VMO Pre-test 7 Solutions

Sayan Mukherjee

January 9, 2011

1 Day 1

Problem 1. Let a, b, c be positive real numbers satisfying $a+b+c=6$ and $a^2+b^2+c^2=14$. Prove that

$$(a-b)(b-c)(c-a) \leq 2.$$

Solution
WLOG assume $c \geq b \geq a$. The given condition leads to

$$6=(a-b)^2+(b-c)^2+(c-a)^2.$$

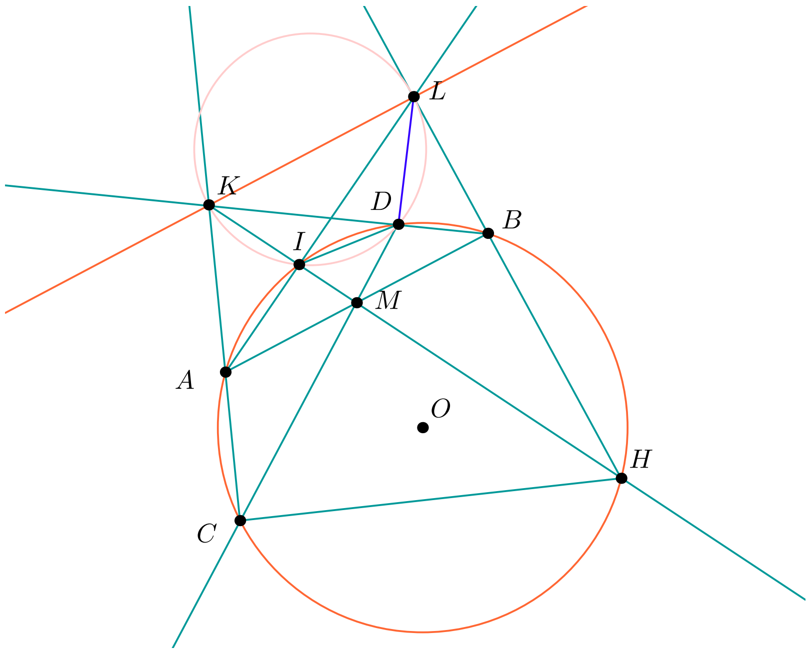
Now, applying the Cauchy-Schwarz inequality we have $(b-a)^2+(c-b)^2 \geq \frac{1}{2}(c-a)^2$; and therefore $6 \geq \frac{3}{2}(c-a)^2 \implies c-a \leq 2$.
Now, we have

$$(c-a)[4(b-a)(c-b)] \leq (c-a) \cdot [b-a+c-b]^2=(c-a)^3 \leq 8;$$

And therefore $(a-b)(b-c)(c-a) \leq 2$.
We are done. Equality holds iff $(a, b, c) \sim (1,2,3)$ and its relevant permutations. $\square$

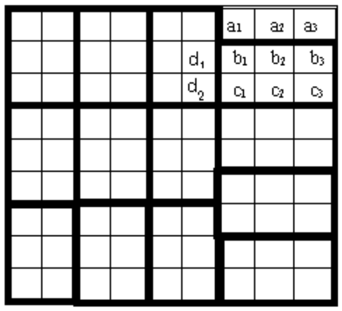
Problem 2. Let circle (O) have two distinct points A, B on its circumference. Let M be the midpoint of AB. A chord passes through M such that $AC \cap BD=K; KM \cap (O)=I, H; AI \cap BH=L$. Show that K, I, D, L are concyclic provided that I is closer to K than H.

Solution
Consider the quadrilateral $AIBH$. Its diagonals meet at M and $AI \cap BH=L$. So it follows that L lies on the polar of M with respect to (O). Similarly we have, considering $ACBD$ that $AB \cap CD=M$ and $AC \cap BD=K$, so again, K must lie on the polar of M with respect to (O). Therefore we deduce that KL is the polar of M wrt (O).
And so $OM \perp KL$ is forced. But we already know that $OM \perp AB$. Using these we see that $AB \parallel KL$.
Therefore after a simple angle-chasing we obtain our desired result. $\square$



Problem 3. Consider a 9×11 rectangle which is divided into 99 unit squares. Colour each square of the rectangle using white or back, such that in an arbitrary 2×3 or 3×2 sub-rectangle, we would have exactly two squares black in colour. Show that there are exactly 33 black squares.

Solution
Firstly we observe that for a 9×2 rectangular division we must have exactly 6 squares coloured black.
Now we are left with a 9×9 square, and it is sufficient to show that this must contain exactly 27 black squares.
Now we divide this 9×9 square as in the diagram, so we will be done, considering each sub-rectangles.
After colouring as shown in the figure, we see that the top right corners have been left out. We have 13 3×2 or 2×3 covering the rest of the board so that there are exactly 26 black squares in the rest of the board. Denote the top-right corner squares as $a_1, a_2, a_3; b_1, b_2, b_3$; and c_1, c_2, c_3 as shown in the diagram. We will show that there will be exactly one black square amongst a_1, a_2, a_3.



CASE I. *All of* a_1, a_2, a_3 *are White.*
In this case we note that considering $(a_1a_2a_3b_1b_2b_3)$, two amongst b_1, b_2, b_3 must be black in colour. Again if we consider $(b_1b_2b_3c_1c_2c_3)$, it is forced that c_1, c_2, c_3 are all white.
Therefore if we consider $(a_1a_2b_1b_2c_1c_2)$ we should have b_1, b_2 to be black. Similarly b_2, b_3 are also black. Then this leads to an obvious contradiction because all three of b_i can never be black.

CASE II.*Excatly two of* a_1, a_2, a_3 *are black.*
In this case, considering $(a_1a_2a_3b_1b_2b_3$, we infer that b_1, b_2, b_3 are all white. Also if a_1 and a_2; or a_2 and a_3 were to be black, then we would have had c_1 and c_2; or c_2 and c_3 to be while, which would have led to a contradiction from $(b_1b_2b_3c_1c_2c_3)$.
Therefore a_1 and a_3 have to be black and a_2 is forced to be white. Therefore considering $(a_1a_2b_1b_2c_1c_2)$ and $(a_2a_3b_2b_3c_2c_3)$ we see that c_1, c_3 have to be back, and hence c_2 must be white. Then the three-member column to the left of $(a_1b_1c_1)$ also has to be white. Let the two lowermost squares be named d_1, d_2. So we have d_1, d_2, b_1, b_2, c_3 to be white so that from $(d_1b_1b_2d_2c_1c_2)$ we get only one black square; a contradiction.

Hence we are done. □

2 Day 2

4. Prove that there do not exist any polynomials $P(x) \in \mathbb{R}[x]$ with degree 2010 and which satisfies

$$P(x)^2 - 1 = P\left(x^2 + 1\right) \quad \forall x \in \mathbb{R}.$$

Solution
Replacing x by $-x$ we see that $P(x)^2 = P(-x)^2$, so that P can be either odd or even. Since P has degree 2010, therefore P must be an even polynomial. Therefore we let

$$P(x) = a_{1005}x^{2010} + a_{1004}x^{2008} + \cdots + a_0.$$

Now we let $x^2 + 1 = z$; so that the given relation leads to

$$P(z) = P\left(\sqrt{z-1}\right)^2 - 1 \quad \forall z \geq 1;$$

Or,

$$a_{1005}z^{2010} + a_{1004}z^{2008} + \cdots + a_0 = \left(a_{1005}(z-1)^{1005} + \cdots + a_0\right)^2 - 1.$$

Comparing the coefficient of z^{2010} in both sides we obtain,

$$a_{1005} = a_{1005}^2 \overset{a_{1005}\neq 0}{\Longrightarrow} a_{1005} = 1.$$

Again, comparing the coefficient of z^{2008} in both sides we get,

$$a_{1004} = (1005a_{1005} + a_{1004})^2\,; \overset{a_{1005}=1}{\Longrightarrow} a_{1004}^2 + 2009a_{1004} + 1005^2 = 0.$$

Obviously the discriminant of this quadratic is $2009^2 - 2010^2$, so that this does not have any real solution. So no such polynomial exists, and we finish our proof here. $\square$

5. Let $\triangle ABC$ be a scalene triangle with $\angle A = 60°$. Let BD and CE be the two internal angle bisectors from B and C, respectively. The circle with centre B and radius BD meets AB at F; and the circle with centre C and radius CE intersects AC at G. Show that we must have $GF \parallel BC$.

Solution

We rephrase the problem into the following equivalent form:

Let $\triangle ABC$ be a scalene triangle with $\angle A = 60°$. Let BD and CE be the two internal angle bisectors from B and C, respectively. Let F be a point on AB with $BD = BF$; and the line through F, parallel to BC intersects AC at G. Then we have $CE = CG$.

Refer to the diagram. We denote, by a, b, c the sides BC, CA, AB respectively of the triangle ABC. Then we know that

$DC = \dfrac{ab}{c+a}$, and $BE = \dfrac{ca}{a+b}$. Now using the Thale's theorem we have $\dfrac{AF}{FB} = \dfrac{AG}{GC}$, leading to $GC = \dfrac{AG}{AF} \cdot FB \overset{FB=BD}{=} \dfrac{AG}{AF} \cdot BD$. Hence it is sufficient to show that

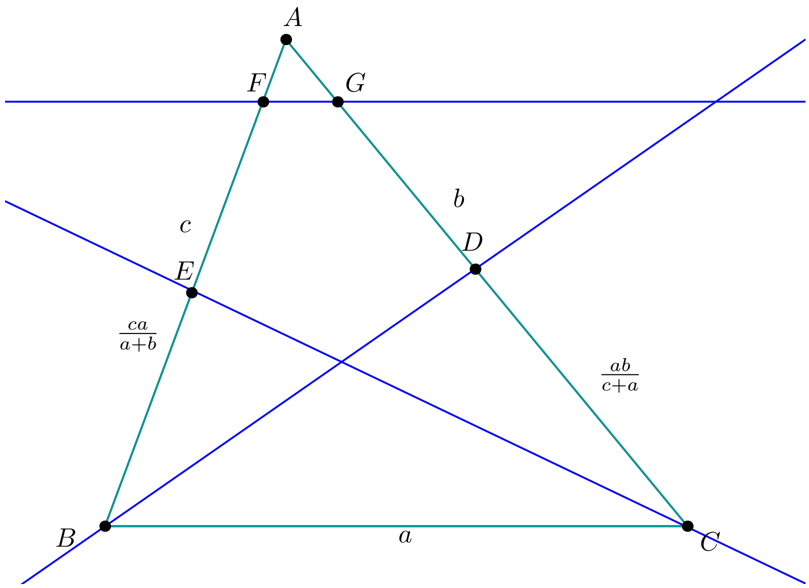
$$\frac{AF}{AG} = \frac{BD}{CF} \overset{\triangle AFG \sim \triangle ABC}{\Longleftrightarrow} \frac{AB}{AC} = \frac{BD}{CE}.$$

Now using the Sine rule in $\triangle BEC$, we have $\dfrac{BD}{\sin C} = \dfrac{ab}{(c+a)\sin\dfrac{B}{2}}$.

Again, using the Sine rule in $\triangle BDC$ we obtain $\dfrac{CE}{\sin B} = \dfrac{ca}{(a+b)\sin\dfrac{C}{2}}$.

Dividing these relations one gets

$$\begin{aligned}\frac{BD}{CE} &= \frac{\sin C}{\sin B} \cdot \frac{ab}{(c+a)\sin\dfrac{B}{2}} \cdot \frac{(a+b)\sin\dfrac{C}{2}}{ca} \\ &= \frac{a+b}{c+a} \cdot \frac{\sin\dfrac{C}{2}}{\sin\dfrac{B}{2}}.\end{aligned}$$



Again, using the sine rule in $\triangle AEC$ we have, $CE = \dfrac{\sqrt{3} \cdot b}{2 \sin\left(\dfrac{C}{2} + 60^\circ\right)}$.

Hence it is sufficient to show that

$$\sin\left(\frac{C}{2} + 60^\circ\right) = \sin\left(\frac{B}{2} + 60^\circ\right);$$

Which is obvious since $\left(\dfrac{B}{2} + 60^\circ\right) + \left(\dfrac{C}{2} + 60^\circ\right) = 120^\circ + \dfrac{180^\circ - 60^\circ}{2} = 180^\circ$.

Therefore we are done. $\square$

6. Prove that there exist three numbers $a, b, c \in \mathbb{N}_{>1}$ **satisfying** $b|(a^2-1), c|(b^2-1), a|(c^2-1)$ **and** $a+b+c > 2011$.

Solution

Let us assume a to be an odd number $2m+1$ sufficiently large. Let $b = 4m$, and $c = 4m+1$. Then we automatically have the following consequences.

$$\begin{cases} \bullet\ 4m|(2m-1)^2 - 1 = 2m(2m-2); \\ \bullet\ 4m+1|16m^2 - 1 = (4m+1)(4m-1); \\ \bullet\ 2m+1|(4m+1)^2 - 1 = 4m(4m+2). \end{cases}$$

We can choose m to be sufficiently large such that $a+b+c$ exceeds 2011. Hence we are done. $\square$