Real Analysis Exam: Part I (Spring 2001)

Do all five problems.

1. Let X be a metric space.
   (a) Suppose X is separable. Show that if G is an open cover of X, then G has a countable subcover.
   (b) (Converse of (a)). Suppose every open cover of X has a countable subcover. Prove that X is separable.

2. Let T be the set of real numbers x with the following property. For every $k < \infty$, there exist integers $N > k$ and a such that

$$\left| x - \frac{a}{10^N} \right| \leq \frac{1}{20^N}.$$

   (a) Prove that T is uncountable.
   (b) What is the Lebesgue measure of T?

3. Let X be a compact metric space. Let $C(X)$ be the space of all continuous real-valued functions on X. Suppose $F : C(X) \to \mathbf{R}$ is a continuous map such that

$$\begin{aligned} F(u+v) &= F(u) + F(v), \\ F(uv) &= F(u)F(v), \\ F(1) &= 1. \end{aligned}$$

Prove that there is an $x \in X$ such that $F(u) = u(x)$ for every $u \in C(X)$.

4. Prove:
   (a) The continuous image of a connected set is connected.
   (b) If X is compact, Y is Hausdorff, and $f : X \to Y$ is one-to-one and continuous, then f^{-1} is continuous.
   (c) The product of two compact spaces is compact.

5. Let S be a subspace of $C[0,1]$. Suppose S is closed as a subspace of $\mathcal{L}^2[0,1]$. Prove:
   (a) S is a closed subspace of $C[0,1]$.
   (b) For $f \in S$, $\|f\|_2 \leq \|f\|_\infty \leq M\|f\|_2$.
   (c) For every $y \in [0,1]$, there is a $K_y \in L_2[0,1]$ such that

$$f(y) = \int_0^1 K_y(x) f(x)\, dx$$

   for every $f \in S$.

Real Analysis Exam: Part II (Spring 2001)

1. Suppose $f_n(x)$ is a sequence of non-decreasing functions on $[0,1]$ that converge pointwise to a continuous function $g(x)$. Prove that the convergence is actually uniform on $[0,1]$.

2. Let A and B be closed linear subspaces of a Hilbert space H such that

$$\inf\{\|x-y\| : x \in A, y \in B, \|x\| = \|y\| = 1\} > 0.$$

Prove that $A + B = \{x + y : x \in A, y \in B\}$ is complete.

3. Let A be the space of Fourier transforms of functions in $\mathcal{L}^1(\mathbf{R})$:

$$A = \{\hat{f} : f \in L^1(\mathbf{R})\}.$$

Let $C_0(\mathbf{R})$ be the space of continuous functions f on $\mathbf{R}$ such that

$$\lim_{|x|\to\infty} f(x) = 0.$$

Prove

(a) $A \neq C_0(\mathbf{R})$. (Hint: use the open mapping theorem.)
(b) A is a dense subset of $C_0(\mathbf{R})$.

4. Let Q be the unit square in $\mathbf{R}^2$. Consider functions $f_n \in L^1(Q)$ such that (as $n \to \infty$)

$$f_n \to f \text{ almost everywhere in } Q$$

and

$$\int_Q |f_n| \to \int_Q |f| < \infty.$$

(a) Prove that $\int_A |f_n| \to \int_A |f|$ for every measurable subset A of Q.
(b) Prove that $f_n \to f$ in L^1.

5. Let f and g be continuous periodic functions with period 1. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)\,dx = \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx.$$