n\}$.*

Solution. We show that the required pairs are $(p, p-1)$, where $p \ge 3$ is a prime number. It is clear by Fermat's theorem that such a pair $(p, p-1)$ is a solution of the problem. Let (m, n) be a solution pair and p a prime divisor of m. Since $p|m$ and $m \mid a^n-1$ we obtain $p \mid a^n-1$, for every $a=1,2,\ldots,n$. We cannot have $n \geq p$ since $p| \ p^n-1$ gives a contradiction. Therefore, $p \geq n+1$.

We consider the integer polynomial $f(X)=\prod_{i=1}^{n}(X-i)-(X^n-1)$ and let $g(X)$ be its reduced polynomial modulo p. We have $\deg(g) \leq n-1$ and , by the hypothesis, g has the roots 1, 2, ..., n (mod p). All these residue classes are distinct. Then, by Lagrange's theorem g is the identical zero polynomial. The leading coefficient of g is $n(n+1)/2$, therefore $p|n(n+1)$. Since $p \geq n+1$, it follows that $p=n+1$. We proved that $n=p-1 \geq 2$ and $m=p^k$.

We show that $k=1$. If $k \geq 2$, we have $p^2 \mid a^{p-1}-1$, for all $a=1,2,\ldots,p-1$. Then $p^2 \mid (p-1)^{p-1}-1$ from which it follows that $p^2 \mid p(p-1)$ and we get a contradiction.

Problem 12. *Prove that there is no function $f:(0,\infty)\to(0,\infty)$ such that*

$$f(x+y)\geq f(x)+yf(f(x)),$$

for every $x, y\in(0,\infty)$.

Solution. Suppose that there is a function which satisfies the condition of the problem. Put $x=1$ and obtain $f(1+y) \geq f(1)+yf(f(1))$. It follows that $\lim_{y\to\infty} f(y)=\infty$. Since $f(x+1) \geq f(x)+f(f(x))$ one also obtains $f(x+1)-f(x) \geq f(f(x))$ and it follows that $\lim_{y\to\infty}(f(x+1)-f(x))=\infty$. Then there exists $x\in\mathbf{R}$ such that $f(x+i)-f(x+i-1) > 2$, for all $i \geq 1$. Therefore $f(x+n)-f(x) > 2n$, and we obtain that $f(x+n) > x+n+1$, if $n \geq x+1$. Then:

$$f(f(x+n)) \geq f(x+n+1)+[f(x+n)-(x+n+1)]f(f(x+n+1)) >$$
$$> f(x+n+1) \geq f(x+n)+f(f(x+n)) > f(f(x+n)),$$

which is a contradiction.

Problem 13. *The tangents at A and B to the circumcircle of the acute triangle ABC intersect the tangent at C in the points D and E, respectively. The line AE intersects BC in P and the line BD intersects AC in R. Let S be the midpoint of the segment AP. Show that the angles $\angle ABQ$ and $\angle BAS$ are equal.*

Solution. First, we remark that $\alpha=\angle SAB$ can be computed in terms of $A=\angle BAC$ and $r=AB/AR$. By the sine law, one has:

$$\frac{SB}{\sin\alpha}=\frac{AB}{\sin ASB},\quad \frac{SR}{\sin(A-\alpha)}=\frac{AR}{\sin ASR}.$$

Since $SB=SR$ and $\angle ASB=180°-\angle ASR$, it follows that

$$\frac{AB}{AR}=\frac{\sin(A-\alpha)}{\sin\alpha}=\sin A\cot\alpha-\cos A.$$

Therefore,

$$\cot\alpha=\frac{r+\cos A}{\sin A}.$$

Next, AR can be computed in terms of sides of triangle ABC:

$$\frac{AR}{RC}=\frac{area(ABD)}{area(CBD)}=\frac{AB\cdot AD\cdot\sin(BAD)}{BC\cdot CD\cdot\sin(BCD)}=\frac{AB\sin(180°-C)}{BC\sin(180°-A)}\frac{c\sin C}{a\sin A}=\frac{c^2}{a^2},$$

therefore $\frac{AR}{AR+RC}=\frac{c^2}{a^2+c^2}$, whence $AR=\frac{bc^2}{a^2+c^2}$.

It follows that

$$\cot(SAB)=\frac{1}{\sin A}\left(\frac{AB}{AR}+\cos A\right)=\frac{1}{\sin A}\left(\frac{a^2+c^2}{bc}+\frac{b^2+c^2-a^2}{2bc}\right)=$$
$$=\frac{R(a^2+b^2+3c^2)}{abc},$$

where R is the circumradius of the triangle. Since the value of $\cot(QBA)$ is obtained from the last formula by interchanging a and b we get $\cot(QBA)=\cot(SAB)$, therefore $\angle SAB=\angle QBA$.

Problem 14. *Let P be a convex polyhedron, with vertices $V_1, V_2,\ldots, V_p$. The distinct vertices V_i and V_j are called neighbours if they belong to the same face of the polyhedron. In each vertex V_i an integer number $v_i(0)$ is written and next, the sequences $(v_i(n))_{n\geq 0}$ are defined as*

follows: $v_i(n+1)$ is the arithmetic mean of the numbers $v_j(n)$, for all vertices V_j which are neighbours with V_i.
Prove that if all $v_i(n)$, $1\le i\le p$, $n\in N$, are integer numbers, then there exists $M\in N$ and $k\in \mathbf{Z}$ such that $v_i(n)=k$, for every $n\ge M$ and every $i=1, 2, \ldots, p$.

Solution. Let $a_n=\min\{v_1(n),\ldots,v_p(n)\}$ and $b_n=\max\{v_1(n),\ldots,v_p(n)\}$. When $a_n=b_n$, one may take $N=n$ and $a_n=k$. We shall prove that when $a_n<b_n$, there exists $j>0$ such that:

$$a_n < a_{n+j} \le b_{n+j} < b_n.$$

From this, we obtain that $a_m=b_m$ for some $m\ge n+j$, and we may take $a_m=k$.

In order to prove the statement above, observe that the set sequence $A_{n+j}=\{V_i \mid v_i(n+j)=a_n\}$ is decreasing for $j\ge 1$. If $A_{n+j}=\varnothing$, then $A_{n+j+1}=\varnothing$; if $V_i\in A_{n+j+1}$, since $v_i(n+j-1)=a_n$, all linked vertices to V_i verify the condition $v(n+j-1)=a_n$. Let V_l be a linked vertex to V_i; in the same face with V_i and V_l there exists a third vertex V_s such that $v_s(n+j-1)=a_n$ and since V_s was arbitrarely chosen, $v_i(n+j)=a_n$. Thus, we have proved $A_{n+j+1}\subset A_{n+j}$. In the case $A_{n+j}\ne\varnothing$, the above inclusion is strict. Since P is a convex polyhedron, it follows that all vertices are contained in A_{n+j}, which contradicts $a_n<b_n$.

The conclusion is that there exists j, $j\ge 0$, such that $A_{n+j}=\varnothing$;this gives $a_n<a_{n+j}$. In the same way, one can prove that there exists $j, j\ge 0$, such that $b_{n+j}<b_n$.

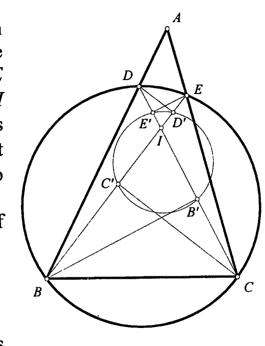
Remark. For a tetrahedron with vertices a, b, c, d, one has

$$a(m-1)+b(m-1)+c(m-1)=3d(m)=3k, \text{ and}$$
$$a(m-1)+b(m-1)+d(m-1)=3c(m)=3k.$$

Hence $c(m-1)=d(m-1)$ and also $a(m-1)=b(m-1)=c(m-1)=d(m-1)=k$. The conclusion is: only constant sequences work in this case.

Selection Examinations for the 5th JBMO
Solutions

Problem 1. *Let ABC be an arbitrary triangle. A circle passes through B and C and intersects the lines AB and AC in D and E, respectively. The projections of the points B and E on CD are denoted by B' and E', respectively. The projections of the points D and C on BE are denoted by D' and C', respectively. Prove that the points B', D', E' and C' lie on the same circle.*



Solution. Let I be the intersection point of the lines BE and CD. The quadrilaterals $BD'B'D$ and $CE'C'E$ are cyclic, hence $\angle BDB'=\angle B'D'I$ and $\angle CEC'=\angle IE'C'$. Since $BDEC$ is also cyclic, $\angle BDB'=\angle CEC'$. It follows that $\angle B'D'I=\angle IE'C'$, so $B'D'E'C'$ is a cyclic quadrilateral.

Second solution. Using the power of a point theorem, one has:

$$IB'\cdot ID=ID'\cdot IB$$
$$IC'\cdot IE=IE'\cdot IC$$
$$IE\cdot IB=ID\cdot IC$$

From these one easily obtains $IB'\cdot IE'=ID'\cdot IC'$, which proves that the quadrilateral $B'D'E'C'$ is cyclic, using the reciprocal of the the power of a point theorem.

Problem 2. *Find $n\in \mathbf{Z}$ such that the number $\sqrt{\dfrac{4n-2}{n+5}}$ is rational.*

Solution. Suppose $\dfrac{4n-2}{n+5}=\dfrac{a^2}{b^2}$, where a and b are coprime integers. One obtains $n=\dfrac{2b^2+5a^2}{4b^2-a^2}=-5+\dfrac{22b^2}{4b^2-a^2}$. Since $\gcd(b^2, 4b^2-a^2)=1$, it follows that $4b^2-a^2$ divides 22.

Observe that $4b^2-a^2\equiv 0$ or $4b^2-a^2\equiv 3 \pmod 4$, hence we have either $4b^2-a^2=-1$, or $4b^2-a^2=11$. The first case leads to $b=0$, which is impossible. In the second case, we obtain $2b-a=1$ and $2b+a=11$, hence $a=5$, $b=3$ and $n=13$.

Problem 3. *In the interior of a circle centered in O a number of 1200 points $A_1, A_2, \ldots, A_{1200}$ are considered, such that for every i, j with $1\le i\le j\le 1200$, the points O, A_i and A_j are not collinear. Prove that there exist the points M and N on the circle, with $m(\angle MON)=30°$, such that in the interior of the angle $\angle MON$ lie exactly 100 points.*

Solution. Divide the interior of the circle into 12 congruent sectors such that each marked point lies in the interior of some sector. If one of them contains exactly 100 marked points, we are done. If not, we can find a sector A containing less than 100 points and a sector B containing more than 100 points (remark that it is not possible that all sectors contain less than 100 points or more than 100 points).

Rotate sector A towards sector B. At each moment at most one marked point gets in or out sector A. It follows that there exists a moment in which the rotating sector contains exactly 100 marked points.

Problem 4. *Three students write on the blackboard next to each other three two–digit squares. In the end, they observe that the 6–digit number thus obtained is also a square. Find this number!*

Solution. Suppose that the number obtained is $n^2=\overline{abcdef}$, where $\overline{ab},\overline{cd},\overline{ef}\in\{16, 25, 36, 49, 64, 81\}$. Since $161616\le n^2\le 818181$, it follows that $402\le n^2\le 904$. Thus, $n=100x+10y+z$ and $x\ge 4$, $z\ge 1$. By the squaring algorithm we obtain $x^2=\overline{ab}$. Also:

$$(100x+10y+z)^2=10^4\overline{ab}+10^3c+10^2d+10e+f,$$

hence

$$2\cdot 10^3xy+2\cdot 10^2xz+10^2y^2+2\cdot 10yz+z^2=10^3c+10^2d+10e+f.$$

So we get the following possibilities:

a) $x=4$, $y\in\{0,1\}$;

When $y=1$, $\overline{ab}=16, \overline{cd}=81$, by using the above equality we get a contradiction.

When $y=0$, $\overline{ab}=16$ and $8z\cdot 10^2+z^2=10^2\overline{cd}+\overline{ef}$. By the unicity of the representation of a number we get $\overline{cd}=8z$ and since $\overline{ef}$ is a two digit number, it follows that $8z\in\{16,64\}$. Therefore, $z=8$ and $n=408$.

b) $x>4$ and $y=0$. We obtain $(200x+z)z=10^2\overline{cd}+\overline{ef}$. Using again the unicity of the representation of a number we obtain $\overline{cd}=2xz$, $\overline{ef}=z^2$, $z\ge 4$ and it follows that $x=8$, $z=4$, hence $n=804$.

In conclusion, the students can obtain the numbers 408^2 or 804^2.

Problem 5. *Let ABCD be a rectangle. We consider the points $E\in CA$, $F\in AB$, $G\in BC$ such that $DE\perp CA$, $EF\perp AB$ and $EG\perp BC$. Solve in the set of rational numbers the equation $AC^x=EF^x+EG^x$.*

Solution. Denote $AD=a$, $AB=b$. We have $AC^2=a^2+b^2$, $CE=b^2/AC$, $AE=a^2/AC$ and $EF/a=AE/AC=a^2/AC$. It follows that $EF=\frac{a^3}{AC^2}$ and, analogously, $EG=\frac{b^3}{AC^2}$. Thus, the equation is equivalent to

$$(a^2+b^2)^{3x}=(a^{3x}+b^{3x})^2.$$

The solution $x=2/3$ is easy to see. If $a=b$ it is clearly the only solution. If $a>b$, denote $b/a=k\in(0,1)$; we obtain $(1+k^2)^{3x}=(1+k^3)^x$. If $x<2/3$, then: $k^{3x}>k^2\Rightarrow 1+k^{3x}>1+k^2>1\Rightarrow(1+k^{3x})^2>(1+k^2)^2>(1+k^2)^{3x}$. If $x<2/3$, we use a similar argument. Thus, the only solution is $x=2/3$.

Problem 6. *Let A be a non–empty subset of* **R** *with the property that for every real numbers x, y, if $x+y\in A$, then $xy\in A$. Prove that $A=$***R***.*

Solution. Let $a\in A$. Then $a+0\in A$, hence $0=0\cdot a\in A$. For any real number b, $b+(-b)=0\in A$, hence $-b^2\in A$. Thus, A contains all negative numbers. Let $c>0$; we have $-\sqrt{c}-\sqrt{c}<0$, so $-\sqrt{c}-\sqrt{c}\in A$. It follows that $c=(-\sqrt{c})(-\sqrt{c})\in A$, hence $A=\mathbf{R}$.

Problem 7. *Let ABCD be a quadrilateral inscribed in the circle O. For a point $E\in O$, its projections K, L, M, N on the lines DA, AB, BC, CD, respectively, are considered. Prove that if N is the orthocenter of the triangle KLM for some point E, different from A, B, C, D, then this holds for every point E of the circle O.*

Solution. Let F and G be the projections of E on the diagonals BD and AC. From Simson's theorem, it follows that the triplets of points (K, L, F), (M, N, F), (K, G, N) and (M, L, G) are collinear. The point N is the orthocenter of the triangle KLM if and only if $KL\perp MN$ and $ML\perp KN$. Let F' and G' the points in which EF and EG intersect the second time the circle. We have $KF\|AF'$ and $MG\|CF'$. Thus $KL\perp MN$ is equivalent to $AF'\perp CF'$ and then to $O\in AC$. Similarly, $ML\perp KN$ is equivalent to $O\in BD$. Thus, $ABCD$ is a rectangle. It is easy to see that in this case, N is the orthocenter of the triangle KLM for any position of the point E.

Problem 8. *Determine positive integers $a<b<c<d$ with the property that each of them divides the sum of the other three.*

Solution. Since $a+b+c<3d$ and $d\mid a+b+c$, it follows that $a+b+c=d$ or $a+b+c=2d$. Suppose first that $a+b+c=d$. Since $a\mid b+c+d=2d-a$, it follows that $a|2d$ and, similarly, $b|2d$, $c|2d$. Let $2d=ax=by=cz$, where $x>y>z>2$. We obtain $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}$.

If $z\geq 6$, then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}<\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2}$, so there are no solutions.

If $z=5$, then $\frac{1}{x}+\frac{1}{y}=\frac{3}{5}$, and we obtain $y=3$, again not possible.

If $z=4$, then $\frac{1}{x}+\frac{1}{y}=\frac{1}{4}$, and we obtain the solutions $(k, 4k, 5k, 10k)$ and $(k, 2k, 3k, 6k)$, with $k\in\mathbf{N}$.

If $z=3$, then $\frac{1}{x}+\frac{1}{y}=\frac{1}{6}$, and we obtain the solutions $(k, 6k, 14k, 21k)$ $(k, 3k, 8k, 12k)$, $(k, 2k, 6k, 9k)$ and $(2k, 3k, 10k, 15k)$, with $k\in\mathbf{N}$.

Now, suppose that $a+b+c=2d$. Analogously, we obtain that $a,b,c\mid 3d$, hence $3d=ax=by=cz$ with $x>y>z>3$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{2}{3}$. Then $z\geq4$, $y\geq5$, $x\geq6$, thus $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\leq\frac{1}{6}+\frac{1}{5}+\frac{1}{4}=\frac{37}{60}<\frac{2}{3}$, so there are no solutions in this case.

Problem 9. *Let n be a non–negative integer. Find the non–negative integers a, b, c, d such that*

$$a^2+b^2+c^2+d^2=7\cdot 4^n.$$

Solution. For $n=0$, we have $2^2+1^2+1^2+1^2=7$, hence $(a,b,c,d)=(2,1,1,1)$ and all permutations. If $n\geq 1$, then $a^2+b^2+c^2+d^2\equiv 0 \pmod 4$, hence the numbers have the same parity. We analyse two cases.

a) The numbers a, b, c, d are odd. We write $a=2a'+1$, etc. We obtain :

$$4a'(a'+1)+4b'(b'+1)+4c'(c'+1)+4d'(d'+1)=4(7\cdot4^{n-1}-1).$$

The left hand side of the equality is divisible by 8, hence $7\cdot4^{n-1}-1$ must be even. This happens only for $n=1$. We obtain $a^2+b^2+c^2+d^2=28$, with the solutions (3, 3, 3, 1) and (1, 1, 1, 5).

b) The numbers a, b, c, d are even. Write $a=2a'$, etc. We obtain

$$a'^2+b'^2+c'^2+d'^2=7\cdot4^{n-1},$$

so we proceed recursively.

Finally, we obtain the solutions $(2^{n+1}, 2^n, 2^n, 2^n)$, $(3\cdot2^n, 3\cdot2^n, 3\cdot2^n, 2^n)$, $(2^n, 2^n, 2^n, 5\cdot2^n)$, $n\in\mathbf{N}$, and the respective permutations.

Problem 10. *Let ABCDEF be a hexagon with $AB\|DE$, $BC\|EF$, $CD\|FA$ and in which the diagonals AD, BE and CF are congruent. Prove that the hexagon can be inscribed in a circle.*

Solution. We first notice that $ABDE$ is an isosceles trapezoid. The segments AB and DE have the same perpendicular bisector. Let O and R the center and radius of the circumcircle of the triangle ABC. One can see that the perpendicular bisectors of DE and CF also pass through O, hence O is the center of the circle circumscribed around DCF, with radius R'. Finally, since $ACDF$ is an isosceles trapezoid, it follows that $R=R'$.

Problem 11. *Let $n\geq 2$ be a positive integer. Find the positive integers x such that*

$$\sqrt{x+\sqrt{x+\ldots+\sqrt{x}}} < n,$$

for any number of radicals.

Solution. Clearly $x \leq n^2$, so let $x=n^2-p$, with $p > 0$. If the number of radicals is 2, we obtain that $x \leq n^2-n$. It is easy to check using induction that all $x \leq n^2-n$ verify the inequality regardless the number of radicals.

Problem 12. *Determine a right parallelepiped with minimal area, if its volume is strictly greater than 1000, and the lengths of its sides are integer numbers.*

Solution. Let $a \leq b \leq c$ the lengths of the parallelepiped's sides. We have $abc \geq 1001$ and $c \geq 11$. By analysing the cases $c\in\{11,\ldots,21\}$ one finds that $a=8$, $b=9$ and $c=14$ is the solution.

CONTENTS

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