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Functional Equations

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0. Basic concepts about functions

Definition. For sets X and Y, a function f from X to Y (also written $f\colon X \to Y$) is a rule that assigns each element x in X a unique element $y = f(x)$ in Y. The set of elements of Y that are $f(x)$ for some $x \in X$ is called the image of f and is denoted by $Im(f)$.

i) A function f is called injective, or one-to-one, if $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$. In other words, to two different x correspond two different y.

ii) A function f is called surjective, or onto, if each $y \in Y$ can be written as $f(x)$ for some $x \in X$.

These two properties are of extreme importance, and should be the basic instruments in your toolkit when you tackle functional equations. Let us see some simple examples of how they apply.

Problem 1. Prove that there is no function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(f(x)+y) = x$, for all x and y in $\mathbb{R}$.

Solution. If $f(x_1) = f(x_2)$, then $f(f(x_1) + y) = f(f(x_2) + y)$, so $x_1 = x_2$. Hence f is injective. On the other hand, $f(f(x) + 0) = f(f(x) + 1) = x$, but $f(x) + 0 \neq f(x) + 1$, a contradiction.

Problem 2. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x + f(y)) = f(x)y$, for all x and y in $\mathbb{R}$.

Solution. If $f(x) = 0$ for all x, then the conditions are satisfied. Now assume $f(x_0) \neq 0$. Then $f(x_0 + f(y)) = f(x_0)y$. As $f(x_0)y$ is surjective, so is f. Thus there exists b such that $f(b) = 0$. Then setting $y = b$ we get $f(x + f(b)) = 0$. But $x + f(b)$ is surjective, hence $f(x) = 0$, a contradiction. So $f(x) = 0$ is the only solution.

Another property of functions that can be exploited is monotonicity. We say that a function is nondecreasing if $f(x) \geq f(y)$ for $x \geq y$ (if $f(x) > f(y)$ for $x > y$ the function is called increasing). Similarly, a function is called nonincreasing if $f(x) \leq f(y)$ for $x \geq y$ (and if $f(x) < f(y)$ for $x > y$ the function is decreasing). A function which is either increasing or decreasing is called monotone. Here is a quick example to illustrate the use of this concept:

Problem 3. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(2) = 2$, $f(x^2 + y^2) = f(x)^2$ and $f(f(x)) = x$, for all x and y in $\mathbb{R}$.

Solution. If $a > b$, then there exist x, y such that $x^2 + y^2 = a, y^2 = b$. Applying the second property of the function, we get that $f(a) = f(b) + f(x)^2$, which means $f(a) > f(b)$, so the function is increasing. Now if $f(x) > x$, then as f is increasing, then $f(f(x)) > f(x) > x$ and if $f(x) < x$, then $f(f(x)) < f(x) < x$, so $f(f(x)) = x$ can hold only for $f(x) = x$.

With all these preliminaries in mind, we are ready to develop some problem solving skills.

1. Problems involving constructions

Some functional equations can have solutions which are easy to write, such as $f(x) = x$. But there are other problems that require an answer which is difficult to express in a closed form, and hard to guess. These problems are called "constructive", because the reader is usually required to come up with a complicated function, or sometimes produce a simple function, but using instead a complex argument.

Many constructive problems involve building a function by induction, if the function is on $\mathbb{N}$.

Problem 4. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(n+1) - f(n) \equiv n (\text{mod } 2).$$

Solution. The function f is determined by $f(1)$, because we can find all the values of f inductively. So let us set $f(1) = k$. Then $f(2) = k+1, f(3) = k+1, f(4) = k+2, f(5) = k+2$. So when computing the actual values of f we see the pattern: the function increases by 1, then stays constant, then again increases by 1, then stays constant, and so on. Thus when computing $f(n)$ we would have roughly $\frac{n}{2}$ jumps. We can now conjecture that $f(n) = k + \lfloor \frac{n}{2} \rfloor$ and then prove it by induction. Indeed, the base case $n = 1$ is true. Then, if $n = 2m + 1$, we get

$$f(n) = f(2m+1) = f(2m) + 2m(\text{mod } 2) = f(2m) = m + k = k + \lfloor \frac{2m+1}{2} \rfloor,$$

and for $n = 2m$,

$$f(n) = f(2m) = f(2m-1) + (2m-1)(\text{mod } 2) = m-1+k+1 = m+k = k + \lfloor \frac{2m}{2} \rfloor.$$

Problem 5. Find all functions $f : \mathbb{N} \to \mathbb{R} \backslash \{0\}$ satisfying

$$f(1) + f(2) + \ldots + f(n) = f(n)f(n+1)$$

for all n.

Solution. If we try to set $f(x) = cx$, we see that $c = \frac{1}{2}$. However, the condition of the problem provides a recursive relation for f, therefore there are as many solutions as possible values for $f(1)$. So set $f(1) = a$. Then setting $n = 1$ in the condition, we have $a = af(2)$ and as $a \neq 0$ we obtain $f(2) = 1$. Then setting $n = 2$ we get $f(3) = a+1$. Setting $n = 3$ we get $f(4)(a+1) = a+1+(a+1)$ so $f(4) = 2$ as $a + 1 = f(3) \neq 0$. Now we see a pattern: for even numbers k, $f(k) = \frac{k}{2}$ as desired, whereas for odd numbers k we have an additional a, and we can suppose that

$$f(k) = \lfloor \frac{k}{2} \rfloor + (k \text{ mod } 2)a = \frac{k}{2} + (k \text{ mod } 2)(a - \frac{1}{2}).$$

Let us now prove this by induction on k. Clearly, we have to consider two cases, according to the parity of k.

a) $k = 2n$. Then

$$f(1) + f(2) + \ldots + f(k) = f(k)f(k+1),$$

or

$$\frac{1}{2} + \frac{2}{2} + \ldots + \frac{2n}{2} + n(a - \frac{1}{2}) = nf(2n+1).$$

Thus

$$\frac{2n(2n+1)}{4} + na - \frac{n}{2} = nf(2n+1),$$

which gives us $f(2n+1) = n + a$, as desired.

b) $k = 2n+1$. This case is completely analogous.

Hence all desired functions are of form $f(k) = \lfloor \frac{k}{2} \rfloor + (k \bmod)a$ for some a. They clearly satisfy the conditions of the problem provided that a is not a negative integer (in which case $f(-2a+1) = 0$).

Problem 6. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

a) $f(0) = 1$

b) $f(f(n)) = n$

c) $f(f(n+2)+2) = n$.

Solution. By applying f to both sides of c) we get $f(f(f(n+2)+2)) = n$. But from b), $f(f(f(n+2)+2)) = f(n+2)+2$. So $f(n+2)+2 = f(n)$. From here we get $f(2k) = f(0) - 2k, f(2k+1) = f(1) - 2k$. But $f(0) = 1$ and $f(1) = f(f(0)) = 0$, so $f(2k) = 1 - 2k, f(2k+1) = -2k$. We conclude $f(x) = 1 - x$.

The functions in mathematical problems are as diverse as are the whims of the mathematicians. Try to guess, for example, the answer to the following equation.

Problem 7. Find all functions $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ satisfying $f(0) = 1$ and

$$f(n) = f(\left\lfloor \frac{n}{a} \right\rfloor) + f(\left\lfloor \frac{n}{a^2} \right\rfloor)$$

for all n.

Solution. Partition $\mathbb{N}$ into sets $S_k = \{a^k, a^k+1, \ldots, a^{k+1}-1\}$. We see that if $n \in S_k$, then $\lfloor \frac{n}{a} \rfloor \in S_{k-1}, and \lfloor \frac{n}{a^2} \rfloor \in S_{k-2}$ (for $k \geq 2$). Next we see that if $k \in S_0$, then $f(k) = 2$ and if $k \in S_1$, then $f(k) = 3$. So we can easily prove by induction that f is constant on each S_k. If we let $g(k)$ be the value of f on S_k, then $g(k) = g(k-1) + g(k-2)$ for $k \geq 2$. It is clear now that $g(k) = F_{k+2}$ where $(F_n)_{n \in N_0}$ is the Fibonacci sequence. So $f(n) = F_{\lfloor \log_a n \rfloor + 2}$ for $n \geq 1$.

In creating functional equations, one could take a special function, state a relation for it, and then let the reader find it. Let us look at this example

Problem 8. Find all functions $f\colon \mathbb{N} \to \mathbb{R}$ such that $f(1) = 1$ and

$$\sum_{d|n} f(d) = n$$

for all n in $\mathbb{N}$.

Solution. Basic mathematical culture helps us: an example of such a function is Euler's totient function ϕ. So let us try to prove that $f = \phi$. As ϕ is multiplicative, let us first show that f is multiplicative, i.e. $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. We do this by induction on $m + n$. Note that when one of m,and n is 1, this is clearly true. Now assume that $m, n > 1$, and $gcd(m, n) = 1$. Then the condition written for mn gives us $\sum_{d|mn} f(d) = mn$. But any $d \mid mn$ can be written uniquely as $d = d_1 d_2$, where $d_1 \mid m$ and $d_2 \mid n$. If $d < mn$, then $d_1 + d_2 < m + n$ and, by the induction hypothesis, $f(d) = f(d_1 d_2) = f(d_1)f(d_2)$ for $d < mn$. Therefore

$$\begin{aligned} mn &= \sum_{d|mn} f(d) \\ &= \sum_{d|mn,\ d<mn} f(d) + f(mn) \\ &= \sum_{d_1|m,\ d_2|n} f(d_1)f(d_2) - f(m)f(n) + f(mn) \\ &= \left(\sum_{d|m} f(d)\right)\left(\sum_{d|n} f(d)\right) \\ &= mn - f(m)f(n) + f(mn), \end{aligned}$$

so $f(mn) = f(m)f(n)$, as desired. So it suffices to compute f for powers of primes. Let p be a prime. Then writing the condition for $n = p^k$ we get $f(1) + f(2) + \ldots + f(p^k) = p^k$. Subtracting this for the analogous condition for $n = p^{k+1}$ we get $f(p^{k+1}) = p^{k+1} - p^k = \phi(p^{k+1})$, and now the relation $f = \phi$ follows from the multiplicativity. It remains to verify that $\sum_{d|n} \phi(d) = n$. There are many proofs of this. One of the shortest is evaluating the numbers of subunitary (and unitary) non-zero fractions with denominator n. On one hand, this number is clearly n. On the other hand, if we write each fraction as $\frac{k}{l}$ in lowest terms, then $l \mid n$ and the number of fractions with denominator l is $\phi(l)$ - the number of numbers not exceeding l which are coprime with l. So this number is also $\sum_{d|n} \phi(d)$.

Let us now look at a different equation. The variety of solutions comes from constructing the function from different initial values.

Problem 9. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f(m+n) + f(mn-1) = f(m)f(n)$$

for all m and n in $\mathbb{Z}$.

Solution. If $f = c$ is constant we have $2c = c^2$ so $c = 0$ or $c = 2$. If f is not constant, setting $m = -1$ gives us $f(n)(1 - f(0)) = -f(-1)$, that is possible only for $f(-1) = 0, f(0) = 1$. Then set $m = -1$ to get $f(n-1) + f(-n-1) = 0$, and set $m = 1$ to get $f(n+1) + f(n-1) = f(1)f(n)$. This is a quadratic recurrence in $f(n)$ with associated equation $x^2 - f(1)x + 1 = 0$.

If $f(1) = 0$ we get $f(n-1) + f(n+1) = 0$ which implies $f(n+2) = -f(n)$ so $f(2k) = (-1)^{2k} f(0) = (-1)^k, f(2k+1) = (-1)^k f(1) = 0$. This function does satisfy the equation.

Indeed, if m and n are both odd, then $mn-1-(m+n) = (m-1)(n-1)-2 \equiv 2 \pmod 4$. Thus $m+n$ and $mn-1$ are even integers which give different residues modulo 4. Hence $f(m+n) + f(mn-1) = 0$, implying $f(m)f(n) = 0$.

If one of m, n is odd and the other even then $m+n$ and $mn-1$ are both odd, hence $f(m+n) + f(mn-1) = f(m)f(n) = 0$.

Finally, if m, n are even, then $f(mn-1) = 0$. We have $f(m+n) = 1$ if $4 \mid m-n$ and -1 otherwise, and the same for $f(m)f(n)$.

If $f(1) = -1$, then we get $f(n) = (n-1) \bmod 3 - 1$ for all n by induction on n. It also satisfies the condition as we can check by looking at m, n modulo 3.

If $f(1) = 2$, then $f(n+1) - 2f(n) + f(n-1) = 0$ and $f(n) = n+1$ by induction on $|n|$. It also satisfies the condition as $(m+n+1) + mn = (m+1)(n+1)$.

If $f(1) = 1$, then $f(n+1) + f(n-1) = f(n)$. Hence $f(-2) + f(0) = f(-1)$ so $f(-2) = -1$. Then $f(-3)+f(-1) = f(-2)$ so $f(-3) = -1$. $f(-4)+f(-2) = f(-3)$ so $f(-4) = -2$. Also $f(0) + f(2) = f(1)$ so $f(2) = 0$. But then $f(2) + f(-4) \neq 0$, a contradiction.

If $f(1) = -2$, then $f(n+1)+f(n-1)+2f(n) = 0$ and $f(-2)+2f(-1)+f(0) = 0$ so $f(-2) = -1$ and then $f(-3) + 2f(-2) + f(-1) = 0$ so $f(-3) = 3$ and we have $f(-3) + f(1) \neq 0$, again a contradiction.

Finally, if $f(1) \neq 0, 1, -1, 2, -2$, then the equation $x^2 - f(1)x + 1 = 0$ has two solutions $\frac{f(1)\pm\sqrt{f^2(1)-4}}{2}$, one of which is greater than 1 in absolute value and one is smaller. If we solve the recurrence, we find that $f(n) = cr^n + ds^n$, where $c, d \neq 0$ and without loss of generality $|r| > 1, |s| < 1$. In this case we have $f(n) \sim cr^n$ for $n \to \infty$. Then $f(m+n)+f(mn-1) = f(m)f(n)$ cannot hold, because the left-hand side is asymptotically equivalent to cr^{mn-1} for $m = n \to \infty$, while the right-hand side is asymptotically equivalent to c^2r^{m+n} and $mn-1$ is much greater than $m+n$.

Now, we have only seen examples of functions on integers. And it seems natural, because to construct a function one needs some kind of inductive argument.

However, there are plenty of examples of functions on reals that can also be "constructed."

Problem 10. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

- $f(-x) = -f(x)$ for all real numbers x;
- $f(x+1) = f(x) + 1$ for all real numbers x and
- $f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$ for all non-zero real numbers x.

Solution. All the conditions are in one variable: x. In this case, some graph theory helps us understand the path to the solution. Consider the reals as vertices of a graph, and connect x with $x+1, -x, \frac{1}{x}$. The conditions link two values of the function in two vertices joined by an edge. So if we pick x_0, we can deduce from $f(x_0)$ the values of f on C, where C is the set of numbers connected to x_0 by some chain of edges. Now we can get a contradiction if and only if there is a cycle somewhere. So finding a cycle would impose a condition on $f(x_0)$ and maybe would exactly find the value of $f(x_0)$.

Let us try to construct such a cycle for any x. After some tries we see that

$$x \to x+1 \to \frac{1}{x+1} \to -\frac{1}{x+1} \to 1 - \frac{1}{x+1} = \frac{x}{x+1} \to \frac{x+1}{x} = 1 + \frac{1}{x} \to \frac{1}{x} \to x.$$

Set $f(x) = y$. Then

$$f(x+1) = y+1, \quad f(\frac{1}{x+1}) = \frac{y+1}{(x+1)^2}, \quad f(-\frac{1}{x+1}) = -\frac{y+1}{(x+1)^2},$$

$$f\left(\frac{x}{x+1}\right) = \frac{x^2+2x-y}{(x+1)^2}, \quad f\left(\frac{x+1}{x}\right) = \frac{x^2+2x-y}{x^2}, \quad f\left(\frac{1}{x}\right) = \frac{2x-y}{x^2},$$

and $f(x) = 2x - y$. So $y = 2x - y$, thus $y = x$.

Note that we need to have $x \neq 0, -1$ in order not to divide by zero. This is not a problem for us, as $f(0) + 1 = f(1)$, and we know that $f(1) = 1$ so $f(0) = 0$. Also $f(-1) = -f(1) = 1$, hence $f(x) = x$ for all x, and it satisfies the condition.

When we have an inequality in the condition of the problem, we can guess the solution, then we can prove it is unique by constructing a counter-example to the inequality. Look at this simple example:

Problem 11. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(2x) = 2f(x)$ and $|x - f(x)| \leq 1$ for all x in $\mathbb{R}$.

Solution. It is clear that $f(x) = x$ is a solution. Now if we assume $f(x_0) \neq x_0$ then $|x_0 - f(x_0)| > 0$. It follows that there is some k for which $2^k|x_0 - f(x_0)| > 1$. However, $2^k|x_0 - f(x_0)| = |2^k f(x_0) - 2^k x_0| = |f(2^k x_0) - 2^k x_0|$ and so $2^k x_0$ violates the inequality in the condition.

The following problem is similar, but more tricky.

Problem 12. Find all functions $f\colon [1,\infty) \to [1,\infty)$ such that

$$f(x) \leq 2(1+x)$$

and

$$xf(x+1) = f^2(x) - 1$$

for all $x \geq 1$.

(China)

Solution. We can guess the solution $f(x) = x+1$ and now we will prove that this is the only one. As in many other situations, we assume that $f(x_0) \neq x_0 + 1$ and we try to obtain an x such that $f(x) < 1$ or $f(x) > 2(1+x)$.

Indeed, we observe that $xf(x+1) = f^2(x) - 1$ can be interpreted as a recurrence on $a_n = f(n+x_0)$ by $a_{n+1} = \frac{a_n^2-1}{n+x_0}$. Consider now $b_n = \frac{a_n}{n+1+x_0}$. Then

$$b_{n+1} = \frac{(n+1+x_0)^2 b_n^2 - 1}{(n+x_0)(n+2+x_0)} = b_n^2 + \frac{b_n^2 - 1}{(n+2)(n+2+x_0)}.$$

If $b_0 > 1$, then we prove by induction that $b_n > 1$, and then $b_{n+1} > b_n^2$, which implies $b_n > 2$, for some n. Hence $f(n+x_0) > 2(1+n+x_0)$, a contradiction.

If $b_0 < 1$, then we prove by induction that $b_n < 1$ and therefore $b_{n+1} < b_n^2$. Thus $b_n < b_0^{2^n}$ and $\frac{1}{b_n} > (\frac{1}{b_0})^{2^n}$. However, $\frac{1}{b_n} = \frac{n+1+x_0}{f(n+x_0)} < n+1+x+0$ and as $b_0 < 1$, $(\frac{1}{b_0})^{2^n} > n+1+x_0$, a contradiction.

Thus $b_0 = 1$, hence $f(x_0) = x_0 + 1$. As x_0 was picked at random, $f(x) = x+1$.

Finally, we have a pure construction problem on $\mathbb{R}$.

Problem 13. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(1+x^2) = f(x)$$

for all x in $\mathbb{R}$.

Solution. Set $g(x) = 1 + x^2$. As g is even, we see that $f(x) = f(g(x)) = f(-x)$, so f is even. Thus we need to find f only on $[0,\infty)$. Now we know that $f(g_k(x)) = f(x)$. Also $g(x)$ is increasing and $g(x) > x$ as $1 + x^2 > x$ for $x \geq 0$. Set $x = 0$ to get $f(0) = f(1)$. Next set $x_k = g_k(0)$, so that $x_0 = 0, x_1 = 1$. Then g maps $[x_{k-1}, x_k]$ into $[x_k, x_{k+1}]$, so g_k maps $[0,1]$ into $[x_k, x_{k+1}]$. As $g(x) > 0$ is increasing and $g(x) > 1$, we cannot establish any condition between $f(x)$ and $f(y)$ for $0 < x < y < 1$, because we cannot link x and y by operating with g. If $g_k(x) = g_l(y)$, then as $g_k(x) \in (x_k, x_{k+1}); g_l(y) \in (x_l, x_{l+1})$, we conclude $k = l$ and by injectivity $x = y$. Thus we may construct f as follows: let f be a continuous function on $[0,1]$ with $f(0) = f(1)$ and extend f to $\mathbb{R}^+$ by setting $f(g_k(x)) = f(x)$

and $f(-x) = -f(x)$. Indeed, f satisfies $f(1+x^2) = f(x)$. Moreover, it is continuous: the graphs of f on $[x_k, x_{k+1}]$ are continuous as they are the composition of the continuous functions f on $[0,1]$ and g_k^{-1} on $[x_k, x_{k+1}]$. As $f(x_k) = f(x_{k+1})$, the continuous graphs of f on intervals $[x_k, x_{k+1}]$ unite to form a continuous curve, and reflecting it with respect to the y axis we get a continuous graph of f.

2. Binary (and other) bases

A popular way of concocting special and interesting functions is to look at bases. One could take, for example, the decimal expansion of n and let $f(n)$ be the number read backwards, or one could take the ternary expansion of n and set $f(n)$ be the sum of the digits of n, and so on. The most used is the binary base, because it has only two digits and is simpler to state conditions on the functions.

The conditions on such functions usually connect $f(x)$ with $f(kx)$ or $f(kx+1)$, etc. Generally, the rule of thumb is this: if you see a condition linking $f(x)$ with $f(kx)$, look at the expansion of x in base k.

Problem 14. Find all functions $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(0) = 0$ and

$$f(2n+1) = f(2n) + 1 = f(n) + 1$$

for all $n \in \mathbb{N}_0$.

Solution. The statement suggests that we look at the binary expansion of f. As $f(2n+1) = f(n)+1$ and $f(2n) = f(n)$, it is straightforward to observe and check that $f(n)$ is the number of ones (or the sum of digits) of the binary representation of n.

The next problem, as the statement suggests, should somehow combine bases 2 and 3.

Problem 15. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(1) = 1, f(2n) < 6f(n)$ and

$$3f(n)f(2n+1) = f(2n)(3f(n) + 1)$$

for all n in $\mathbb{N}$.

(China)

Solution. Rewrite the main condition as $\frac{f(2n+1)-f(2n)}{f(2n)} = \frac{1}{3f(n)}$.

It follows that $f(2n+1) - f(2n) > 0$ and $3f(n)(f(2n+1) - f(2n)) = f(2n)$. As $f(2n) < 6f(n)$, we deduce $f(2n+1) - f(2n) < 2$. Thus the only possibility is $f(2n+1) - f(2n) = 1$ and $f(2n) = 3f(n)$. This is clearly a recurrence to compute $f(n)$ according to its binary expansion, whose solution is: $f(n)$ is the number obtained by writing n in base 2 and reading the result in base 3.

We conclude this section with a hard problem from IMO Shortlist 2000.

Problem 16. The function f on the non-negative integers takes non-negative integer values and satisfies $f(4n) = f(2n)+f(n), f(4n+2) = f(4n)+1, f(2n+1) = f(2n)+1$ for all n. Prove that the number of non-negative integers n such that $f(4n) = f(3n)$ and $n < 2^m$ is $f(2^{m+1})$.

(IMO Shortlist, 2000)

Solution. The condition suggests looking at the binary representation of n. First, as $f(4n) = f(2n) + f(n)$, we can easily deduce that $f(2^k) = F_{k+1}$, where $(F_n)_{n\in N}$ is the Fibonacci sequence. Indeed, setting $n = 0$ we get $f(0) = 0$, thus $f(1) = 1, f(2) = 1$. Now the conditions $f(4n+2) = f(4n)+1 = f(4n)+2, f(2n+1) = f(2n) + 1$ may suggest some sort of additivity for f, at least $f(a + b) = f(a) + f(b)$ when a does not share digits in base 2. And this is indeed the case if we look at some small values of f. So we conjecture this assertion, which would mean that $f(n)$ is actually n transferred from base 2 into "Fibonacci base", i.e. $f(b_k2^k + \ldots + b_0) = b_kF_{k+1} + \ldots + b_0$. This is easily accomplished by induction on n: if $n = 4k$, then $f(n) = f(2k) + f(k)$, if $n = 2k + 1$, then $f(n) = f(2k) + 1$, and if $n = 4k + 2$, then $f(n) = f(4k) + 1$, and the verification is direct.

Now, as we found f, let us turn to the initial question. It asks when $f(4n) = f(3n)$. Actually f should be some sort of increasing function, so we could suppose $f(3n) \leq f(4n)$. Indeed this holds true if we check some particular cases, with equality sometimes. Now what connects $4n$ and $3n$? The condition says that $f(4n) = f(2n) + f(n)$ but we have $3n = 2n + n$. So we can suppose that $f(a + b) \leq f(a) + f(b)$ and look for equality cases.

We work in binary base. The addition of two binary numbers can be thought of as adding their corresponding digits pairwise, and then repeating a number of times the following operation: if we reached a 2 in some position, replace it by a zero and add a 1 to the next position. (Note that we will never have digits greater than two if we eliminate the 2 at the highest level at each step). For example $3 + 9 = 11_2 + 1001_2 = 1012_2$, then we remove the 2 to get 1020_2 and again to get $1100_2 = 10$ so $3+7 = 10$. We can extend f to sequences of 0's, 1's and 2's by setting $f(b_k, \ldots, b_0) = b_kF_{k+1} + b_{k-1}F_k + \ldots + b_0$. Then we can see that if S is the sequence obtained by adding a and b componentwise (as vectors), then $f(s) = f(a) + f(b)$. And we need to prove that the operation of removing a 2 does not increase f.

Indeed, if we remove a 2 from position k and add a 1 to position $k + 1$ the f changes by $F_{k+2} - 2F_{k+1}$. This value is never positive and is actually zero only for $k = 0$. So f is not increased by this operation (which guarantees the claim that $f(a + b) \leq f(a) + f(b)$), and moreover it is not decreased by it only if the operation consists of removing the 2 at the units position. So $f(a + b) = f(a) + f(b)$ if and only if by adding them componentwise we either reach no transfer of unity, or have only one transfer at the lowest level. Hence $f(4n) = f(3n)$ if and only if adding

$2n+n$ we can reach at most a transfer at the lowest level. This cannot occur as the last digit of $2n$ is 0. So $f(4n) = f(3n)$ if and only if by adding $2n$ and n we have no transfer i.e. $2n$ and n do not share a unity digit in the same position. But as the digits of $2n$ are just the digits of n shifted one position, this is possible if and only if n has no two consecutive unities in its binary representation. So we need to prove that there are exactly $f(2^{m+1}) = F_{m+2}$ such numbers less than 2^m.

Let $g(m)$ be these numbers. Then $g(0) = 1, g(1) = 2$. Now note that if n is such a number and $n \geq 2^{m-1}$, then $n = 2^{m-1} + n'$, where $n' < 2^{m-2}$ (as it cannot have a unity in position $m-1$ that would conflict with the leading unity), so we have $g(m-2)$ possibilities for this case. For $n < 2^{m-1}$ we have $g(m-1)$ possibilities. Thus $g(m) = g(m-1) + g(m-2)$ and an induction finishes the proof.

3. Iterations and orbits

There is a class of functional equations, most of them on $\mathbb{N}$, which involve repeated applications of the unknown function, such as $f(f(x)) = g(x)$. They can be solved by constructing the "orbits" of x: $O(x) = (x, g(x), g(g(x), \ldots))$ and investigating the relations determined by f on these orbits. This type of equations will be exemplified here.

Problem 17. Let k be an even positive integer. Find the number of functions $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(f(n)) = n + k$$

for each $n \in \mathbb{N}_0$.

Solution. We can guess a particular solution: $f(n) = n + \frac{k}{2}$, but the statement of the problem itself suggests that there are many solutions, and those are not so easy to find. We have

$$f(n+k) = f(f(f(n))) = f(n) + k$$

and it follows by induction on m that

$$f(n+km) = f(n) + km,$$

for all $n, m \in \mathbb{N}_0$.

Now take an arbitrary integer p, $0 \leq p \leq k-1$, and let $f(p) = kq + r$, where $q \in \mathbb{N}_0$ and $0 \leq r \leq k-1$. Then

$$p + k = f(f(p)) = f(kq + r) = f(r) + kq.$$

Hence either $q = 0$ or $q = 1$ and therefore

$$f(p) = r,\ \ f(r) = p + k \quad \text{or} \quad f(p) = r + k,\ \ f(r) = p.$$

In both cases we have $p \neq r$, which shows that f defines a pairing of the set $A = \{0, 1, \dots, k\}$. Note that different functions define different pairings of A.

Conversely, any pairing of A defines a function $f \colon \mathbb{N}_0 \to \mathbb{N}_0$ with the given property in the following way. We define f on A by setting $f(p) = r,\ \ f(r) = p + k$ for any pair (p, r) of the given pairing and $f(n) = f(q) + ks,$ for $n \geq k + 1$, where q and s are respectively the quotient and the remainder when n in divided by k.

Thus the number of the functions with the given property is equal to that of all pairings of the set A. It is not difficult to see that this number is equal to $\dfrac{k!}{(k/2)!}$.

Remark. The above solution shows that if k is an odd positive integer, then there are no functions $f \colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(f(n)) = n + k$$

for all $n \in N_0$. For $k = 1987$ this problem was given at the IMO 1987.

Note that in order to find out information about the function f, we looked at $x + mk = f(f(\dots f(x)))$, based on the idea presented in the beginning of the paragraph. Let us look at a very similar example.

Problem 18. Find all functions $f \colon \mathbb{N} \to \mathbb{N}$ for which there is a positive integer a such that $f(f(n)) = an$ for all n in $\mathbb{N}$.

Solution. If $a = 1$, then $f(f(x)) = x$, so f is an involution and it is obtained by paring all the natural numbers and mapping one element of a pair into another. Next, suppose $a > 1$. If $f(x) = y$, then $f(y) = ax$, $f(ax) = f(f(y)) = ay$ and we prove by induction on k the following statement: $(*)$ $f(a^k x) = a^k y, f(a^k y) = a^{k+1} x$. Let S be the set of all numbers not divisible by a. Every positive integer can be represented uniquely as $a^k b$, where $b \in S$. Now let $s \in S$ and $f(s) = a^k t$, where $t \in S$. If we set $u = f(t)$, then using $(*)$ we get $f(a^k t) = a^k u$. But $f(a^k t) = f(f(s)) = as$, therefore $a^k u = as$. Thus as s is divisible by u, we get either $k = 1, u = s$ or $k = 0, u = as$. In the first case, $f(t) = s, f(s) = at$ and in the second case, $f(s) = t, f(t) = as$. In any case, f maps one of s, t into another. Therefore S separates into pairs (x, y) that satisfy $f(x) = y, f(y) = ax$, hence by $(*)$, $f(a^k x) = a^k y, f(a^k y) = a^{k+1} x$. It is clear that all such functions satisfy our requirements.

Finally, these two problems can be generalized to the following harder problem.

Problem 19. Let n be an integer greater than 1 and let $a, b \in \mathbb{Z}$, $a \notin \{0, 1\}$. Prove that there exist infinitely many functions $f \colon \mathbb{Z} \to \mathbb{Z}$ such that $f_n(x) = ax + b$ for all $x \in \mathbb{Z}$, where f_n is the n-th iterate of f. Prove that for $a = 1$ there exists b, such that $f_n(x) = ax + b$ has no solutions.

(Romanian TST, 1991)

Solution. The second part of the problem is already known to us when $n = 2$, n is odd, and the procedure is the same. If $b = n - 1$ and we let $a_i = f_i(0)$ ($a_{i+n} = a_i + b$), then for some $0 \leq i < j \leq n$ we have $a_i \equiv a_j (\bmod\ n)$. Thus $a_i = a_j + hb$, hence $a_{i+hn} = a_j$, and therefore $a_{r+hn+i-j} = a_r$ for all sufficiently large r. As $hn + i - j \neq 0$, this implies $a_{r+n(hn+i-j)} = a_r$, which contradicts the fact that $a_{r+n(hn+i-j)} = a_r + b(hn + i - j)$. Actually, with one more effort, one can prove that f exists if and only if $n \mid b$.

Now let us turn to the first part, which seems more challenging but bears also some similarity to the simpler case $a = 1, n = 2$. Let $g(x) = ax + b$. First consider the case $a \neq -1$ (it is special because $g(g(x)) = x$ in this case, whereas in the general case $|g_n(x)|$ tends to infinity for almost all x). We see that $g_n(x) = a^n(x - \frac{b}{a-1}) + \frac{b}{a-1}$. In particular, this proves our claim that $|g_n(x)|$ tends to infinity for almost all x: all x except, perhaps, $x = \frac{b}{a-1}$ (if it is an integer).

Let $C(x) = \{x, g(x), g_2(x), \ldots, g_n(x), \ldots\}$ be the chain generated by x and call a chain maximal if it is not a proper subchain of another chain (that is, if $x \neq g(y)$ for $y \in \mathbb{Z}$). We claim that maximal chains form a partition of $\mathbb{N} \setminus \{-\frac{b}{a-1}\}$. Indeed, first pick a number $n \neq -\frac{b}{a-1}$. Then $n = g_k(m)$ is equivalent to $n = a^k(m + \frac{b}{a-1}) - \frac{b}{a-1}$, or $(a-1)n + b = a^k((a-1)m + b)$. So take k to be the greatest power of a dividing $(a-1)n + b$ and let $s = \frac{(a-1)m+b}{a^k}$. Then s is not divisible by a and moreover $s - b$ is divisible by $a - 1$. Hence if we set $m = \frac{s-b}{a-1} + b$, then m is an integer and the equation $g(t) = m$ has no solutions in $\mathbb{N}$ (because otherwise $at = m - b = \frac{s-b}{a-1}$, so $s - b$ is divisible by a). Thus $C(m)$ is the desired maximal chain.

Next, let us prove that two distinct maximal chains do not intersect. If $C(x)$ and $C(y)$ intersect for $x \neq y$, then $g_m(x) = g_n(y)$ for some $m \neq n$. Without loss of generality, $m \geq n$. Then, as g is invertible on $\mathbb{R}$, we deduce $g_{m-n}(x) = y$, hence $C(y) \subset C(x)$, contradicting the fact that $C(y)$ is a maximal chain. Now consider all maximal chains (there are infinitely many of them since every element x for which the equation $g(y) = x$ has no solutions in $\mathbb{N}$ generates such a chain). We can group then into n-tuples. Now we define f on each of the n-tuples. Let $(C(x_1), C(x_2), \ldots, C(x_n))$ be such an n-tuple. Then we define $f(g_k(x_i)) = g_k(x_{i+1})$ for $i = 1, 2, \ldots, n-1$ and $f(g_k(x_n)) = g_{k+1}(x_1)$. Define also $f(-\frac{b}{a-1}) = -\frac{b}{a-1}$. Then f satisfies our requirements.

Let us investigate the case $a = 1$. In this case, $\mathbb{N} \setminus \{\frac{b}{2}\}$ splits into infinitely many disjoint pairs (x, y) with $x + y = b$. Again, we can group the pairs into n-tuples and define f on each n-tuple $(x_1, y_1), \ldots, (x_n, y_n)$ as $f(x_i) = x_{i+1}, f(y_i) = y_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $f(x_n) = y_1, f(y_n) = x_1$. Define $f(\frac{b}{2}) = \frac{b}{2}$, if necessary. Again, we see that f satisfies the conditions.

Finally, in both cases, as we can group the chains or the pairs into n-tuples in infinitely many ways, we have infinitely many such functions. It can be also proved that all functions with the desired property are of this form.

4. Approximating with linear functions

There are some weird functional equations on $\mathbb{N}$ that seem untouchable. But we can sometimes prove that they are unique. In this case guessing the function would be very helpful, and very often, the solutions are linear, thus it is natural to try $f(x) = cx$. But sometimes c can be rational or even irrational, and we can have formulas such as $f(x) = \lfloor cx \rfloor$. To overcome this difficulty, we write $f(x) \sim cx$, meaning that $|f(x) - cx|$ is bounded, or $\frac{f(x)}{cx}$ is close to 1, or whatever intuitive condition we mean, as long as cx is the most important part in expressing $f(x)$. Now we can guess c from the condition and then look at some initial case to guess the exact formula. Examples are given below.

Problem 20. Find all increasing functions $f\colon \mathbb{N} \to \mathbb{N}$ such that the only positive integers that are not in the image of f are those of form $f(n) + f(n+1), n \in \mathbb{N}$.

Solution. First assume that $f(x) \sim cx$. Let find c. If $f(n) = m$, then there are exactly $m - n$ positive integers not exceeding m that are not values of f. Therefore we conclude that they are exactly

$$f(1) + f(2), \ldots, f(m-n) + f(m-n+1).$$

Hence $f(m-n)+f(m-n+1) < m < f(m-n+1)+f(m-n+2)$. Now as $f(x) \sim cx$, we conclude that $m \sim cn$. Thus $2c(m-n) \sim m$, or $2c(c-1)n \sim cn$. It follows that $2c - 2 = 1$, so $c = \frac{3}{2}$. Hence we make the assumption that $f(x) = \lfloor \frac{3}{2}x + a \rfloor$ for some a. Let us search for a.

Clearly, $f(1) = 1, f(2) = 2$, as 1 and 2 must necessarily belong to Imf. Then 3 does not belong to Imf, hence $f(3) \geq 4$, so $f(2) + f(3) \geq 6$. Thus 4 belongs to Imf and $f(3) = 4$. We continue to $f(4) = 5, f(5) = 7$, and so on. So $\lfloor \frac{3}{2} + a \rfloor = 1, \lfloor 3+a \rfloor = 2$, which implies $a \in [-\frac{1}{2}; 0)$. And we see that for any a, b in this interval, $\lfloor \frac{3}{2}x + a \rfloor = \lfloor \frac{3}{2}x + b \rfloor$. So we can assume $a = -\frac{1}{2}$ and conjecture that $f(n) = \lfloor \frac{3n-1}{2} \rfloor$. First we wish to show that $\lfloor \frac{3n-1}{2} \rfloor$ satisfies the conditions.

Indeed, $\lfloor \frac{3n-1}{2} \rfloor + \lfloor \frac{3(n+1)}{2} \rfloor = \lfloor \frac{3n-1}{2} \rfloor + 1 + \lfloor \frac{3n}{2} \rfloor = 3n + 1$ by Hermite's Identity, and we need to prove that the only numbers that are not of form $\lfloor \frac{3n-1}{2} \rfloor$ are those that give residue 1 to division by 3. Indeed, if $n = 2k$, then $\lfloor \frac{3n-1}{2} \rfloor = 3k - 1$ and if $n = 2k + 1$, then $\lfloor \frac{3n-1}{2} \rfloor = 3k$. The conclusion is straightforward.

The fact that $f(n) = \lfloor \frac{3n-1}{2} \rfloor$ stems now from the inductive assertion that f is unique. Indeed, if we determined $f(1), f(2), \ldots, f(n-1)$, then we determined all $f(1)+f(2), f(2)+f(3), \ldots, f(n-2)+f(n-1)$. Then $f(n)$ must be the least number which is greater than $f(n-1)$ and not among $f(1) + f(2), f(2) + f(3), \ldots, f(n-2)+f(n-1)$. This is because if m is this number and $f(n) \neq m$, then $f(n) > m$ and m does not belong to $Im(f)$ or to the set $\{f(n) + f(n+1) | n \in \mathbb{N}\}$, a contradiction. Hence $f(n)$ is determined uniquely from the previous values of f.

The next problem was the hardest at the IMO 1979, but if you "guess" the answer, you have a chance to do the computations and solve the problem.

Problem 21. Find all increasing functions $f\colon \mathbb{N} \to \mathbb{N}$ with the property that all positive integers that are not in the image of f are those of form $f(f(n))+1, n \in \mathbb{N}$.

(IMO, 1979)

Solution. Again, f is unique. If $f(x) \sim cx$, then we conclude that $m \sim c^2(m-n)$, where $m = f(n)$. Thus $c = c^2(c-1)$ or $c^2 - c - 1 = 0$. We get $c = \frac{1+\sqrt{5}}{2} \sim 1.618$, the positive root of the above quadratic equation. We try to set $f(x) = \lfloor cx + d \rfloor$ for some constant d. Now we compute $f(1) = 1$, $f(2) = 3$, $f(3) = 4, f(4) = 6, f(5) = 8$, and we can try to put $d = 0$, so $f(n) = \lfloor cn \rfloor$. Let us prove that it satisfies the hypothesis.

If $f(n) = m$, then $m < cn < m+1$, so $\frac{m}{c} < n < \frac{m+1}{c}$. As $\frac{1}{c} = c - 1$, we get $(c-1)m < n < (c-1)(m+1)$, and so m is in $Im(f)$ if an only if the interval $(cm, cm+c-1)$ contains an integer which is equivalent to the fact that $\{cm\} > 2-c$. If $f(f(n))+1 = m$, then $\lfloor c\lfloor cn \rfloor\rfloor = m-1$. It follows that

$$\lfloor cn \rfloor \in ((m-1)(c-1), m(c-1)),$$

so

$$n \in ((m-1)(c-1)^2, m(c-1)^2 + (c-1)) = ((2-c)m + c - 2, (2-c)m + c - 1),$$

or

$$n = \lfloor (2-c)m + c - 1 \rfloor = 2m - \lfloor c(m-1) \rfloor - 2.$$

Therefore $m = f(f(n)) + 1$ if and only if the number $n = 2m - \lfloor c(m-1) \rfloor - 2$ satisfies the condition $f(f(n)) + 1 = m$. We set $u = \{c(m-1)\}$. Then
$n = (2-c)m + c - 2 + u$, so

$$f(n) = \lfloor c(2-c)m + cu - 2c + c^2 \rfloor = \lfloor (c-1)m + cu - c + 1 \rfloor = \lfloor (c-1)(m-1) + cu \rfloor =$$

$$= \lfloor c(m-1) - m + 1 + cu \rfloor = c(m-1) - m + 1 + cu - \{u(c+1)\}.$$

Set $s = \{u(c+1)\}$. Then $f(f(n)) = \lfloor c(c-1)(m-1) + c^2u - cs \rfloor = \lfloor m - 1 + (c+1)u - cs \rfloor$. So $f(f(n)) + 1 = m$ if and only if $0 < (c+1)u - cs < 1$.

If $t = u(c+1) \in (0, 1+c)$ this is equivalent to $t - c\{t\} \in (0,1)$.

When $t < 1$, this is false, as the requested value is negative.

When $1 < t < 2$, we have $t - c\{t\} = t - c(t-1) = c - (c-1)t \in (0,1)$.

When $t > 2$,

$$t - c\{t\} = t - c(t-2) = 2c - (c-1)t > 2c - (c-1)(c+1) = 2c - c^2 + 1 = c > 1.$$

Our condition is equivalent to $t \in (1,2)$ or $u \in (\frac{1}{c+1}, \frac{2}{c+1}) = (2-c, 4-2c)$, so $\{cm - c\} \in (2-c, 4-2c)$ or $\{cm\} \in \{0, 2-c\}$. Hence this condition is equivalent to $\{cm\} < 2 - c$.

Thus we see that the condition $m = f(n)$ is equivalent to $\{cm\} > 2 - c$ and the condition $m = f(f(n)) + 1$ is equivalent to $\{cm\} < 2 - c$. The proof is finished.

5. Cauchy's equation

The Cauchy equation is like a threshold for functional equations: anyone who studies it will eventually master it, and this is what distinguishes a novice in this field from an experienced solver. The importance of this equations stems from its simple and natural statement on one side, and ingenious proof on the other (whose reasoning is mirrored while solving many other, far more complicated problems).

Problem 22. Find all monotonic (or continuous) functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy the condition

$$f(x+y) = f(x) + f(y)$$

for all x in $\mathbb{R}$.

(Cauchy's equation)

Solution. If $f(1) = c$, then $f(2) = f(1+1) = c + c = 2c$. Thus $f(3) = f(2+1) = f(2) + f(1) = 2c + c$, and by induction $f(n) = cn$ for $n \in \mathbb{N}$. Because $f(0+1) = f(0) + f(1)$, we get $f(0) = 0$ and then as $0 = f(n + (-n)) = nc + f(-n)$, we find $f(-n) = cn$ for $n \in \mathbb{N}$.

We have computed f on $\mathbb{N}$. To compute other values of f, note that $f(mx) = mf(x)$ by induction on $m \in \mathbb{N}$ (this is exactly the same way as we proved that $f(n) = cn$). Thus, if $x \in \mathbb{Q}$, then $x = \frac{p}{q}$ for $p \in \mathbb{Z}, q \in \mathbb{N}$. Hence $pc = f(p) = f(qx) = qf(x)$, so $f(x) = c\frac{p}{q} = cx$ for $x \in \mathbb{Q}$.

Now, as we have shown $f(x) = cx$ for $x \in \mathbb{Q}$, it is natural to assume $f(x) = cx$ for $x \in \mathbb{R}$. This is where we use the fact that f is either continuous or monotonic.

If f is continuous, then as $\mathbb{Q}$ is dense in $\mathbb{R}$, we can select $x_n \in \mathbb{Q}$ such that $\lim_{n\to\infty} x_n = x$ and then $f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} cx_n = cx$.

If f is monotonic, assume that $f(x) \neq cx$. If $c = 0$, then there are $u < x < v$ such that $u, v \in \mathbb{Q}$. Hence $f(u) = f(v) = 0$ and the monotonicity of f implies that $f(x) = 0$, too. If $c > 0$, then f is increasing. Now if $f(x) = cy$ and $y > x$, we can take $z \in \mathbb{Q}$ such that $x < z < y$ and then $f(x) \leq f(z) = cz < cy = f(x)$, a contradiction. If $y < x$, choose $z \in \mathbb{Q}$ such that $y < z < x$ and then $f(x) \geq f(z) = cz > cy = f(x)$, again a contradiction. If $c < 0$, then f is decreasing and the reasoning mirrors the one for $c > 0$.

A function f for which $f(x+y) = f(x) + f(y)$ for all x and y is called additive. The additional assumptions on f, either monotonicity or continuity, are crucial: additive functions that are not linear can be constructed, but we will not touch this advanced topic here. There is another equation which resembles Cauchy:

Problem 23. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(xy) = f(x)f(y)$ for all x and y in $\mathbb{R}$ (such a function is called multiplicative).

Solution. If we let $g(x) = \ln(x)$, then $g(xy) = g(x) + g(y)$ and if we set $x = e^a, y = e^b$, we get $g(e^{a+b}) = g(e^a) + g(e^b)$. Now if we set $h(x) = g(e^x)$, we get $g(x+y) = g(x) + g(y)$. We know how to solve this equation, which gives us $h(x) = cx$. It follows that $g(e^x) = cx$, thus $g(x) = c \ln x$ and from here $f(x) = x^c$.

All is well until we realize that ln is defined only for positive real numbers. However this can be fixed. We note that $f(x^2) = f(x)^2$ so f is non-negative on $\mathbb{R}^+$. If for some $a \neq 0$ we have $f(a) = 0$, then we would have $f(x) = f(a\frac{x}{a}) = f(x) = 0$, so f would be identically zero. Thus aside for the identically zero solution, all other functions satisfying the condition are positive on $\mathbb{R}^+$. Therefore $f(x) = x^c$ for some c. As f is continuous at 0, we must have $f(0) = \lim_{x\to 0} x^c$. This tells us that $c \geq 0$ (otherwise the limit is $+\infty$). If $c > 0$, we get $f(0) = 0$, and if $c = 0$, we get $f(0) = 1$. Finally, we are to handle the negative numbers. As $f(-1)^2 = f(1) = 1$, we get $f(-1) = \pm 1$. If $f(-1) = 1$, then $f(-x) = f(x)$, hence $f(-x) = x^c$ for $x \geq 0$ thus $f(x) = |x^c|$. This is clearly a solution to the equation. If $f(-1) = 1$, we get $f(-0) = -f(0)$, so $f(0) = 0$, which implies $c > 0$. Then $f(-x) = -f(x)$ hence $f(x) = sgn(x)x^c$, which is also a solution to our problem.

Here is another application of Cauchy's equation.

Problem 24. Find all pairs of continuous functions $f, g\colon \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x) + f(y) = g(x+y)$$

for all x and y in $\mathbb{R}$.

Solution. We remark that $f(x+y) + f(0) = f(x) + f(y) = g(x+y)$. Thus $f(x) - f(0) + f(y) - f(0) = f(x+y) - f(0)$, so $f(x) - f(0)$ is an additive function. Therefore $f(x) = ax + c$ for some $c = f(0)$ and hence $g(x) = ax + 2c$.

Let us now combine the multiplicative and the additive Cauchy equation into a "power" Cauchy equation.

Problem 25. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x^y) = f(x)^{f(y)}$$

for all $x, y \in \mathbb{R}^+$.

(Russia, 1993)

Solution. We will prove that the functions $f(x) = x$ and $f(x) = 1$ are the only solutions to the problem. Suppose that $f(a) \neq 1$ for some $a > 0$. Then

$$f(a)^{f(xy)} = f(a^{xy}) = f(a^x)^{f(y)} = f(a)^{f(x)f(y)},$$

that is $f(xy) = f(x)f(y)$. Hence

$$f(a)^{f(x+y)} = f(a^{x+y}) = f(a^x)f(a^y) = f(a)^{f(x)+f(y)},$$

that is $f(x + y) = f(x) + f(y)$. But f is non-decreasing, since for $x - y > 0$, we have $f(x) - f(y) = f(x - y) = f(\sqrt{x-y}^2) = f(\sqrt{x-y})^2 \geq 0$. Hence $f(x) = cx$ and $cx^y = (cx)^{cy}$. In particular $c = c^{cy}$ for all $y > 0$, so $c = 1$.

This equation may have many variations. Consider the following example:

Problem 26. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$\{f(x + y)\} = \{f(x) + f(y)\}$$

for all x and y in $\mathbb{R}$ ({a} means the fractional part of a)

Solution. Basically, the problem says that $f(x+y) - f(x) - f(y)$ is an integer. Now, the difficulty is that it does not say which one. This inconvenience can be overcome by looking at the function $g_y(x) = f(x+y) - f(x) - f(y)$ for a fixed y. It must be continuous and integer-valued, which of course can happen only when it is constant. Thus $g_y(x) = g_0(x) = f(y) - f(0) - f(y) = -f(0)$. Hence $f(x+y) - f(x) - f(y) = -f(0)$ and then we just have $f(x+y) - f(0) = (f(x) - f(0)) + (f(y) - f(0))$. Thus $f(x) = cx + f(0) = cx + d$ where $d \in \mathbb{Z}$.

Not all applications of Cauchy's equation are so evident. In fact, the difficulty often lies in molding the condition such that we are able to use Cauchy's equation. We sometimes need to be really patient, as in the following example:

Problem 27. Prove that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + y - xy) + f(xy) = f(x) + f(y)$$

for all x and y in $\mathbb{R}$ is linear.

(Hosszu's functional equation)

Solution. This equation is of the form $f(a) + f(b) = f(c) + f(d)$, where $a + b = c + d$. The inconvenience is in that (a, b) and (c, d) are linked to each other, so we cannot state that $f(a) + f(b) = f(c) + f(d)$ whenever $a + b = c + d$. We try to eliminate this inconvenience by adding a new variable and symmetrizing:

$$f(x)+f(y)+f(z) = f(x+y-xy)+f(xy)+f(z) = f(x+y-xy)+f(xy+z-xyz)+f(xyz).$$

By symmetry, it also equals to $f(x+z-xz)+f(xz+y-xyz)+f(xyz)$ and $f(y+z-yz)+f(yz+x-xyz)+f(xyz)$. Thus we deduce that $f(x+y-xy)+f(xy+z-xyz)=f(x+z-xz)+f(xz+y-xyz)=f(y+z-yz)+f(yz+x-xyz)$.

This is again an equation of the form $f(a)+f(b)=f(c)+f(d)$, where $a+b=c+d$, but this time the constraints are milder. Indeed, if we set $g(t)=f(1-t)$, we can rewrite the equation as

$$\begin{aligned} g((1-x)(1-y))+g((1-xy)(1-z)) &= g((1-x)(1-z))+g((1-y)(1-xz)) \\ &= g((1-y)(1-z))+g((1-x)(1-yz)). \end{aligned}$$

Now, let us find for which a,b,c,d with $a+b=c+d$ we can find x,y,z with $(1-x)(1-y)=a,(1-z)(1-xy)=b,(1-x)(1-z)=c$. For convenience, we set $u=1-x,v=1-y,w=1-z$ to get $uv=a,w(u+v-uv)=b,uw=c$. Hence $w=\frac{c}{u},v=\frac{a}{u}$, and $\frac{c}{u}(u+\frac{a}{u}-a)=b$ or $(c-b)u^2-acu+ac=0$.

For this equation to have a non-zero solution, we need to have a nonnegative discriminant. Thus $a^2c^2-4ac(c-b)\geq 0$, or $(ac-2c)^2+4(abc-c^2)\geq 0$. When $abc>c^2$, this is certainly true. Therefore if $abc>c^2$ we have $g(a)+g(b)=g(c)+g(d)$ for $a+b=c+d$. Now consider (a,b) and (c,d) with $a+b=c+d$. If ab and cd have the same sign, then we can find an e sufficiently small in absolute value such that $abe>e^2$ and $cde>e^2$. Setting $e_1=a+b-e=c+d-e$, we get $g(a)+g(b)=g(e)+g(e_1)$, as $abe>e^2$. Similarly, $g(e)+g(e_1)=g(c)+g(d)$, so $g(a)+g(b)=g(c)+g(d)$. Thus $g(a)+g(b)=g(c)+g(d)$, when $a+b=c+d$ and $abcd>0$.

Now take $a,b\neq 0$ and $a+b\neq 0$. Then for all sufficiently small c such that c has the same sign as $ab(a+b)$, we have $abc(a+b-c)>0$. Hence $g(a)+f(b)=g(a+b-c)+g(c)$. By taking $c\to 0$ and passing to the limit we get $g(a)+g(b)=g(a+b)+g(0)$ whenever $ab(a+b)\neq 0$. The restriction $ab(a+b)\neq 0$ is not a restriction to us: for any a,b we can find a_n tending to a, b_n tending to b such that $a_nb_n(a_n+b_n)\neq 0$. Then we have $g(a_n)+g(b_n)=g(a_n+b_n)+g(0)$ and by passing to the limit we get $g(a)+g(b)=g(a+b)+g(0)$. But then $g(x)-g(0)$ satisfies the Cauchy equation. Hence $g(x)$ is linear, so is $f(x)=g(1-x)$. Conversely, any linear function satisfies Hosszu's equation.

Let us look at another subtle use of Cauchy's equation.

Problem 28. Find all functions $f\colon\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x)^2+y)=x^2+f(y).$$

for all x and y in $\mathbb{R}$.

Solution. If $f(x_1)=f(x_2)$, then setting $x=x_1,x_2$ we get $x_1^2=x_2^2$, or $x_2=\pm x_1$. Now consider the function $h\colon\mathbb{R}^+\to\mathbb{R}^+$ defined by $h(x)=f^2(\sqrt{x})$. We can rewrite the condition as $f(h(x)+y)=x+f(y)$ for $x>0$. Then

$$f(h(u)+h(v)+y)=u+f(h(v)+y)=u+v+f(y)=f(h(u+v)+y).$$

Therefore $h(u)+h(v)+y = \pm(h(u+v)+y)$. But $h(u)+h(v)+y = -(h(u+v)+y)$ cannot hold for all y, so for at least one y we have $h(u)+h(v)+y = h(u+v)+y$.

Thus h is additive. As h is non-negative by definition, $h(x) = cx$, where $c \geq 0$. We can deduce that $f(x) = \pm\sqrt{c}x$. Hence $f(cx^2+y) = x^2 + f(y)$. If $f(y) = -\sqrt{c}y, y \neq 0$, then $f(cx^2+y) = x^2 - \sqrt{c}y$. It follows that

$$f^2(cx^2+y) = (x^2 - \sqrt{c}y)^2 \neq (cx^2+y)^2$$

for at least some x, because $(x^2 - \sqrt{c}y)^2 = (cx^2+y)^2$ is equivalent to the not identically zero polynomial equation $(c^2-1)x^4 + 2(c+\sqrt{c})yx^2 + (1-c)y^2 = 0$. So $f(y) = \sqrt{c}y$. In this case we get analogously $(x^2+\sqrt{c}y)^2 = (cx^2+y)^2$, which for $x = 0$ becomes $cy^2 = y^2$, $c = 1$. Hence $f(x) = x$ for all x. The identity function satisfies the equation.

6. Substitutions

This method is very common in almost every area of mathematics. But it is especially useful in functional equations, since after all, every such equation has to be solved by substituting some values into the equation and then drawing the conclusion. In fact, many of the problems in other sections rely heavily on substitutions and could be easily be placed here. However, we present here the ones which emphasize the role of substitutions most. The skills learned in this section will prove useful in all the other sections.

Let us start with a warm-up example.

Problem 29. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ that satisfy

$$f(xy-x-y) = f(xy) - f(x+y)$$

for all x and y in $\mathbb{R}$.

Solution. Just substitute $y = 1$ to get $f(-1) = f(x) - f(x+1)$. Hence $f(x+1) = f(x) - f(-1)$ and from here we immediately conclude that $f(x) = -xf(-1) + f(0)$. Now if we set $x = -1$, we get $f(-1) = f(-1) + f(0)$, so $f(0) = 0$. Thus if we set $f(-1) = -a$, we get $f(x) = ax$ and clearly any such function satisfies the condition in the problem.

Problem 30. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) - f(x-y) = f(x)f(y)$$

for all x and y in $\mathbb{R}$.

Solution. Set $x = y = 0$ to get $f(0)^2 = 0$, therefore $f(0) = 0$. If we set $y \to -y$ we get

$$f(x-y) - f(x+y) = f(x)f(-y) = -f(x)f(y).$$

In particular, $f(y)f(-y) = -f(y)^2$ and $f(-y)^2 = -f(y)f(y)$. Hence

$$f(y)(f(y) + f(-y)) = f(-y)(f(y) + f(-y)) = 0,$$

so either $f(y)+f(-y) = 0$, or $f(y) = f(-y) = 0$. We conclude that $f(y)+f(-y) = 0$, for all y, thus f is odd. Now set $y = x$ to get $f(2x) = f(x)^2$. Then $f(-2x) = f(-x)^2 = f(x)^2 = f(2x)$. As $f(-2x) = -f(2x,)$ we deduce that $f(2x) = 0$ and so f is identically zero.

Substitutions are not only about clever choices of variables, but also about efficient selection of functions.

Problem 31. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$, continuous at zero, satisfying $f(x + y) - f(x) - f(y) = xy(x + y)$, for all x and y in $\mathbb{R}$.

Solution. We can guess the solution $\frac{x^3}{3}$, thus $g(x) = f(x) - \frac{x^3}{3}$ is additive. Now we claim $f(x) = cx$ for $c = f(1)$. Indeed, assume that $d = f(t) \neq ct$. If t is irrational, then we can find $m, n \in \mathbb{Z}$ with $|m + nt| < \epsilon$ for any $\epsilon > 0$. Then $f(m + nt) = mc + nd = c(m + nt) + n(d - ct)$. But now if we take ϵ small enough we force n to be as large as we want. Thus $|f(m + nt)| > n|d - ct| - c\epsilon$ increases to infinity, which contradicts the continuity of f in 0. So $f(x) = \frac{x^3}{3} + cx$.

Remark: We have encountered a stronger version of the continuous Cauchy's equation here. If f is additive and continuous at one point, then it is linear.

Problem 32. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$$

for all x and y in $\mathbb{R}$.

Solution. We know that $f(x) = \tan x$ satisfies this equation. Therefore if we set $g(x) = \arctan f(x)$, then $g(x + y) = g(x) + g(y) \pm 2k\pi$. Setting $h(x) = \frac{g(x)}{2\pi}$, we get $\{h(x) + h(y)\} = \{h(x + y)\}$. Thus $h(x) = cx + d$, where $d \in \mathbb{Z}$ (we have met this problem in the previous section). Then $g(x) = ax + 2\pi k$ for some $k \in \mathbb{Z}$, hence $f(x) = \tan(ax + 2\pi k) = \tan(ax)$. These functions satisfy the equation.

Problem 33. Find all polynomials $P(x, y) \in \mathbb{R}^2[x, y]$ satisfying

$$P(x + a, y + b) = P(x, y),$$

where a, b are some reals, not both zero.

Solution. Suppose $b \neq 0$. Consider the polynomial $R \in \mathbb{R}^2[x, y]$ defined by $R(x, y) = P(x + \frac{a}{b}y, y)$. Observe that $P(x, y) = R(x - \frac{a}{b}y, y)$.

Thus $P(x + a, y + b) = P(x, y)$ can be rewritten in terms of R as $R((x + a) - \frac{a}{b}(y + b), y + b) = R(x - \frac{a}{b}y, y)$ or $R(x - \frac{a}{b}y, y + b) = R(x - \frac{a}{b}y, y)$.

If we set $x \to (x - \frac{a}{b}y)$, we get $R(x, y) = R(x, y + b)$. Then by induction on n

$$R(x, y) = R(x, y + b) = \ldots = R(x, y + nb).$$

Set $Q_x(y) = R(x, y)$. Then $Q_x(y) = Q_x(y + b) = \ldots = Q_x(y + nb)$ and taking $n > \deg Q_x$ we get that Q_x is constant so $Q_x(y) = Q_x(0) = R(x, 0)$.

As $R(x, 0)$ is a polynomial in x, R is a polynomial in x, so $R(x, y) = Q(x)$, for some polynomial $Q \in \mathbb{R}[x]$. Then $P(x, y) = R(x - \frac{a}{b}y, y) = Q(x - \frac{a}{b}y) = Q(bx - ay)$. Each such polynomial satisfies the condition, as

$$P(x + a, y + b) = Q(b(x + a) - a(y + b)) = Q(bx - ay) = P(x, y).$$

If $b = 0$, then $a \neq 0$, and we repeat the reasoning by replacing b with a and y with x to get again $P(x, y) = Q(bx - ay)$.

The next problem is very beautiful, but in some sense quite hard (try to do it without substitutions).

Problem 34. Give an example of a bijection $f : \mathbb{N}_0 \to \mathbb{N}_0$ satisfying the equation $f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n)$.

(IMO Shortlist, 1996)

Solution. If we denote $g(3k + 1) = f(k)$, then the condition becomes

$$g((3m + 1)(3n + 1)) = 4g(3m + 1)g(3n + 1) + g(3m + 1) + g(3n + 1).$$

Next, if we denote $4g(x) + 1 = h(x)$, the condition above can be rewritten as $h((3m + 1)(3n + 1)) = h(3m + 1)h(3n + 1)$. Now we understand that we need to construct a multiplicative bijection of A into B, where $A = \{3k+1 \mid k \in \mathbb{N}\}, and B = \{4k + 1 \in \mathbb{N}\}$.

We can set $h(1) = 1$. Let U be the set of all primes of the form $3k - 1$, V the set of all primes of the form $3k + 1$, X the set of all primes of the form $4k - 1$, and Y the set of all primes of the form $4k + 1$. All these four sets are infinite. So we can provide a bijection h between U and X and between V and Y. We extend it by multiplicativity to the whole set A. We prove this is the required bijection.

Indeed, assume that $3k + 1 = \prod p_i^{a_i} \prod q_i^{b_i}$ where $p_i \in U, q_i \in B$. Then p_i are -1 mod 3, q_i are 1 mod 3, so $\sum a_i$ must be even. Then $h(3k+1) = \prod h(p_i)^{a_i} \prod h(q_i)^{b_i}$, where $h(p_i) \in X, h(q_i) \in Y$. As $h(p_i)$ is -1 mod 4 and $h(q_i)$ is 1 mod 4, but $\sum a_i$ is even, we conclude that $h(3k + 1)$ is 1 mod 4, so $h(3k + 1) \in B$.

We can analogously prove the converse: assume that $4k+1 = \prod p_i^{a_i} \prod q_i^{b_i}$, where $p_i \in X$, and $q_i \in Y$. As p_i are -1 mod 4 and q_i are 1 mod 4, $\sum a_i$ must be even. Then $x = \prod h^{-1}(p_i)^{a_i} \prod h^{-1}(q_i)^{b_i}$ satisfies $h(x) = 4k+1$. Moreover, as $h^{-1}(p_i)$ are -1 mod 3, $h^{-1}(q_i)$ are 1 mod 3, and $\sum a_i$ is even, we conclude that x is 1 mod 3, so $x \in A$. Finally, h is injective, because of the uniqueness of the prime factorization.

Problem 35. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

(IMO, 1999)

Solution. Setting $x = f(y)$ in the given equation gives

$$f(x) = \frac{c+1-x^2}{2}, \tag{1}$$

where $c = f(0)$. On the other hand, if $y = 0$ we get $f(x-c) - f(x) = f(c) + cx - 1$. Hence $f(-c) - c = f(c) - 1$, which shows that $c \neq 0$. Thus for any $x \in \mathbb{R}$ there is $t \in \mathbb{R}$ such that $x = y_1 - y_2$, where $y_1 = f(t-c)$ $y_2 = f(t)$. Now using the given equation we get

$$\begin{aligned} f(x) = f(y_1 - y_2) &= f(y_2) + y_1 y_2 + f(y_1) - 1 \\ &= \frac{c+1-{y_2}^2}{2} + y_1 y_2 + \frac{c+1-{y_1}^2}{2} - 1 \\ &= c - \frac{(y_1-y_2)^2}{2} = c - \frac{x^2}{2}. \end{aligned}$$

This together with (1) gives $c = 1$ and $f(x) = 1 - \dfrac{x^2}{2}$. Conversely, it is easy to check that this function satisfies the given equation.

7. Fixed points

An important information about a function is given by its fixed points. There are a lot of important theorems concerning fixed points in more advanced mathematics, such as Brower's Theorem. We will illustrate how the idea of looking at fixed points helps us in solving some interesting problems.

Problem 36. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

(*i*) $f(xf(y)) = yf(x)$ for all $x, y \in \mathbb{R}^+$;

(*ii*) $\lim_{x \to +\infty} f(x) = 0$.

(IMO, 1983)

Solution. From (i) it follows that $f(xf(x)) = xf(x)$ for all $x > 0$. By induction on n we have that if $f(a) = a$ for some $a > 0$, then $f(a^n) = a^n$ for all $n \in \mathbb{N}$. Note also that $a \leq 1$, since otherwise

$$\lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} a^n = +\infty,$$

in contradiction to (ii).

On the other hand, $a = f(1 \cdot a) = f(1 \cdot f(a)) = af(1)$. Hence

$$1 = f(1) = f(a^{-1}a) = f(a^{-1}f(a)) = af(a^{-1}),$$

implying $f(a^{-1}) = a^{-1}$. Thus we have (as above) $f(a^{-n}) = a^{-n}$ for all $n \in \mathbb{N}$ and so $a^{-1} \leq 1$.

In conclusion, the only $a > 0$ such that $f(a) = a$ is $a = 1$. Hence the identity $f(xf(x)) = xf(x)$ implies $f(x) = \dfrac{1}{x}$ for all $x > 0$. It is easy to check that this function satisfies the conditions (i) and (ii) of the problem.

Problem 37. Let S be the set of all real numbers greater than -1. Find all functions $f\colon S \to S$ such that

(i) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all $x, y \in S$;

(ii) $\dfrac{f(x)}{x}$ is strictly increasing in the intervals $(-1, 0)$ and $(0, +\infty)$.

(IMO, 1994)

Solution. If $x = y > -1$ we have from (i) that

$$f(x + (1 + x)f(x)) = x + (1 + x)f(x). \qquad (1).$$

On the other hand, (ii) implies that the equation $f(x) = x$ has at most one solution in each of the intervals $(-1, 0)$ and $(0, +\infty)$.

Suppose that $f(a) = a$ for some $a \in (-1, 0)$. Then (1) implies $f(a^2 + 2a) = a^2 + 2a$ and therefore $a^2 + 2a = a$, because $a^2 + 2a = (a + 1)^2 - 1 \in (-1, 0)$. Hence $a = -1$ or $a = 0$, a contradiction. The same arguments show that the equation $f(x) = x$ has no solutions in the interval $(0, +\infty)$.

Then we conclude from (1) that $x + (1 + x)f(x) = 0$, that is $f(x) = -\dfrac{x}{1 + x}$ for all $x > -1$. It is easy to check that this function satisfies the conditions (i) and (ii) of the problem.

Problem 38. Prove that there is no function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) = x^2 - 1996$$

for all $x \in \mathbb{R}$.

(Tournament of the towns, 1996)

Solution. We will prove the following more general result.

Lemma Let $g(x)$ be a quadratic function such that the equation $g(g(x)) = x$ has at least three different real roots. Then there is no function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) = g(x) \tag{1}$$

for all $x \in \mathbb{R}$.

Proof. The fixed points of $g(x)$ are also fixed points of the fourth degree polynomial $h(x) = g(g(x))$. Hence by the given conditions it follows that $g(x)$ has one or two real fixed points. Denote them by x_1 and x_2. Then $h(x)$ has one or two real fixed points, different from x_1 and x_2. Denote them by x_3 and x_4. The identity

$$f(g(x)) = f(f(f(x))) = g(f(x))$$

implies that $\{f(x_1), f(x_2)\} = \{x_1, x_2\}$. On the other hand,

$$f(f(g(x))) = f(g(f(x))) \text{ and } f(f(f(g(x)))) = f(f(g(f(x)))),$$

that is $f(h(x)) = h(f(x))$. Hence $\{f(x_3), f(x_4)\} \in \{x_1, x_2, x_3, x_4\}$. Suppose that $f(x_l) = x_k$ for some $k \in \{1, 2\}$ and $l \in \{3, 4\}$. Then

$$x_l = h(x_l) = f(f(f(f(x_l)))) = f(g(x_k)) = f(x_k) \in \{x_1, x_2\},$$

a contradiction. Hence $f(x_3) = x_3$, if $x_3 = x_4$, and $\{f(x_3), f(x_4)\} = \{x_3, x_4\}$, if $x_3 \neq x_4$. In both cases we have $g(x_3) = f(f(x_3)) = x_3$, a contradiction. Thus the lemma is proved. Turning back to the problem, we note that the equation $g(g(x)) = (x^2 - 1996)^2 - 1996 = x$ has four different real roots since

$$(x^2 - 1996)^2 - 1996 - x = (x^2 - 1996 - x)(x^2 + x - 1995).$$

Remark. Set $g(x) = ax^2 + bx + c$. Then

$$g(g(x)) - x = (ax^2 + (b-1)x + c)(a^2x^2 + a(b+1)x + ac + b + 1).$$

Therefore the four roots of the equation $g(g(x)) = x$ are equal to:

$$\frac{1 - b + \sqrt{D}}{2a}, \ \frac{1 - b - \sqrt{D}}{2a}, \ \frac{-1 - b + \sqrt{D-4}}{2a} \text{ and } \frac{-1 - b - \sqrt{D-4}}{2a},$$

where $D = (b-1)^2 - 4ac$. All these roots are real if and only if $D \geq 4$. If $D > 4$ then all the roots are different whereas for $D = 4$, one of them is equal to $\dfrac{3-b}{2a}$ and the other three are equal to $-\dfrac{1+b}{2a}$.

The lemma proved above says that if $D > 4$, then there are no functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = g(x)$ for all $x \in \mathbb{R}$. On the other hand, for $D = 4$ there are infinitely many continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the above equation.

8. Polynomials

The most commonly used functions are probably the polynomials, especially the ones in one variable. We know a lot about them: we know they have roots, and the number of roots is exactly the degree of the polynomial (in $\mathbb{C}$), and so on. It is the multitude of information that allows mathematicians to create a lot of diverse problems involving polynomials, from easy to very difficult. Let us see some of the most typical examples here.

Problem 39. Find all polynomials $P \in \mathbb{R}[X]$ such that $P(P(x)) = x^2P(x+1)$ for all x.

Solution. If $p(x) = 0$, then the relation clearly holds. Otherwise, if the degree of the polynomial is n, then $P(P(x))$ has degree n^2, whereas $x^2P(x+1)$ has degree $n+2$. Hence $n^2 = n+2$, so $n = 2$, or $n = -1$. We see that $n = -1$ clearly does not work, so $P(x) = ax^2 + bx + c$.

Thus the leading coefficient of $P(P(x))$ is a^3 and the leading coefficient of $x^2P(x+1)$ is a, so we get $a^3 = a$. It follows that $a = \pm 1$, as $a \neq 0$. Next set $x = 0$ to get $P(c) = 0$. Then $x - c \mid P(x)$, so P has real roots. As their product is $\frac{c}{a}$, we get $P(x) = (x-c)(x-1)$ or $P(x) = -(x-c)(x+1)$. We now just have to check these possibilities.

If $P(x) = (x-c)(x-1)$, then

$$\begin{aligned} P(P(x)) &= (P(x)-c)(P(x)-1) \\ &= (x^2-(c+1)x)(x^2-(c+1)x+(c-1)) \\ &= x(x-c-1)(x^2-(c+1)x+c-1). \end{aligned}$$

On the other side, it is equal to $x^2P(x+1) = x^3(x+c-1)$. Thus $x^3(x+c-1) = x(x-c-1)(x^2-(c+1)x+c-1)$ or $x^2(x+c-1) = (x-c-1)(x^2-(c+1)x+c-1)$. Then $x \mid x^2-(c+1)x+c-1$, hence $c = 1$ and the relation becomes $x^3 = (x-2)^2x$, which is impossible.

If $P(x) = -(x-c)(x+1)$, then

$$\begin{aligned} P(P(x)) &= -(P(x)-c)(P(x)+1) \\ &= -(-x^2+(1-c)x)(-x^2+(1-c)x+c+1) \\ &= -x(x+c-1)(x^2+(1-c)x+c+1). \end{aligned}$$

On the other side, it is equal to

$$x^2P(x+1) = -x^2(x+1-c)(x+2).$$

So $(x+c-1)(x^2+(1-c)+c+1) = x(x+1-c)(x+2)$. Plugging $x = 0$, we get $c-1 = 0$, or $c+1 = 0$. Thus $c = 1$ or $c = -1$. If $c = 1$ the relation becomes $x(x^2+2) = x^2(x+2)$ and if $c = -1$, $(x-2)(x^2+2x) = x(x+2)^2$, which again fails. Therefore we have no polynomials of this kind except $P(x) = 0$.

This problem exemplifies some important steps in solving functional equations for polynomials: we try to find the degree of the polynomial, then we seek as many roots of the polynomial as possible, try to cancel as many factors on both sides, plug in cleverly chosen values of x, or find the leading coefficient or free coefficient of the polynomial.

Another "computational" example is as follows:

Problem 40. Find all polynomials $P \in \mathbb{C}[X]$ satisfying $P(2x) = P'(x)P''(x)$.

Solution. If $\deg(P) = k > 0$, then the degree of $P'P''$ is $2k-3$ (unless $k = 1$, when it is zero), so $k = 3$. Now if the leading coefficient of P is a, then the leading coefficients of $P(2x), P'(x), P''(x)$ are $8a, 3a, 6a$, respectively.

It follows that $8a = 18a^2$, hence $a = \frac{4}{9}$. Now let $P'(x) = \frac{4}{3}(x-4a)(x-4b)$. Then $P''(x) = \frac{8}{3}(x-2a-2b)$ and hence $P(2x) = \frac{4}{9}(x-4a)(x-4b)(x-2a-2b)$. Thus

$$P(x) = \frac{4}{9}(x-2a)(x-2b)(x-a-b),$$

and

$$P'(x) = \frac{4}{9}(3x^2 - 6(a+b)x + 2a^2 + 2b^2 + 8ab).$$

As $P'(x) = \frac{4}{3}(x-4a)(x-4b) = \frac{4}{9}(3x^2 - 12(a+b)x + 48ab)$, we conclude that $b = -a$ and $-4a^2 = -48a^2$. Therefore $a = b = 0$ and $P(x) = \frac{4}{9}x^3$.

The idea of looking at the roots is more visible in the following problem:

Problem 41. Find all polynomials P satisfying the equation

$$P(x^2 - y^2) = P(x-y)P(x+y).$$

Solution. Suppose w is a root of P. Then $x - y = w$ implies that $x^2 - y^2 = w(x+y) = w(2x-w)$ is also a root of w. Now if $w \neq 0$, $x^2 - y^2$ can take any value, being a non-constant linear function in x, so P is identically zero. Thus P is either identically zero or has all roots zero.

If $P = cx^n$, then $c(x^2-y^2)^n = c(x-y)^n c(x+y)^n = c^2(x^2-y^2)^n$, so $c = 1$. Hence all solutions are $P(x) = 0$ and $P(x) = x^n$ for $n \geq 0$.

To solve the following example, one needs to study its roots more carefully:

Problem 42. Find all nonconstant polynomials P satisfying the equation

$$P(x)P(x+1) = P(x^2 + x + 1).$$

Solution. If P is non-constant, let w be its root of maximal absolute value. Denote $x = w$ to conclude that $w_1 = w^2 + w + 1$ is a root of P and let $x = w - 1$

to conclude that $w_2 = w^2 - w + 1$ is a root of P. Then $|w_1 - w_2| = 2|w|$. But $|w_1 - w_2| \leq |w_1| + |w_2| = 2|w|$. The equality can hold only if $w_1 + w_2 = 0$, so $w^2 + 1 = 0$, hence $w = \pm i$. In this case $(w_1, w_2) = (i, -i)$. Thus $x^2 + 1 \mid P(x)$. However $Q(x) = x^2 + 1$ satisfies $Q(x)Q(x+1) = Q(x^2+x+1)$, therefore $\frac{P}{Q}$ satisfies the same condition. We can repeat the same operation until we reach a constant polynomial, so $P(x) = c(x^2+1)^n$, which clearly satisfies the condition.

Let us now pass to harder functional equations, that exemplify more subtle methods.

Problem 43. Find all polynomials $P \in \mathbb{R}[x]$ such that

$$P(x)P(2x^2 - 1) = P(x^2)P(2x - 1)$$

for all $x \in \mathbb{R}$.

(Romania, 2001)

First solution. It is clear that the constant polynomials are solutions. Suppose now that $\deg P = n \geq 1$. Then $P(2x-1) = 2^n P(x) + R(x)$, where either $R \equiv 0$ or $\deg R = m < n$. Assume that $R \not\equiv 0$. It follows from the given identity that

$$P(x)(2^n P(x^2) + R(x^2)) = P(x^2)(2^n P(x) + R(x)),$$

that is $P(x)R(x^2) = P(x^2)R(x)$ for all $x \in R$. Hence $n + 2m = 2n + m$, i.e. $n = m$, a contradiction. Thus $R \equiv 0$ and $P(2x-1) = 2^n P(x)$. Set $Q(x) = P(x+1)$. Then

$$Q(2x) = 2^n Q(x) \tag{1}$$

for all $x \in \mathbb{R}$. Let

$$Q(x) = \sum_{k=0}^{n} a_k x^{n-k}.$$

Then comparing the coefficients of $x^n - k$ on both sides of (1) gives $a_k 2^{n-k} = 2^n a_k$, that is $a_k = 0$ for $k \geq 1$. Hence $Q(x) = a_0 x^n$ and therefore $P(x) = a_0(x-1)^n$.

Second solution. Suppose that $P \not\equiv 0$ and set

$$P(x) = \sum_{k=0}^{n} a_k x^{n-k},$$

where $n = \deg P$ and $a_0 \neq 0$. Then

$$\sum_{k=0}^{n} a_k x^{n-k} \sum_{k=0}^{n} a_k (2x^2-1)^{n-k} = \sum_{k=0}^{n} a_k x^{2(n-k)} \sum_{k=0}^{n} a_k (2x-1)^{n-k}.$$

Comparing the coefficients of $x^{3n-k}, k \geq 1$, on both sides gives

$$a_k a_0 + R_1(a_0, \ldots, a_{k-1}) = a_0 a_k 2^{n-k} + R_2(a_0, \ldots, a_{k-1}),$$

where R_1 and R_2 are polynomials in $k-1$ variables. Hence a_k is determined uniquely by $a_0, \ldots, a_{k-1}$. This shows that for given a_0 and n there is at most one polynomial satisfying the condition in the problem. On the other hand, it is easy to check that the polynomials $P(x) = a_0(x-1)^n$ are solutions.

Third solution. Suppose that the polynomial $P(x)$ has a complex root $\alpha \neq 1$. Of all these roots take that for which the number $|\alpha - 1| \neq 0$ is the least possible. Let β be a complex number such that $\alpha = 2\beta^2 - 1$. Setting $x = \pm\beta$ in the given equation we see that either $P\left(\frac{\alpha+1}{2}\right) = 0$ or $P(2\beta - 1) = P(-2\beta - 1) = 0$. The inequality $\left|\frac{\alpha+1}{2} - 1\right| < |\alpha - 1|$ shows that $P\left(\frac{\alpha+1}{2}\right) \neq 0$, that is $P(2\beta-1) = P(-2\beta-1) = 0$. Then

$$2|(\beta-1)(\beta+1)| = |\alpha - 1| \leq \min(|(2\beta-1)-1|, |(-2\beta-1)-1|)$$

and $\beta \neq \pm 1$ imply that $\max(|\beta - 1|, |\beta + 1|) \leq 1$, i.e. $\beta = 0$. Hence $\alpha = -1$ and therefore $P(x) = (x+1)^k Q(x)$, where $k \geq 1$ and $Q(-1) \neq 0$. Substituting in the given equation gives

$$(x+1)^k x^k Q(x) Q(2x^2 - 1) = (x^2 + 1) Q(x^2) Q(2x - 1).$$

Setting $x = 0$ in this identity gives $Q(0) = 0$, since $Q(-1) \neq 0$. Thus $P(0) = 0$, which contradicts the choice of $\alpha = -1$, since $|-1-1| > |0-1|$. Hence all the roots of the polynomial P are equal to 1 and therefore $P(x) = a_0(x-1)^n$ for some real constant a_0.

Problem 44. Let $\{P_n\}_{n=1}^{\infty}$ be the sequence of polynomials defined by:

$$P_1(x) = x, P_{n+1}(x) = P_n^2(x) + 1, \quad n \geq 1.$$

Prove that a polynomial P satisfies the identity

$$P(x^2 + 1) = P^2(x) + 1$$

for all $x \in \mathbb{R}$ if and only if P belongs to the above sequence.

Solution. Let P satisfy the given identity. Then $P^2(x) = P^2(-x)$, hence for all x either $P(x) = P(-x)$ or $P(x) = -P(-x)$. It follows that $P(x) \equiv P(-x)$ or $P(x) \equiv -P(-x)$. In the second case we get $P(0) = 0$ and an easy induction shows that $P(n) = n$ for all $n \in \mathbb{N}$. Hence $P(x) = x$ for all $x \in \mathbb{R}$ and this polynomial belongs to the given sequence. In the first case it follows easily that $P(x) = Q(x^2)$, where Q is a polynomial. Then

$$Q((x^2+1)^2) = P(x^2+1) = P^2(x) + 1 = Q^2(x^2) + 1$$

and setting $R(x) = Q(x-1)$ we see that $R(y^2+1) = R^2(y) + 1$ for $y = x^2 + 1$. Hence $R(y^2+1) = R^2(y) + 1$ for all $y \in \mathbb{R}$. Thus

$$P(x) = R(x^2+1) = R^2(x) + 1$$

where $\deg R = \dfrac{\deg R}{2}$ and the polynomial R satisfies the given condition. Conversely, if R is a polynomial satisfying the given identity, then the same is true for the polynomial $P(x) = R(x^2+1)$. Now the statement of the problem follows by induction on the degree of P.

Problem 45. Find all polynomials P with real zeroes only satisfying

$$P(x)P(-x) = P(x^2-1).$$

Solution. If r is a root of P, then by setting $x = r$ we conclude that $g(r) = r^2 - 1$ is also a root of P. Then $g(g(r))$ is also a root of P and so on. As we may have only a finite number of roots, we will encounter a root for a second time, so $g(g(\ldots(s))) = s$ for some s in the sequence.

Now let us find r. We have $g(r) - r = (r-u)(r-v)$, where $u = \frac{-1-\sqrt{5}}{2}$, $v = \frac{-1+\sqrt{5}}{2}$, and

$$g(g(r)) - r = r(r+1)(r-u)(r-v).$$

If $r < -1$, then set $x = \sqrt{1+r}$ to obtain that $\pm\sqrt{1+r}$ is a root of P. But such a root is not real, so this case is not possible.

If $r = -1$, then $g(r) = 0, g(g(r)) = -1$, so $x(x+1) \mid P$.

If $r \in (-1, u)$, then $g(r) \in (u; 0)$ and $g(g(r)) \in (-1, u)$. But

$$g(g(r)) - r = r(r+1)(r-u)(r-v) < 0,$$

so $g(g(r)) < r$. We repeat the reasoning with $g(g(r))$ and so on to obtain and infinite decreasing sequence of roots of P in $(-1, u)$, a contradiction.

If $r = u$, then $u - x \mid P$.

If $r \in (u, 0)$, then $g(r) \in (-1, u)$ and we have shown that no root can occur in $(-1, u)$.

If $r = 0$, then $g(r) = -1$ and $x(x+1) \mid P$.

If $0 < r < v$, then $\pm\sqrt{1+r}$ is a root of P. As P has no roots less than -1, $\sqrt{1+r}$ is a root of P. Also $r < \sqrt{1+r}, \sqrt{1+r} < v$ and we can build an increasing sequence of roots of P in $(0, v)$.

If $r = v$, then $v - x \mid P$.

If $r > v$, then $g(r) > v$ is a root of P, and continuing this operation we get an infinite increasing set of roots of P greater than v.

Therefore all roots can be $-1, 0, u$ or v. As $x(x+1), u - x$ and $v - x$ all satisfy the condition, we can divide P by any of them and repeat the reasoning to get that $P(x) = x^m(x+1)^m(u-x)^q(v-x)^q$. If P is a constant, then $P = 0$ or $P = 1$.

9. Iterations and Recursive Relations

Problem 46. A function $f\colon \mathbb{N} \to \mathbb{R}$ satisfies for some positive integer m the conditions $f(m) = f(1995), f(m+1) = 1996, f(m+2) = 1997$ and $f(n+m) = \frac{f(n)-1}{f(n)+1}$. Prove that $f(n+4m) = f(n)$ and find the least m for which this function exists.

(Nordic Contest, 1999)

Solution. If $h(x) = \frac{x-1}{x+1}$, then $f(n+m) = h(f(n))$, so $f(n+4m) = h_4(f(n))$, where h_k is h iterated k times. We need to check that $h_4(x) = x$.

Indeed $h_2(x) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{-1}{x}$, and therefore $h_4(x) = h_2(h_2(x)) = x$. We have solved the first part of the problem. The least possible value of m is 1.

If $m = 1$, then $f(n+4) = f(n)$, so $f(1997) = f(5) = f(1)$. But we know that $f(1997) = f(3) = h_2(f(1)) = \frac{-1}{f(1)}$. Thus $f(1) = \frac{-1}{f(1)}$, so $f(1)^2 = -1$, a contradiction.

Similarly, $m = 2$ gives $f(1995) = f(3) = f(2), f(1996) = f(4) = f(3), f(1997) = f(5) = f(4)$, thus $f(2) = f(3) = f(4) = f(5)$. Then $h(f(2)) = f(2)$. But the equation $\frac{x-1}{x+1} = x$ gives $x - 1 = x^2 + x$ and again $x^2 = -1$, a contradiction.

Finally, if $m = 3$, then for any value of $f(1), f(2), f(3)$ we can compute f inductively. Because $12 \mid 1992$, we get $f(1995) = f(3) = f(m)$ and so $m = 3$ is the answer.

Problem 47. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$

for all $n \in \mathbb{N}$.

Solution. Set $a_k = f(f(\dots f(n)))$ where f is iterated k times. We get the recursive relation

$$a_{k+3} + a_{k+2} + a_{k+1} = 3a_k,$$

with characteristic equation $x^3 + x^2 + x = 3$. Its roots are equal to 1 and $-1 \pm \sqrt{2}$. It follows that

$$a_k = c_0 + c_1(-1+\sqrt{2})^k + c_2(-1-\sqrt{2})^k, \quad k \geq 0.$$

Because $a_k > 0$ and $|-1-\sqrt{2}| > 1 > |-1+\sqrt{2}|$, we conclude as in the solutions of the previous two problems that $c_2 = 0$. From where we get $c_1 = 0$. Hence $a_1 = a_0$, and $f(n) = n$ for all $n \in \mathbb{N}$.

Problem 48. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$2n + 2000 \le f(f(n)) + f(n) \le 2n + 2002$$

for all $n \in \mathbb{N}$.

(Balkan MO, 2002)

Solution. Fix an n and set

$$a_0 = n,\ \ a_{k+1} = f(a_k),\ \ c_k = a_{k+1} - a_k - 667, \quad k \ge 0.$$

Then

$$2a_k + 2001 \le a_{k+2} + a_{k+1} \le 2a_k + 2002,$$

$$0 \le c_{k+1} + 2c_k \le 1, \quad k \ge 0.$$

We will prove that $c_0 = 0$. Assume the contrary. Then we may assume that $c_0 \ge 1$, since otherwise $c_1 \ge -2c_0 \ge 2$ and we consider the sequence $c_1, c_2, \ldots$ We have

$$c_{2k+2} \ge -2c_{2k+1} \ge 4c_{2k} - 2 \ge 2c_{2k}$$

and it follows by induction that $c_{2k} \ge 2^k$, $k \ge 0$. Hence

$$a_{2k+2} = a_{2k} + c_{2k} + c_{2k+1} + 1334 \le a_{2k} + 1335 - c_{2k} \le$$

$$\le a_{2k} + 1335 - 2^k, \quad k \ge 0.$$

Summing up these inequalities we obtain

$$a_{2k} \le a_0 + 1335k - 2^k, \quad k \ge 0.$$

This inequality shows that $a_{2k} \le 0$ for all sufficiently large k, a contradiction. Thus $c_0 = 0$ and $f(n) = n + 667$ for all n. It is easy to check that this function satisfies the given conditions.

Problem 49. Prove that if the function $f\colon \mathbb{R} \to \mathbb{R}$ is such that $|f(x)| \le 1$ and

$$f(x) + f\left(x + \frac{13}{42}\right) = f\left(x + \frac{1}{6}\right) + f\left(x + \frac{1}{7}\right)$$

for all $x \in \mathbb{R}$, then $f(x)$ is periodic.

(IMO Shortlist, 1997)

Solution. We have

$$f\left(x+\frac{1}{6}+\frac{1}{7}\right)-f\left(x+\frac{1}{7}\right)=f\left(x+\frac{1}{6}\right)-f(x),$$

which implies

$$f\left(x+\frac{k}{6}+\frac{1}{7}\right)-f\left(x+\frac{k-1}{6}+\frac{1}{7}\right)=f\left(x+\frac{k}{6}\right)-f\left(x+\frac{k-1}{6}\right)$$

for $1 \le k \le 6$. Summing up these equalities gives

$$f\left(x+1+\frac{1}{7}\right)-f\left(x+\frac{1}{7}\right)=f(x+1)-f(x).$$

Set $g(x)=f(x+1)-f(x)$. Then $g\left(x+\frac{1}{7}\right)=g(x)$, which implies

$$g(x)=g\left(x+\frac{1}{7}\right)=g\left(x+\frac{2}{7}\right)=\cdots=g(x+1).$$

Hence $g(x)=g(x+n)$ for all $n \in \mathbb{N}$. Then

$$f(x+n)-f(x)=(f(x+n)-f(x+n-1))+\cdots+(f(x+1)-f(x))=$$

$$=g(x+n-1)+\cdots+g(x)=ng(x).$$

This is

$$f(x+n)-f(x)=ng(x)$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Hence

$$n|g(x)|=|f(x+n)-f(x)|\le|f(x+n)|+|f(x)|\le 2,$$

i.e. $n|g(x)|\le 2$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$. This shows that $g(x)=0$ for all $x \in \mathbb{R}$, that is $f(x+1)=f(x)$.

Problem 50. Let $0<a_1<a_2<\ldots<a_k$ be integer numbers, $b_0,b_2,b_3,\ldots,b_k$ be real numbers such that $b_k=\pm 1$ and $b_0+b_1x^{a_1}+\ldots+b_kx^{a_k}$ has all roots of absolute value 1. Let f be a bounded function such that

$$b_0f(x)+b_1f(x+a_1)+\ldots+b_kf(x+a_k)=0.$$

Prove that f is periodic.

Solution. Observe that this is a generalization of the previous problem:

Set $g(x)=f(\frac{x}{42})$ to obtain $g(x+13)+g(x)=g(x+6)+g(x+7)$ and the polynomial $x^{12}-x^6-x^7+1=(x^6-1)(x^7-1)$ has all roots of absolute value 1.

However, the method is hard to generalize, as here we have a very vague and complex relation. The fact that a_i are rational can help us reduce the problem to a polynomial recurrence. Now we employ two lemmas which will clearly help us.

Lemma 1. If $w_1, w_2, \ldots, w_k$ have absolute value 1 and $a_n = w_1^n + w_2^n + \ldots + w_k^n$ is not identically zero, then there is an $\epsilon > 0$ such that $|a_n| > \epsilon$ for infinitely many n.

Proof. Let $w_i = e^{2\pi i a_i}$, where $a_i \in R$. For each n, consider the k-tuple

$$(\{na_1\}, \{na_2\}, \ldots, \{na_k\}).$$

If we divide $[0,1)^k$ into N^k boxes $[\frac{i}{n}; \frac{i+1}{n}) \times [\frac{j}{n}; \frac{j+1}{n}) \times \ldots$, then taking $n > N^k$ we deduce that for some $i, j < n$ the k-tuples $(\{ia_1\}, \{ia_2\}, \ldots, \{ia_k\})$ and $(\{ja_1\}, \{ja_2\}, \ldots, \{ja_k\})$ will fall into the same box. This means that $|\{ia_m\} - \{ja_m\}| < \frac{1}{N}$. Therefore $\langle (i-j)a_m \rangle < \frac{1}{N}$, where $\langle x \rangle = \min(\{x\}, 1 - \{x\})$.

We thus conclude that $|w_m^i - w_m^j| = |1 - w_m^{i-j}| < |1 - e^{\frac{2\pi i}{N}}| = 2\sin\frac{\pi}{N} < \frac{2\pi}{N}$. Hence if we let $r = i - j$, we get $|a_i - a_{i+r}| < \frac{2\pi k}{N}$. Now if we take a_i such that $a_i \neq 0$ we can set $\epsilon = \frac{|a_i|}{2}$.

Taking N_1 such that $\frac{2\pi k}{N_1} < \frac{\epsilon}{2}$ we find r_1 such that $|a_{i+r_1} - a_i| < \frac{\epsilon}{2}$. Thus $|a_{i+r_1}| > (1 + \frac{1}{2})\epsilon$. Analogously we find r_2 such that $|a_{i+r_1+r_2}| > (1 + \frac{1}{4})\epsilon$. Reasoning by induction, we find $r_1, r_2, \ldots, r_l$ such that $|a_{i+r_1+\ldots+r_l}| > (1 + \frac{1}{2^l})\epsilon$ and this proves the claim.

Lemma 2. If $P \in \mathbb{Z}[X]$ is a monic polynomial which has all roots of absolute value 1, then these roots are roots of unity.

Proof. Let $P(X) = (x - w_1)(x - w_2)\ldots(x - w_n)$. Let

$$P_k(X) = (x - w_1^k)(x - w_2^k)\ldots(w - w_n^k).$$

As P_k is symmetric in $w_1, w_2, \ldots, w_n$, its coefficients express as integer polynomials in the symmetric sums of $w_1, w_2, \ldots, w_n$. These sums are integers, as $P \in \mathbb{Z}[X]$, thus $P_k \in \mathbb{Z}[X]$. However,

$$[x^m]P_k(x) = |\sum_{1 \le i_1 < i_2 < \ldots < i_m \le n} w_{i_1}^k w_{i_2}^k \ldots w_{i_m}^k| \le \binom{n}{m},$$

as $|w_i| = 1$. Thus the coefficients of $P_k(X)$ are bounded, hence for some $k < l$ we have $P_k(X) = P_l(X)$. This means that $(w_1^k, w_2^k, \ldots, w_n^k)$ is a permutation of $(w_1^l, w_2^l, \ldots, w_n^l)$. Hence $w_i^k = w_{i_1}^l$. Then $w_{i_1}^k = w_{i_2}^l$, so $w_i^{k^2} = w_{i_2}^{l^2}$. Reasoning inductively we get $w_i^{k^j} = w_{i_j}^{l^j}$. Eventually, we return to i ($i_j = i$) so we get $w_i^{k^j} = w_i^{l^j}$, this is $w_i^{l^j - k^j} = 1$, so w_i is a root of unity.

Now we return to the problem. If we set $c_n = f(x+n)$, then this is a polynomial recurrence with associated polynomial $b_0 + b_1x^{a_1} + \ldots + b_kx^{a_k}$. Then $c_n = \sum_{i=1}^{l} p_i(n)w_i^n$, where w_i are the roots of the equation. Now we claim that p_i are constants.

Indeed, assume the contrary. Then $c_n = (d_0(n)n^m + d_1(n)n^{m-1} + \ldots + d_m(n))$, where $d_0, d_1, \ldots, d_m$ are simple polynomial recurrences in $w_1, w_2, \ldots$. Now if k is the least such that d_k is not identically zero, then applying Lemma 1 we get infinitely many n for which $|d_k(n)| > \epsilon$. Then

$$\frac{c_n}{r^n n^k} = d_k(n) + \frac{d_{k+1}(n)}{n} + \ldots.$$

Also $|d_i(n)|$ is bounded, because w_i have absolute value 1. Now it is clear that for sufficiently large n we have $|\frac{c_n}{n^k} - d_k()| < \frac{\epsilon}{2}$. Thus for infinitely many n we have $\frac{c_n}{n^k} > \frac{\epsilon}{2}$, which contradicts the boundedness of f, unless $k = 0$. Therefore $c_n = d_0(n)$, and this guarantees the claim. Now as w_i are roots of unity, according to Lemma 2 we have N such that $w_i^N = 1$, hence $c_n = c_{n+N}$. Because N does not depend on x, we get $f(x) = f(x+N)$, as desired.

10. Symmetrization and additional variables

We sometimes have a condition in x and y, say $u(x,y) = v(x,y)$, such that one side of it is symmetric in x and y, but the other is not (or we can obtain such a condition by an appropriate substitution). Then swapping x with y we get a new condition, which might prove helpful. For example if $u(x,y) = u(y,x)$, then as $u(x,y) = v(x,y)$ and $u(y,x) = v(y,x)$, we have $v(x,y) = v(y,x)$. In other cases, we might need to add one additional variable to get one side of the equation symmetric. See the examples below.

Problem 51. Find all continuous functions $f, g, h\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+y) + g(xy) = h(x) + h(y).$$

Solution. Set $y = 0$ to get $f(x) = h(x) + h(0) - g(0)$. The condition rewrites as $h(x+y) - h(x) - h(y) = g(xy)$, where we replace g by $g - g(0) - h(0)$ for simplicity. Thus

$$h(x+y+z) = h(x) + h(y+z) + g(xy+xz) = h(x) + h(y) + h(z) + g(yz) + g(xy+xz).$$

Symmetrizing this we conclude that

$$g(yz) + g(xy+xz) = g(xz) + g(xy+yz) = g(xy) + g(xz+yz).$$

As for $a, b, c > 0$ we can find x, y, z with $yz = a, xz = b, xy = c$, we get $g(a) + g(b + c) = g(b) + g(a + c) + g(c) + g(a + b)$ and taking $c \to 0^+$ we get $g(a + b) + g(0) = g(a) + g(b)$. Next if we take $a > 0, b < 0, c < 0$, we can also find x, y, z with $yz = a, xz = b, xy = c$ so $g(a) + g(b + c) = g(b) + g(a + c) + g(c) + g(a + b)$. Taking $c \to 0^-$ we get $g(a) + g(b) = g(0) + g(a + b)$.

Finally, if we take $a < 0, b < 0, c > 0$ and take $c \to 0^+$ we get $g(a)+g(b) = g(a+b)$ in this case, too.

It follows that $g(a + b) + g(0) = g(a) + g(b)$ holds for all non-zero a and b by continuity and then $f(x) = ax + b$ is linear. So $h(x + y) - h(y) - h(z) = axy + b$. If we consider $H(x) = h(x) - \frac{a}{2}x^2 + b$, then we see that $H(x) + H(y) = H(x + y)$, so $H(x) = cx$. Therefore we find a representation $h(x) = ux^2 + vx + w, g(x) = 2ux - w$, and we are done.

Problem 52. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f((x - y)^2) = f(x)^2 - 2xf(y) + y^2.$$

Solution. Symmetrize the condition to get

$$f((x - y)^2) = f(x)^2 - 2xf(y) + y^2 = x^2 - 2f(x)y + f(y)^2$$

and the equality of the last two expressions can be written as

$$(f(x) + y)^2 = (f(y) + x)^2.$$

One can guess that only the functions $f(x) = x + a$, and $f(x) = -x$ satisfy the condition. Indeed, assume that $f(a) \neq -a$. Let $f(a) = b$. Pick another c and let $f(c) = d$. We wish to prove that $d = c + b - a$. Indeed, we have $(a + d)^2 = (b + c)^2$, so either $d = c + b - a$, or $d = -a - b - c$. If it is the latter, pick any x. We have $(f(x) + a)^2 = (x + b)^2$, so either $f(x) = x + b - a$, or $f(x) = -x - b - a$. We also have $(f(x) + c)^2 = (x - a - b - c)^2$, so either $f(x) = x - a - b - 2c$ or $f(x) = a + b - x$.

It follows that the sets $\{x + b - a, -x - a - b\}$ and $\{x - a - b - 2c, a + b - x\}$ must intersect. We can pick such an x that satisfies $x + b - a \neq a + b - x$ and also $-x - a - b \neq x + a - b - 2c$. Then either $x + b - a = x - a - b - 2c$, or $-x - a - b = a + b - x$. Thus either $b + c = 0$ or $a + b = 0$. But $a + b \neq 0$, as $f(a) \neq a$. Hence $b + c = 0$, and in this case $d = -a - b - c = c + b - a$. Therefore $d = c + b - a$, so $f(c) = c + b - a$. As c is arbitrary, we get $f(x) = x + b - a$. This proves our claim, so $f(x) = -x$, or $f(x) = x + a$.

It remains only to check which of them satisfies the condition.

If $f(x) = -x$, then $f(x - y)^2 = -(x - y)^2$, while $f^2(x) - 2xf(y) + y^2 = x^2 + 2xy + y^2 = (x + y)^2$, and the condition is not satisfied.

If $f(x) = x + a$, then $f((x - y)^2) = x^2 - 2xy + y^2 + a$, while $f^2(x) - 2xf(y) + y^2 = (x + a)^2 - 2x(y + a) + y^2 = x^2 - 2xy + y^2 + a^2$. This holds if and only if $a^2 = a$. Thus $f(x) = x$ and $f(x) = x + 1$ are the solutions to the problem.

11. Functional equations without solutions

Problem 53. Do there exist functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x^2$ and $g(f(x)) = x^3$?

Solution. The intuitive answer is no. Indeed, if there were such functions, they might well be monotonic on $\mathbb{R}^-$. But then $f(g(x))$ and $g(f(x))$ would also be monotonic, and would be either both decreasing or increasing, whereas in our case one is increasing and the other decreasing. With this idea in mind, we postpone constructing an example and begin with searching for a prove that such functions do not exist.

Indeed, assume $f(g(x)) = x^2$ and $g(f(x)) = x^3$. Note that as $g(f(x))$ is injective, $f(x)$ should also be injective. Now, let us apply f to the relation $g(f(x)) = x^3$. We get $f(g(f(x))) = f(x^3)$, which can be rewritten as $f(x)^2 = f(x^3)$. In particular, for $x = x^3$ we get $f(x)^2 = f(x)$, thus $f(x) \in \{0, 1\}$. But there are three x for which $x = x^3$, they are $0, 1, -1$. Hence $f(0), f(1), f(-1)$ must belong to the set $\{0, 1\}$. Therefore two of them are equal. But this contradicts the injectivity of f.

Problem 54. Prove that there is no function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f^2(x) \geq f(x+y)(f(x)+y)$$

for all $x, y \in \mathbb{R}^+$.

(Bulgaria, 1998)

Solution. Suppose that there is a function f with this property. Then

$$f(x) - f(x+y) \geq \frac{f(x)y}{f(x)+y}, \tag{1}$$

which shows that f is an increasing function. Given an $x \in \mathbb{R}^+$ we choose an $n \in \mathbb{N}$ such that $nf(x+1) \geq 1$. Then

$$f\left(x + \frac{k}{n}\right) - f\left(x + \frac{k+1}{n}\right) \geq \frac{f(x+\frac{k}{n}).\frac{1}{n}}{f(x+\frac{k}{n}) + \frac{1}{n}} > \frac{1}{2n},$$

for all $k \in \mathbb{N}$. Note that $nf\left(x + \dfrac{k}{n}\right) > nf(x+1) > 1$. Summing up these inequalities for $k = 0, 1, \ldots, n-1$ we get

$$f(x) - f(x+1) > \frac{1}{2}.$$

Now take a positive integer m such that $m \geq 2f(x)$. Then

$$f(x) - f(x+m) = (f(x) - f(x+1)) + \cdots + (f(x+m-1) - f(x+m)) > \frac{m}{2} \geq f(x).$$

Hence $f(x+m) < 0$, a contradiction.

Problem 55. Prove that there is no function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(0) > 0$ and

$$f(x+y) \geq f(x) + yf(f(x)) \tag{1}$$

for all $x, y \in \mathbb{R}$.

Solution. Suppose there is such a function f. If $f(f(x)) \leq 0$ for all $x \in \mathbb{R}$, then

$$f(x+y) \geq f(x) + yf(f(x)) \geq f(x)$$

for all $y \leq 0$ and the function f is decreasing. The inequalities $f(0) > 0 \geq f(f(x))$ imply $f(x) > 0$ for all x, which contradicts $f(f(x)) \leq 0$. Hence there exists z such that $f(f(z)) > 0$. Then the inequality

$$f(z+x) \geq f(z) + xf(f(z))$$

shows that $\lim\limits_{x\to+\infty} f(x) = +\infty$ and therefore $\lim\limits_{x\to\infty} f(f(x)) = +\infty$.

In particular, there exist $x, y > 0$ such that

$$f(x) \geq f(f(x)) > 1, \ \ y \geq \frac{x+1}{f(f(x)) - 1}, \ \ f(f(x+y+1)) \geq 0.$$

Then

$$f(x+y) \geq f(x) + yf(f(x)) \geq x+y+1,$$

and therefore

$$f(f(x+y)) \geq f(x+y+1) + (f(x+y) - (x+y+1))f(f(x+y+1)) \geq$$

$$\geq f(x+y+1) \geq f(x+y)+f(f(x+y)) \geq f(x)+yf(f(x))+f(f(x+y) > f(f(x+y)),$$

a contradiction.

Remark. Note that the only function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ satisfying inequality (1) is the constant 0. Indeed, as in the second part of the solution above we conclude that $f(f(x)) \leq 0$ for all $x \in \mathbb{R}$. On the other hand, setting $x = 0$ in (1) gives $f(y) \geq 0$ for all x. Hence $f(x+y) \geq f(x)$ for all $x, y \in \mathbb{R}$ which easily implies $f(x) = 0$ for all x.

Easy Problems

E1. Prove that the functions $f\colon \mathbb{R}\setminus\{1\}\to\mathbb{R}$ such that

$$f(x) = f\left(\frac{x}{x-1}\right)$$

for all $x\neq 1$ are exactly those that can be written as

$$f(x) = g(x) + g\left(\frac{x}{x-1}\right)$$

for some function $g\colon \mathbb{R}\setminus\{1\}\to\mathbb{R}$.

E2. Is there a function $f\colon \mathbb{R}\setminus\{0\}\to\mathbb{R}\setminus\{0\}$ such that

$$f(f(x)) = \frac{1}{x}$$

for all $x\neq 0$?

E3. Find all continuous functions $f\colon \mathbb{R}\to\mathbb{R}$ such that

$$f(\sqrt{x^2+y^2}) = f(x)f(y)$$

for all x and y in $\mathbb{R}$.

E4. Find a function $f\colon \mathbb{R}^+\to\mathbb{R}^+$ such that

$$f^2(x) = 1 + xf(x+1)$$

for all $x\in\mathbb{R}$.

E5. Find all continuous functions $f\colon \mathbb{R}\to\mathbb{R}$ satisfying

$$f(x+y) - f(x-y) = 4xy$$

for all x and y in $\mathbb{R}$.

E6. Find all non-decreasing functions $f\colon \mathbb{Z}\to\mathbb{Z}$ satisfying

$$f(k) + f(k+1) + \ldots + f(k+n-1) = k,$$

for each $k\in\mathbb{Z}$, and a fixed n.

E7. Find all functions $f\colon \mathbb{R}\backslash\{0,1\} \to \mathbb{R}$ satisfying

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}$$

for all x in the domain of f.

E8. Find all functions $f\colon \mathbb{N} \to \mathbb{R}$ such that $f(1) = 1$ and

$$\sum_{d|n} f(d) = 0$$

whenever $n \geq 2$.

E9. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ satisfying $f(0) = 0$ and

$$f(n) = 1 + f(\lfloor \frac{n}{k} \rfloor)$$

for all $n \in N$.

E10. Find all functions $f\colon \mathbb{N} \to \mathbb{R}$ such that $f(1) = 1$ and

$$f(2n+1) = f(2n) + 1 = 3f(n) + 1$$

for all $n \in \mathbb{N}$.

E11. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$

for all $n \in \mathbb{N}$.

E12. Let $g\colon \mathbb{C} \to \mathbb{C}$ be a given function, $a \in \mathbb{C}$, and w the primitive cubic root of unity. Find all functions $f\colon \mathbb{C} \to \mathbb{C}$ such that

$$f(z) + f(wz + a) = g(z)$$

for all z in $\mathbb{C}$.

E13. Find all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that

$$f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{2}$$

for all non-zero x, y in $\mathbb{Q}$.

E14. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$xf(x) - yf(y) = (x-y)f(x+y)$$

for all $x, y \in \mathbb{R}$.

E15. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xf(z) + y) = zf(x) + y$$

for all $x, y, z \in \mathbb{R}$.

E16. Let n be an integer greater than 2. Find all continuous functions $f\colon [0,1] \to \mathbb{R}$ for which $f(x_1) + f(x_2) + \ldots + f(x_n) = 1$ whenever $x_1, x_2, \ldots, x_n \in [0,1]$ and $x_1 + x_2 + \ldots + x_n = 1$.

E17. Let $n \in \mathbb{N}$. Find all polynomials $P \in \mathbb{R}[X]$ such that

$$P(P(x)) = P(x^n).$$

E18. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all $x, y \in \mathbb{R}$.

E19. Find all functions $f\colon \mathbb{N}^2 \to \mathbb{N}$ satisfying

a) $f(n,n) = n$;

b)$f(m,n) = f(n,m)$;

c)$\frac{f(m,n+m)}{f(m,n)} = \frac{n+m}{n}$.

E20. Let p be a prime number. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

a) if $p \mid m - n$, then $f(m) = f(n)$.

b) $f(mn) = f(m)f(n)$ for all integers m and n.

Medium Problems

M1. Prove that there is a unique function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(f(x)) = 6x - f(x)$$

for all x in $\mathbb{R}^+$

M2. Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy the equations

$$f(x^2) = (f(x))^2$$

and $f(x+1) = f(x) + 1$. Prove that $f(x) = x$ for all x in $\mathbb{R}$.

M3. Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xy) = f(x+y)$$

for all $x, y \in \mathbb{R}$. Prove that f is constant.

M4. Prove that an additive function $f\colon \mathbb{R} \to \mathbb{R}$ which is bounded below on an interval must be of the form $f(x) = cx$.

M5. Let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy

$$|f(x) - f(y)| \le |x-y|^2$$

for all x, $y \in \mathbb{R}$ Prove that f is constant.

M6. Prove that there is no function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$\frac{f(x)+f(y)}{2} \ge f\left(\frac{x+y}{2}\right) + |x-y|$$

for all $x, y \in \mathbb{R}$.

M7. Find all polynomials $P \in \mathbb{R}[X]$ such that

$$P(x-y) + P(y-z) + P(z-x) = 2P(x+y+z)$$

whenever $xy + yz + zx = 0$.

M8. Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) = n^2$$

for all n in $\mathbb{N}$?

M9. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(f(k+1)+3) = k$$

for all k in $\mathbb{Z}$.

M10. Find all surjective functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $m \mid n$ if and only if $f(m) \mid f(n)$ for all $m, n \in \mathbb{N}$.

M11. Let f be an increasing function on $\mathbb{N}$ such that $f(f(n)) = 3n$. Find $f(2007)$.

M12. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ that satisfy $f(1) = 1, f(3) = 3$, and

$$f(2n) = f(n)$$

$$f(4n+1) = 2f(2n+1) - f(n),$$
$$f(4n+3) = 3f(2n+1) - 2f(n)$$

for all $n \in \mathbb{N}$.

M13. Find all functions $f : \mathbb{Z} \to \mathbb{N}_0$ for which

$$6f(k+3) - 3f(k+2) - 2f(k+1) - f(k) = 0$$

for all k in $\mathbb{Z}$.

M14. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) + f(n+1) = n + 2$$

for all $n \in \mathbb{N}$.

M15. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ for which

$$\frac{1}{2}f(xy) + \frac{1}{2}f(xz) - f(x)f(yz) \geq \frac{1}{4}$$

for all $x, y, z \in \mathbb{R}$.

M16. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) = \frac{f(x) + f(y) + 2f(x)f(y)}{1 - f(x)f(y)}$$

for all x, y, z in $\mathbb{R}$.

M17. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(f(x) + yz) = x + f(y)f(z)$$

for all x, y, z in $\mathbb{R}$.

M18. Suppose that f is a rational function in x that satisfies

$$f(x) = f\left(\frac{1}{x}\right)$$

for all $x \neq 0$. Prove that f is a rational function in $x + \frac{1}{x}$.

M19. Find all polynomials P and Q with real coefficients such that for infinitely many $x \in \mathbb{R}$

$$\frac{P(x)}{Q(x)} - \frac{P(x+1)}{Q(x+1)} = \frac{1}{x(x+2)}$$

M20. Find all differentiable functions $f \colon \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x+y) - f(x-y) = y(f'(x+y) + f'(x-y))$$

for all x and y in $\mathbb{R}$.

Hard Problems

H1. Let $a \in \mathbb{R} \setminus \{0\}$ and let $f\colon \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}$$

for all x in $\mathbb{R}$. Prove that f is periodic. Find all such continuous functions f.

H2. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that for each $x \in (0,1)$ there exist $y, z \in (0,1), y \neq z$, for which $x = \frac{y+z}{2}$ and

$$f(x) = \frac{f(y) + f(z)}{2}.$$

Prove that f is linear.

H3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous and injective function such that $f(1) = 1$ and

$$f(2x - f(x)) = x$$

for all real x in $\mathbb{R}$. Prove that $f(x) = x$ for all x.

H4. For which a, b, c, p, q, r is there a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(ax + by + c) = pf(x) + qf(y) + r,$$

and what is the general form of the solution?

H5. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xf(x) + f(y)) = f(x)^2 + y$$

for all $x, y \in \mathbb{R}$.

H6. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f(y))(f(u) + f(v)) = f(xu - yv) + f(xv + yu)$$

for all $x, y, u, v \in \mathbb{R}$.

H7. Is there a function $s\colon \mathbb{Q} \to \{-1, 1\}$ such that if x and y are distinct rational numbers satisfying $xy = 1$ or $x + y \in \{0, 1\}$, then

$$s(x)s(y) = -1.$$

H8. Find all functions $g\colon \mathbb{R} \to \mathbb{R}$ for which there is a strictly monotonic function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+y) = f(x)g(y) + f(y).$$

H9. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function satisfying

$$f(f(n)) = 4n - 3$$

and

$$f(2^n) = 2^{n+1} - 1,$$

for all n in $\mathbb{N}$. Find $f(1993)$. Can you find explicitly the value of $f(2007)$? What values can $f(1997)$ take?

H10. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(a^3 + b^3 + c^3) = f(a)^3 + f(b)^3 + f(c)^3$$

whenever $a, b, c \in \mathbb{Z}$.

H11. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that

$$f(n+1) > f(f(n))$$

for all $n \in \mathbb{N}$. Prove that $f(n) = n$ for all $n \in \mathbb{N}$.

H12. Determine all functions $f\colon \mathbb{N} \to \mathbb{N}$ satisfying $f(1) = 1, f(n+1) = f(n) + 2$ if $f(f(n) - n + 1) = n$, and $f(n+1) = f(n) + 1$ otherwise.

H13. Find all functions $f, g\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x + g(y)) = xf(y) - yf(x) + g(x)$$

for all x and y in $\mathbb{R}$.

H14. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y + f(y)) = 2y + f(x)^2$$

for all x, y in $\mathbb{R}$.

H15. Find all continuous functions $f\colon \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$f(x + \frac{1}{x}) + f(y + \frac{1}{y}) = f(x + \frac{1}{y}) + f(y + \frac{1}{x})$$

for all $x, y \in \mathbb{R}^+$.

H16. Suppose $P \in \mathbb{Z}[X]$ is a polynomial such that for each positive integer n the equation $P(x) = 2^n$ has at least one integer root. Prove that P is linear.

H17. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2$$

for all $x, y \in \mathbb{R}$.

H18. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y)+f(y+z)+f(z+x)=f(x+y+z)+f(x)+f(y)+f(z).$$

for all x, y, z in $\mathbb{R}$.

H19. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y)f(x-y)=f(x)^2 f(y)^2,$$

for all $x, y \in \mathbb{R}$.

H20. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, solutions to the equation

$$f(x+y)+f(xy)=f(x)+f(y)+f(xy+1).$$