Spring 2001

Ph.D. Qualifying Examination

Algebra

Part I

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. If $p < q < r$ are primes and G is a group of order pqr, show that G contains a normal subgroup of order r. [Hint: First show that G contains some normal Sylow subgroup.]

2. (a) If $I \subset A$ is an ideal in a commutative Noetherian ring and if $ab \in I$ for some a, b with $a \notin I$ and $b^n \notin I$ for all n, show that $I = (I, b^m) \cap (I, a)$ for some m. [Hint: first show $xb^{m+1} \in I$ implies that $xb^m \in I$ for some m.

(b) Let A be a commutative ring and E be a finitely generated A-module. If $\{e_1, \cdots, e_r\} \subset E$ is a finite subset whose images span E/mE as an A/m vector space for all maximal ideals $m \subset A$, show that $\{e_1, \cdots, e_r\}$ generate E as an A-module.

3. (a) How many similarity classes of 10×10 matrices over $\mathbb{Q}$ are there with minimal polynomial $(x+1)^2(x^4+1)$?

(b) Give an example of a 10×10 matrix over $\mathbb{R}$ with minimal polynomial $(x+1)^2(x^4+1)$ which is not similar to a matrix with rational coefficients.

4. Let G be a finite group and H be a subgroup of index k. Let (π, V) be an irreducible complex representation of G, and let U be a nonzero H-invariant subspace. Prove that the dimension of U is at least $\frac{1}{k}\dim(V)$. If its dimension is exactly $\frac{1}{k}\dim(V)$, prove U is irreducible over H and that there is no other H-invariant subspace of V isomorphic to U as an H-module.

5. (a) Find $[E : \mathbb{Q}]$ where E is the splitting field of $x^6 - 4x^3 + 1$ over $\mathbb{Q}$.

(b) Show that $\mathrm{Gal}(E/\mathbb{Q})$ is nonabelian and contains an element of order 6.

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Part II

General Directions: Work all problems in separate bluebooks. Give reasons for your assertions and state precisely any theorems that you quote.

1. Determine the number of isomorphism classes of groups of order $273 = 3 \cdot 7 \cdot 13$.

2. Suppose that E/F is an algebraic extension of fields of characteristic zero. Suppose that every polynomial in $F[x]$ has at least one root in E.

(a) Show that E/F is normal.

(b) Show that E is algebraically closed.

3. Suppose that E is the degree three field extension of the rational function field $\mathbb{Q}(x)$ defined by $E = \mathbb{Q}(x)[Y]/(Y^3 + x^2 - 1)$. Let y be the image of Y in E and let $B \subset E$ denote the integral closure of $A = \mathbb{Q}[x]$ in E. It is known—and you may assume—that B is the ring $\mathbb{Q}[x, y]$ generated by x and y over $\mathbb{Q}$. For each of the prime ideals P of A below, describe the factorization of the ideal PB of B.

(i) $P = (x)$.

(ii) $P = (x - 1)$.

(iii) $P = (x^2 + 3)$.

4. Suppose k is a field and V is a module over the polynomial ring $k[T]$ which is finite dimensional as a vector space over k. Define a $k[T]$ module structure on the dual vector space V^* by $(T\alpha)v = \alpha(Tv)$, $\alpha \in V^*$, $v \in V$. Show that $V \cong V^*$ as $k[T]$ modules.

5. Let G be the nonabelian group of order 39 with generators and relations

$$\left\langle x, y | x^3 = y^{13} = 1, xyx^{-1} = y^3 \right\rangle .$$

Find its conjugacy classes and compute its character table.