Solutions to the 66th William Lowell Putnam Mathematical Competition
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A1 We proceed by induction, with base case $1 = 2^0 3^0$. Suppose all integers less than $n-1$ can be represented. If n is even, then we can take a representation of $n/2$ and multiply each term by 2 to obtain a representation of n. If n is odd, put $m = \lfloor \log_3 n \rfloor$, so that $3^m \leq n < 3^{m+1}$. If $3^m = n$, we are done. Otherwise, choose a representation $(n-3^m)/2 = s_1 + \cdots + s_k$ in the desired form. Then

$$n = 3^m + 2s_1 + \cdots + 2s_k,$$

and clearly none of the $2s_i$ divide each other or 3^m. Moreover, since $2s_i \leq n - 3^m < 3^{m+1} - 3^m$, we have $s_i < 3^m$, so 3^m cannot divide $2s_i$ either. Thus n has a representation of the desired form in all cases, completing the induction.

Remarks: This problem is originally due to Paul Erdős. Note that the representations need not be unique: for instance,

$$11 = 2 + 9 = 3 + 8.$$

A2 We will assume $n \geq 2$ hereafter, since the answer is 0 for $n = 1$.

First solution: We show that the set of rook tours from $(1,1)$ to $(n,1)$ is in bijection with the set of subsets of $\{1, 2, ..., n\}$ that include n and contain an even number of elements in total. Since the latter set evidently contains 2^{n-2} elements, so does the former.

We now construct the bijection. Given a rook tour P from $(1,1)$ to $(n,1)$, let $S = S(P)$ denote the set of all $i \in \{1, 2, \dots, n\}$ for which there is either a directed edge from $(i,1)$ to $(i,2)$ or from $(i,3)$ to $(i,2)$. It is clear that this set S includes n and must contain an even number of elements. Conversely, given a subset $S = \{a_1, a_2, \dots, a_{2r} = n\} \subset \{1, 2, \dots, n\}$ of this type with $a_1 < a_2 < \cdots < a_{2r}$, we notice that there is a unique path P containing $(a_i, 2 + (-1)^i), (a_1, 2)$ for $i = 1, 2, \dots, 2r$. This establishes the desired bijection.

Second solution: Let A_n denote the set of rook tours beginning at $(1,1)$ and ending at $(n,1)$, and let B_n denote the set of rook tours beginning at $(1,1)$ and ending at $(n,3)$.

For $n \geq 2$, we construct a bijection between A_n and $A_{n-1} \cup B_{n-1}$. Any path P in A_n contains either the line segment P_1 between $(n-1,1)$ and $(n,1)$, or the line segment P_2 between $(n,2)$ and $(n,1)$. In the former case, P must also contain the subpath P_1' which joins $(n-1,3)$, $(n,3)$, $(n,2)$, and $(n-1,2)$ consecutively; then deleting P_1 and P_1' from P and adding the line segment joining $(n-1,3)$ to $(n-1,2)$ results in a path in A_{n-1}. (This construction is reversible, lengthening any path in A_{n-1} to a path in A_n.) In the latter case, P contains the subpath P_2' which joins $(n-1,3)$, $(n,3)$, $(n,2)$, $(n,1)$ consecutively; deleting P_2' results in a path in B_{n-1}, and this construction is also reversible. The desired bijection follows.

Similarly, there is a bijection between B_n and $A_{n-1} \cup B_{n-1}$ for $n \geq 2$. It follows by induction that for $n \geq 2$, $|A_n| = |B_n| = 2^{n-2}(|A_1| + |B_1|)$. But $|A_1| = 0$ and $|B_1| = 1$, and hence the desired answer is $|A_n| = 2^{n-2}$.

Remarks: Other bijective arguments are possible: for instance, Noam Elkies points out that each element of $A_n \cup B_n$ contains a different one of the possible sets of segments of the form $(i,2), (i+1,2)$ for $i = 1, \dots, n-1$. Richard Stanley provides the reference: K.L. Collins and L.B. Krompart, The number of Hamiltonian paths in a rectangular grid, *Discrete Math.* **169** (1997), 29–38. This problem is Theorem 1 of that paper; the cases of $4 \times n$ and $5 \times n$ grids are also treated. The paper can also be found online at the URL `kcollins.web.wesleyan.edu/vita.htm`.

A3 Note that it is implicit in the problem that p is nonconstant, one may take any branch of the square root, and that $z = 0$ should be ignored.

First solution: Write $p(z) = c \prod_{j=1}^n (z - r_j)$, so that

$$\frac{g'(z)}{g(z)} = \frac{1}{2z} \sum_{j=1}^n \frac{z + r_j}{z - r_j}.$$

Now if $z \neq r_j$ for all j,then

$$\frac{z + r_j}{z - r_j} = \frac{(z + r_j)(\overline{z} - \overline{r_j})}{|z - r_j|^2} = \frac{|z|^2 - 1 + 2\mathrm{Im}(\overline{z} r_j)}{|z - r_j|^2},$$

and so

$$\mathrm{Re}\, \frac{z g'(z)}{g(z)} = \frac{|z|^2 - 1}{2} \left(\sum_j \frac{1}{|z - r_j|^2} \right).$$

Since the quantity in parentheses is positive, $g'(z)/g(z)$ can be 0 only if $|z| = 1$. If on the other hand $z = r_j$ for some j, then $|z| = 1$ anyway.

Second solution: Write $p(z) = c \prod_{j=1}^n (z - r_j)$, so that

$$\frac{g'(z)}{g(z)} = \sum_{j=1}^n \left(\frac{1}{z - r_j} - \frac{1}{2z} \right).$$

We first check that $g'(z) \neq 0$ whenever z is real and $z > 1$. In this case, for $r_j = e^{i\theta_j}$, we have $z - r_j = (z - \cos(\theta_j)) + \sin(\theta_j)i$, so the real part of $\frac{1}{z-r_j} - \frac{1}{2z}$ is

$$\begin{aligned} &\frac{z - \cos(\theta_j)}{z^2 - 2z\cos(\theta_j) + 1} - \frac{1}{2z} \\ &= \frac{z^2 - 1}{2z(z^2 - 2z\cos(\theta_j) + 1)} > 0. \end{aligned}$$

Hence $g'(z)/g(z)$ has positive real part, so $g'(z)/g(z)$ and hence $g(z)$ are nonzero.

Applying the same argument after replacing $p(z)$ by $p(e^{i\theta}z)$, we deduce that g' cannot have any roots outside the unit circle. Applying the same argument after replacing $p(z)$ by $z^n p(1/z)$, we also deduce that g' cannot have any roots inside the unit circle. Hence all roots of g' have absolute value 1, as desired.

Third solution: Write $p(z) = c\prod_{j=1}^n (z - r_j)$ and put $r_j = e^{2i\theta_j}$. Note that $g(e^{2i\theta})$ is equal to a nonzero constant times

$$\begin{aligned} h(\theta) &= \prod_{j=1}^n \frac{e^{i(\theta+\theta_j)} - e^{-i(\theta+\theta_j)}}{2i} \\ &= \prod_{j=1}^n \sin(\theta + \theta_j). \end{aligned}$$

Since h has at least $2n$ roots (counting multiplicity) in the interval $[0, 2\pi)$, h' does also by repeated application of Rolle's theorem. Since $g'(e^{2i\theta}) = 2ie^{2i\theta}h'(\theta)$, $g'(z^2)$ has at least $2n$ roots on the unit circle. Since $g'(z^2)$ is equal to z^{-n-1} times a polynomial of degree $2n$, $g'(z^2)$ has all roots on the unit circle, as then does $g'(z)$.

Remarks: The second solution imitates the proof of the Gauss-Lucas theorem: the roots of the derivative of a complex polynomial lie in the convex hull of the roots of the original polynomial. The second solution is close to problem B3 from the 2000 Putnam. A hybrid between the first and third solutions is to check that on the unit circle, $\mathrm{Re}(zg'(z)/g(z)) = 0$ while between any two roots of p, $\mathrm{Im}(zg'(z)/g(z))$ runs from $+\infty$ to $-\infty$ and so must have a zero crossing. (This only works when p has distinct roots, but the general case follows by the continuity of the roots of a polynomial as functions of the coefficients.) One can also construct a solution using Rouché's theorem.

A4 **First solution:** Choose a set of a rows $r_1, \dots, r_a$ containing an $a \times b$ submatrix whose entries are all 1. Then for $i, j \in \{1, \dots, a\}$, we have $r_i \cdot r_j = n$ if $i = j$ and 0 otherwise. Hence

$$\sum_{i,j=1}^a r_i \cdot r_j = an.$$

On the other hand, the term on the left is the dot product of $r_1 + \cdots + r_a$ with itself, i.e., its squared length. Since this vector has a in each of its first b coordinates, the dot product is at least $a^2 b$. Hence $an \geq a^2 b$, whence $n \geq ab$ as desired.

Second solution: (by Richard Stanley) Suppose without loss of generality that the $a \times b$ submatrix occupies the first a rows and the first b columns. Let M be the submatrix occupying the first a rows and the last $n - b$ columns. Then the hypothesis implies that the matrix MM^T has $n - b$'s on the main diagonal and $-b$'s elsewhere. Hence the column vector v of length a consisting of all 1's satisfies $MM^T v = (n - ab)v$, so $n - ab$ is an eigenvalue of MM^T. But MM^T is semidefinite, so its eigenvalues are all nonnegative real numbers. Hence $n - ab \geq 0$.

Remarks: A matrix as in the problem is called a *Hadamard matrix*, because it meets the equality condition of Hadamard's inequality: any $n \times n$ matrix with ± 1 entries has absolute determinant at most $n^{n/2}$, with equality if and only if the rows are mutually orthogonal (from the interpretation of the determinant as the volume of a paralellepiped whose edges are parallel to the row vectors). Note that this implies that the columns are also mutually orthogonal. A generalization of this problem, with a similar proof, is known as *Lindsey's lemma*: the sum of the entries in any $a \times b$ submatrix of a Hadamard matrix is at most $\sqrt{abn}$. Stanley notes that Ryser (1981) asked for the smallest size of a Hadamard matrix containing an $r \times s$ submatrix of all 1's, and refers to the URL `www3.interscience.wiley.com/cgi-bin/abstract/110550861/ABSTRACT` for more information.

A5 **First solution:** We make the substitution $x = \tan\theta$, rewriting the desired integral as

$$\int_0^{\pi/4} \log(\tan(\theta) + 1)\, d\theta.$$

Write

$$\begin{aligned} &\log(\tan(\theta) + 1) \\ &\quad = \log(\sin(\theta) + \cos(\theta)) - \log(\cos(\theta)) \end{aligned}$$

and then note that $\sin(\theta) + \cos(\theta) = \sqrt{2}\cos(\pi/4 - \theta)$. We may thus rewrite the integrand as

$$\frac{1}{2}\log(2) + \log(\cos(\pi/4 - \theta)) - \log(\cos(\theta)).$$

But over the interval $[0, \pi/4]$, the integrals of $\log(\cos(\theta))$ and $\log(\cos(\pi/4 - \theta))$ are equal, so their contributions cancel out. The desired integral is then just the integral of $\frac{1}{2}\log(2)$ over the interval $[0, \pi/4]$, which is $\pi\log(2)/8$.

Second solution: (by Roger Nelsen) Let I denote the desired integral. We make the substitution $x = (1 -$

$u)/(1+u)$ to obtain

$$\begin{aligned} I &= \int_0^1 \frac{(1+u)^2 \log(2/(1+u))}{2(1+u^2)} \frac{2\,du}{(1+u)^2} \\ &= \int_0^1 \frac{\log(2) - \log(1+u)}{1+u^2}\,du \\ &= \log(2) \int_0^1 \frac{du}{1+u^2} - I, \end{aligned}$$

yielding

$$I = \frac{1}{2}\log(2) \int_0^1 \frac{du}{1+u^2} = \frac{\pi \log(2)}{8}.$$

Third solution: (attributed to Steven Sivek) Define the function

$$f(t) = \int_0^1 \frac{\log(xt+1)}{x^2+1}\,dx$$

so that $f(0) = 0$ and the desired integral is $f(1)$. Then by differentiation under the integral,

$$f'(t) = \int_0^1 \frac{x}{(xt+1)(x^2+1)}\,dx.$$

By partial fractions, we obtain

$$\begin{aligned} f'(t) &= \left. \frac{2t\arctan(x) - 2\log(tx+1) + \log(x^2+1)}{2(t^2+1)} \right|_{x=0}^{x=1} \\ &= \frac{\pi t + 2\log(2) - 4\log(t+1)}{4(t^2+1)}, \end{aligned}$$

whence

$$f(t) = \frac{\log(2)\arctan(t)}{2} + \frac{\pi \log(t^2+1)}{8} - \int_0^t \frac{\log(t+1)}{t^2+1}\,dt$$

and hence

$$f(1) = \frac{\pi \log(2)}{4} - \int_0^1 \frac{\log(t+1)}{t^2+1}\,dt.$$

But the integral on the right is again the desired integral $f(1)$, so we may move it to the left to obtain

$$2f(1) = \frac{\pi \log(2)}{4}$$

and hence $f(1) = \pi \log(2)/8$ as desired.

Fourth solution: (by David Rusin) We have

$$\int_0^1 \frac{\log(x+1)}{x^2+1}\,dx = \int_0^1 \left(\sum_{n=1}^\infty \frac{(-1)^{n-1}x^n}{n(x^2+1)} \right)\,dx.$$

We next justify moving the sum through the integral sign. Note that

$$\sum_{n=1}^\infty \int_0^1 \frac{(-1)^{n-1}x^n\,dx}{n(x^2+1)}$$

is an alternating series whose terms strictly decrease to zero, so it converges. Moreover, its partial sums alternately bound the previous integral above and below, so the sum of the series coincides with the integral.

Put

$$J_n = \int_0^1 \frac{x^n\,dx}{x^2+1};$$

then $J_0 = \arctan(1) = \frac{\pi}{4}$ and $J_1 = \frac{1}{2}\log(2)$. Moreover,

$$J_n + J_{n+2} = \int_0^1 x^n\,dx = \frac{1}{n+1}.$$

Write

$$\begin{aligned} A_m &= \sum_{i=1}^m \frac{(-1)^{i-1}}{2i-1} \\ B_m &= \sum_{i=1}^m \frac{(-1)^{i-1}}{2i}; \end{aligned}$$

then

$$\begin{aligned} J_{2n} &= (-1)^n(J_0 - A_n) \\ J_{2n+1} &= (-1)^n(J_1 - B_n). \end{aligned}$$

Now the $2N$-th partial sum of our series equals

$$\begin{aligned} &\sum_{n=1}^N \frac{J_{2n-1}}{2n-1} - \frac{J_{2n}}{2n} \\ &= \sum_{n=1}^N \frac{(-1)^{n-1}}{2n-1}(J_1 - B_{n-1}) - \frac{(-1)^n}{2n}(J_0 - A_n) \\ &= A_N(J_1 - B_{N-1}) + B_N(J_0 - A_N) + A_N B_N. \end{aligned}$$

As $N \to \infty$, $A_N \to J_0$ and $B_N \to J_1$, so the sum tends to $J_0 J_1 = \pi \log(2)/8$.

Remarks: The first two solutions are related by the fact that if $x = \tan(\theta)$, then $1 - x/(1+x) = \tan(\pi/4 - \theta)$. The strategy of the third solution (introducing a parameter then differentiating it) was a favorite of physics Nobelist (and Putnam Fellow) Richard Feynman. Noam Elkies notes that this integral is number 2.491#8 in Gradshteyn and Ryzhik, *Table of integrals, series, and products*. The *Mathematica* computer algebra system (version 5.2) successfully computes this integral, but we do not know how.

A6 **First solution:** The angle at a vertex P is acute if and only if all of the other points lie on an open semicircle. We first deduce from this that if there are any two acute angles at all, they must occur consecutively. Suppose the contrary; label the vertices $Q_1, \dots, Q_n$ in counterclockwise order (starting anywhere), and suppose that the angles at Q_1 and Q_i are acute for some i with $3 \leq i \leq n-1$. Then the open semicircle starting at Q_2 and proceeding counterclockwise must contain all of $Q_3, \dots, Q_n$, while the open semicircle starting at Q_i and proceeding counterclockwise must contain $Q_{i+1}, \dots, Q_n, Q_1, \dots, Q_{i-1}$. Thus two open semicircles cover the entire circle, contradiction.

It follows that if the polygon has at least one acute angle, then it has either one acute angle or two acute angles occurring consecutively. In particular, there is a unique pair of consecutive vertices Q_1, Q_2 in counterclockwise order for which $\angle Q_2$ is acute and $\angle Q_1$ is not acute. Then the remaining points all lie in the arc from the antipode of Q_1 to Q_1, but Q_2 cannot lie in the arc, and the remaining points cannot all lie in the arc from the antipode of Q_1 to the antipode of Q_2. Given the choice of Q_1, Q_2, let x be the measure of the counterclockwise arc from Q_1 to Q_2; then the probability that the other points fall into position is $2^{-n+2} - x^{n-2}$ if $x \leq 1/2$ and 0 otherwise.

Hence the probability that the polygon has at least one acute angle with a *given* choice of which two points will act as Q_1 and Q_2 is

$$\int_0^{1/2} (2^{-n+2} - x^{n-2})\,dx = \frac{n-2}{n-1} 2^{-n+1}.$$

Since there are $n(n-1)$ choices for which two points act as Q_1 and Q_2, the probability of at least one acute angle is $n(n-2)2^{-n+1}$.

Second solution: (by Calvin Lin) As in the first solution, we may compute the probability that for a particular one of the points Q_1, the angle at Q_1 is not acute but the following angle is, and then multiply by n. Imagine picking the points by first choosing Q_1, then picking $n-1$ *pairs* of antipodal points and then picking one member of each pair. Let $R_2, \dots, R_n$ be the points of the pairs which lie in the semicircle, taken in order away from Q_1, and let $S_2, \dots, S_n$ be the antipodes of these. Then to get the desired situation, we must choose from the pairs to end up with all but one of the S_i, and we cannot take R_n and the other S_i or else $\angle Q_1$ will be acute. That gives us $(n-2)$ good choices out of 2^{n-1}; since we could have chosen Q_1 to be any of the n points, the probability is again $n(n-2)2^{-n+1}$.

B1 Take $P(x,y) = (y-2x)(y-2x-1)$. To see that this works, first note that if $m = \lfloor a \rfloor$, then $2m$ is an integer less than or equal to $2a$, so $2m \leq \lfloor 2a \rfloor$. On the other hand, $m+1$ is an integer strictly greater than a, so $2m+2$ is an integer strictly greater than $2a$, so $\lfloor 2a \rfloor \leq 2m+1$.

B2 By the arithmetic-harmonic mean inequality or the Cauchy-Schwarz inequality,

$$(k_1 + \cdots + k_n)\left(\frac{1}{k_1} + \cdots + \frac{1}{k_n}\right) \geq n^2.$$

We must thus have $5n - 4 \geq n^2$, so $n \leq 4$. Without loss of generality, we may suppose that $k_1 \leq \cdots \leq k_n$.

If $n=1$, we must have $k_1 = 1$, which works. Note that hereafter we cannot have $k_1 = 1$.

If $n = 2$, we have $(k_1, k_2) \in \{(2,4), (3,3)\}$, neither of which work.

If $n = 3$, we have $k_1 + k_2 + k_3 = 11$, so $2 \leq k_1 \leq 3$. Hence $(k_1, k_2, k_3) \in \{(2,2,7), (2,3,6), (2,4,5), (3,3,5), (3,4,4)\}$, and only $(2,3,6)$ works.

If $n = 4$, we must have equality in the AM-HM inequality, which only happens when $k_1 = k_2 = k_3 = k_4 = 4$.

Hence the solutions are $n = 1$ and $k_1 = 1$, $n = 3$ and (k_1, k_2, k_3) is a permutation of $(2,3,6)$, and $n = 4$ and $(k_1, k_2, k_3, k_4) = (4,4,4,4)$.

Remark: In the cases $n = 2, 3$, Greg Kuperberg suggests the alternate approach of enumerating the solutions of $1/k_1 + \cdots + 1/k_n = 1$ with $k_1 \leq \cdots \leq k_n$. This is easily done by proceeding in lexicographic order: one obtains $(2,2)$ for $n=2$, and $(2,3,6), (2,4,4), (3,3,3)$ for $n = 3$, and only $(2,3,6)$ contributes to the final answer.

B3 **First solution:** The functions are precisely $f(x) = cx^d$ for $c, d > 0$ arbitrary except that we must take $c = 1$ in case $d = 1$. To see that these work, note that $f'(a/x) = dc(a/x)^{d-1}$ and $x/f(x) = 1/(cx^{d-1})$, so the given equation holds if and only if $dc^2a^{d-1} = 1$. If $d \neq 1$, we may solve for a no matter what c is; if $d = 1$, we must have $c = 1$. (Thanks to Brad Rodgers for pointing out the $d = 1$ restriction.)

To check that these are all solutions, put $b = \log(a)$ and $y = \log(a/x)$; rewrite the given equation as

$$f(e^{b-y})f'(e^y) = e^{b-y}.$$

Put

$$g(y) = \log f(e^y);$$

then the given equation rewrites as

$$g(b-y) + \log g'(y) + g(y) - y = b - y,$$

or

$$\log g'(y) = b - g(y) - g(b-y).$$

By the symmetry of the right side, we have $g'(b-y) = g'(y)$. Hence the function $g(y) + g(b-y)$ has zero derivative and so is constant, as then is $g'(y)$. From this we deduce that $f(x) = cx^d$ for some c, d, both necessarily positive since $f'(x) > 0$ for all x.

Second solution: (suggested by several people) Substitute a/x for x in the given equation:

$$f'(x) = \frac{a}{xf(a/x)}.$$

Differentiate:

$$f''(x) = -\frac{a}{x^2 f(a/x)} + \frac{a^2 f'(a/x)}{x^3 f(a/x)^2}.$$

Now substitute to eliminate evaluations at a/x:

$$f''(x) = -\frac{f'(x)}{x} + \frac{f'(x)^2}{f(x)}.$$

Clear denominators:

$$xf(x)f''(x) + f(x)f'(x) = xf'(x)^2.$$

Divide through by $f(x)^2$ and rearrange:

$$0 = \frac{f'(x)}{f(x)} + \frac{xf''(x)}{f(x)} - \frac{xf'(x)^2}{f(x)^2}.$$

The right side is the derivative of $xf'(x)/f(x)$, so that quantity is constant. That is, for some d,

$$\frac{f'(x)}{f(x)} = \frac{d}{x}.$$

Integrating yields $f(x) = cx^d$, as desired.

B4 **First solution:** Define $f(m,n,k)$ as the number of n-tuples $(x_1, x_2, \dots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \leq m$ and exactly k of $x_1, \dots, x_n$ are nonzero. To choose such a tuple, we may choose the k nonzero positions, the signs of those k numbers, and then an ordered k-tuple of positive integers with sum $\leq m$. There are $\binom{n}{k}$ options for the first choice, and 2^k for the second. As for the third, we have $\binom{m}{k}$ options by a "stars and bars" argument: depict the k-tuple by drawing a number of stars for each term, separated by bars, and adding stars at the end to get a total of m stars. Then each tuple corresponds to placing k bars, each in a different position behind one of the m fixed stars.

We conclude that

$$f(m,n,k) = 2^k \binom{m}{k} \binom{n}{k} = f(n,m,k);$$

summing over k gives $f(m,n) = f(n,m)$. (One may also extract easily a bijective interpretation of the equality.)

Second solution: (by Greg Kuperberg) It will be convenient to extend the definition of $f(m,n)$ to $m,n \geq 0$, in which case we have $f(0,m) = f(n,0) = 1$.

Let $S_{m,n}$ be the set of n-tuples $(x_1, \dots, x_n)$ of integers such that $|x_1| + \cdots + |x_n| \leq m$. Then elements of $S_{m,n}$ can be classified into three types. Tuples with $|x_1| + \cdots + |x_n| < m$ also belong to $S_{m-1,n}$. Tuples with $|x_1| + \cdots + |x_n| = m$ and $x_n \geq 0$ correspond to elements of $S_{m,n-1}$ by dropping x_n. Tuples with $|x_1| + \cdots + |x_n| = m$ and $x_n < 0$ correspond to elements of $S_{m-1,n-1}$ by dropping x_n. It follows that

$$\begin{aligned} &f(m,n) \\ &= f(m-1,n) + f(m,n-1) + f(m-1,n-1), \end{aligned}$$

so f satisfies a symmetric recurrence with symmetric boundary conditions $f(0,m) = f(n,0) = 1$. Hence f is symmetric.

Third solution: (by Greg Martin) As in the second solution, it is convenient to allow $f(m,0) = f(0,n) = 1$. Define the generating function

$$G(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m,n) x^m y^n.$$

As equalities of formal power series (or convergent series on, say, the region $|x|, |y| < \frac{1}{3}$), we have

$$\begin{aligned} G(x,y) &= \sum_{m \geq 0} \sum_{n \geq 0} x^m y^n \sum_{\substack{k_1, \dots, k_n \in \mathbb{Z} \\ |k_1| + \cdots + |k_n| \leq m}} 1 \\ &= \sum_{n \geq 0} y^n \sum_{k_1, \dots, k_n \in \mathbb{Z}} \sum_{m \geq |k_1| + \cdots + |k_n|} x^m \\ &= \sum_{n \geq 0} y^n \sum_{k_1, \dots, k_n \in \mathbb{Z}} \frac{x^{|k_1| + \cdots + |k_n|}}{1-x} \\ &= \frac{1}{1-x} \sum_{n \geq 0} y^n \left(\sum_{k \in \mathbb{Z}} x^{|k|} \right)^n \\ &= \frac{1}{1-x} \sum_{n \geq 0} y^n \left(\frac{1+x}{1-x} \right)^n \\ &= \frac{1}{1-x} \cdot \frac{1}{1 - y(1+x)/(1-x)} \\ &= \frac{1}{1-x-y-xy}. \end{aligned}$$

Since $G(x,y) = G(y,x)$, it follows that $f(m,n) = f(n,m)$ for all $m,n \geq 0$.

B5 **First solution:** Put $Q = x_1^2 + \cdots + x_n^2$. Since Q is homogeneous, P is divisible by Q if and only if each of the homogeneous components of P is divisible by Q. It is thus sufficient to solve the problem in case P itself is homogeneous, say of degree d.

Suppose that we have a factorization $P = Q^m R$ for some $m > 0$, where R is homogeneous of degree d and not divisible by Q; note that the homogeneity implies that

$$\sum_{i=1}^{n} x_i \frac{\partial R}{\partial x_i} = dR.$$

Write ∇^2 as shorthand for $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$; then

$$\begin{aligned} 0 &= \nabla^2 P \\ &= 2mnQ^{m-1}R + Q^m \nabla^2 R + 2 \sum_{i=1}^{n} 2mx_i Q^{m-1} \frac{\partial R}{\partial x_i} \\ &= Q^m \nabla^2 R + (2mn + 4md) Q^{m-1} R. \end{aligned}$$

Since $m > 0$, this forces R to be divisible by Q, contradiction.

Second solution: (by Noam Elkies) Retain notation as in the first solution. Let P_d be the set of homogeneous polynomials of degree d, and let H_d be the subset of P_d of polynomials killed by ∇^2, which has dimension $\geq \dim(P_d) - \dim(P_{d-2})$; the given problem amounts to showing that this inequality is actually an equality.

Consider the operator $Q\nabla^2$ (i.e., apply ∇^2 then multiply by Q) on P_d; its zero eigenspace is precisely H_d. By the calculation from the first solution, if $R \in P_d$, then

$$\nabla^2(QR) - Q\nabla^2 R = (2n+4d)R.$$

Consequently, $Q^j H_{d-2j}$ is contained in the eigenspace of $Q\nabla^2$ on P_d of eigenvalue

$$(2n + 4(d-2j)) + \cdots + (2n + 4(d-2)).$$

In particular, the $Q^j H^{d-2j}$ lie in distinct eigenspaces, so are linearly independent within P_d. But by dimension counting, their total dimension is at least that of P_d. Hence they exhaust P_d, and the zero eigenspace cannot have dimension greater than $\dim(P_d) - \dim(P_{d-2})$, as desired.

Third solution: (by Richard Stanley) Write $x = (x_1, \dots, x_n)$ and $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. Suppose that $P(x) = Q(x)(x_1^2 + \cdots + x_n^2)$. Then

$$P(\nabla)P(x) = Q(\nabla)(\nabla^2)P(x) = 0.$$

On the other hand, if $P(x) = \sum_\alpha c_\alpha x^\alpha$ (where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$), then the constant term of $P(\nabla)P(x)$ is seen to be $\sum_\alpha c_\alpha^2$. Hence $c_\alpha = 0$ for all α.

Remarks: The first two solutions apply directly over any field of characteristic zero. (The result fails in characteristic $p > 0$ because we may take $P = (x_1^2 + \cdots + x_n^2)^p = x_1^{2p} + \cdots + x_n^{2p}$.) The third solution can be extended to complex coefficients by replacing $P(\nabla)$ by its complex conjugate, and again the result may be deduced for any field of characteristic zero. Stanley also suggests Section 5 of the arXiv e-print `math.CO/0502363` for some algebraic background for this problem.

B6 **First solution:** Let I be the identity matrix, and let J_x be the matrix with x's on the diagonal and 1's elsewhere. Note that $J_x - (x-1)I$, being the all 1's matrix, has rank 1 and trace n, so has $n-1$ eigenvalues equal to 0 and one equal to n. Hence J_x has $n-1$ eigenvalues equal to $x-1$ and one equal to $x+n-1$, implying

$$\det J_x = (x+n-1)(x-1)^{n-1}.$$

On the other hand, we may expand the determinant as a sum indexed by permutations, in which case we get

$$\det J_x = \sum_{\pi \in S_n} \mathrm{sgn}(\pi) x^{\nu(\pi)}.$$

Integrating both sides from 0 to 1 (and substituting $y = 1-x$) yields

$$\begin{aligned}
\sum_{\pi \in S_n} \frac{\mathrm{sgn}(\pi)}{\nu(\pi)+1} &= \int_0^1 (x+n-1)(x-1)^{n-1}\,dx \\
&= \int_0^1 (-1)^{n+1}(n-y)y^{n-1}\,dy \\
&= (-1)^{n+1}\frac{n}{n+1},
\end{aligned}$$

as desired.

Second solution: We start by recalling a form of the principle of inclusion-exclusion: if f is a function on the power set of $\{1, \dots, n\}$, then

$$f(S) = \sum_{T \supseteq S} (-1)^{|T|-|S|} \sum_{U \supseteq T} f(U).$$

In this case we take $f(S)$ to be the sum of $\sigma(\pi)$ over all permutations π whose fixed points are exactly S. Then $\sum_{U \supseteq T} f(U) = 1$ if $|T| \geq n-1$ and 0 otherwise (since a permutation group on 2 or more symbols has as many even and odd permutations), so

$$f(S) = (-1)^{n-|S|}(1-n+|S|).$$

The desired sum can thus be written, by grouping over fixed point sets, as

$$\begin{aligned}
&\sum_{i=0}^n \binom{n}{i}(-1)^{n-i}\frac{1-n+i}{i+1} \\
&= \sum_{i=0}^n (-1)^{n-i}\binom{n}{i} - \sum_{i=0}^n (-1)^{n-i}\frac{n}{i+1}\binom{n}{i} \\
&= 0 - \sum_{i=0}^n (-1)^{n-i}\frac{n}{n+1}\binom{n+1}{i+1} \\
&= (-1)^{n+1}\frac{n}{n+1}.
\end{aligned}$$

Third solution: (by Richard Stanley) The *cycle indicator* of the symmetric group S_n is defined by

$$Z_n(x_1, \dots, x_n) = \sum_{\pi \in S_n} x_1^{c_1(\pi)} \cdots x_n^{c_n(\pi)},$$

where $c_i(\pi)$ is the number of cycles of π of length i. Put

$$F_n = \sum_{\pi \in S_n} \sigma(\pi) x^{\nu(\pi)} = Z_n(x, -1, 1, -1, 1, \dots)$$

and

$$f(n) = \sum_{\pi \in S_n} \frac{\sigma(\pi)}{\nu(\pi)+1} = \int_0^1 F_n(x)\,dx.$$

A standard argument in enumerative combinatorics (the Exponential Formula) gives

$$\sum_{n=0}^{\infty} Z_n(x_1,\dots,x_n)\frac{t^n}{n!} = \exp\sum_{k=1}^{\infty} x_k\frac{t^k}{k},$$

yielding

$$\begin{aligned}\sum_{n=0}^{\infty} f(n)\frac{t^n}{n!} &= \int_0^1 \exp\left(xt - \frac{t^2}{2} + \frac{t^3}{3} - \cdots\right)\,dx \\ &= \int_0^1 e^{(x-1)t+\log(1+t)}\,dx \\ &= \int_0^1 (1+t)e^{(x-1)t}\,dx \\ &= \frac{1}{t}(1-e^{-t})(1+t).\end{aligned}$$

Expanding the right side as a Taylor series and comparing coefficients yields the desired result.

Fourth solution (sketch): (by David Savitt) We prove the identity of rational functions

$$\sum_{\pi\in S_n}\frac{\sigma(\pi)}{\nu(\pi)+x} = \frac{(-1)^{n+1}n!(x+n-1)}{x(x+1)\cdots(x+n)}$$

by induction on n, which for $x=1$ implies the desired result. (This can also be deduced as in the other solutions, but in this argument it is necessary to formulate the strong induction hypothesis.)

Let $R(n,x)$ be the right hand side of the above equation. It is easy to verify that

$$\begin{aligned}R(x,n) &= R(x+1,n-1) + (n-1)!\frac{(-1)^{n+1}}{x} \\ &\quad + \sum_{l=2}^{n-1}(-1)^{l-1}\frac{(n-1)!}{(n-l)!}R(x,n-l),\end{aligned}$$

since the sum telescopes. To prove the desired equality, it suffices to show that the left hand side satisfies the same recurrence. This follows because we can classify each $\pi\in S_n$ as either fixing n, being an n-cycle, or having n in an l-cycle for one of $l=2,\dots,n-1$; writing the sum over these classes gives the desired recurrence.