Notation:

$\mathbb{Q}$ denotes the field of rational numbers,

$\mathbb{Z}$ denotes the ring of ordinary integers,

$\mathbb{R}$ denotes the field of real numbers,

$\mathbb{C}$ denotes the field of complex numbers,

$\mathbb{F}_q$ denotes the finite field with q elements.

If R is any ring then $\mathrm{Mat}_n(R)$ denotes the ring of $n \times n$ matrices with coefficients in R.

If R is any ring then $GL_n(R)$ denotes the group of invertible $n \times n$ matrices in $\mathrm{Mat}_n(R)$.

If A is any ring then $A[t]$ denotes the ring of polynomials with coefficients in A.

Fall 2002

Ph.D. Qualifying Examination

Algebra

Part I

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. (a) Let F be a field, V a finite-dimensional vector space over F and $T : V \to V$ a linear transformation. Suppose that all roots of the characteristic polynomial of T are in F. Show that with respect to some basis of V the matrix of T is upper triangular.

(b) Suppose that V is a four dimensional vector space over the field $\mathbb{R}$ of real numbers and $T : V \to V$ a linear transformation. Show that with respect to some basis of V the matrix of T has the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}.$$

2. Classify those finite groups of order $351 = 3^3 \cdot 13$ that have an abelian 3-Sylow subgroup containing no elements of order 9.

3. Let K be the splitting field of the polynomial $x^6 - 3 = 0$ over $\mathbb{Q}$. Compute $\mathrm{Gal}(K/\mathbb{Q})$.

4. (*Chinese remainder theorem.*) Let A be a commutative ring with unit and let I, J be ideals of A such that $A = I + J$. Prove that $IJ = I \cap J$ and that there is a ring isomorphism

$$A/IJ \cong (A/I) \times (A/J).$$

5. A nonabelian group G of order 36 has generators x, y and z subject to the relations:

$$x^3 = y^3 = 1, \qquad xy = yx, \qquad z^4 = 1, \qquad zxz^{-1} = y, \qquad zyz^{-1} = x^2.$$

Find the conjugacy classes of G and compute its character table.

Fall 2002

Ph.D. Qualifying Examination

Algebra, Part II

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Let p be a prime.

(a) Consider the action of $GL(4, \mathbb{F}_p)$ on the set of two-dimensional vector subspaces of $\mathbb{F}_p^4$. Let $U = \{(0, 0, x, y) \mid x, y \in \mathbb{F}_p\}$. Describe the subgroup of $g \in GL(4, \mathbb{F}_p)$ such that $gU = U$ and compute its order.

(b) Compute the number of two dimensional vector subspaces of $\mathbb{F}_p^4$.

2. Let p and q be primes with $q \geq p$. Prove that there exists a nonabelian group of order pq^2 if and only if p divides one of $q - 1$, q or $q + 1$.

3. If A is a commutative ring with unit and $I \subset A$ is a proper ideal, prove that there exists a prime ideal $P \subset A$ which is "minimal over I." This means that $I \subseteq P$ and if Q is prime with $I \subseteq Q \subseteq P$ then $Q = P$. [**Hint:** Zorn's Lemma.]

4. Let p be a prime and let E/F be a cyclic Galois extension of degree p. Let σ be a generator of $\mathrm{Gal}(E/F)$.

(a) Suppose the characterisic of E and F is p. Show that there exists $\alpha \in E$ such that $\alpha \notin F$ but $\sigma(\alpha) - \alpha \in F$.

(b) Show that if the characteristic of E and F is *not* p then $\sigma(\alpha) - \alpha \in F$ if and only if $\alpha \in F$.

Hint for both parts: It may help to think of E as a vector space over F, and σ as a linear transformation.

5. Let R be a commutative ring containing $\mathbb{C}$, and let M be a simple R-module. (Recall that this means that M has no submodules except $\{0\}$ and M itself.) Suppose that $\dim_{\mathbb{C}}(M) < \infty$.

(a) Prove that if $r \in R$ there exists $\alpha \in \mathbb{C}$ such that $rm = \alpha m$ for all $m \in M$. (**Remark:** If you use some version of Schur's Lemma, you must prove it.)

(b) Prove that $\dim_{\mathbb{C}}(M) = 1$.